MATH381 Assignment 1

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Question 1

Note: AOL is an abbreviation for Algebra Of Limits

(i) $$\lim_{n\to\infty} \frac{n^3 + n\sqrt{n} + i\left(2 - \frac{1}{n^3}\right)}{2n^3 - 3n^2}$$

We rewrite the sequence by taking the "dominant" n^3 out $\implies$

$$\lim_{n\to\infty} \frac{\left(n^3 + n\sqrt{n} + i\left(2 - \frac{1}{n^3}\right)\right) \cdot \frac{1}{n^3}}{\left(2n^3 - 3n^2\right) \cdot \frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{1}{n^{3/2}} + i\left(2 - \frac{1}{n^3}\right)}{2 - \frac{3}{n}}$$

By AOL, limit of quotient = quotient of limits $\implies$

$$\frac{\lim_{n\to\infty} \left(1 + \frac{1}{n^{3/2}} + i\left(2 - \frac{1}{n^3}\right)\right)}{\lim_{n\to\infty} \left(2 - \frac{3}{n}\right)}$$

By AOL, limit of sum = sum of limits $\implies$

$$\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n^{3/2}} + \lim_{n\to\infty} i\left(2 - \frac{1}{n^3}\right)}{\lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{3}{n}}$$

By AOL, limit of constant = constant. Also, for $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$, $\implies$

$$\frac{1 + 0 + i \cdot \lim_{n\to\infty} \left(2 - \frac{1}{n^3}\right)}{2 - 0}$$

By AOL, limit of difference = difference of limits $\implies$

$$\frac{1 + i \cdot \left(\lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{1}{n^3}\right)}{2}$$

By AOL, limit of constant = constant and for $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$, $\implies$

$$\frac{1 + i \cdot (2 - 0)}{2} = \frac{1 + 2i}{2} = \frac{1}{2} + i \quad \square$$

(ii)
$$\lim_{n\to\infty} \frac{3^n + 2^n + i\left(3^n - \sin(n^2)\right)}{3^{n+2} - 3^{\frac{n}{2}}}$$

We rewrite the sequence by taking the "dominant" 3^n out $\implies$

$$\lim_{n\to\infty} \frac{\left(3^n + 2^n + i\left(3^n - \sin(n^2)\right)\right) \cdot \frac{1}{n^3}}{\left(3^{n+2} - 3^{\frac{n}{2}}\right) \cdot \frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1 + \left(\frac{2}{3}\right)^n + i\left(1 - \frac{\sin(n^2)}{3^n}\right)}{3^2 - \left(\frac{1}{\sqrt{3}}\right)^n}$$

By AOL, limit of quotient = quotient of limits $\implies$

$$\frac{\lim_{n\to\infty} \left(1 + \left(\frac{2}{3}\right)^n + i\left(1 - \frac{\sin(n^2)}{3^n}\right)\right)}{\lim_{n\to\infty} \left(3^2 - \left(\frac{1}{\sqrt{3}}\right)^n\right)}$$

By AOL, limit (or difference) of sum = sum (or difference) of limits $\implies$

$$\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(\frac{2}{3}\right)^n + \lim_{n\to\infty} i\left(1 - \frac{\sin(n^2)}{3^n}\right)}{\lim_{n\to\infty} 3^2 - \lim_{n\to\infty} \left(\frac{1}{\sqrt{3}}\right)^n}$$

By AOL, limit of constant = constant, For $|p| < 1$, $\lim_{n\to\infty} p^n = 0 \implies$

$$\frac{1 + i \cdot \lim_{n\to\infty} \left(1 - \frac{\sin(n^2)}{3^n}\right)}{9}$$

By AOL, limit of difference = difference of limits $\implies$

$$\frac{1 + i \cdot \left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{\sin(n^2)}{3^n}\right)}{9}$$

By AOL, limit of constant = constant $\implies$

$$\frac{1 + i \cdot \left(1 - \lim_{n\to\infty} \frac{\sin(n^2)}{3^n}\right)}{9}$$

We now apply the squeeze theorem.

