Problems

For problems 1 and 2, X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} 3x^2, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

1. Let $Y = \sqrt{X}$. What is f_Y?
2. Let $Z = X^2$. What is f_Z?

For problems 3 through 7, let $X \sim \text{Unif}(0,1)$ and $Y \sim \text{Unif}(-1,1)$ be independent random variables.

3. What is the joint distribution $f_{X,Y}(x,y)$?
4. What is the expected value $E[X]$?
5. What is the expected value of $\sqrt{X}$, $E[\sqrt{X}]$?
6. What is the expected value of XY, $E[XY]$?
7. Let $U = \frac{X}{2}$ and $V = X + Y$. What is the joint distribution $f_{U,Y}$?

For problems 8 and 9, let $X \sim \text{Exp}(\lambda = 1)$.

8. What is the moment generating function $M_X(t)$ of X?
9. Use the moment generating function of X to find the variance $Var(X) = E[X^2] - E[X]^2$.
10. Suppose that I'm watching a cat parade where the average number of rainbow cats that pass by every minute is 15. If I think that the number of rainbow cats that pass by in a given minute can be modeled by a Poisson random variable, what is the probability that less than 3 rainbow cats will pass me during the coming minute. (Note that the expected value of a Poisson(λ) distribution is λ.)
11. Every time I commit a math heresy, there is a 5% chance that a butterfly dies as a result. If I commit 20 math heresies, what is the probability that I will cause the death of more than one butterfly?

For problems 12 and 13, let X and Y be jointly distributed with joint distribution function

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & x,y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

12. What is the marginal distribution function f_X?
13. What is the expected value of $X + Y$?
14. Calculate the probability $P\{X > Y\}$.
15. Show that the expected value of a binomial distribution with $n = 3$ and $p = 2/3$ is 2.

Solutions

1. First, we find F_Y, then we differentiate with respect to y to find f_y:

$$F_Y(y) = P\{Y \leq y\} = P\{X \leq y^2\} = F_X(y^2)$$

$$f_y(y) = \frac{d}{dy}F_y(y) = \frac{d}{dy}F_x(y^2) = f_X(y^2)(2y)$$

Plugging into f_X, we get

$$f_Y(y) = \begin{cases} 6y^5, & y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

2. The strategy is the same.

$$F_Y(y) = P\{Y \leq y\} = P\{X \leq \sqrt{y}\} = F_X(\sqrt{y})$$

(Note that we don't have to consider the case where $X \geq -\sqrt{y}$ because X is always positive.)

$$f_y(y) = \frac{d}{dy}F_y(y) = \frac{d}{dy}F_x(\sqrt{y}) = \frac{f_X(\sqrt{y})}{2\sqrt{y}}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}\sqrt{y}, & y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

3. The distributions of X and Y are

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{2}, & y \in [-1,1] \\ 0, & \text{otherwise.} \end{cases}$$

Since X and Y are independent, $f_{X,Y}$ is simply the product of f_X and f_Y:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{2}, & x \in [0,1] \text{ and } y \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

4. If it's not apparent that the expected value of a uniform random variable is simply the center of the interval its distributed on, we can use the standard formula for expected value:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 xdx = \frac{1}{2}$$

5. Remember that for a continuous function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

In this case,

$$E[\sqrt{X}] = \int_{-\infty}^{\infty} \sqrt{x} f_X(x) dx = \int_0^1 \sqrt{x} dx = \frac{2}{3}$$

6. For a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$,

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy dx$$

so in this case, using the answer from question 3,

$$E[XY] = \int_0^1 \int_{-1}^1 \frac{1}{2} xy \; dy dx = 0$$

Note that since X and Y are independent, you could also solve this by computing the expected value for each and multiplying them together ($E[XY] = E[X]E[Y]$).

7. First, put X and Y in terms of U and V:

$$X = 2U, \quad Y = V - 2U$$

The Jacobian for the transformation from (U, V) to (X, Y) is

$$\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$$

and its determinant is therefore 2. Applying the change of variables formula and using the answer from problem 3,

$$f_{U,V}(u,v) = f_{X,Y}(2u, v - 2u) = \begin{cases} 1, & u \in [0, \frac{1}{2}] \text{ and } v - 2u \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

8. The p.d.f. of X is

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

So the moment generating function is

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} e^{-x} = \frac{1}{1-t}$$

(note that we must have $t < 1$, otherwise the integral diverges.)

9. $M_X'(t) = \frac{1}{(1-t)^2}$ and $M_X''(t) = \frac{2}{(1-t)^3}$, so $E[X] = M_X'(0) = 1$, $E[X^2] = M_X''(0) = 2$, and therefore $Var(X) = 2 - 1 = 1$.

10. Let's model number of passing rainbow cats with a random variable $X \sim$ Poisson(15). The probability function of X is

$$f_X(x) = P\{X = x\} = \begin{cases} \frac{e^{-15}15^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

We want to find $P\{X \in \{0, 1, 2\}\}$. This is just

$$\frac{e^{-15}15^0}{0!} + \frac{e^{-15}15^1}{1!} + \frac{e^{-15}15^2}{2!}$$

11. The number of butterflies that die can be modeled with a binomial distribution. The probability that 0 or 1 butterflies die is

$$\binom{20}{0}(0.05^0)(0.95^{20}) + \binom{20}{1}(0.05^1)(0.95^19)$$

the probability that more than one butterfly dies is just 1 minus that.

12. To find f_X, we just integrate $f_{X,Y}$ over all y. For $x \in [0, 1]$,

$$f_X(x) = \int_0^1 4xy \; dy = 2x$$

So

$$f_X(x) = \begin{cases} 2x, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

13. Similar to problem 6,

$$E[X + Y] = \int_0^1 \int_0^1 (x + y)(4xy) dx dy = \frac{4}{3}$$

14. We just choose the appropriate limits of integration and go for it:

$$P\{X > Y\} = \int_0^1 \int_0^x 4xy \; dy dx = \frac{1}{2}$$

15. In general, the expected value of a discrete random variable is

$$\sum_x x f_X(x)$$

Here, we have

$$
\begin{aligned}
E[X] = 0 \times \binom{3}{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{3} + 1 \times \binom{3}{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{2} + \\
2 \times \binom{3}{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{1} + 3 \times \binom{3}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{0} \\
= (1)(3)\frac{2}{27} + (2)(3)\frac{4}{27} + (3)(1)\frac{8}{27} \\
= \frac{2}{9} + \frac{8}{9} + \frac{8}{9} = \frac{18}{9} = 2
\end{aligned}
$$