Problems

Functions of random variables

Let X be a standard normal random variable. The probability density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

1. Let $Y = X^2$. What is the probability density function f_Y of Y?
2. Let $Z = X^3$. What is the probability density function f_Z of Z?
3. Let $W = |X|$. What is the probability density function f_W of W?

Jointly distributed random variables

Let X and Y be independently distributed random variables, both uniformly distributed on the interval $[-1, 1]$.

1. What is $P\{X > Y > 0\}$?
2. Let $U = XY$, $V = e^X$. What is the joint distribution function $f_{U,V}$?
3. Now let $U = X + Y$. What is the probability density function f_U?

Solutions

Functions of random variables

1. First, find the cumulative density function F_y:

$$\begin{aligned} F_Y(y) = P\{Y \leq y\} = P\{X^2 \leq y\} &= P\{-\sqrt{y} \leq X \leq \sqrt{y}\} \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

   Now, differentiate $F_y(y)$ with respect to y to get the p.d.f. $f_Y(y)$:

$$\begin{aligned} f_Y(y) = \frac{d}{dy}F_y(y) &= \frac{d}{dy}F_X(\sqrt{y}) - \frac{d}{dy}F_X(-\sqrt{y}) \\ &= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \\ \text{(plug into } f_X\text{)} \qquad &= \frac{e^{-(\sqrt{y})^2/2} + e^{-(-\sqrt{y})^2/2}}{2\sqrt{2\pi y}} \\ &= \frac{e^{-y/2}}{\sqrt{2\pi y}} \end{aligned}$$

   Note that Y can only assume *positive* values, since it is the square of X. Thus the correct answer is technically

$$f_Y(y) = \begin{cases} \frac{e^{-y/2}}{\sqrt{2\pi y}}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

2. The strategy here is the same as before, although this time things are a bit simpler since the cubic function is one-to-one:

$$F_Z(z) = P\{Z \leq z\} = P\{X^3 \leq z\} = P\{X \leq z^{1/3}\} = F_X(z^{1/3})$$

$$f_Z(z) = \frac{d}{dz}F_X(z^{1/3}) = \frac{f_X(z^{1/3})}{3z^{2/3}} = \frac{e^{-(z^{2/3})/2}}{3\sqrt{2\pi}z^{2/3}}$$

3. The only real trick here is to remember that W can only take positive values (like Y in part 1). For $w \geq 0$:

$$\begin{aligned} F_W(w) = P\{W \leq w\} = P\{|X| \leq w\} &= P\{-w \leq X \leq w\} \\ &= F_X(w) - F_X(-w) \end{aligned}$$

$$f_W(w) = f_X(w) + f_X(-w) = \frac{2e^{w^2/2}}{\sqrt{2\pi}}$$

   The complete answer is

$$f_W(w) = \begin{cases} \frac{2e^{w^2/2}}{\sqrt{2\pi}} & w \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Jointly distributed random variables

1. The probability density functions for X and Y are

$$f_X(x) = \begin{cases} 1/2, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} 1/2, & y \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

   Therefore (keeping in mind that X and Y are independent)

$$\begin{aligned} P\{X > Y > 0\} &= \int_0^1 \int_0^x f_{X,Y}(x,y)dydx \\ &= \int_0^1 \int_0^x f_X(x)f_Y(y)dydx \\ &= \int_0^1 \int_0^x \frac{1}{4}dydx \\ &= \int_0^1 \frac{1}{4}xdx = \left[\frac{1}{8}x^2\right]_0^1 = \frac{1}{8} \end{aligned}$$

   This problem is simple enough that you could probably just solve it by drawing a picture, but it's good to practice the math for more complicated cases where that won't work.

2. We need to start by writing X and Y in terms of U and V:

$$X = \ln V$$

$$Y = \frac{U}{\ln V}$$

   Next we find the Jacobian matrix for the transformation from (u,v) to (x,y):

$$J = \begin{pmatrix} 0 & \frac{1}{v} \\ \frac{1}{\ln v} & -\frac{1}{v(\ln v)^2} \end{pmatrix}$$

   The absolute value of the determinant of J is $1/(v \ln v)$ (note that V is always positive). Therefore, keeping in mind that X and Y are independent,

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}\left(\ln v, \frac{u}{\ln v}\right)\frac{1}{v\ln v} \\ &= f_X(\ln v)\, f_Y\left(\frac{u}{\ln v}\right)\frac{1}{v\ln v} \\ &= \begin{cases} \frac{1}{4v\ln v}, & \ln v \in [-1,1] \text{ and } u/\ln v \in [-1,1] \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

3. One way to solve this is introduce another random variable V, which we simply set equal to one of X or Y. Let's say $V = X$. Then, we can start by finding the joint distribution $f_{U,V}$ like in the previous problem:

$$X = V$$

$$Y = U - V$$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \det(J) = -1$$

$$\begin{aligned} f_{U,V}(u,v) = f_{X,Y}(v, u-v) = f_X(v) f_Y(u-v) \\ = \begin{cases} 1/4, & v \in [-1,1] \text{ and } u - v \in [-1,1] \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

We can then find f_U by integrating over all possible values of V. We have to address multiple cases depending on what u is. If $u \in [-1, 1]$, then

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v) dv \\ &= \int_{-1}^{1} \frac{1}{4} dv = \frac{1}{8} \end{aligned}$$

If $u \in [-2, -1)$, then

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v) dv \\ &= \int_{-1}^{u+1} \frac{1}{4} dv = \frac{1}{4}(u+2) \end{aligned}$$

Finally, if $u \in (1, 2]$, then

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v) dv \\ &= \int_{u-1}^{1} \frac{1}{4} dv = \frac{1}{4}(2-u) \end{aligned}$$