1. About CAT(-1) surfaces

In this section we present Theorem 1.1 with an ommission of several details, the main reference is [1].

1.1. **The main result.** Let Σ be a non-simply connected closed surface, we denote by $\mathcal{A}_\Sigma$ the collection of CAT(-1) surface M, which is homeomorphic to Σ.

As stated in [1], $\mathcal{A}_\Sigma$ is compact when equipped a certain topology, and thus the supremum of the systole is attained inside $\mathcal{A}_\Sigma$, that is the motivation to consider Alexandrov surfaces. We will give more details later.

Theorem 1.1 ([1])**.** *Let Σ be a non-simply connected closed surface, $U \subset \mathcal{A}_\Sigma$ be an open subset, then there exists a hyperbolic surface $M \in U$, s.t.*

$$\operatorname{sys}(M) = \sup_{M' \in U} \operatorname{sys}(M')$$

Thus we have the following corollary, which is used in the proceeding section.

Corollary 1.2 ([1])**.** *Let Σ be a non-simply connected closed surface, then the maximal systole of the CAT(-1) metrics on Σ is attained by a hyperbolic metric.*

1.2. **Some definitions and results.** We consider only closed surfaces.

Definition 1.3 (simple singularity and conical singularity)**.** Let (M, g) be a closed surface with metric g.

(1) We say g has a simple singularity of order β at $p \in M$, if g can be written as
$$g = e^{2u(z)}|z|^{2\beta}|\mathrm{d}z|^2$$
on some coordinate neighborhood centered at p, where $u : \mathbb{C} \to \mathbb{R}$ is continuous, $\beta \in \mathbb{R}$.

(2) if (1) occurs with $\beta > -1$, then we call p a *conical singularity* of *total angle* $\theta_p = 2\pi(\beta + 1)$.

As mentioned in[2], if M is compact, the only possible singularity is conical. The total angle means that, if we rotate a point around 0 for 2π in $\mathbb{C}$, then the corresponding point will rotate around p for $2\pi(\beta + 1)$. If a metric g does not have any singularity, then it is hyperbolic.

In this report, we regard *Alexandrov surfaces* as those closed surfaces M with a metric g which may admit singularities. It is plausible, since according to [1], every closed Alexandrov surface M can be approximated by a piecewise hyperbolic surface with conical singularities.

The term *CAT(-1) surfaces* refers to Alexandrov surfaces of *Alexandrov curvature* at most -1. As in [1], a closed piecewise hyperbolic surface with conical singularities of total angle at least 2π belongs to CAT(-1) surfaces, and conversely, for a CAT(-1) surfaces, all the total angles for singularities are not less than 2π.

Definition 1.4 (topology on $\mathcal{A}_\Sigma$, bilipschitz distance)**.** For $M, M' \in \mathcal{A}_\Sigma$, define

$$d_{\mathrm{Lip}}(M, M') = \inf_f \max\{\log \|f\|_{\mathrm{Lip}}, \log \|f^{-1}\|_{\mathrm{Lip}}\},$$

where the infimum is taken over all the bilipschitz homeomorphism $f : M \to M'$.

Proposition 1.5 (compactness)**.** *Fix a non-negative integer N and a positive real number s, let Σ be a non-simply connected closed surface. Then the space of piecewise hyperbolic surfaces $M \cong \Sigma$ of CAT(-1) surface with at most N conical singularities and systoles at least s is compact.*

Definition 1.6 (large and small singularity)**.** A conical singularity $p \in M$ is said to be

(1) large, if the total angle at p is at least 3π;
(2) small, otherwise.

Proposition 1.7 (large conical singularity)**.** *Let M be a closed piecewise hyperbolic CAT(-1) surface, then there are less than $2|\chi(M)|$ large conical singularities on M.*

Proof. Suppose the conical singularities are $\{p_i\}$, use Gauss-Bonnet for M,

$$\begin{aligned} 2\pi|\chi(M)| &= \left| \int_M K \mathrm{d}A - \sum_{i=1}^N (\theta_{p_i} - 2\pi) \right| \\ &= \left| \int_M K \mathrm{d}A \right| + \left| \sum_{i=1}^N (\theta_{p_i} - 2\pi) \right| \\ &\geqslant \sum_{p_i \text{ large}} (\theta_{p_i} - 2\pi) = \pi \cdot \#\{p_i \text{ large}\} \end{aligned}$$

The result follows. □

1.3. **Systolic decomposition and kite excision.** For a closed CAT(-1) surface M, there are at most finitely many systolic loops. As stated in [1, 3], every pair of intersecting systolic loops meet exactly at one or two points, or along a line.

