MATH381 Assignment 4

Ben Vickers

Due: 11:55 PM, Thursday 3 October 2024

Q1. Use Cauchy's integral formula and its generalisation (from Taylor's Theorem for open disks) to evaluate the following integrals.

(i)

$$\int_{C(0;1)} \frac{\sin(z)}{z^n}\, dz \text{ for each } n = 1, 2, 3 \ldots$$

Notice that $\sin(z)$ is holomorphic on all of $\mathbb{C} \ni \{z_0 = 0\}$.

Now, because $\forall R \in (0, \infty]\,, D(0; R) \subseteq \mathbb{C}$ we can apply **Theorem 10.8** :

$$\implies \sin(z) = c_0 + \sum_{n=1}^{\infty} c_n (z-0)^n \quad \forall z \in D(0; R)$$

$$\text{With coefficients } c_{n-1} = \frac{f^{(n-1)}(z_0)}{(n-1)!} \quad \text{for all } n \geq 1$$

Now because we have $r = 1 \in (0, \infty]\,,$ this gives:

$$\int_{C(z_0=0;1)} \frac{\sin(z)}{z^n}\, dz = 2\pi i c_{n-1} = 2\pi i \frac{\sin^{(n-1)}(0)}{(n-1)!}$$

$$= \frac{2\pi i}{(n-1)!} \cdot \sin\left((n-1)\frac{\pi}{2}\right) \text{ for each } n = 1, 2, 3 \ldots$$

I suppose we could also re-write this as a non-trigonometric sequence if we felt inclined:

$$\frac{\pi i}{(n-1)!}\left(\left(1 + (-1)^{n-2}\right)(-1)^{\frac{n}{2}-1}\right)$$

(ii)

$$\int_{C(2;3)} \frac{z^{10}}{z^4 - 8z^2 + 16}\, dz$$

Using partial fraction decomposition we have:

$$\int_{C(2;3)} \frac{z^{10}}{z^4 - 8z^2 + 16}\, dz = \int_{C(2;3)} \frac{z^{10}}{(z-2)^2 (z+2)^2}\, dz$$

$$\text{Let } f(z) := \frac{z^{10}}{(z+2)^2}$$

Now notice that $f(z)$ is holomorphic on $D(2;3) \ni \{z_0 = 2\}$.

$$\implies f(z) = c_0 + \sum_{n=1}^{\infty} c_n (z-2)^n \quad \forall z \in D(2;3)$$

$$\text{With coefficients } c_n = \frac{f^{(n)}(z_0)}{(n)!} \quad \text{for all } n \geq 0$$

Now because we have $R = 3 \in (0, \infty]$, this gives:

$$\int_{C(2;3)} \frac{z^{10}}{z^4 - 8z^2 + 16}\, dz = \int_{C(2;3)} \frac{f(z)}{(z-2)^2}\, dz$$

$$= 2\pi i c_1 = 2\pi i f'(2)$$

$$= 2\pi i \left[\frac{(z+2)^2 \cdot 10z^9 - 2(z+2) \cdot z^{10}}{(z+2)^4} \right]_{z=2}$$

$$= 2\pi i * 288 = 576\pi i$$

Q2. Let f be holomorphic in an open disk D and let γ be a closed curve in D. Prove that, for every $n \in \mathbb{N}$ and $z_0 \in D \setminus \gamma$,

$$\mathbf{n}(\gamma; z_0)\, f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z_0)^{n+1}}\, dw$$

We immediately notice that the conditions for **Theorem 10.4 - Cauchy's integral formula for open disks** are satisfied. As f is holomorphic in the open disk D and γ is closed in D.

$$\implies \mathbf{n}(\gamma; z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \quad \text{for every } z \in D \setminus \gamma$$

The proof now follows by differentiating both sides with respect to z.

Let us firstly observe, however, that the derivate of $\mathbf{n}(\gamma; z)$ w.r.t $z = 0$.

