MATH381 Assignment 3

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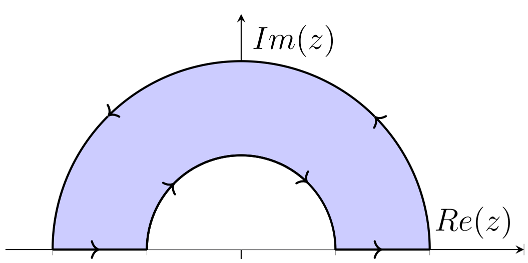
Due: 11:55 PM, Thursday 19 September 2024

Q1. Evaluate $\int_\gamma z\overline{z}\, dz$ where γ is a closed curve that traverses the boundary of the half annulus $\{z : 1 < |z| < 2 \text{ and } \operatorname{Im}(z) > 0\}$ once in an anti-clockwise direction.
Hint: The integrand is not holomorphic, so most integral theorems don't apply and you'll need to use the definition of an integral over a curve.



We take γ to be the joining of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ defined as follows:

1. $\gamma_1 = t$, $t \in [1, 2]$
2. $\gamma_2 = 2e^{it}$, $t \in [0, \pi]$
3. $\gamma_3 = t$, $t \in [-2, -1]$
4. $\gamma_4 = e^{it}$, $t \in [\pi, 0]$ technically this notation doesn't make sense, I would write e.g. $\gamma_4(t) = e^{i(\pi - t)}$, $t \in [0, \pi]$

Now by **Proposition 7.20** we can take the integral over γ to be the sum of the integrals over the four component curves of γ as $f(z) = z\overline{z} = |z|^2$ is continuous on $\mathbb{C}$ and more specifically on γ.

Namely we can apply the following:

1. $\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$
2. $\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \cdots + \int_{\gamma_4} f(z)\,dz$ ✓

Which obtains the following result:

$$\int_{\gamma_1} f(z)\,dz = \int_1^2 f(t)\cdot 1\,dt = \int_1^2 t\cdot t\,dt = \int_1^2 t^2\,dt = \frac{7}{3}$$ ✓

$$\int_{\gamma_2} f(z)\,dz = \int_0^\pi f(2e^{it})\cdot 2ie^{it}\,dt = \int_0^\pi 2e^{-it}\cdot 2e^{it}\cdot 2ie^{it}\,dt = 8i\int_0^\pi e^{it}$$

$$= 8i\cdot\frac{1}{i}\left[e^{it}\right]_0^\pi = 8\cdot[-1-1] = 8\cdot[-2] = -16$$ ✓

$$\int_{\gamma_3} f(z)\,dz = \int_{-2}^{-1} f(t)\,dt = \int_{-2}^{-1} t\cdot t\,dt = \int_{-2}^{-1} t^2\,dt = \frac{7}{3}$$ ✓

$$\int_{\gamma_4} f(z)\,dz = \int_\pi^0 f(e^{it})\cdot ie^{it}\,dt = \int_\pi^0 e^{-it}\cdot e^{it}\cdot ie^{it}\,dt = i\int_\pi^0 e^{it} = -i\int_0^\pi e^{it} = -i\cdot 2i = 2$$ ✓

$$\implies \int_\gamma f(z)\,dz = \frac{7}{3} - 16 + \frac{7}{3} + 2 + \frac{7}{3} = \frac{14}{3} - 14 = \frac{-28}{3}$$ ✓

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Q2. Evaluate the following integrals:

(i) $\int_\gamma 2z^3 - 5z^2 + z + 4\,dz$ where $\gamma = \overline{[2, 3i]}$.

(ii) $\int_\gamma (z-3)^8\,dz$ where γ is the clockwise circular arc from $3+2i$ to $3-2i$ having radius $\sqrt{13}$ centred at the origin.

(iii) $\int_\gamma \cos(z)e^{i\pi\sin(z)}\,dz$ where $\gamma(t) = \frac{\pi}{2}t + it(1-t)$ for $t \in [0,1]$.

Hint: Use the Fundamental Theorem of Calculus for curves where possible.