Squeeze Theorem: Suppose that there exists N such that $x_n \leq y_n \leq z_n$ for all $n \geq N$, and that $x_n \to x$ and $z_n \to x$ as $n \to \infty$. Then $y_n \to x$ as $n \to \infty$.

Notice that $\sin(n) \in [-1, 1] \implies \sin(n^2) \in [-1, 1]$ also.

$$\implies \frac{\sin(n^2)}{3^n} \in \left[\frac{-1}{3^n}, \frac{1}{3^n}\right] \implies \frac{-1}{3^n} \leq \frac{\sin(n^2)}{3^n} \leq \frac{1}{3^n}$$

$$\lim_{n\to\infty} \left(\frac{-1}{3}\right)^n = 0 \text{ as } \left|\frac{-1}{3}\right| < 1$$

$$\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0 \text{ as } \left|\frac{1}{3}\right| < 1$$

$$\text{So, by the squeeze theorem, } \lim_{n\to\infty} \frac{\sin(n^2)}{3^n} = 0$$

Which by AOL now gives us: $\dfrac{1 + i \cdot (1 - 0)}{9} = \dfrac{1+i}{9} = \dfrac{1}{9} + \dfrac{i}{9}$ $\square$ ✓ 5/5

Question 2

Let $z_n = \dfrac{n^2 + 2 + 3in}{2n^2 - n}$ for $n \in \mathbb{N}$

Prove from the definition that $z_n \to \dfrac{1}{2}$ as $n \to \infty$

$$\text{Given } \epsilon > 0, \text{ take } N \in \mathbb{N} \text{ such that } N > \frac{\epsilon + 8}{2\epsilon}$$

$$\implies \forall n \geq N, \text{we have: } n > \frac{\epsilon + 8}{2\epsilon}$$

$$\implies 2n\epsilon > \epsilon + 8$$

$$\implies 2n\epsilon - \epsilon > 8$$

$$\implies \epsilon(2n - 1) > 8$$

$$\implies \epsilon > \frac{8}{2n - 1} \quad (\text{As } 2n - 1 > 0 \text{ for } n \geq 1)$$

$$\implies \epsilon > \frac{16n}{4n^2 - 2n}$$ ✓

$$\implies \epsilon > \frac{\sqrt{37n^2 + 8n + 16}}{4n^2 - 2n} \quad (\text{For } n \geq 1)$$

The proof of the last step is omitted, however, can easily be verified by solving the quadratic $(16n)^2 = 37n^2 + 8n + 16$, which has positive solution, $n \approx 0.289$. Then, as $y = 256n^2 - 37n^2 - 8n - 16$ is an increasing parabola, $16n > \sqrt{37n^2 + 8nx + 16}$ for $n > 0.289$. As $n \geq N \geq 1$, this equality holds for each n in our sequence.

We continue by factorising the denominator $\implies$

$$
\begin{aligned}
\epsilon &> \frac{\sqrt{37n^2 + 8n + 16}}{2n\,(2n-1)} \\
&= \sqrt{\frac{37n^2 + 8n + 16}{4n^2\,(2n-1)^2}} \\
&= \sqrt{\frac{(n+4)^2 + 36n^2}{4n^2\,(2n-1)^2}} \\
&= \sqrt{\frac{(n+4)^2}{4n^2\,(2n-1)^2} + \frac{9}{(2n-1)^2}} \\
&= \sqrt{\left(\frac{n+4}{2n\,(2n-1)}\right)^2 + \left(\frac{3}{2n-1}\right)^2} \\
&= \left|\left(\frac{n+4}{2n\,(2n-1)}\right) + i \cdot \left(\frac{3}{2n-1}\right)\right| \\
&= \left|\left(\frac{2n^2 + n + 4 - 2n^2}{2n\,(2n-1)}\right) + i \cdot \left(\frac{3}{2n-1}\right)\right| \\
&= \left|\left(\frac{n^2}{n\,(2n-1)}\right) + \left(\frac{2}{n\,(2n-1)}\right) - \left(\frac{n(2n-1)}{2n\,(2n-1)}\right) + i \cdot \left(\frac{3}{2n-1}\right)\right| \\
&= \left|\left(\frac{n^2}{n\,(2n-1)}\right) + \left(\frac{2}{n\,(2n-1)}\right) - \frac{1}{2} + i \cdot \left(\frac{3}{2n-1}\right)\right| \\
&= \left|\left(\frac{n^2+2}{n\,(2n-1)} + i \cdot \left(\frac{3n}{n(2n-1)}\right)\right) - \frac{1}{2}\right| \\
&= \left|\left(\frac{n^2 + 2 + 3in}{n\,(2n-1)}\right) - \frac{1}{2}\right| \\
&= \left|z_n - \frac{1}{2}\right|
\end{aligned}
$$