We have:

$$\frac{d}{dz}\mathbf{n}(\gamma; z) = \frac{d}{dz}\int_\gamma \frac{dw}{w-z} = \int_\gamma \frac{d}{dz}\left(\frac{dw}{w-z}\right) = \int_\gamma \frac{dw}{(w-z)^2} = \int_\gamma \frac{d}{dw}\left(\frac{1}{(w-z)^2}\right) dw$$

Note we can move the derivative within the integral as $z \in D \setminus \gamma \implies z \neq w$. This also allows us to apply **Corollary 8.4** which tells us that if f is continuous on an open set U such that $f : U \to \mathbb{C}$ and $f = F'$ for some $F : U \to \mathbb{C}$ then:

$$\int_\gamma f(z)\,dz = 0 \quad \text{for all closed } \gamma \in U$$

$$\implies \int_\gamma \frac{d}{dw}\left(\frac{1}{(w-z)^2}\right) dw = 0 = \frac{d}{dz}\mathbf{n}(\gamma; z)$$

Of course this makes sense intuitively as informally for any $z \in D \setminus \gamma$ we can get sufficiently close to z such that $D(z; r) \cap \gamma = \emptyset$ for some $r > 0$. I.e. a sufficiently small disk around z does not intersect γ so the winding number of any z_0 in that disk will be the same as that of z itself.

Let us now return to the following statement, **Theorem 10.4**:

$$\mathbf{n}(\gamma; z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \quad \text{for every } z \in D \setminus \gamma$$

We can use proof by induction to show the desired result holds $\forall\, n \in \mathbb{N}$.
By differentiating both sides we have:

$$\mathbf{n}'(\gamma; z)f(z) + \mathbf{n}(\gamma; z)f'(z) = \frac{d}{dz}\left[\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw\right]$$

However, recall that we have shown that $\mathbf{n}'(\gamma; z) = 0$ for every $z \in D\backslash\gamma$. Also using the same logic as applied in the above steps, we can move the differential operator inside the integral. Combining these two facts we obtain:

$$\mathbf{n}(\gamma; z)f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\left(\frac{f(w)}{w-z}\right)\,dw = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}\,dw$$

So for $n = 1$ we have shown that we have:

$$\mathbf{n}(\gamma; z)f^{(1)}(z) = \frac{1!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{1+1}}\,dw$$

We now make the inductive assumption that this holds for all $n \in \mathbb{N}$. I.e. make the inductive assumption that:

$$\mathbf{n}\left(\gamma; z\right)f^{(n)}\left(z\right) = \frac{n!}{2\pi i}\int_\gamma \frac{f\left(w\right)}{(w-z)^{n+1}}\,dw$$

Differentiating both sides with respect to z we obtain:

$$\mathbf{n}'(\gamma; z)f^{(n)}(z) + \mathbf{n}(\gamma; z)f^{(n+1)}(z) = \frac{d}{dz}\left[\frac{n!}{2\pi i}\int_\gamma \frac{f\left(w\right)}{(w-z)^{n+1}}\,dw\right]$$

Again we apply the fact that $\mathbf{n}'(\gamma; z) = 0$ and as $z \in D\setminus\gamma \implies z \neq w$ which means we can bring the differential operator within the integral. So we now have:

$$\begin{aligned}
\mathbf{n}(\gamma; z)f^{(n+1)}(z) &= \frac{n!}{2\pi i}\int_\gamma \frac{d}{dz}\left(\frac{f\left(w\right)}{(w-z)^{n+1}}\right)\,dw \\
&= \frac{n!}{2\pi i}\int_\gamma \frac{d}{dz}\left(f\left(w\right)(w-z)^{-(n+1)}\right)\,dw \\
&= \frac{n!}{2\pi i}\int_\gamma (n+1)\,f\left(w\right)(w-z)^{-(n+1)-1}\,dw \\
&= \frac{n!\cdot(n+1)}{2\pi i}\int_\gamma \cdot f\left(w\right)(w-z)^{-(n+2)}\,dw
\end{aligned}$$

$$= \frac{(n+1)!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+2}}\, dw$$

Which is nothing but our inductive assumption at $n \to n+1$. Therefore the proof by induction is complete and we have shown that the inductive step holds. I.e.