(i) By the FTC (**Theorem 8.3**) we have that as f is continuous on $\mathbb{C} \supset \gamma$ (by the continuity of polynomials), and there exists F such that $F' = f$ then $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$

We have:

$$F(z) = \int_0^z 2s^3 - 5s^2 + s + 4\,ds = \frac{1}{2}z^4 - \frac{5}{3}z^3 + \frac{1}{2}z^2 + 4z$$ ✓

$$\implies \int_\gamma 2z^3 - 5z^2 + z + 4\,dz = \left[\frac{1}{2}z^4 - \frac{5}{3}z^3 + \frac{1}{2}z^2 + 4z\right]_2^{3i}$$
$$= \left[\frac{81}{2} + \frac{135i}{3} - \frac{9}{2} + 12i\right] - \left[8 - \frac{40}{3} + 2 + 8\right]$$
$$= \frac{94}{3} + \frac{171i}{3}$$
$$= \frac{94}{3} + 57i$$

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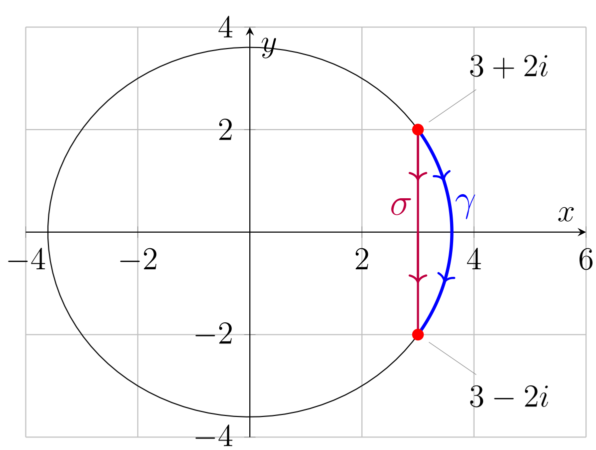
(ii) We can notice that as f is a polynomial in z, it is holomorphic in $\mathbb{C}$ and for the sake of the next step, holomorphic in say the open disk, $D(0,10) \subset \mathbb{C}$.

We can now apply **Corollary 8.2** which states that if f is holomorphic in an open disk D and there exists two curves, γ and σ in D with the same start and end points then:

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz.$$



Let us define $\sigma := \overline{[3+2i, 3-2i]}$ which notice has the same orientation and endpoints as γ.

We can parameterise the line segment σ as:

$$\sigma(t) := 3 + 2i + t\,(3 - 2i - (3 + 2i)) \quad (t \in [0,1])$$

✓

$$= 3 + 2i - t\,(4i) \quad (t \in [0, 1])$$

Now from **Corollary 8.2** we obtain:

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz = \int_{\overline{[3+2i,3-2i]}} f(z)\,dz = \int_{\overline{[3+2i,3-2i]}} (z-3)^8\ dz$$

$$= \int_0^1 ((3+2i-4ti)-3)^8 \cdot (-4i)\ dt$$

$$= -4i \int_0^1 (2i-4ti)^8\ dt$$

$$= -4i \int_0^1 (2-4t)^8\ dt$$

$$= 4i \left[\frac{(2-4t)^9}{9}\right]_0^1$$

$$= 4i \left[\frac{(-2)^9}{9} - \frac{2^9}{9}\right]$$

$$= -\frac{1024}{9} i$$ ✓

Alternatively we can again apply the FTC (**Theorem 8.3**), by noticing that $f = (z-3)^8$ is continuous on $\mathbb{C} \supset \gamma$ by the continuity of polynomials. There also exists F such that $F' = f$ and therefore, $\int_\gamma (z-3)^8\ dz = F(3+2i) - F(3-2i) = \left[\frac{(z-3)^9}{9}\right]_0^1 = -\frac{1024}{9} i$ as we found above. ✓

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(iii) We begin by noticing that f is continuous on $\mathbb{C} \supset \gamma$ as it is composed of a composition of continuous functions. Our first aim is to find F such that $F' = f$ as we can then we can apply the FTC (**Theorem 8.3**) which tells us $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$.