Definition 1.8 (systolic decomposition)**.** The *systolic decomposition* of M is the collection of open domains (called *systolic domains*) defined as the connected components of the complementary set in M of the systolic loops. It can be regarded as a polygon, where

(1) the *vertices* are the intersection points and the endpoints of intersecting lines;
(2) the *edges* are the separated geodesic arcs of the boundary of the domains by the vertices.

Proposition 1.9. *Let M be a closed piecewise hyperbolic CAT(-1) surface, then the number of domains, edges and vertices in the systolic decomposition of M have an upper bound which depends only on the topology of M.*

Proof. Take Q to be the maximal number of pairwise non-homotopic simple loops on M, which is finite and depends only on the topology of M. Then the number of systolic loops is small than Q. Set $N = 8\binom{Q}{2}$, then the number of domains, edges and vertices are all smaller than N. $\square$

Let $p, q \in M$ be two conical singularities, s.t. the geodesic arc $[p, q]$ has no interior singularity. Take $r \in M$, s.t. the triangle $\triangle pqr$ is hyperbolic with *acute* angle at p, q. Define the *kite* $K = prqr'$ as the union of two symmetric hyperbolic triangles. The vertices p, q are called the *main vertices* of K, and

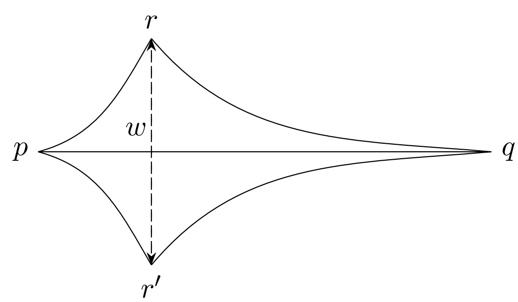


FIGURE 1. A kite K with width w

the length of diagonal $[r, r']$ is called the *width* of K, denoted by w. We say K is *exact* at p, if

(1) p is a small singularity;
(2) $\measuredangle rpr' = \theta_p - 2\pi < \pi, \measuredangle rqr' \leqslant \min\{\theta_q - 2\pi, \pi\}$.

Definition 1.10 (excised surface)**.** Let $K_w \subset M$ be a kite of width w. We define the excised surface by

$$M_w = (M \backslash K_w)/\sim$$

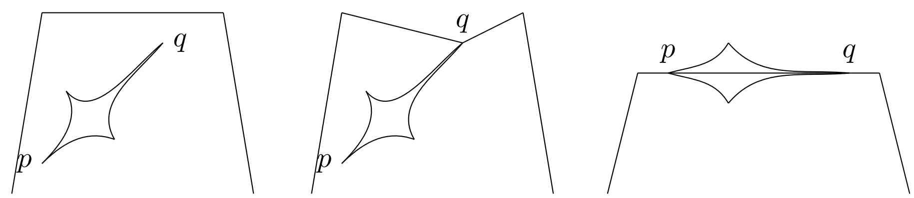
where $\sim$ means identifying $[p, r], [q, r]$ with $[p, r'], [q, r']$ respectively.

Proposition 1.11. *Let K_w be an exact kite, then M_w is also a CAT(-1) surface, with the same number of conical singularities as M.*

Proof. Suppose $K_w = prqr'$ is a kite which is exact at p. Then the total angle at $r = r'$ is $4\pi - \measuredangle prq - \measuredangle pr'q > 2\pi$, thus $r = r'$ is a conical singularity. But the total angles at p, q are $\theta_p - \measuredangle rpr' = 2\pi, \theta_q - \measuredangle rqr' \geqslant 2\pi$, thus p is no longer a conical singularity. And all the total angles remain no less than 2π, thus M_w is still a CAT(-1) surface. $\square$

Proposition 1.12. *Consider an exact kite K_w at p with main diagonal $[p, q]$. Then the excised surface M_w converges to M w.r.t. the bilipschitz distance as w tends to zero.*

We now consider systolic decomposition with kite excision. There are three typical cases of position for a kite with main diagonal $[p, q]$ within a systolic domain D.



