

# Algebraic Geometry(rv)

(24fall)quiddite

This is a very very brief note based on a course lectured by Prof. Zhang, which covers roughly the first 2 chapters of [1], with more examples. Good references are [1, 2, 3, 4, 5, 6, 7](the first three books are used frequently) & [8]. I type it in order to review & tide up my mind, so don't blame me for the abundant typos & mis-usages of symbols, terms blabla.

## 目录

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Varieties</b>                               | <b>3</b>  |
| <b>2</b> | <b>Schemes</b>                                 | <b>3</b>  |
| 2.1      | Schemes . . . . .                              | 3         |
| 2.2      | Properties of schemes & morphisms I . . . . .  | 6         |
| 2.3      | Properties of schemes & morphisms II . . . . . | 9         |
| 2.4      | Quasi-coherent sheaves . . . . .               | 9         |
| 2.5      | Projective sheaves . . . . .                   | 9         |
| <b>A</b> | <b>Category theory</b>                         | <b>10</b> |
| A.1      | colimit & limit . . . . .                      | 10        |
| <b>B</b> | <b>Commutative algebra</b>                     | <b>12</b> |
| B.1      | Valuation rings . . . . .                      | 12        |
| B.2      | Jacobson rings . . . . .                       | 14        |
| B.3      | Nakayama's lemma . . . . .                     | 15        |



# 1 Varieties

## 2 Schemes

### 2.1 Schemes

**Definition 2.1** (spectrum of a ring). *Let  $A$  be a ring, the spectrum is a ringed space  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$  given by*

- (1)  $\text{Spec } A$  with the Zariski topology;
- (2)  $\mathcal{O}_{\text{Spec } A}(U) = \{s : u \rightarrow \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mid s(\mathfrak{p}) \in A_{\mathfrak{p}}\}$ .

Luckily, the tedious construction above is used not that often. We always simply use the properties suggested by the following proposition.

**Proposition 2.2** (\*). *Let  $A$  be a ring,*

- (1) *for any  $\mathfrak{p} \in \text{Spec } A$ ,  $\mathcal{O}_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ ;*
- (2) *for any  $f \in A$ ,  $\mathcal{O}_{\text{Spec } A}(D(f)) \cong A_f$ ;*
- (3) *as a result of (2),  $\mathcal{O}_{\text{Spec } A}(\text{Spec } A) \cong A$ .*

**Definition 2.3** (ringed spaces and morphisms).

- (1) *A ringed space is a pair  $(X, \mathcal{O}_X)$ ;*
- (2) *A locally ringed space is a r.s. whose stalks  $\mathcal{O}_{X,P}$  are local rings  $\forall P \in X$ ;*
- (3) *A morphism between r.s.'s  $(X, \mathcal{O}_X) \mathcal{E} (Y, \mathcal{O}_Y)$  is a pair  $(f, f^\#)$ , where  $f : X \xrightarrow{\text{conti}} Y \mathcal{E} f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ ;*
- (4) *A morphism between l.r.s.'s is a morphism  $X \xrightarrow{f} Y$  between r.s.'s, which induces **local** homomorphisms  $f_p^\# : \mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$ , i.e.  $(f_p^\#)^{-1}$  preserves the maximal ideal.*

**Proposition 2.4.**

- (1)  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$  is a l.r.s.;
- (2) The set of morphisms  $(f, f^\#)$  between l.r.s.'s  $(\text{Spec } B, \mathcal{O}_{\text{Spec } B})$  &  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$  consists exactly of the morphisms induced by some  $\varphi : A \xrightarrow{\text{homo}} B$ .

Now we can define schemes.

**Definition 2.5** (schemes).

- (1) An affine scheme is a l.r.s.  $(X, \mathcal{O}_X)$  which is isomorphic to some  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ ;
- (2) A scheme is a l.r.s.  $(X, \mathcal{O}_X)$  which is locally affine, i.e.  $\exists$  an open cover  $\{U\}$  s.t. each  $(U, \mathcal{O}_X|_U)$  is an affine scheme;
- (3) A morphism of schemes is a morphisms of l.r.s.'s.

**Example 2.6** (schemes). In these examples,  $k = \text{alg.cl } k$ .

- (1) If  $R$  is a d.v.r., then  $\text{Spec } R = \{\circ, \bullet\}$ , where  $\circ$  is a generic point and  $\bullet$  is a closed point(see [1]P.74 for detailed explanation);
- (2)  $\mathbb{A}_k^1 = \text{Spec } k[x] = \{\circ\} \cup k$ , where  $\{\circ\}$  is a generic point and points in  $k$  are all closed;
- (3)  $\mathbb{A}_k^2 = \text{Spec } k[x, y] = \{(0)\} \cup \{f \in k[x, y] \mid f \text{ is irreducible}\} = \{\circ\} \cup k^2 \cup \{f \in k[x, y] \mid f \text{ is irreducible, } \deg f \geq 2\}$ . The first part is the generic point, the second part consists of closed points, and the third part consists of generic points of such curves  $f(x, y) = 0$ .
- (4) ( $\ast$ affine line with a doubled point) Let  $X_1 = X_2 = \mathbb{A}_k^1, U_1 = U_2 = \mathbb{A}_k^1 \setminus \{0\}$ . Glueing  $X_1$  &  $X_2$  along  $U_1$  &  $U_2$  via the identity map  $U_1 \rightarrow U_2$ , nothing is done except for  $\{0\}$ .

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this gives a non-affine scheme.