$$\mathbf{n}(\gamma; z_0)\, f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\, dw \quad \text{for every } z_0 \in D \setminus \gamma \qquad \square$$

Q3. Let $f(z)$ be holomorphic on an open set U, z_0 be a root of $f(z)$, and $D = D(z_0; R) \subset U$ for some $R > 0$. Show that if there is a sequence $z_1, z_2, \dots \in D \setminus \{z_0\}$ converging to z_0 for which $f(z_j) = 0$ for each z_j, then $f(z) = 0$ for every $z \in D$.

We immediately notice that the conditions for **Theorem 10.8 - Taylor's Theorem for holomorphic functions** are satisfied. As f is holomorphic in the open set U and $D(z_0; R) \subset U$ for some $R > 0$.

$$\implies f(z) = c_0 + \sum_{n=1}^{\infty} c_n (z-z_0)^n \qquad \text{for every } z \in D(z_0; R) = D.$$

Where we defined the coefficients as follows:

$$c_n = \frac{f^{(n)}(z_0)}{n!} \qquad \text{for all } n \geq 0$$

Now notice that for whatever $R > 0$ we take, as $z_n \to z_0$, we can find a term z_j in the sequence such that $z_j \in D(z_0; R)$.

$$\implies f(z_j) = c_0 + \sum_{n=1}^{\infty} c_n (z_j - z_0)^n$$

We start with the observation that $f(z_0) = 0 \implies c_0 = 0$

$$\implies f(z_j) = \sum_{n=1}^{\infty} c_n (z_j - z_0)^n$$

But we know that $f(z_j) = 0$ for each z_j.

$$\implies f(z_j) = \sum_{n=1}^{\infty} c_n (z_j - z_0)^n = 0$$

We will now assume by contradiction that there exists an $m > 0$ defined to be the smallest m such that $c_m \neq 0$.

$$\implies f(z_j) = (z_j - z_0)^m \sum_{n=m}^{\infty} c_n \, (z_j - z_0)^{n-m}$$

We will denote $g(z_j) := \sum_{n=m}^{\infty} c_n \, (z_j - z_0)^{n-m}$

$$\implies f(z_j) = (z_j - z_0)^m \, g(z_j)$$

Notice now that we are given that the sequence $z_n \in D \setminus z_0 \implies z_j \neq z_0 \, \forall \, j$.

From the contradictive assumption and the above statement, we must then have that $g(z_j) \neq 0$ and $(z_j - z_0)^m \neq 0$.

But then that would imply that $f(z_j) = (z_j - z_0)^m \, g(z_j) \neq 0$ which contradicts our original assumption that $f(z_j) = 0$ for each z_j as per the question.

So we have then that no m exists such that $c_m \neq 0$ which means all coefficients c_n must be 0.

Recalling our prior expression for $f(z_j)$ and having shown that $c_n = 0 \, \forall n \geq 0$:

$$f(z) = \sum_{n=1}^{\infty} c_n \, (z - z_0)^n \qquad \text{for every } z \in D.$$

$$\implies f(z) = \sum_{n=1}^{\infty} 0 \cdot (z - z_0)^n \qquad \text{for every } z \in D.$$

$$\implies f(z) = 0 \qquad \text{for every } z \in D. \quad \square$$

Q4. Describe the family of Möbius transformations which map the exterior of the unit disk at the origin, that is $\mathbb{C} \setminus \overline{D}(0; 1)$, onto the upper half plane $\{z \in \mathbb{C} : Im(z) > 0\}$.

Our aim is to find any $f(z)$ that is a Möbius transformation from the exterior of the unit disk at the origin to the upper half plane. We will then generalise this transformation to describe the family of transformations that satisfy the same mapping, which we will denote as $M(z)$.

