Stochastic processes

Wiener process

A Wiener process $W(t)$ or Standard Brownian Motion is a stochastic process with the following properties :

1. $W(0) = 0$.
2. Non-overlapping increments are independent : $0 \leq t < T \leq s < S$, the increments $W(T) - W(t)$ and $W(S) - W(s)$ are independent random variables.
3. For $t \leq s$ the increment $W(s) - W(t)$ is a normal random variable, with zero mean and variance $s - t$.
4. $\forall \omega \in \Omega$ the path $t \to W(t)$ is a continuous function

For each $t > 0$ the random variable $W(t) = W(t) - W(0)$ is the increment in $[0; t]$: it is normally distributed with zero mean, variance t and density

$$f(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. This is known as Donsker's theorem. Unlike the random walk, it is scale invariant

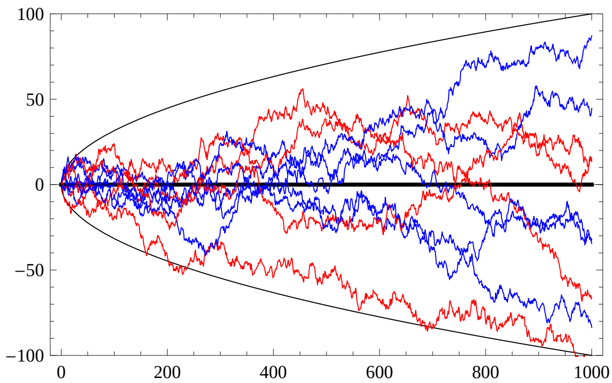

FIGURE 1 – Wiener process

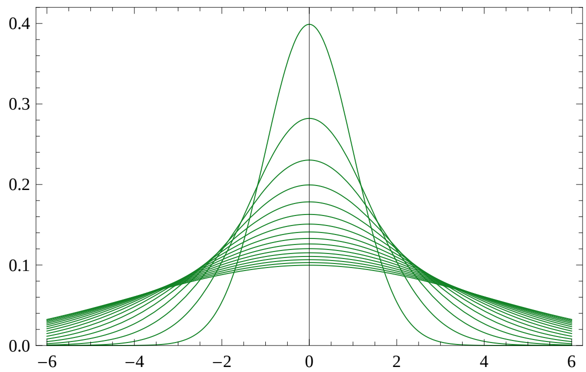

FIGURE 2 – Distribution wave in Wiener process

Brownian Motion

A Brownian Motion BM or a generalised Wiener process $X(t)$ is the solution of an stochastic differential equation SDE with constant drift and diffusion coefficients The distribution function of the generalised Wiener process is a decreasing bell-shaped curve.

$$dX(t) = \mu dt + \sigma dW(t)$$

with initial value $X(0) = x_0$. By direct integration

$$X(t) = x_0 + \mu t + \sigma W(t)$$

and hence $X(t)$ is normally distributed, with mean $x_0 + \mu t$ and variance $\sigma^2 t$. Its density function is

