MATH381 Assignment 2

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Due: 11:55 PM, Thursday 5 September 2024

Question 1 (4 marks) Prove using the sequential characterisation of closed sets that the intersection of an *arbitrary* collection of closed subsets of $\mathbb{C}$ is again closed. That is, prove that if F_j $(j \in J)$ are closed subsets of $\mathbb{C}$ where J is an arbitrary set of indices, then $\bigcap_{j\in J} F_j$ is closed.

Solution:

$$\text{Set } F = \bigcap_{j\in J} F_j = F_1 \cap F_2 \cap \cdots \cap F_m$$

(J could be uncountable, in which case the sets couldn't be listed in this way)

The specifity of the word arbitrary in relation to the collection of subsets implies m could be infinite. I.e. J is any arbitrary set of indices that is not necessarily finite.

To prove that F is closed using the sequential characterisation of closed sets, we can take a sequence, $(z_n)_{n=1}^{\infty} \in F$ such that $z_n \to z \in \mathbb{C}$. We aim to prove that $z \in F$.

We know $\forall\, j \in J$:

- $z_n \in F_j$ as $z_n \in F = \bigcap_{j\in J} F_j$ ✓
- F_j is closed (as per the question)

It then follows from the sequential characterisation of closed sets that as z_n is convergent to some $z \in \mathbb{C}$, $z_n \in F_j$ $(\forall j \in J)$ and F_j is closed that $z \in F_j$. ✓

Then by the properties of the intersection of an arbitrary collection of subsets of $\mathbb{C}$, as $z \in F_j$ $(\forall j \in J)$, we have $z \in F = \bigcap_{j\in J} F_j$

As the following conditions have now been shown to hold: **1)** $z_n \to z \in \mathbb{C}$ **2)** $z_n \in F$ **3)** $z \in F$, it follows from the sequential characterisation of closed sets that $\bigcap_{j \in J} F_j$ $(= F)$ is closed. $\square$ ✓

4/4

Question 2 (5 marks) Let $f(z) := ln|z| + iArg(z)$, where $\ln : (0, \infty) \to \mathbb{R}$ is the natural logarithm and $Arg(z)$ is the principal argument of z. Prove that f satisfies the Cauchy-Riemann equations on the open upper half-plane.

$$U := \{z : \mathrm{Im}\, z > 0\} \quad \text{which is identified with} \quad \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

by writing $\mathrm{Arg}(x + iy)$ as $\mathrm{arccot}\left(\frac{x}{y}\right)$ when $y > 0$. Recall that the inverse cotangent, arcot is a differentiable function from $\mathbb{R}$ into $(0, \pi)$ with $\mathrm{arccot}'(\mathrm{t}) = -\frac{1}{1+t^2}$ for $t \in \mathbb{R}$.

Solution:

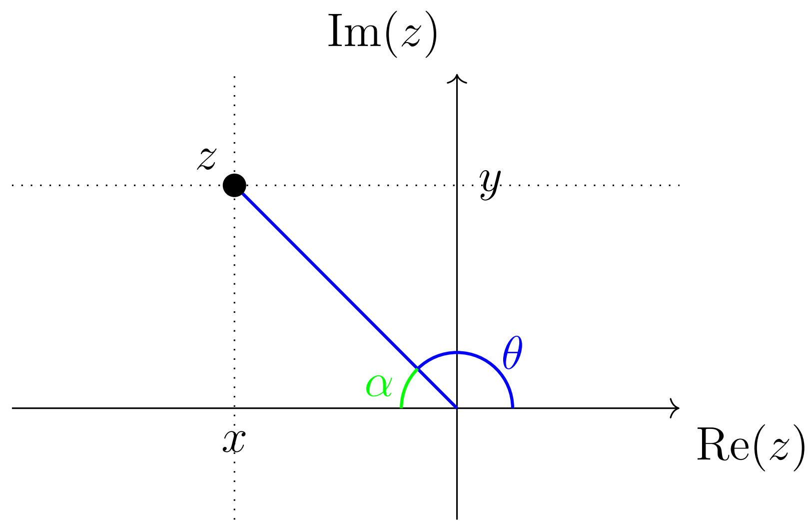
We consider the open upper half-plane, $U := \{z : \mathrm{Im}\, z > 0\}$ as three regions. One region either side of the imaginary axis and another on the imaginary axis. All with imaginary parts greater than 0.