We begin by making the u-substitution $u = \sin(s) \implies du = \cos(s)ds$

$$\implies \int_a^z \cos(s)e^{i\pi \sin(s)}\, ds = \int_{sin(a)}^{sin(z)} e^{i\pi u}\, du = \left[\frac{-i}{\pi}e^{i\pi u}\right]_{sin(a)}^{sin(z)}$$

$$= \frac{-i}{\pi}e^{i\pi \sin(z)} + \frac{i}{\pi}e^{i\pi \sin(a)} = \frac{-i}{\pi}e^{i\pi \sin(z)} + a_0 = F(z) \quad \left(\text{where } \frac{i}{\pi}e^{i\pi \sin(a)} = a_0 \in \mathbb{C}\right) \quad \checkmark$$

Notice that this F satisfies the condition $F' = f$

We have $\gamma(0) = 0$ and $\gamma(1) = \frac{\pi}{2}$

$$\implies \int_\gamma \cos(z)e^{i\pi \sin(z)}\, dz = F(\gamma(1)) - F(\gamma(0)) = F(\frac{\pi}{2}) - F(0)$$

Now applying Euler's identity, we get:

$$F(\frac{\pi}{2}) - F(0) = \frac{-1}{\pi} \cdot -1 + \frac{i}{\pi} \cdot 1 = \frac{i}{\pi} + \frac{i}{\pi} = \frac{2i}{\pi} \quad \checkmark$$

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Q3. Evaluate the following integrals:

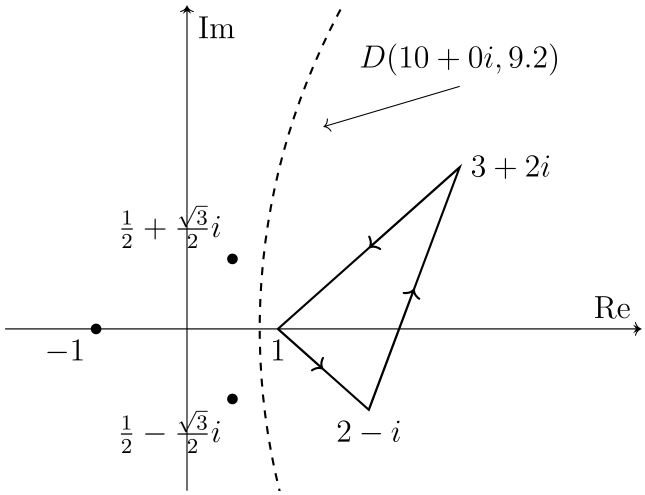
(i) $\int_\gamma \frac{e^z}{z^3+1}\, dz$ where γ is a curve going once anti-clockwise around a triangle with vertices 1, $2 - i$, and $3 + 2i$.

(ii) $\int_\gamma \frac{1}{\cos(z)}\, dz$ where γ is a curve going once anti-clockwise around the boundary of the coordinate rectangle R having opposite corners $-1-3i$ and $1 + 3i$.

Note: The intention is that you only use lecture content up to and including section 8 for this question.

(i) Notice $z^3 + 1 = 0 \iff z = -1, \frac{1-\sqrt{3}}{2}i, \frac{1+\sqrt{3}}{2}i$ ✓

So f is holomorphic in $\mathbb{C} \setminus \{-1, \frac{1-\sqrt{3}}{2}i, \frac{1+\sqrt{3}}{2}i\}$ or more specifically we can say that f is holomorphic in the open disk, $D(10 + 0i, 9.2)$.



Nice!