$$f(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}}$$

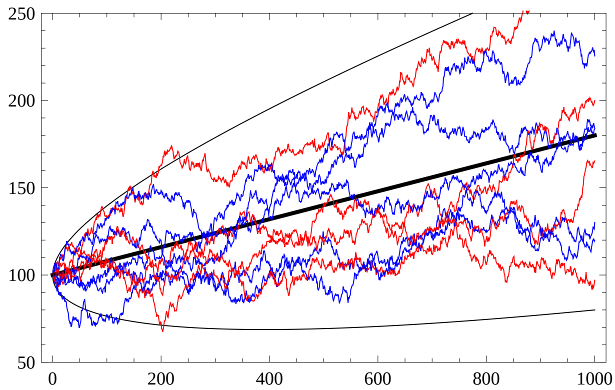

FIGURE 3 – Brownian motion

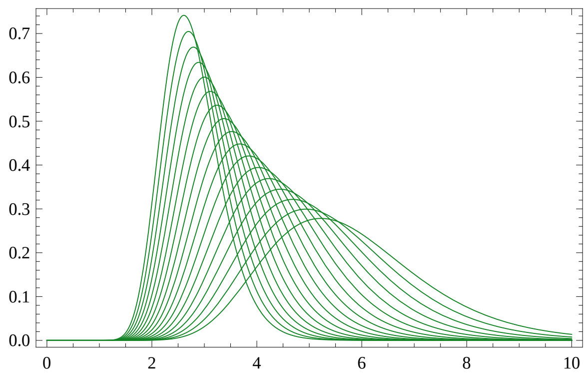

FIGURE 4 – Distribution wave in Brownian motion

Geometric Brownian Motion

A Geometric Brownian Motion GBM $X(t)$ is the solution of an SDE with linear drift and diffusion coeffcients

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

with initial value $X(0) = x_0$. A straightforward application of Itô's lemma (to $F(X) = log(X)$) yields the solution

$$X(t) = x_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

and hence $X(t)$ is lognormally distributed, with

$$\begin{array}{lrcl} \text{mean} & E\left(X(t)\right) & = & x_0 e^{\mu t} \\ \text{variance} & Var\left(X(t)\right) & = & x_0^2 e^{2\mu t}\left(e^{\sigma^2} - 1\right) \\ \text{density} & f(t,x) & = & \frac{1}{\sigma x \sqrt{2\pi t}} e^{-\left(\log x - \log x_0 - \left(\mu - 1/2\sigma^2\right)^2 / 2\sigma^2 t\right)} \end{array}$$

A geometric Brownian motion, occasionally called exponential Brownian motion, is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion.It is applicable to mathematical modelling of some phenomena in financial markets. It is used particularly in the field of option pricing because a quantity that follows a GBM may take any positive value, and only the fractional changes of the random variate are significant. This is a reasonable approximation of stock price dynamics except for rare events

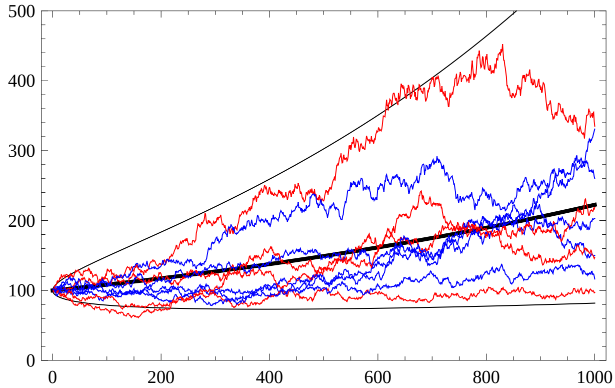

FIGURE 5 – Geometric Brownian motion

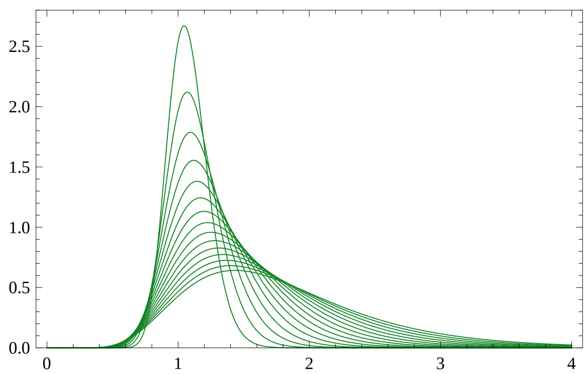

FIGURE 6 – Distribution wave in Geometric Brownian motion

Itô's Lemma

Brownian Motion $dX(t) = \mu dt + \sigma dW(t)$ and geometric Brownian Motion $dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$ are generalized to an *Itô's (drift diffusion) process*

$$dX(t) = \mu(X,t)dt + \sigma(X,t)dW(t)$$

where the coefficients $\mu(X,t)$ and $\sigma(X,t)$ are functions of X and t. Itô's Lemma shows that functions $f(X,t)$, where one of the variables is an Itô's process, are also Itô's process with drifted coefficients. To simplify we write the equation $dX(t) = \mu(X,t)dt + \sigma(X,t)dW(t)$ in the short form $dx = \mu dt + \sigma dw$.