I.e. $U = L \cup I \cup R$ where:

- $L := \{(x, y) : y > 0, x < 0\}$
- $R := \{(x, y) : y > 0, x > 0\}$
- $I := \{(x, y) : y > 0, x = 0\}$

No need to split up since $\mathrm{Arg}(z) = \mathrm{arccot}(x/y)$ everywhere in U.

Case 1: on the open upper left half-plane, $L := \{(x, y) : y > 0, x < 0\}$



$$\cot(\alpha) = \frac{-x}{y}$$

$$\implies \alpha = cot^{-1}\left(\frac{-x}{y}\right)$$

$$\theta = \operatorname{Arg} z = \pi - \alpha = \pi - cot^{-1}\left(\frac{-x}{y}\right) = cot^{-1}\left(\frac{x}{y}\right)$$

So we now have: $f(z) = ln|z| + i\,Arg\,(z) = ln\left(\sqrt{x^2+y^2}\right) + i\,arccot\left(\frac{x}{y}\right)$

[Notice $|z| \subset (0,\infty)$ as $y > 0$ so the domain of ln is satisfied.]

$$f(z) = Re\,(f(z)) + i \cdot Im\,(f(z)) = u(x,y) + i \cdot v(x,y)$$

So we have $u(x,y) = ln\left(\sqrt{x^2+y^2}\right)$ and $v(x,y) = arccot\left(\frac{x}{y}\right)$

Now differentiating the real and imaginary parts partially with respect to both x and y using the chain rule and given derivative of $arccot$, we obtain:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\frac{1}{2} \cdot 2x \cdot (x^2+y^2)^{\frac{-1}{2}}}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{\frac{1}{2} \cdot 2y \cdot (x^2+y^2)^{\frac{-1}{2}}}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v(x,y)}{\partial x} = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v(x,y)}{\partial y} = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{-x}{y^2}\right) = \frac{x}{x^2+y^2}$$

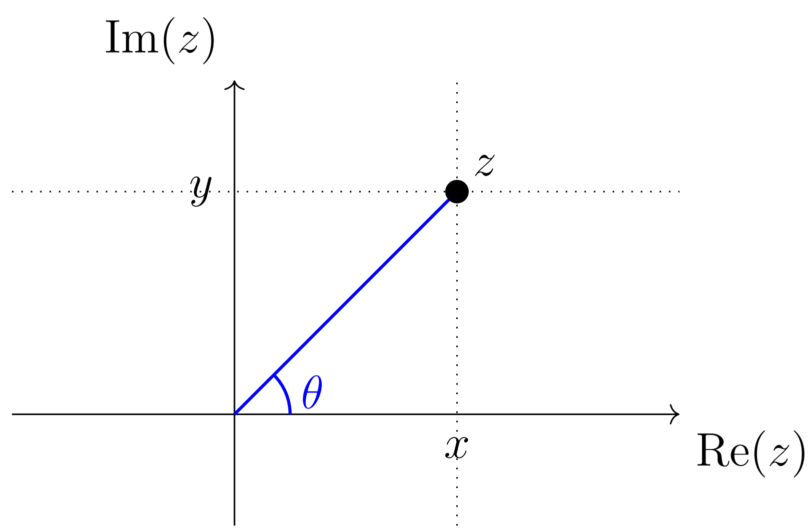
So the Cauchy-Riemann equations:

✓

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

have been satisfied by f on the open upper left half-plane.