**Proposition 2.7** (generic points). *Let  $X$  be a scheme, then every non-empty irreducible closed subset  $Y$  of  $X$  has a unique generic point, i.e. a point  $p \in Y$  s.t.  $\overline{\{p\}} = Y$*

Let  $U = \text{Spec } A$  be an affine open subset of  $X$  s.t.  $U \cap Y \neq \emptyset$ , then  $U \cap Y$  is an irreducible closed subset of  $U$  (i.e. “reduced” to affine case), thus  $U \cap Y = V(p)$  for some  $p \in \text{Spec } A$ . Obviously,  $U \cap Y = \overline{\{p\}}^U = \overline{\{p\}} \cap U$ . At the same time,  $U \cap Y \neq \emptyset$  is open in  $Y$ , from the irreducibility,  $\overline{U \cap Y}^Y = Y$ , so  $Y \subset \overline{\{p\}}$ , i.e.  $\overline{\{p\}} = Y$ . For the uniqueness, if  $y = \overline{\{p\}} = \overline{\{p'\}}$ , then  $V(p) = U \cap Y = V(p')$ , thus  $p = p'$ .

Now we are going to a criterion for affine-ness (see [1]P.81 or [3]P.28).

**Procedure 2.8** (Construction of  $X_f$ ). *Let  $X$  be a scheme,  $f \in \mathcal{O}_X(X)$*

1.  $X_f = \{p \in X \mid f_p \notin \mathfrak{m}_p = \mathfrak{m}_{\mathcal{O}_p} (\text{equivalently, } f_p \text{ is invertible in } \mathcal{O}_p)\};$

2. *properties of  $X_f$ :*

(a)  $X_f$  **is open in  $X$** ;

(b)  $X_f \cap X_g = X_{fg}$ ;

(c) **if  $X$  has a finite cover  $\{U_i\}$ , s.t. each  $U_i \cap U_j$  is q.c., then  $\mathcal{O}_X(X_f) = (\mathcal{O}_X(X))_f$ .**

**Proposition 2.9** ( $\star$ criterion for affine-ness). *Let  $X$  be a scheme, then  $X$  is affine  $\iff \exists$  finitely many  $\{f_i\}$  s.t.*

(1)  $X_{f_i}$  *are affine*;

(2)  $\{f_i\}$  *generates  $\mathcal{O}_X(X)$ .*

**Definition 2.10** (residue field). *Let  $X$  be a scheme,  $(\mathcal{O}_x, \mathfrak{m}_x)$  be the local ring at  $x \in X$ .  $k(x) = \mathcal{O}_x / \mathfrak{m}_x$  is called the residue field of  $x$ .*

**Remark 2.11.** *In order to define a morphism  $f : \text{Spec } K \rightarrow X$ , where  $K$  is a field, it suffices to identify a point  $x \in X$  & an inclusion  $k(x) \hookrightarrow K$ . e.g.  $k(x) \xrightarrow{\text{id}} k(x) \hookrightarrow \text{Spec } k(x) \hookrightarrow X$ .*

## 2.2 Properties of schemes & morphisms I

Let's begin with an annoying table of definitions.

**Definition 2.12** (some special schemes). *A scheme  $X$  is called*

- (1) *quasi-compact, if  $\text{sp}(X)$  is q.c.;*
- (2) *connected, if  $\text{sp}(X)$  is connected;*
- (3) *irreducible, if  $\text{sp}(X)$  is irreducible;*
- (4) *reduced, if  $\forall U \overset{\text{open}}{\subset} X$ ,  $\mathcal{O}_X(U)$  is reduced, i.e.  $\text{nil}(\mathcal{O}_X(U)) = \{0\}$ ;*
- (5) *integral, if  $\forall U \overset{\text{open}}{\subset} X$ ,  $\mathcal{O}_X(U)$  is a domain;*
- (6) *locally noetherian, if  $\forall U = \text{Spec } A \overset{\text{open}}{\subset} X$ ,  $A$  is noetherian;*
- (7) *noetherian, if  $X$  is l.n. & q.c.;*

**Remark 2.13.** *The condition (6) can be replaced with “ $\exists$  a cover  $\{U_i\}$  of  $X$ , where  $U_i = \text{Spec } A_i \overset{\text{open}}{\subset} X$ , each  $A_i$  is noetherian”. The equivalence between “ $\forall U$ ” & “ $\exists$  a cover  $\{U_i\}$ ” also holds for (1)(2)(3)(5).*

Here's some connections between these definitions.

**Proposition 2.14.** *Let  $X$  be a scheme,*

- (1)  *$X$  is integral  $\iff X$  is reduced & irreducible;*
- (2) *if  $X = \text{Spec } A$  is affine, then  $X$  is noetherian  $\iff A$  is noetherian;*

Let's continue with an annoying table of definitions.

**Definition 2.15** (some special morphisms). *Let  $f : X \rightarrow Y$  be a morphism between schemes,  $f$  is (called)*

- (1) *locally of finite type, if  $\forall V = \text{Spec } B \overset{\text{open}}{\subset} Y, \exists$  a cover  $\{U_j\}$  of  $f^{-1}(V)$ , where  $U_j = \text{Spec } A_j \overset{\text{open}}{\subset} X$ , each  $A_j$  is a f.g.  $B$ -algebra;*

- (2) of finite type, if  $\forall V = \operatorname{Spec} B \xrightarrow{\text{open}} Y, \exists$  a **finite** cover  $\{U_j\}$  of  $f^{-1}(V)$ , where  $U_j = \operatorname{Spec} A_j \xrightarrow{\text{open}} X$ , each  $A_j$  is a f.g.  **$B$ -algebra**;;
- (3) finite, if  $\forall V = \operatorname{Spec} B \xrightarrow{\text{open}} Y, f^{-1}(V) = \operatorname{Spec} A \xrightarrow{\text{open}} X$ , where  $A$  is a f.g.  **$B$ -module**;
- (4) quasi-finite, if  $\forall y \in Y, f^{-1}(y)$  is a finite set;
- (5) quasi-compact, if  $\forall V = \operatorname{Spec} B \xrightarrow{\text{open}} Y, f^{-1}(V)$  is q.c..