Now we can apply Cauchy's theorem (**Theorem 8.1**) and state that as f is holomorphic in D and γ is closed in D, we have:

$$\int_\gamma \frac{e^z}{z^3+1}\,dz = 0$$ ✓

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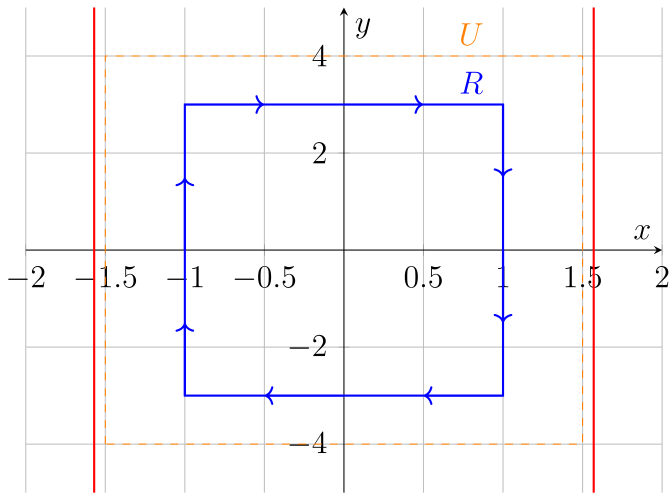
(ii) We have $\cos(z) = 0 \iff z = \frac{\pi}{2} + n\pi \quad \forall n \in \mathbb{Z}$ ✓

So f is holomorphic in $\mathbb{C}\setminus\{\frac{\pi}{2}+n\pi \quad \forall n \in \mathbb{Z}\}$ or more specifically we can say that f is holomorphic in the open set U := Rec $(-1.5-4i, 1.5, 4i)$ as $U \cap \mathbb{C} \setminus \{\frac{\pi}{2} + n\pi\, \forall n \in \mathbb{Z}\} = \emptyset$.

Note that the choice of U is made arbitrarily to satisfy an open set containing R with $|Re(z)| < \frac{\pi}{2}\, \forall z \in U$.

Notice that the coordinate rectangle $R :=$ Rec$(-1-3i, 1+3i)$ is included in U. I.e $R \subset U$.

Let us observe the plot of R and U on $\mathbb{C}$:



We can now apply Goursatt's Lemma (**Lemma 8.10**) which tells us that if we have a coordinate rectangle R in $\mathbb{C}$ and f is holomorphic in an open set $U \supset R$ then:

$$\int_{\partial R} f(z)\, dz = 0$$

We have shown that f is holomorphic in the open set $U \supset R$ and therefore we can conclude by Goursatt's Lemma (**Lemma 8.10**) that: ✓

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$$\int_{\partial R} \frac{1}{\cos(z)}\, dz = 0$$

Q4. Let $p(z)$ be a quadratic polynomial with real coefficients and no real roots. Let γ be a closed curve that goes around both roots in an anti-clockwise direction the same number of times. Prove that

$$\int_\gamma \frac{dz}{p(z)} = 0.$$

We know that $p(z)$ has two non-real roots, let us denote these as z_1 and z_2.

From this we can express $p(z)$ as:

$$p(z) := a\,(z - z_1)\,(z - z_2) \quad (a \in \mathbb{R})$$ ✓

We now seek to evaluate:

$$\int_\gamma \frac{dz}{p(z)} = \frac{1}{a} \cdot \int_\gamma \frac{dz}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{a} \cdot \int_\gamma \left[\frac{a_1}{(z-z_1)} + \frac{a_2}{(z-z_2)} \right] dz \quad (a_1, a_2 \in \mathbb{C})$$

We use a partial fraction decomposition to find a_1, a_2:

$$\frac{1}{(z-z_1)(z-z_2)} = \frac{a_1}{(z-z_1)} + \frac{a_2}{(z-z_2)}$$

$$\implies 1 = a_1(z-z_2) + a_2(z-z_1)$$

$$\implies 1 = z(a_1+a_2) - (a_1 z_2 + a_2 z_1)$$

$$\implies \begin{cases} a_1 + a_2 = 0 \\ a_1 z_2 + a_2 z_1 = -1 \end{cases}$$

From this we see $a_1 = -a_2$ and therefore:

$$a_2(z_1 - z_2) = -1$$

$$\implies a_2 = \frac{1}{z_2 - z_1}$$

$$\implies a_1 = \frac{1}{z_1 - z_2}$$

Substituting these coefficients into our expression for $\frac{1}{p(z)}$ we get:

$$\frac{1}{p(z)} = \frac{1}{a} \cdot \left[\frac{\frac{1}{z_1 - z_2}}{z - z_1} + \frac{\frac{1}{z_2 - z_1}}{z - z_2} \right]$$

$$= \frac{1}{a} \cdot \frac{1}{z_1 - z_2} \cdot \left[\frac{1}{z - z_1} - \frac{1}{z - z_2} \right]$$

