Definitions

S Stock price
X Strike price
T Time to Maturity
r Riskfree interest rate
t Time

P (synthetic) Portfolio
f Price of derivative instrument (put or call option)
μ Return of Stock
σ Volatility of Stock

Conditions

Stock price as an Itô process (geometric Brownian motion)

$$dS = \mu S dt + \sigma S dz$$

Itô's Lemma applied to any function f(S,t)

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$

Derivation

Portfolio to eliminate Wiener process (delta hedge portfolio)

$$P = -f + \frac{\partial f}{\partial S} S \qquad dP = -df + \frac{\partial f}{\partial S} dS$$

Substitute df and dS in left side of equation

$$dP = -\left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt - \frac{\partial f}{\partial S} \sigma S dz + \frac{\partial f}{\partial S} (\mu S dt + \sigma S dz)$$

$$dP = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt$$

Portfolio to eliminate arbitrage condition (riskfree portfolio)

$$dP = rP dt = \left(-rf + \frac{\partial f}{\partial S} rS \right) dt$$

Black Scholes partial differential equation

Combination of the two last equations to eliminate dP and dt leads to

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 - rf = 0$$

Boundary Condition

$$f = \max(S - X, 0) \qquad t = T$$