Here's a famous & useful trick.

**Proposition 2.16** (Nike's trick). *Let  $X$  be a scheme,  $\operatorname{Spec} A, \operatorname{Spec} B \xrightarrow{\text{open}} X$ , then  $\operatorname{Spec} A \cap \operatorname{Spec} B$  is covered by (principle) open  $\{\operatorname{Spec} C\}$ , which is open both in  $\operatorname{Spec} A$  &  $\operatorname{Spec} B$ .*

$\forall p \in \operatorname{Spec} A \cap \operatorname{Spec} B$ , take  $f \in A, g \in B$  s.t.  $p \in D_{\operatorname{Spec} B}(g) \subset D_{\operatorname{Spec} A}(f) \subset \operatorname{Spec} A \cap \operatorname{Spec} B$ .

Let  $g' = g|_{D_{\operatorname{Spec} A}(f)} \in \mathcal{O}_{\operatorname{Spec} A}(D_{\operatorname{Spec} A}(f)) = A_f$  (since  $D_{\operatorname{Spec} A}(f) \subset \operatorname{Spec} B$ , this can be done). Then we write  $g' = \frac{h}{f^n}$ , where  $h \in A, n \in \mathbb{N}$ .

$$D_{\operatorname{Spec} B}(g) = D_{\operatorname{Spec} A_f}(g') = \operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fh},$$

where “=” holds on the “set” level. Thus  $D_{\operatorname{Spec} B}(g)$  is open in  $\operatorname{Spec} B$ .

**Remark 2.17.** *As for intersections of the form  $U \cap \operatorname{Spec} A$ , where  $U$  is an arbitrary open set, the result is easier (since openness is “weaker”):  $U \cap \operatorname{Spec} A$  is covered by open  $\{D_{\operatorname{Spec} A}(f)\}$ .*

**Proposition 2.18** (closed points). *Let  $X$  be a scheme which is of finite type over a field  $k$ , then the set of closed points is dense in  $X$ .*

According to the condition, we have a finite cover  $\{U_i\}$  of  $X$  ( $\text{Spec } k$  is a singleton), where  $U_i = \text{Spec } A_i \xrightarrow{\text{open}} X$ , each  $A_i$  is a f.g.  $k$ -algebra. Only need to prove that,  $\forall U = \text{Spec } B \xrightarrow{\text{open}} X$ , it contains a closed point of  $X$ . Let  $p$  be a closed point in  $U$ , and consider  $U_i \ni p$ . Using 2.16, take a principle open set  $p \in D(f) \neq \emptyset$  in  $U_i \cap U$ . The inclusion  $i : D(f) \rightarrow U_i$  induces  $i^\# : A_i \rightarrow (A_i)_f$  between Jacobson rings, so  $p = i(p)$  is closed in  $U_i$ . Thus  $p$  is closed in  $X$ . The existence of such  $p \in U$  follows the existence of maximal ideals (reduce to affine case).

**Remark 2.19.** 2.18 fails generally, e.g.  $(1)X = \{\circ, \bullet\}$ .

problem 3.3 & 3.13

**Definition 2.20** (open & closed immersions). Let  $f : X \rightarrow Y$  be a morphism of schemes,  $f$  is called a

- (1) *open immersion*, if  $(X, \mathcal{O}_X) \xrightarrow{f} (Z, \mathcal{O}_Z)$ , for some open subscheme  $(Z, \mathcal{O}_Z)$  of  $Y$ ;
- (2) *closed immersion*, if  $\text{sp}(X) \xrightarrow{f} \text{sp}(Z) \hookrightarrow \text{sp}(Y)$  &  $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  is surjective;
- (3) *immersion*, if  $f$  can be factorized as  $h \circ g : X \rightarrow U \rightarrow Y$ , where  $g : X \rightarrow U$  is a closed imm. &  $h : U \rightarrow Y$  is an open imm..
- (4) 2 closed imm.'s  $f_1 : X_1 \rightarrow Y, f_2 : X_2 \rightarrow Y$  are equivalent if  $\exists$  an isom.  $g : X_1 \rightarrow X_2$  s.t. the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y \\ \sim \downarrow & \nearrow f_2 & \\ X_2 & & \end{array}$$

The following proposition characterizes closed immersions in affine case.



**Proposition 2.21.** *Let  $A$  be a ring,  $X$  be a scheme.  $X \rightarrow \operatorname{Spec} A$  is a closed imm.  $\iff (X, \mathcal{O}_X) \cong (\operatorname{Spec} A/\mathfrak{a}, \mathcal{O}_{\operatorname{Spec} A})$  for some ideal  $\mathfrak{a}$  of  $A$ .*

**Definition 2.22** (fiber product & fiber).

(1) *Let  $X, Y$  be schemes over  $S$ , the fiber product  $X \times_S Y$  is defined by the following diagram of morphisms:*

$$\begin{array}{ccccc}
 & & Z & \xrightarrow{\quad} & \\
 & \searrow & \downarrow & \nearrow & \\
 & & X \times_S Y & \xrightarrow{\operatorname{pr}_1} & X \\
 & & \downarrow \operatorname{pr}_2 & & \downarrow \\
 & & Y & \longrightarrow & S
 \end{array}$$

(2) *the fiber of  $f : X \rightarrow Y$  at  $y$  is defined by  $X_y = X \times_Y \operatorname{Spec} k(y)$*

$$\begin{array}{ccc}
 X \times_Y \operatorname{Spec} k(y) & \xrightarrow{\operatorname{pr}_1} & X \\
 \downarrow \operatorname{pr}_2 & & \downarrow \\
 \operatorname{Spec} k(y) & \hookrightarrow & Y
 \end{array}$$

where “ $\hookrightarrow$ ” exists in the sense of 2.11.