Case 2: on the open upper right half-plane, $R := \{(x,y) : y > 0, x > 0\}$



$$\cot(\theta) = \frac{x}{y}$$

$$\implies \theta = \operatorname{Arg} z = cot^{-1}\left(\frac{x}{y}\right)$$

So we now have: $f(z) = ln|z| + i\,Arg\,(z) = ln\left(\sqrt{x^2+y^2}\right) + i\,arccot\left(\frac{x}{y}\right)$ ✓

(And you have already done the Cauchy-Riemann computations for this f(z))

[Notice $|z| \subset (0, \infty)$ as $y > 0$ so the domain of ln is satisfied.]

$$f(z) = Re\,(f(z)) + i \cdot Im\,(f(z)) = u(x,y) + i \cdot v(x,y)$$

So we have $u(x,y) = ln\left(\sqrt{x^2+y^2}\right)$ and $v(x,y) = arccot\left(\dfrac{x}{y}\right)$

Now differentiating the real and imaginary parts partially with respect to both x and y using the chain rule and given derivative of $arccot$, we obtain:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\frac{1}{2} \cdot 2x \cdot (x^2+y^2)^{\frac{-1}{2}}}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{\frac{1}{2} \cdot 2y \cdot (x^2+y^2)^{\frac{-1}{2}}}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v(x,y)}{\partial x} = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right) = \frac{-y}{x^2+y^2}$$

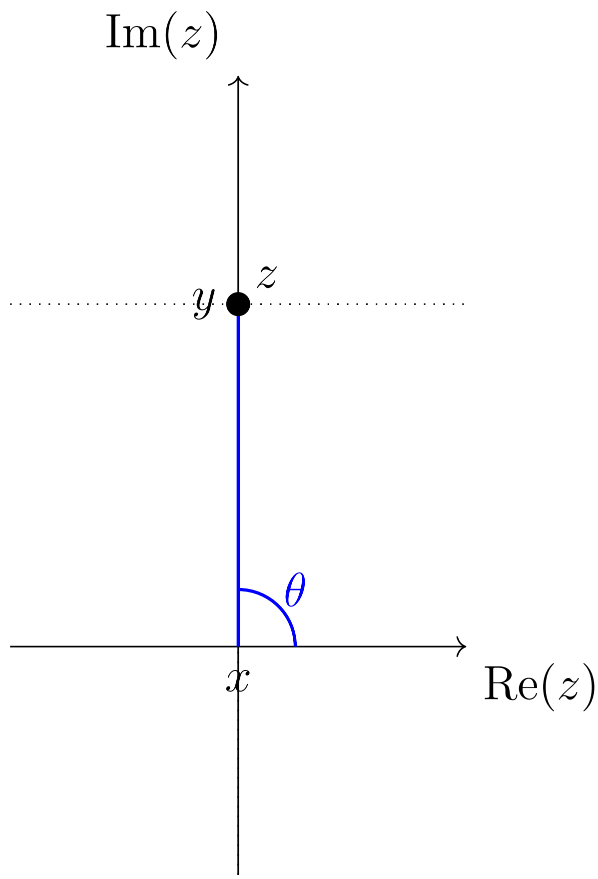
$$\frac{\partial v(x,y)}{\partial y} = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{-x}{y^2}\right) = \frac{x}{x^2+y^2}$$

So the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

have been satisfied by f on the open upper right half-plane.

Case 3: The imaginary axis for $y > 0$, $I := \{(x,y) : y > 0, x = 0\}$



We have already shown that for all $y > 0$, if $x < 0$ or $x > 0$ we have:

$$f(z) = ln|z| + i\,Arg\,(z) = ln\left(\sqrt{x^2 + y^2}\right) + i\,arccot\left(\frac{x}{y}\right)$$

So although at $x = 0$ we have:

$$\mathrm{Arg}z = \theta = \cot^{-1}\left(\frac{x}{y}\right) = \cot^{-1}(0) = \frac{\pi}{2}$$

We need still need to consider $\mathrm{Arg}z = arccot\left(\frac{x}{y}\right)$ to allow for differentiating with respect to x. As when differentiating with respect to x we need to allow for changes in x around the neighbourhood of $x = 0$.