Itô's Lemma
For x following an Itô's process

$$dx = \mu dt + \sigma dw$$

and a twice differentiable function $f(x,t)$ then f follows the Itô's process

$$df = \left(\frac{\partial f}{\partial x}\mu + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right)dt + \frac{\partial f}{\partial x}\sigma dw$$

Proof
The Taylor series of $df = f(x+dx, t+dt) - f(x,t)$ is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial x \partial t}dxdt + \frac{\partial^2 f}{\partial t^2}dt^2 + \dots$$

Since dw is a Wiener process we have $E(dw) = 0$ and $Var(dw) = dt$ so $dw = O(\sqrt{Var}) = O(\sqrt{dt})$ where $O(.)$ is the Landau symbol. This means that dz is growing in the order of $\sqrt{dt}$. We have then

$$\begin{aligned}
dx^2 &= (\mu dt + \sigma dw)^2 \\
&= \mu^2 dt^2 + 2\mu\sigma dt dw + \sigma^2 dz^2 \\
&= \sigma^2 dt + O(dt^{3/2}) \\
dxdt &= O(dt^{3/2}) \\
dt^2 &= O(dt^2)
\end{aligned}$$

Replacing $dx^2, dxdt$ and dt^2 in df gives

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma dt + O(dt^{3/2})$$

Replacing $dx = \mu dt + \sigma dw$ in df gives

$$df = \left(\frac{\partial f}{\partial x}\mu + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right)dt + \frac{\partial f}{\partial x}\sigma dw$$

□

Black Scholes model

The Black–Scholes model is a mathematical description of financial markets and derivative investment instruments. The model develops partial differential equations whose solution, the Black–Scholes formula, is widely used in the pricing of European-style options.

The Black–Scholes model of the market for a particular equity makes the following explicit assumptions :

- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- The price follows a *Geometric Brownian motion* with constant drift and volatility. It follows from this that the *return* has a *Normal distribution* and the *price of the underlying* has a *Log-normal distribution*.
- There are no transaction costs or taxes.
- All securities are infinitely divisible.
- There are no restrictions on short selling.

To understand the Black–Scholes model we table the variables of the model

DEFINITIONS

S	Stock price
K	Strike price
T	Time to maturity
t	Time
P	(synthetic) Portfolio
f	Price of option
μ	Mean return of stock
σ	Volatility of stock

CONDITIONS

As per the assumptions, the price of the underlying asset is a geometric Brownian motion

$$dS = \mu S dt + \sigma S dw$$

Then by Itô's Lemma, setting $x = S$ and taking μ and σ as linear functions in the variable $S : \mu(S,t) = \mu S$ and $\sigma(S,t) = \sigma S$, we get

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial x}\sigma S dw$$

DERIVATION

To find the Black Scholes differential equation, the no arbitrage theorem is used. We consider an synthetic portfolio $P = -f + \alpha S$ combined of a short position in an Option with price f and a long position in the underlying stock with price S. The weight of S is chosen as the (constant) value $\alpha = \partial f / \partial S$:

$$P = -f + \frac{\partial f}{\partial S}S \quad dP = -df + \frac{\partial f}{\partial S}dS$$

Substituting df and dS in the equation on the left hand leads to

$$\begin{aligned} dP &= -\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial x}\sigma S dw + \frac{\partial f}{\partial S}\left(\mu S dt + \sigma S dw\right) \\ &= \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt \end{aligned}$$

To eliminate arbitrage, the portfolio change dP is hedged with a riskfree security :

$$dP = rPdt = \left(-rf + \frac{\partial f}{\partial S}rS\right) dt$$

Combining these equations and eliminating dP and dt leads to the
BLACK SCHOLES PARTIAL DIFFERENTIAL EQUATION

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - rf = 0$$