(a) Case1 (b) Case2 (c) Case3

FIGURE 2. Three exact kite configurations

(Case1) For $[p, q] \subset D$, take K_w exact at p of sufficiently small width, s.t. it lies inside D;
(Case2) For $[p, q) \subset D$ with $q \in \partial D$, take K_w exact at p of sufficiently small width, s.t. $K_w \backslash q$ lies in D;
(Case3) For $[p, q] \subset \partial D$, take K_w exact at p.

Proposition 1.13. *Let M be a closed piecewise hyperbolic CAT(-1) surfaces. Consider a kite $K_w \subset M$ exact at p and satisfying one of the three cases. Then for sufficiently small width w, we have*

$$\operatorname{sys}(M_w) \geqslant \operatorname{sys}(M).$$

Sketch of proof. blabla □

Theorem 1.14. *Let Σ be a non-simply connected closed surface. Let $U \subset \mathcal{A}_\Sigma$ be an open set. Then there exists a const N which depends only on the topology of Σ and a piecewise hyperbolic surface $M \in U$ with at most N conical singularities, s.t.*

$$\operatorname{sys}(M) = \sup_{M' \in U} \operatorname{sys}(M').$$

Proof. By approximation, we can consider only piecewise hyperbolic surfaces with conical singularities. For every $\varepsilon > 0$ small enough, there exists a such surface M s.t.

$$\operatorname{sys}(M) > \max_{M' \in U} \operatorname{sys}(M') - \varepsilon > \max_{M' \in \partial U} \operatorname{sys}(M').$$

By the compactness (Proposition 1.5), there exists some surfaces in U, which attain the maximal systole. We assume $M_1 \in U$ is of minimal area in these surfaces.

Lemma 1.15.

(1) *Every domain of the systolic decomposition of M_1 contains at most one small conical singularity.*

(2) *The interior of every edge of a domain of M_1 contains at most one small conical singularity.*

According to Proposition 1.9, we can take N_Σ to be an upper bound of the number of domains, edges, vertices in the systolic decomposition of M_1. WLOG, we assume $N_0 \geqslant 2|\chi(M|$. Combine Proposition 1.7 and Lemma 1.15, M_1 has at most $N = 4N_\Sigma$ conical singularities.

By compactness(Proposition 1.5), there exists a surface $M_0 \in U$ with maximal systole among all piecewise hyperbolic surfaces in U, with at most N conical singularities. Note that M_0 does not depend on ε (since N does not), and

$$\mathrm{sys}(M_0) \geqslant \mathrm{sys}(M_1) \geqslant \mathrm{sys}(M) > \max_{M' \in U} \mathrm{sys}(M') - \varepsilon, \forall \varepsilon.$$

Thus $\mathrm{sys}(M_0) = \sup_{M' \in U} \mathrm{sys}(M')$. □

1.4. **Kite insertion and deformation.** Let $M \in \mathcal{A}_\Sigma$, $m \in M$ be a conical singularity with total angle $\theta_m > 2\pi$, (p, q) be a geodesic arc passing through m, s.t. m is the only conical singularity on the segment $[p, q]$.

For $0 < \alpha < \frac{1}{2}(\theta_m - 2\pi)$, take $q_\alpha \in M$ with $|mq_\alpha| = |mq|$ and $\measuredangle pmq_\alpha = \pi + \alpha$. Denote by M' the surface by cutting along $[p, m], [m, q_\alpha]$, the boundary of M' is the geodesic quadrilateral $pmq_\alpha m'$.