Note this holds as we have now shown that on L and R the expression $arccot\left(\frac{x}{y}\right) = \text{Arg}z$.

Now differentiating the real and imaginary parts partially with respect to both x and y using the chain rule and given derivative of *arccot*, we obtain as previously seen:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\frac{1}{2} \cdot 2x \cdot (x^2+y^2)^{\frac{-1}{2}}}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{\frac{1}{2} \cdot 2y \cdot (x^2+y^2)^{\frac{-1}{2}}}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v(x,y)}{\partial x} = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{y}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v(x,y)}{\partial y} = \frac{-1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{-x}{y^2}\right) = \frac{x}{x^2+y^2}$$

Note that as $y > 0$ all of these terms will be defined.

So the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

have been satisfied by f on the open region I and hence f satisfies the Cauchy Riemann equations on U as $U = L \cup R \cup I$. $\square$ ✓ 5/5

Question 3 (5 marks) Let $f : \mathbb{C} \to \mathbb{C}$, and define $g(z) := \overline{f(\overline{z})}$ for every $z \in \mathbb{C}$. Suppose that f is complex-differentiable at some $c \in \mathbb{C}$. Prove from the definition that g is complex-differentiable at $\overline{c}$.

Solution:

By the complex-differentiability of f at $c \in \mathbb{C}$ we have:

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c} \to L = f'(c) \quad ✓$$

To show that g is complex-differentiable at $\overline{c}$, we need to show:

$$\lim_{z \to \overline{c}} \frac{g(z) - g(\overline{c})}{z - \overline{c}} = g'(\overline{c}) \text{ exists}$$

Given $g(z) := \overline{f(\overline{z})}$:

$$g(z) - g(\overline{c}) = \overline{f(\overline{z})} - \overline{f(\overline{\overline{c}})} = \overline{f(\overline{z})} - \overline{f(c)}$$

So we seek to evaluate the limit:

$$\lim_{z \to \overline{c}} \frac{\overline{f(\overline{z})} - \overline{f(c)}}{z - \overline{c}}$$ ✓

$$= \lim_{z \to \overline{c}} \overline{\left(\frac{f(\overline{z}) - f(c)}{\overline{z - \overline{c}}} \right)}$$

$$= \lim_{z \to \overline{c}} \overline{\left(\frac{f(\overline{z}) - f(c)}{\overline{z} - c} \right)}$$ ✓

Let us make the substitution, $u = \overline{z} \implies z = \overline{u}$.

By the continuity of the conjugate function, as $z \to \overline{c}$, we have $u \to c$.

So we now have:

$$\lim_{u \to c} \overline{\left(\frac{f(u) - f(c)}{u - c} \right)}$$

Now by the continuity of the conjugate function again, we have:

$$\overline{\left[\lim_{u \to c} \left(\frac{f(u) - f(c)}{u - c} \right) \right]}$$

$$= \overline{f'(c)} \in \mathbb{C}$$ ✓

So we have shown that g is complex differentiable at $\overline{c}$ (and furthermore has derivative, $g'(\overline{c}) = \overline{f'(c)}$). $\square$

5/5

Question 4 (4+2 marks) Consider the power series $\sum_{n=1}^{\infty} \frac{(z-2)^n}{n}$.

(i) Determine the radius of convergence R of this power series. You should find that $R > 0$.

(ii) The general theory learned so far then tells us that this power series defines a holomorphic function λ on $D(2; R)$. Determine $\lambda(x)$ for $x \in (2-R, 2+R)$

Solution (i): To determine the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{(z-2)^n}{n}$$

We take:

$$\mathscr{C} := \left\{ r \in [0, \infty) : \text{the sequence } \left(\frac{r^n}{n} \right)_{n=1}^{\infty} \text{ is bounded} \right\}$$

$$\text{If } r \in [0,1] \text{ then } 0 \le r^n \le 1 \quad \forall n \in \mathbb{N}$$
$$\implies 0 \le \frac{r^n}{n} \le \frac{1}{n} \le 1$$