We need a point on the boundary of the disk that will map to the "point at infinity". I.e.

$$f(e^{i\theta}) = f(-\frac{d}{c})$$

We can take θ to be 0.

$$\implies f(1) = f(-\frac{d}{c})$$

$$\implies c = -d$$

We can now take $\theta = \pi$ to be the point mapping to 0.

$$\implies f(e^{i\pi}) = f(-1) = 0$$

But $f(z) = 0 \implies z = -\frac{b}{a}$.

$$\implies -\frac{b}{a} = -1$$

$$\implies b = a$$

We can take another pair of points on opposite sides of the boundary of the disk which map to $\pm 1 \in \mathbb{R}$.

Let's take these points to be $\pm i$

$$\implies f(i) = 1 = \frac{a \cdot i + a}{c \cdot i - c}$$

$$\implies ci - c = ai + a$$

$$\implies c(i-1) = a(i+1)$$

$$\implies \frac{c}{a} = \frac{i+1}{i-1} = -i$$

$$\implies c = -ia$$

But now notice that:

$$f(-i) = \frac{a \cdot (-i) + a}{-ia \cdot (-i) + ia}$$

$$= \frac{a}{a}\left[\frac{-i+1}{-1+i}\right] = -1 \text{ as required.}$$

However, thus far we have only used the fact that the boundary of the unit disk is mapped to the real axis. So we need to ensure that our f still maps to the upper half plane. We have:

$$f(z) = \frac{az + a}{-iaz + iaz} = \frac{a}{a}\left[\frac{z+1}{-i+iz}\right] = -i\left(\frac{z+1}{z-1}\right)$$

But notice that if we take a point from outside of the unit disk, say $z = 2$ then we have:

$$f(2) = -i\left(\frac{3}{2}\right) = -1$$

Which is in the lower half plane. Which means $f(z) = -i\left(\frac{z+1}{z-1}\right)$ maps points exterior to the unit disk to the lower half plane.

To resolve this we can take $g : f(z) \mapsto -f(z)$. So we have:

$$g(z) = -i\left(\frac{1+z}{1-z}\right)$$

We notice that this is one example of a Möbius transformation that maps points exterior to the unit disk to the upper half plane. However, notice that if we define:

$$g(z) := u(x, y) + i \cdot v(x, y) \text{ and } h : g(z) \mapsto g(z) + a \quad a \in \mathbb{R}$$
$$\implies h(z) = (u(x, y) + a) + i \cdot v(x, y)$$

So notice that for any point z_0 in $\mathbb{C}\backslash\overline{D}(0; 1)$ then $g(z_0) \in \{\text{The upper half plane}\}$

$$\implies h(z_0) = g(z_0) + a = (u(z_0) + a) + i \cdot v(z_0)$$
$$\implies Im(h(z_0)) = Im(g(z_0))$$

So because the condition of being in the upper half plane is only conditional on the Imaginary part, then $h(z_0)$ must also be in $\{\text{The upper half plane}\}$.

So our first generalisation of the family of Möbius transformations is:

$$h(z) = -i\left(\frac{1+z}{1-z}\right) + a \quad a \in \mathbb{R}$$

Now consider that if we take some real functions u_1 and v_1 to be the real and imaginary components of h then defining $k : h(z) \mapsto b \cdot h(z) \quad b \in \mathbb{R}^+$

$$\implies k(z) = -ib\left(\frac{1+z}{1-z}\right) + ab \quad a \in \mathbb{R}, b \in \mathbb{R}^+$$

$$= b \cdot h(z) = b\left(u_1(x,y) + i \cdot v_1(x,y)\right)$$

$$\implies Im(k(z)) = b \cdot Im(h(z))$$

So notice that for any point z_1 in $\mathbb{C} \backslash \overline{D}(0;1)$ then $h(z_1) \in \{\text{The upper half plane}\}$.