## 2.3 Properties of schemes & morphisms II

## 2.4 Quasi-coherent sheaves

## 2.5 Projective sheaves

## A Category theory

### A.1 colimit & limit

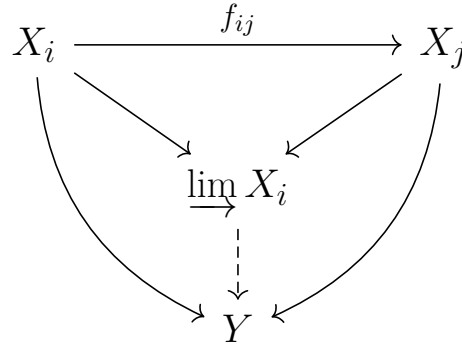
**Definition A.1** (direct system). *Let  $I$  be a directed set, a direct system  $\{X_i, f_{ij}\}$  over  $I$  consists of a family of objects  $\{X_i\}_{i \in I}$  & morphisms  $f_{ij} : X_i \rightarrow X_j$  s.t.*

$$(1) f_{ii} = \text{id}_{X_i}, \forall i;$$

$$(2) f_{ik} = f_{jk} \circ f_{ij}, \forall i \leq j \leq k.$$

“colimit” has many names, including “direct limit”, “inductive limit”.

**Definition A.2** (colimit). *Let  $\{X_i, f_{ij}\}$  be a direct system, then colimit  $\varinjlim X_i$  is defined by the following diagram.*



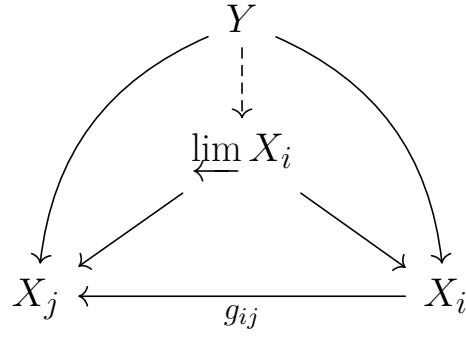
**Definition A.3** (inverse system). *Let  $I$  be a directed set, a inverse system  $\{X_i, g_{ij}\}$  over  $I$  consists of a family of objects  $\{X_i\}_{i \in I}$  & morphisms  $g_{ij} : X_j \rightarrow X_i$  s.t.*

$$(1) g_{ii} = \text{id}_{X_i}, \forall i;$$

$$(2) g_{ik} = g_{ij} \circ g_{jk}, \forall i \leq j \leq k.$$

“limit” has many names, including “inverse limit”, “projective limit”.

**Definition A.4** (limit). *Let  $\{X_i, g_{ij}\}$  be an inverse system, then limit  $\varprojlim X_i$  is defined by the following diagram.*



**Example A.5** (colimit & limit).

(1) Let  $I$  be equipped with the discrete order ( $i \leq j \iff i = j$ ),  $\{X_i\}$  be a family of objects, then

(a)  $\varinjlim X_i = \coprod X_i$ , it's called the sum or coproduct;

(b)  $\varprojlim X_i = \prod X_i$ , it's called the product.

(2) Let  $I = \emptyset$ ,

(a) the colimit coincides with the initial object;

(b) the limit coincides with the terminal object.

(3) In the category of  $R$ -algebras,  $A \coprod B = A \otimes_R B$ ;

(4) Let  $I = \{a, b, c\}$ , where  $a \leq b, c$ ,  $\{X_i\}$  be a family of objects, then

$$\varprojlim X_i = X_b \times_{X_a} X_c$$

**Proposition A.6** (with adjoint functors). Let  $\mathcal{C}, \mathcal{D}$  be 2 categories,  $F, G$  be a pair of adjoint functors, i.e.

$$(1) \mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}; \quad (2) \operatorname{Hom}_{\mathcal{C}}(G(-), \star) \cong \operatorname{Hom}_{\mathcal{D}}(-, F(\star)).$$

Let  $I$  be a directed set,

(1)  $\{Y_i\} \subset \mathcal{D}$  be a direct system, then  $G(\varinjlim Y_i) = \varinjlim G(Y_i)$ ;

(2)  $\{X_i\} \subset \mathcal{C}$  be an inverse system, then  $F(\varprojlim X_i) = \varprojlim F(X_i)$ .