We now reconsider our integral:

$$\int_\gamma \frac{dz}{p(z)} = \frac{1}{a} \cdot \frac{1}{z_1 - z_2} \cdot \int_\gamma \left[\frac{1}{z - z_1} - \frac{1}{z - z_2} \right] dz$$

$$= \frac{1}{a} \cdot \frac{1}{z_1 - z_2} \cdot \left[\int_\gamma \frac{dz}{z - z_1} - \int_\gamma \frac{dz}{z - z_2} \right]$$

$$= \frac{1}{a} \cdot \frac{1}{z_1 - z_2} \cdot \left[2\pi i \cdot \mathbf{n}(\gamma; z_1) - 2\pi i \cdot \mathbf{n}(\gamma; z_2) \right]$$

✓

As per **Q4**, we know that the curve γ goes around z_1 and z_2 in an anti-clockwise direction the same number of times. Let us denote this winding number with k.

We now have:

$$\int_\gamma \frac{dz}{p(z)} = \frac{1}{a} \cdot \frac{1}{z_1 - z_2} \cdot [2\pi i \cdot k - 2\pi i \cdot k]$$

$$= \frac{1}{a} \cdot \frac{1}{z_1 - z_2} \cdot 0 = 0 \quad \text{as required.} \quad \square$$

✓

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Q5. Use winding numbers to evaluate the following integrals:

(i) $\int_\gamma \frac{dz}{z^2-1}$ where $\gamma(t) = 1 + t(1-t) + e^{4\pi i t}$ for $t \in [0, 1]$.

(ii) $\int_\gamma \frac{dz}{z^3-z^2+4z-4}$ where γ goes once anti-clockwise around the circle of radius 3 centred at $2 + i$.

(i) We begin by considering the partial fraction decomposition of $\frac{1}{z^2-1}$

$$\frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)} = \frac{a}{z-1} + \frac{b}{z+1}$$

$$\implies 1 = a(z+1) + b(z-1)$$
$$\implies 1 = z(a+b) + (a-b)$$
$$\implies \begin{cases} a + b = 0 \\ a - b = 1 \end{cases}$$
$$\implies a = \frac{1}{2}, \quad b = \frac{-1}{2}$$

✓

So we now have:

$$\int_\gamma \frac{dz}{z^2-1} = \int_\gamma \left[\frac{1}{2\,(z-1)} - \frac{1}{2\,(z+1)}\right] dz$$

$$= \frac{1}{2}\int_\gamma \frac{dz}{z-1} - \frac{1}{2}\int_\gamma \frac{dz}{z+1}$$

$$= \frac{1}{2}\cdot 2\pi i \cdot \mathbf{n}\,(\gamma;1) - \frac{1}{2}\cdot 2\pi i \cdot \mathbf{n}\,(\gamma;-1)$$

$$= \pi i \cdot \mathbf{n}\,(\gamma;1) - \pi i \cdot \mathbf{n}\,(\gamma;-1)$$

Let us now compute these two winding numbers:

$$\mathbf{n}\,(\gamma;1) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-1}\,dz$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{1}{\left(1+t\,(t-1)+e^{4\pi i t}\right)-1}\cdot\left(1-2t+4\pi i t e^{4\pi i t}\right)\,dt$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{1-2t+4\pi i t e^{4\pi i}}{t\,(t-1)+e^{4\pi i t}}\,dt$$

$$= \frac{1}{2\pi i}\left[\ln\left(t\,(t-1)+e^{4\pi i t}\right)\right]_0^1$$

$$= \frac{1}{2\pi i}\left[4\pi i - \ln(1)\right] = \frac{4\pi i}{2\pi i} = 2$$

$$\mathbf{n}\,(\gamma;-1) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z+1}\,dz$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{1}{\left(1+t\,(t-1)+e^{4\pi i t}\right)+1}\cdot\left(1-2t+4\pi i t e^{4\pi i t}\right)\,dt$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{1-2t+4\pi i t e^{4\pi i t}}{2+t\,(t-1)+e^{4\pi i t}}\,dt$$

$$= \frac{1}{2\pi i}\left[\ln\left(2+t\left(t-1\right)+e^{4\pi it}\right)\right]_0^1$$