✓

So, if $r \in [0,1]$ then $\frac{r^n}{n}$ is bounded by $[0,1]$ and hence the interval $[0,1]$ is included in $\mathscr{C}$

We can intuitively see that for $r > 1$, $\frac{r^n}{n}$ will be unbounded as the exponential numerator will rapidly outgrow the linear denominator. ✓

Let us prove this formally by trying to contradict this statement and show the sequence is bounded.

$$\text{If } \frac{r^n}{n} \text{ is bounded then } \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, \left| \frac{r^n}{n} \right| \le M$$

$$\text{For } r > 1, n \in \mathbb{N}, \frac{r^n}{n} > 0 \implies \frac{r^n}{n} \le M$$

$$\implies r^n \le nM$$

$$\text{Set } r := 1 + \alpha \text{ so } \alpha > 2$$

$$\text{Then } r^n = (1+\alpha)^n > \frac{n(n-1)}{2}\alpha^2 \quad \forall\, \alpha > 2 \text{ and } n \in \mathbb{N}$$

It is again trivial from this inequality that $\frac{r^n}{n} > (n-1) \cdot \frac{\alpha^2}{2}$ and hence $\frac{r^n}{n}$ is unbounded. However, again let us be slightly more formal.

If we can show that there exists an n such that $\frac{n(n-1)}{2}\alpha^2 > nM$ then we will have a contradiction to the statement that the sequence is bounded.

Rearranging we get:

$$n > \frac{2M}{\alpha^2} + 1 = \frac{2M}{(r-1)^2} + 1 \quad (n \in \mathbb{N})$$

So for whatever $r > 1$ we take, then for whatever bound, M you try to place on the sequence, you can find a term in the sequence such that $\frac{r^n}{n}$ is greater than M.

$$\text{Hence } \frac{r^n}{n} \text{ is unbounded.}$$

$$\text{So we get } \mathscr{C} = \{r \in [0,1]\} \implies R = 1 \quad \square$$

✓ 4/4

Alternatively, we could use the ratio test result obtained in tutorial 7, question 3 and compute:

$$\begin{aligned} R &= lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|} \\ &= lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} \\ &= lim_{n\to\infty}\frac{n+1}{n} = 1 \end{aligned}$$

✓

Solution (ii): We now find $\lambda(x)$ for $x \in (2-R, 2+R) = (1,3)$:

Since the power series has radius of convergence 1, it defines a holomorphic function on $D(2;1)$

$$\lambda(z) := \sum_{n=1}^{\infty} \frac{(z-2)^n}{n} = (z-2) + \frac{(z-2)^2}{2} + \frac{(z-2)^3}{3} \dots \qquad \textbf{(1)}$$

So for $z \in D(2;1)$,

$$\lambda'(z) = 1 + z - 2 + (z-2)^2 + \dots = S = 1 + \sum_{n=1}^{\infty} (z-2)^n$$

Notice this is a geometric series as $|z-2| < 1$, so we have:

$$S = 1 + \frac{z-2}{1-(z-2)} = 1 + \frac{z-2}{3-z}$$
$$= \frac{3-z}{3-z} + \frac{z-2}{3-z} = \frac{1}{3-z}$$

Restricting λ to $\mathbb{R}$ we have $x \in D(2;1) \cap \mathbb{R} = (1,3)$

$$\implies \lambda(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} \dots \in \mathbb{R}$$

So $\lambda : (1,3) \to \mathbb{R}$ is a real valued function with:

$$\lambda'(x) = \frac{1}{3-x}$$

$$\implies \lambda(x) = -ln\,|3-x| + C \quad (x \in (1,3))$$

Notice that from **(1)** we have: $\lambda(2) = 0 = -ln\,|1| + C = C \implies C = 0$

$$\implies \lambda(x) = -ln\,|3-x| \quad \text{for } x \in (1,3)$$

(Note: absolute value not needed as $x \in (1,3) \Rightarrow 3-x>0$)

2/2

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