## B Commutative algebra

### B.1 Valuation rings

**Definition B.1** (valuation rings). *Let  $k$  be a field,  $A$  be a subring (thus a domain) of  $k$ . We say  $A$  is a valuation ring of  $k$  if  $\forall x \neq 0 \in k$ , either  $x \in A$  or  $\frac{1}{x} \in k$ .*

**Proposition B.2** (properties of v.r.s). *Let  $A$  be a v.r. of  $k$ .*

- (1)  *$A$  is a local ring, and  $\mathfrak{m}_A = \{x \in A \mid x \text{ is not invertible}\} = \{x \neq 0 \in A \mid \frac{1}{x} \notin A\} \cup \{0\}$ ;*
- (2)  *$A$  is integrally closed in  $k$ ;*
- (3) *if  $B$  is a ring s.t.  $A \subset B \subset k$ , the  $B$  is also a v.r. of  $k$ . Moreover,*
  - (a)  $\mathfrak{m}_B \subset A$ ;
  - (b)  $\mathfrak{m}_B$  is a prime ideal of  $A$ ;
  - (c)  $B = A_{\mathfrak{m}_B}$ , i.e.  $B$  is a local ring of  $A$
- (4)  *$\forall 2$  ideals  $\mathfrak{a}, \mathfrak{b}$  of  $A$ , either  $\mathfrak{a} \subset \mathfrak{b}$  or  $\mathfrak{a} \supset \mathfrak{b}$ . Moreover, if any subring  $B$  of  $k$  with this comparable properties, must be a v.r..*

Now we are going to construct v.r.'s of a field  $k$ .

**Procedure B.3.** *Fix a field  $k$  and an algebraically closed field  $\Omega$ .*

1.  $\Sigma = \{(A, f) \mid A \subset k, f : A \xrightarrow{\text{homo}} \Omega\}$ ;
2. *define a partial order on  $\Sigma$ :*

$$(A, f) \leq (B, g) \iff A \subset B \text{ \& } g|_A = f,$$

*then  $\Sigma$  has at least one maximal element (Zorn's lemma);*

3. *let  $(B, g)$  be a maximal element of  $\Sigma$ , then*
  - (a)  *$(B, g)$  is a local ring &  $\mathfrak{m}_B = \ker g$ ;*

**(b)  $(B, g)$  is a v.r. of  $k$ .**

**Corollary B.4.** *Let  $A$  be a subring of  $k$ , then  $\text{int.cl } A = \cap B$ , where the intersection is taken over  $\{B \mid A \subset B \subset k \text{ \& } B \text{ is a v.r. of } k\}$ .*

- Obviously  $\text{int.cl } A \subset \cap B$ ;
- Conversely, if  $x \in \text{int.cl } A$  but  $x \notin A$ , let  $B = A[\frac{1}{x}]$ , then  $\frac{1}{x}$  is not invertible in  $B$ . Let  $\mathfrak{m}$  be a maximal ideal of  $B$  s.t.  $\frac{1}{x} \in \mathfrak{m}$ , and let  $\Omega = \text{alg.cl } B/\mathfrak{m}$ . The quotient gives a map  $f : B \rightarrow \Omega$ . From B.3,  $(B, f)$  can be extended to some valuation ring  $(C, g)$ . But  $f(\frac{1}{x}) = 0$ , thus  $\frac{1}{x} \in \ker C$ , i.e  $x \notin C$ .

There's another construction which happens to be equivalent to B.3.

**Procedure B.5.** *Fix a field  $k$ .*

1.  $\Sigma = \{(A, \mathfrak{m}) \mid A \subset k \text{ is a local ring with maximal ideal } \mathfrak{m}\}$ ;
2. *define a partial order(called dominance) on  $\Sigma$ :*

$$(A, \mathfrak{m}) \leq (B, \mathfrak{n}) \iff A \subset B \text{ \& } \mathfrak{m} \subset \mathfrak{n},$$

*then  $\Sigma$  has at least one maximal element;*

3.  **$(A, \mathfrak{m})$  is a maximal element of  $\Sigma \iff A$  is a v.r. of  $k$ .**

**Proposition B.6.** *Let  $A \subset B$  be 2 domains,  $B$  f.g. over  $A$ .  $\forall x \neq 0 \in B, \exists u \neq 0 \in A$  s.t. any  $f : A \rightarrow \Omega = \text{alg.cl } \Omega, f(u) \neq 0$  can be extended to  $g : B \rightarrow \Omega$  with  $g(v) \neq 0$ .*

Using B.6, we can prove one form of Hilbert's Nullstellensatz.

**Corollary B.7.** *Let  $k$  be a field and  $B$  a f.g.  $k$ -algebra. If  $B$  is a field, the  $B/k$  is a finite algebraic extension.*

Take  $A = k, v = 1, \Omega = \text{alg.cl } k$ , then we get some  $g : B \rightarrow \Omega$ , which is non-trivial thus injective.

*Explanation:* Consider only the case when  $B = k[x]$ . Take some  $\xi \neq 0 \in \Omega = \text{alg.cl } k$ , we get a homomorphism by sending  $x$  to  $\xi$ .

Finally, we explore the relation between v.r.'s & valuations of a field.

**Definition B.8** (valuations). *Let  $k$  be a field,  $G$  be a totally ordered abelian group. A valuation of  $k$  with values in  $G$  is a mapping  $v : k^* \rightarrow G$  s.t.*

$$(1) v(xy) = v(x)v(y);$$

$$(2) v(x + y) \geq \min\{v(x), v(y)\}, \text{ if } x + y \neq 0.$$

**Procedure B.9.**

1. From a v.r  $A$  of  $k$  to a valuation

$$(a) U = \{\text{units of } A\}, G = k^*/U;$$

(b) define a partial order on  $G$ :

$$[x] \leq [y] \iff \frac{y}{x} \in A,$$

then  $G$  becomes **a totally ordered group**, moreover, the quotient  $v : k \rightarrow G$  is **a valuation with values in  $G$** .