So now as we have that $h(z_1)$ is in the upper half plane this tells us that $Im(h(z_1)) > 0$. Now as $b > 0$ this implies that $Im(k(z_1)) = b \cdot Im(h(z_1)) > 0$.

Which shows that k maps points outside the unit disk to the upper half plane.

Thus far we have shown that we can generalise our transformation by translating and scaling our initial transformation (given certain conditions).

Notice that we cannot rotate our transformation as this would change our map to give points outside the upper half plane.

Finally we can define our M as follows:

$$\text{Let } t(z) := z \cdot e^{i\theta} \quad \theta \in \mathbb{R}$$

$$M(z) := k(t(z)) = -ib\left(\frac{1 + z \cdot e^{i\theta}}{1 - z \cdot e^{i\theta}}\right) + ab \quad a \in \mathbb{R}, b \in \mathbb{R}^+, \theta \in \mathbb{R}$$

Notice that all our M does differently from k is take notice of the fact that if we take a point z_2 in $\mathbb{C} \setminus \overline{D}(0;1)$ then $z_2 \cdot e^{i\theta}$ just rotates z_2 by the angle θ about the origin. So if $z_2 \in \mathbb{C} \setminus \overline{D}(0;1) \implies z_2 \cdot e^{i\theta} \in \mathbb{C} \setminus \overline{D}(0;1) \quad \forall \theta \in \mathbb{R}$.

So we have can describe the family of Möbius transformations that map the exterior of the unit disk at the origin onto the open half plane by:

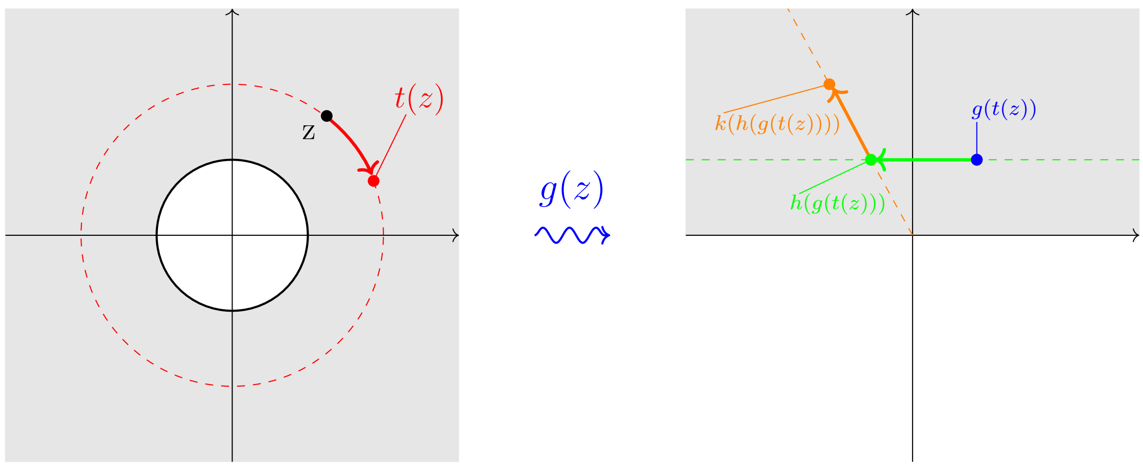
$$M(z) := k(t(z)) = -ib\left(\frac{1 + z \cdot e^{i\theta}}{1 - z \cdot e^{i\theta}}\right) + ab \quad a \in \mathbb{R}, b \in \mathbb{R}^+, \theta \in \mathbb{R}$$

Scale Mobius

$$M(z) = k(h(g(t(z))))$$

Translate Rotate

We can visualise the mapping of the Möbius transformation M on the region exterior to the unit disk at the origin on the below plot:



This plot shows us that starting with any given $z \in \mathbb{C} \setminus \overline{D}(0;1)$ we can apply $M(z)$ to map z to any point in the upper half plane $\{z \in \mathbb{C} : Im(z) > 0\}$

Q5. Consider a thin membrane which minimises its internal energy on $U = D(1;2)$ with surface height described by $u(x,y)$ which satisfies $u(x,y) = y^2 - x^3$ on $C(1;2)$. Determine $u(x,y)$.