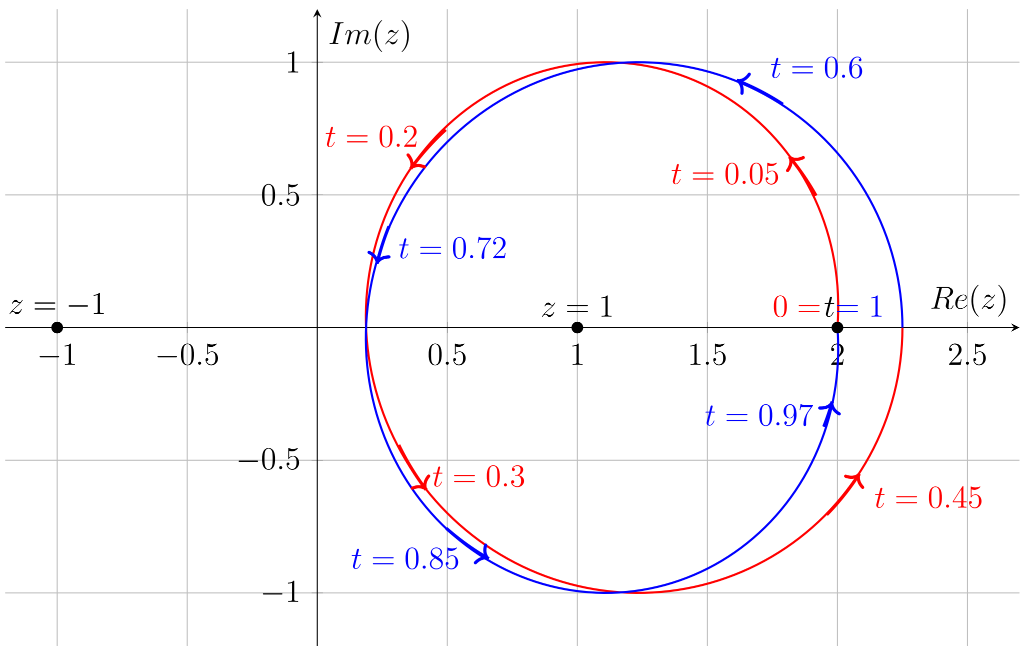
$$= \frac{1}{2\pi i}\left[\ln\left(2+e^{4\pi i}\right)-\ln\left(3\right)\right] = \frac{1}{2\pi i}\left[\ln\left(3\right)-\ln\left(3\right)\right] = 0$$ ✓

So we now have:

$$\int_\gamma \frac{dz}{z^2-1} = \pi i\cdot\mathbf{n}\left(\gamma;1\right) - \pi i\cdot\mathbf{n}\left(\gamma;-1\right) = \left[2\right]\pi i - \left[0\right]\pi i = 2\pi i$$ ✓

Let us observe these winding numbers on the plot of:

$$\gamma(t) = 1 + t(1-t) + e^{4\pi it} \text{ for } t\in[0,1]$$



Notice γ goes around $z=1$ twice in an anticlockwise direction but does not go around $z=-1$.

Hence $\mathbf{n}\left(\gamma;1\right)=2$ and $\mathbf{n}\left(\gamma;-1\right)=0$ as previously shown.