2. From a valuation  $v : k^* \rightarrow G$  of  $k$  to a v.r.

$$(a) A = \{x \in k^* \mid v(x) \geq 0\} \cup \{0\};$$

(b)  **$A$  is a v.r. of  $k$ , which is called the v.r. of  $v$ .**

## B.2 Jacobson rings

**Definition B.10** (Jacobson rings). *We say a ring  $A$  is a Jacobson ring if  $\forall \mathfrak{p} \in \text{Spec } A, \mathfrak{p} = \bigcap \mathfrak{m}$ , where the intersection is taken over  $\{\mathfrak{m} \in \text{Spm} \mid \mathfrak{p} \subset \mathfrak{m}\}$ .*

**Remark B.11.** *In non-commutative cases, Jacobson rings are defined via primitive ideals.*

**Example B.12** (Jacobson rings). *The following rings are Jacobson.*

- (1) *A field  $k$ ;*
- (2) *A polynomial ring  $k[x_1, \dots, x_n]$ ;*
- (3) *A p.i.d.  $A$  with  $\text{Jac}(A) = 0$ ;*
- (4) *A ring of Krull dimension 0, e.g. a ring with only one prime ideal.*

Here's an interesting example.

**Example B.13** ( $\ast$ Jacobson yet not noetherian). *Let  $k$  be a field,  $R = k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$ . The only prime ideal of  $R$  is  $(x_1, x_2, \dots)$ , which is not f.g..*

**Proposition B.14** (properties of Jacobson rings).

- (1) *A ring  $A$  is Jacobson  $\iff A[x]$  is Jacobson ([9]P.18);*
- (2) *As a result of (1), a f.g. algebra over a Jacobson ring is also Jacobson;*
- (3) *Let  $A, B$  be Jacobson,  $f : A \rightarrow B$ , then  $f^{-1}(\mathfrak{m})$  is a maximal ideal of  $A$ ,  $\forall$  maximal ideal  $\mathfrak{m}$  of  $B$ ;*
- (4) *As a result of (3),  $f^\# : \text{Spec } B \rightarrow \text{Spec } A$  maps closed points in  $\text{Spec } B$  to closed points in  $\text{Spec } A$ .*

### B.3 Nakayama's lemma

**Theorem B.15** (Nakayama's lemma). *Let  $M$  be a f.g.  $A$ -module,  $\mathfrak{a}$  be an ideal. If  $\mathfrak{a}M = M$ , then  $\exists x \in A$  s.t.*

- (1)  *$x \equiv 1 \pmod{\mathfrak{a}}$ ;*
- (2)  *$xM = 0$ .*

**Corollary B.16.** *Let  $M$  be a f.g  $A$ -module, ([4]P.556)*

- (1) *if  $u : M \xrightarrow{\text{homo}} M$  is surjective, then  $u$  is bijective;*
- (2) *if  $A$  is local with  $\mathfrak{m}$ , then a subset  $\{m_1, \dots, m_r\}$  generates  $M \iff \{m_1, \dots, m_r\}$  generates  $M/\mathfrak{m}M$  over  $A/\mathfrak{m}$ .*

- (1) Consider  $M$  as  $A[x]$ -module, where  $x \cdot m = u(m)$  ([10]P.9);
- (2) “ $\Rightarrow$ ” is obvious. As for “ $\Leftarrow$ ”, let  $N = (x_1, \dots, x_r)$ , then  $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$  is exact, i.e.  $M = N + \mathfrak{m}M$ .

**Remark B.17.** *Essentially, B.15 generalizes the existence of annihilating polynomial in linear algebra. Thus the idea in (1) is natural.*

## C Sheaves

We consider only sheaves of Abelian groups, thus rings & modules are treated as special cases.

**Definition C.1** (presheaves & sections & morphisms). *Let  $X$  be a topo. space,*

- (1) *a presheaf  $\mathcal{F}$  on  $X$  is a contra. functor from  $\text{Open}(X) \rightarrow \mathbf{Ab}$ ;*
- (2) *a morphism between 2 presheaves  $\mathcal{F}, \mathcal{G}$  is a natural transformation from  $\mathcal{F} \rightarrow \mathcal{G}$ .*

*Or we can adopt human’s language,*

- (1) *a presheaf  $\mathcal{F}$  on  $X$  consists of*

(a)  $\mathcal{F}(U) \in \mathbf{Ab}, \forall U \in \text{Open}(X)$ ;

(b) restriction  $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V), \forall V \subset U \in \text{Open}(X)$ ;

*s.t.*

(a)  $\mathcal{F}(\emptyset) = 0$ ;



(b)  $\rho_{UU} = \text{id}$ ;

(c)  $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ .

(2) any element  $s \in \mathcal{F}(U)$  is called a section of  $\mathcal{F}$  on  $U$ , we sometimes write  $\Gamma(U, \mathcal{F})$  for  $\mathcal{F}(U)$ ;

(3) a morphism  $f : \mathcal{F} \rightarrow \mathcal{G}$  consists of morphisms  $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ ,  $\forall U \in \text{Open}(X)$ , s.t. the following diagram commutes  $\forall V \subset U \in \text{Open}(X)$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array}$$

**Definition C.2** (sheaves). Let  $X$  be a topo. space. A sheaf  $\mathcal{F}$  on  $X$  is a presheaf, s.t. if  $U \in \text{Open}(X)$ ,  $\{V_i\} \subset \text{Open}(X)$  is a cover of  $U$ ,

(1) (factorizing) if  $s \in \mathcal{F}(U)$  s.t.  $s|_{V_i} = 0, \forall i$ , then  $s = 0$ ;

(2) (glueing) if  $s_i \in \mathcal{F}(V_i)$  s.t.  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}, \forall i, j$ , then  $\exists s \in \mathcal{F}(U)$  s.t.  $s|_{V_i} = s_i, \forall i$ .