We aim to find $g(z)$ such that $Re(g(z)) = u(z)$ with $u(z)|_{C(1;2)} = y^2 - x^3$

Notice that on $C(1;2)$ we have $(x-1)^2 + y^2 = 4$

$$\implies y^2 = 4 - (x-1)^2$$

$$\Longrightarrow\ y^2 = -\left(x-3\right)\left(x+1\right)$$

Consider that if we make the ansatz:

$$g(z) = az^3 + bz^2 + cz + d$$

$$\Longrightarrow\ Re(g(z)) = Re(a\left(x+iy\right)^3 + b\left(x+iy\right)^2 + c\left(x+iy\right) + d)$$

$$= Re(ax^3 + i3ax^2y - 3axy^2 - iay^3 + bx^2 + i2bxy - by^2 + cx + icy + d)$$

$$= ax^3 - 3axy^2 + bx^2 - by^2 + cx + d$$

$\Longrightarrow$ On $C(1;2)$ we have:

$$y^2 - x^3 = ax^3 - 3axy^2 + bx^2 - by^2 + cx + d \quad , \quad y^2 = -\left(x-3\right)\left(x+1\right)$$

$$\Longrightarrow\ \left(-\left(x-3\right)\left(x+1\right)\right) - x^3 = ax^3 - 3ax\left(-\left(x-3\right)\left(x+1\right)\right) + bx^2 - b\left(-\left(x-3\right)\left(x+1\right)\right) + cx + d$$

$$\Longrightarrow\ -x^3 - x^2 + 2x + 3 = ax^3 - 3ax\left(-x^2+2x+3\right) + bx^2 - b\left(-x^2+2x+3\right) + cx + d$$

$$\Longrightarrow\ -x^3 - x^2 + 2x + 3 = ax^3 + 3ax^3 - 6ax^2 - 9ax + bx^2 + bx^2 - 2bx - 3b + cx + d$$

$$\Longrightarrow\ -x^3 - x^2 + 2x + 3 = 4ax^3 + x^2\left(2b - 6a\right) + x\left(c - 2b - 9a\right) + d - 3b$$

Now equating the coefficients, we obtain:

x^3) $\quad -1 = 4a \Longrightarrow a = -\frac{1}{4} = -0.25$

x^2) $\quad -1 = 2b - 6a = 2b + \frac{3}{2} \Longrightarrow b = -\frac{5}{4} = -1.25$

x) $\quad 2 = c - 2b - 9a \Longrightarrow c = 2 + 2\left(-1.25\right) + 9\left(-0.25\right) = -\frac{11}{4} = -2.75$

constant) $\quad 3 = d - 3b = d + 3.75 \Longrightarrow d = \frac{-3}{4} = -0.75$

Putting it all together we have:

$$g(z) = -\frac{1}{4}z^3 - \frac{5}{4}z^2 - \frac{11}{4}z - \frac{3}{4}$$

$$\Longrightarrow\ u(x,y) = Re(g(z)) = ax^3 - 3axy^2 + bx^2 - by^2 + cx + d$$

$$= -\frac{1}{4}x^3 + \frac{3}{4}xy^2 - \frac{5}{4}x^2 + \frac{5}{4}y^2 - \frac{11}{4}x - \frac{3}{4}$$

$$= \frac{-x^3 + 3xy^2 - 5x^2 + 5y^2 - 11x - 3}{4}$$

Notice that because g is a polynomial in z, it is holomorphic on $\mathbb{C}$. Therefore, because u is the real part of g, we have that u is harmonic. We also have shown that on ∂U, our solution for $u(x, y) = y^2 - x^3$ by the reverse of the working we used to obtain the coefficients of g. Therefore it must be that $u(x, y)$ minimises its internal energy on U as required.