$$
\implies \left|z_n - \frac{1}{2}\right| < \epsilon
$$

$$\text{So, given } \epsilon > 0, \text{if we choose an integer, } N > \frac{\epsilon+8}{2\epsilon}$$

Then if n $\geq N$, we have:

$$|z_n - \frac{1}{2}| < \frac{8}{2n-1} \leq \frac{8}{2N-1} < \epsilon$$

Therefore, by definition of the convergence of a sequence, we obtain that $(z_n)_{n=1}^{\infty}$ is convergent with limit $\frac{1}{2}$. $\square$

4.5/5

Question 3

Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be sequences of complex numbers. Suppose that $z_n \to z$ and $w_n \to w$ in $\mathbb{C}$. Prove from the definition that $z_n + w_n \to z + w$ as $n \to \infty$.

We know $z_n \to z \implies \forall \epsilon > 0, \exists N_Z$ such that $|z_n - z| < \frac{\epsilon}{2}$ for all $n \geq N_Z$.

Likewise, $w_n \to w \implies \forall \epsilon > 0, \exists N_W$ such that $|w_n - w| < \frac{\epsilon}{2}$ for all $n \geq N_W$.

$\Rightarrow \forall \epsilon > 0, \exists N_Z, N_W$ such that $\forall n \geq N = \max\{N_Z, N_W\}$, we have:

$$|z_n - z| < \frac{\epsilon}{2} \text{ and } |w_n - w| < \frac{\epsilon}{2}.$$

By the triangle inequality, $|(z_n+w_n)-(z+w)| \leq |z_n-z|+|w_n-w| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$.

$\Rightarrow \forall \epsilon > 0, \exists N$ such that $|(z_n+w_n)-(z+w)| < \epsilon$ for all $n \geq N$.

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So, by definition: $z_n + w_n \to z + w$ as $n \to \infty$ $\square$

Question 4

Let $f : S \to \mathbb{C}$, and let $a \in S \subseteq \mathbb{C}$. Suppose that, for every sequence $(z_n)_{n=1}^{\infty}$ in S, if $z_n \to a$ then $f(z_n) \to f(a)$. Prove that the function f is continuous at a.

We use proof by contradiction to show f is continuous at a.

Assume by contradiction that f is not continuous at a.

$\Rightarrow \exists \epsilon > 0$ such that $\forall \delta > 0, \exists z \in S$ with $0 \leq |z-a| < \delta$ but $|f(z) - f(a)| \geq \epsilon$.

$\implies \forall n \in \mathbb{N}$ we can define $\delta := \frac{1}{n}$, which gives $z_n \in S$ such that:

$$0 \leq |z_n - a| < \frac{1}{n} \qquad (4.1)$$

and

$$|f(z_n) - f(a)| \geq \epsilon \qquad (4.2)$$

By (4.1), we can apply the squeeze theorem and since $\frac{1}{n} \to 0$ as $n \to \infty \Rightarrow |z_n - a| \to 0$ ✓

We now have $z_n \to a$ as $n \to \infty$.

What's more, because we have $z_n \in S$ and $z_n \to a \implies f(z_n) \to f(a)$. (as per Q4)

Since for this given $\epsilon > 0$ we have $f(z_n) \to f(a)$,

$\implies \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ |f(z_n) - f(a)| < \epsilon$

However, this contradicts (4.2)

We have therefore shown that the assumption that f is not continuous at a is false

$\implies$ The function f is continuous at a $\square$ ✓ 5/5

This was my first time using Latex so any feedback on the formatting, layout, notation etc. would be greatly appreciated. (After feedback on my actual working of course!)