**Definition C.3** (stalks & germs). Let  $X$  be a topo. space  $p \in X$ ,  $\mathcal{F}$  be a presheaf on  $X$ .

(1) We define the stalk  $\mathcal{F}_p$  at  $p$  by

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U),$$

where the colimit is taken over  $\{U \overset{\text{open}}{\subset} X \mid p \in U\}$ ;

(2) Any element  $s_p \in \mathcal{F}_p$  is called a germ of  $\mathcal{F}$  at  $p$ .

**Remark C.4.** To be more concrete,  $\mathcal{F}_p = \{(U, s) \mid p \in U \overset{\text{open}}{\subset} X, s \in \mathcal{F}(U)\} / \sim$ , where  $(U, s) \sim (V, t) \iff \exists W \overset{\text{open}}{\subset} X$  s.t.

(1)  $p \in W \subset U \cap V$ ;

(2)  $s|_W = t|_W$ .

Thus any germ  $s_p$  comes from some sections.

**Proposition C.5** (sheaves are determined by stalks). *Let  $X$  be a topo. space,  $\mathcal{F}, \mathcal{G}$  be 2 sheaves on  $X$ ,  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism. Then*

(1)  $\varphi$  is injective  $\iff \varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is injective  $\forall p \in X$ ;

(2)  $\varphi$  is surjective  $\iff \varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is surjective  $\forall p \in X$ ;

(3) Thus  $\varphi$  is an isom.  $\iff \varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is an isom.  $\forall p \in X$ ;

**Remark C.6.** This result applies **only** to sheaves.

**Procedure C.7** (sheafification). *Let  $X$  be a topo. space,  $\mathcal{F}$  be a presheaf on  $X$ .*

1.  $\forall U \overset{\text{open}}{\subset} X$ , define  $\mathcal{F}^\dagger(U) = \{s : U \rightarrow \cup_{p \in U} \mathcal{F}_p \mid s \text{ satisfies 1a \& 1b}\}$ :

(a)  $\forall p \in U, s(p) \in \mathcal{F}_p$

(b)  $\forall p \in U, \exists p \in V \subset U, t \in \mathcal{F}(V)$  s.t.  $\forall q \in V, t_q = s(q)$ .

2. properties of  $\mathcal{F}^\dagger$ :

(a)  $\mathcal{F}^\dagger$  is a sheaf;

(b)  $\exists$  a natural morphism  $\theta : \mathcal{F} \rightarrow \mathcal{F}^\dagger$ , which satisfies the following universal property.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^\dagger \\ \downarrow & \swarrow \exists! & \\ \mathcal{G}(\text{sheaf}) & & \end{array}$$

Moreover, the pair  $(\mathcal{F}^\dagger, \theta)$  is unique in this sense.

(c)  $\mathcal{F}_p = \mathcal{F}_p^\dagger, \forall p \in X$ .

**Definition C.8** (kernel, image & cokernel). *Let  $X$  be a topo. space,  $\mathcal{F}, \mathcal{G}$  be 2 sheaves on  $X$ ,  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism.*

$$(1) \ker \varphi(U) := \ker(\varphi(U));$$

$$(2) \text{p.im } \varphi(U) := \text{im}(\varphi(U));$$

$$(3) \text{p.coker } \varphi(U) := \text{p.coker}(\varphi(U))$$

In general,  $\text{p.im } \varphi, \text{p.coker } \varphi$  fail to be sheaves, so we define

$$(1) \text{im } \varphi = (\text{p.im } \varphi)^\dagger;$$

$$(2) \text{coker } \varphi = (\text{p.coker } \varphi)^\dagger.$$

**Definition C.9** (injective & surjective morphisms, exact sequences). *Let  $X$  be a topo. space,*

(1) *a sequence  $\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$  of sheaves on  $X$  is called exact if  $\text{im } \varphi^{i-1} = \ker \varphi^i, \forall i$ ;*

(2)  *$\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is called injective if  $0 \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$  is exact;*

(3)  *$\varphi : \mathcal{F} \rightarrow \mathcal{G}$  is called surjective if  $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \rightarrow 0$  is exact.*

**Proposition C.10** (exactness on the stalk level). *Let  $X$  be a topo. space, a sequence  $\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$  of sheaves on  $X$  is exact  $\iff \dots \rightarrow \mathcal{F}_p^{i-1} \xrightarrow{\varphi_p^{i-1}} \mathcal{F}_p^i \xrightarrow{\varphi_p^i} \mathcal{F}_p^{i+1} \rightarrow \dots$  is exact  $\forall p \in X$ .*

**Proposition C.11** (left-exactness of restriction). *Let  $X$  be a topo. space,  $U \xrightarrow{\text{open}} \subset X$ . Then the functor  $\Gamma(U, \star)$  is left exact, i.e. if  $0 \rightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$  is an exact sequence of sheaves on  $X$ , then  $0 \rightarrow \mathcal{F}'(U) \xrightarrow{\varphi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U)$  is an exact sequence in **Ab**.*

**Remark C.12.** *The problem occurs since  $(\text{im } \varphi)(U) \neq \text{im}(\varphi(U)) = (\text{p.im } \varphi)(U)$  in general. Roughly speaking, the left-exactness comes from (3) of the following proposition, since  $\ker(\psi(U)) = (\ker \psi)(U) = (\text{im } \varphi)(U) \cong \mathcal{F}'(U) \cong (\text{p.im } \varphi)(U) = \text{im}(\varphi(U))$ .*

**Proposition C.13** (on injectivity). *Let  $X$  be a topo. space,  $\mathcal{F}, \mathcal{G}$  be 2 presheaves on  $X$ ,  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism.*