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(ii) We begin by considering the partial fraction decomposition of $\frac{1}{z^3-z^2+4z-4}$

$$\frac{1}{z^3-z^2+4z-4}=\frac{1}{(z-1)(z-2i)(z+2i)}=\frac{a}{z-1}+\frac{b}{z-2i}+\frac{c}{z+2i}$$

$$\implies 1=a(z-2i)(z+2i)+b(z-1)(z+2i)+c(z-1)(z-2i)$$

$$\implies 1=z^2(a+b+c)+z(2ib-b-2ic-c)+(4a-2ib+2ic)$$

$$\implies \begin{cases} a+b+c=0 \\ 2ib-b-2ic-c=0 \\ 4a-2ib+2ic=1 \end{cases}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2i-1 & -2i-1 \\ 4 & -2i & 2i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2i-1 & -2i-1 \\ 4 & -2i & 2i \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} a \\ b \\ c \end{pmatrix}=\frac{1}{20}\begin{pmatrix} 4 & 4 & 4 \\ 8-4i & -2-4i & -2+i \\ 8+4i & -2+4i & -2-i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} a \\ b \\ c \end{pmatrix}=\frac{1}{20}\begin{pmatrix} 4 \\ -2+i \\ -2-i \end{pmatrix}=\begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10}+\frac{i}{20} \\ -\frac{1}{10}-\frac{i}{20} \end{pmatrix}$$

So we now have:

$$\int_\gamma \frac{dz}{z^3-z^2+4z-4}=\int_\gamma\left[\frac{1}{5(z-1)}-\frac{2+i}{20(z+2i)}-\frac{2-i}{20(z-2i)}\right]dz$$

$$= \frac{1}{5}\int_\gamma \frac{dz}{z-1} - \frac{2+i}{20}\int_\gamma \frac{dz}{z+2i}\,dz - \frac{2-i}{20}\int_\gamma \frac{dz}{z-2i}$$

$$= \frac{1}{5}(2\pi i)\,\mathbf{n}(\gamma;1) - \frac{2+i}{20}(2\pi i)\,\mathbf{n}(\gamma;-2i) - \frac{2-i}{20}(2\pi i)\,\mathbf{n}(\gamma;2i)$$

We know γ only goes once around the circle $C(2+i;3)$ and it does so anti-clockwise. Therefore, each winding number will be either 1 or 0 depending on whether the point is contained within $C(2+i;3)$.

Firstly consider the point $z = 1$:

$$|1-(2+i)| = |-1-i| = \sqrt{2} < 3.$$

$$\implies \text{The point } z = 1 \text{ is contained within } C(2+i;3)$$

$$\text{Hence } \mathbf{n}(\gamma;1) = 1$$

Secondly consider the point $z = -2i$:

$$|-2i-(2+i)| = |-2-3i| = \sqrt{13} > 3.$$

$$\implies \text{The point } z = -2i \text{ is not contained within } C(2+i;3)$$

$$\text{Hence } \mathbf{n}(\gamma;-2i) = 0$$

Finally consider the point $z = 2i$:

$$|2i-(2+i)| = |-2+i| = \sqrt{5} < 3.$$

$$\implies \text{The point } z = 2i \text{ is contained within } C(2+i;3)$$

$$\text{Hence } \mathbf{n}(\gamma;2i) = 1$$

Therefore, we can conclude that:

$$\int_\gamma \frac{dz}{z^3-z^2+4z-4} = \frac{1}{5}(2\pi i)\,\mathbf{n}(\gamma;1) - \frac{2+i}{20}(2\pi i)\,\mathbf{n}(\gamma;-2i) - \frac{2-i}{20}(2\pi i)\,\mathbf{n}(\gamma;2i)$$

$$= \frac{1}{5}(2\pi i)\,\mathbf{n}(\gamma;1) - \frac{2-i}{20}(2\pi i)\,\mathbf{n}(\gamma;2i)$$

$$= 2\pi i \left[\frac{1}{5}\mathbf{n}\left(\gamma;1\right) - \frac{2-i}{20}\mathbf{n}\left(\gamma;2i\right)\right]$$

$$= 2\pi i \left[\frac{1}{5} - \frac{2-i}{20}\right]$$

$$= 2\pi i \left[\frac{2+i}{20}\right]$$

$$= \pi i \left[\frac{2+i}{10}\right]$$

$$= \frac{2\pi i}{10} - \frac{\pi}{10}$$

$$= \frac{\pi}{10}\left[2i - 1\right]$$ ✓

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