- (1) *If  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective  $\forall U \overset{\text{open}}{\subset} (X)$ , then the induced morphism  $\varphi^\dagger : \mathcal{F}^\dagger \rightarrow \mathcal{G}^\dagger$  is injective;*

$$\begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\quad \theta_1 \quad} & \mathcal{F}^\dagger \\
 \varphi \downarrow & & \searrow \varphi^\dagger \\
 \mathcal{G} & & \\
 \downarrow \theta_2 & \swarrow & \\
 \mathcal{G}^\dagger & & 
 \end{array}$$

- (2) *As a result of (1), if  $\varphi$  is a morphism of sheaves, then  $\text{im } \varphi$  is naturally a subsheaf of  $\mathcal{G}$ . Moreover,  $\text{im } \varphi \cong \mathcal{F} / \ker \varphi$ .*

$$\mathcal{F}(U) \longrightarrow \text{p.im } \varphi(U) \hookrightarrow \mathcal{G}(U)$$

$$\begin{array}{ccc}
 \mathcal{F} & & \\
 \downarrow & \searrow & \\
 \text{p.im } \varphi & \xrightarrow{\quad} & \text{im } \varphi \\
 \downarrow & \swarrow & \\
 \mathcal{G} & & 
 \end{array}$$

- (3) *As a result of (2), if  $0 \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$  is exact, then  $\mathcal{F} \cong \text{im } \varphi$ .*

**Definition C.14** (direct image & inverse image). *Let  $f : X \rightarrow Y$  be a conti. map of topo. spaces,  $\mathcal{F}, \mathcal{G}$  are sheaves on  $X, Y$  resp.,*

- (1)  $(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$ , *this gives the direct image  $f_*\mathcal{F}$  on  $Y$ ;*  
(2)  $(f^{-1}\mathcal{G})(U) := \varinjlim \mathcal{G}(V)$ , *where the colimit is taken over  $\{V \overset{\text{open}}{\subset} Y \mid f(U) \subset V\}$ , this gives the inverse image  $f^{-1}\mathcal{G}$  on  $X$ .*

**Remark C.15.**

- (1) Calculating  $f_*\mathcal{F}$  is always a crucial problem in algebraic geometry;  
(2)  $f^{-1}\mathcal{G}$  is difficult to define, but easy to use.

**Proposition C.16** ( $f^{-1}(-)$  &  $f_*(\star)$  are adjoint). *Let  $f : X \rightarrow Y$  be a conti. map of topo. spaces,  $\mathcal{F}, \mathcal{G}$  are sheaves on  $X, Y$  resp.,*

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

*More precisely, we have 2 natural maps  $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$  &  $\mathcal{G} \rightarrow f_*f^{-1}\mathcal{G}$ .*

Let's examine some examples.

**Example C.17** (sheaves). *Let  $X$  be a topo. space,  $A$  be an ab. grp..*

- (1) (constant sheaf) *Equip  $A$  with the disc. topo.. Define  $\mathcal{A}(U) = \{f : U \xrightarrow{\text{conti}} A\}, \forall U \overset{\text{open}}{\subset} X$ , then  $\mathcal{A}(U) \cong A$  if  $U$  is connected.*
- (2) (skyscraper sheaf) *For  $p \in X$ , define  $i_p(A)(U) = \begin{cases} A, & \text{if } p \in U \\ 0, & \text{otherwise} \end{cases}$ .*

*Note that  $(i_p(A))_q = \begin{cases} A, & \text{if } q \in \overline{\{p\}} \\ 0, & \text{otherwise} \end{cases}$ . Also, let  $\mathcal{A}$  be the const. sheaf on  $\{p\}$ ,  $j : \{p\} \rightarrow X$ , then  $i_p(A) = j_*\mathcal{A}$ .*

**Definition C.18** (flasque sheaves). *Let  $X$  be a topo. space,  $\mathcal{F}$  be a sheaf on  $X$ .  $\mathcal{F}$  is called flasque if  $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  is surjective  $\forall V \subset U \in \mathrm{Open}(X)$ .*

**Proposition C.19** (properties of flasque sheaves).

- (1) *Let  $X$  be a topo. space,  $0 \rightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \rightarrow 0$  be an exact sequence of sheaves on  $X$*
- (a) *if  $\mathcal{F}'$  is flasque, then  $\Gamma(U, \star)$  is exact, i.e.  $0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$  is an exact sequence in  $\mathbf{Ab}$ ;*
- (b) *if  $\mathcal{F}'$  &  $\mathcal{F}$  are flasque, then  $\mathcal{F}''$  is flasque.*
- (2) *Let  $f : X \rightarrow Y$  be a conti. map of topo. spaces,  $\mathcal{F}$  be a flasque sheaf on  $X$ , then  $f_*\mathcal{F}$  is a flasque sheaf on  $Y$ .*

(3) Let  $X$  be a topo. space,  $\mathcal{F}$  be a sheaf on  $X$ . Define  $\mathcal{F}^\dagger(U) = \{s : U \rightarrow \cup_{p \in U} \mathcal{F}_p \mid \forall p \in U, s(p) \in \mathcal{F}_p\}$ . Then

(a)  $\mathcal{F}^\dagger$  is a flasque sheaf;

(b)  $\exists$  a natural injective morphism  $\mathcal{F} \rightarrow \mathcal{F}^\dagger$ .

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^\dagger \\ \downarrow & \nwarrow \exists! & \\ \mathcal{G}(\text{fls. sh.}) & & \end{array}$$

0-extension of sheaves.

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