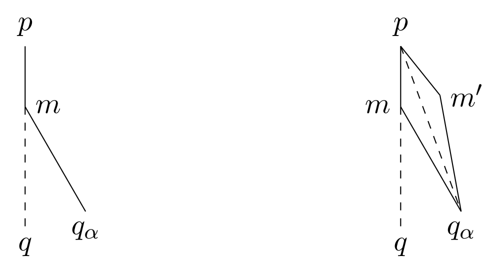


Figure 3. Insertion of a kite

Let $K_\alpha = \bar{p}\bar{m}\bar{q}_\alpha\bar{m}'$ be a kite in $\mathbb{H}^2$, with $|\bar{p}\bar{m}| = |pm|, |\bar{q}_\alpha\bar{m}| = |q_\alpha m|$, $|\bar{p}\bar{m}'| = |pm'|, |\bar{q}_\alpha\bar{m}'| = |q_\alpha m'|$ and $\measuredangle\bar{p}\bar{m}\bar{q}_\alpha = \measuredangle\bar{p}\bar{m}\bar{q}_\alpha = \pi - \alpha$. We attach K_α to M' along the corresponding arcs, to get a surface M_α.

Proposition 1.16. *M_α is also a CAT(-1) surface, with more conical singularities than M.*

Proof. According to the construstion of K_α, the total angles at m, p, q_α, m' are $2\pi, 2\pi + \measuredangle\bar{m}\bar{p}\bar{m}' \geqslant 2\pi, 2\pi + \measuredangle\bar{m}\bar{q}_\alpha\bar{m}' \geqslant 2\pi, \theta_m - 2\alpha > 2\pi$ respectively. Thus M_α is still a CAT(-1) surface. □

Proposition 1.17. *The surface M_α converges to M w.r.t. the bilipschitz distance as α tends to zero.*

Now we consider the deformation of systolic loop with kite insertion.

Definition 1.18. Let M be a closed CAT(-1) surface.

(1) Given a free homotopy class C of M, define $L_M(C)$ as the minimal length among loops in C.
(2) Define $\#_s(M) < \infty$ as the number of systolic loops of M.

Since M is of negative sectional curvature, this minimal length must be attained by some closed geodesic in C.

Let $m \in M$ be a conical singularity with total angle $\theta_m > 2\pi$. Since there are finitely many systolic loops, we can choose a geodesic arc (p,q) passing through m, s.t. **at least one systolic loop of M transversely intersects (p,q) and all the systolic loops of M meeting $[p,q]$ intersects (p,q) only for some** $x \in (p,m]$. Let M_α be the inserted surface.

Proposition 1.19. *Let C be the free homotopy class of a systolic loop γ of M, for $\alpha > 0$ small enough*

(1) *if γ does not transversely intersect $[p,q]$, then*

$$L_{M_\alpha}(C) = L_M(C) = \mathrm{sys}(M);$$

(2) *if γ transversely intersects $[p,q]$, then*

$$L_{M_\alpha}(C) > L_M(C) = \mathrm{sys}(M).$$

Theorem 1.20 (deformation via kite insertion)**.** *Let (M) be a closed piecewise hyperbolic CAT(-1) surface, with a conical singularity m. Then M can be deformed into a closed piecewise hyperbolic CAT(-1) surface M_α, s.t. for $\alpha > 0$ small enough, one of the following statements holds*

(1) $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$*;*
(2) $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ *and* $\#_s(M_\alpha) < \#_s(M)$*.*

Proof. Take a geodesic (p,q) as before, and obtian a surface M_α by inserting a kite K_α along some segment $[p, q_\alpha]$.

Let C be a free homotopy class of M, there are three possible cases:

(Case1) If C is not represented by a systolic loop of M, then from Proposition 1.17, for $\alpha > 0$ small enough,

$$L_{M_\alpha}(C) > \mathrm{sys}(M);$$

(Case2) If C is represented by a systolic loop of M, which does not transversely intersect $[p,q]$, then from Proposition 1.19,

$$L_{M_\alpha}(C) = L_M(C) = \mathrm{sys}(M);$$

(Case3) If C is represented by a systolic loop of M, which transversely intersects $[p,q]$, then from Proposition 1.19

$$L_{M_\alpha}(C) > L_M(C) = \mathrm{sys}(M).$$

In conclusion, $\mathrm{sys}(M_\alpha) \geqslant \mathrm{sys}(M)$. Moreover,

(1) if all the systolic loops of M meeting $[p,q]$ transversely intersect $[p,q]$, as in (Case3), then $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$;

(2) if there is a systolic loop of M as in (Case2), then $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$, for $\alpha > 0$ small enough. In this case, any systolic loop of M transversely intersecting $[p, q]$ (such a loop exists, by the assumption on (p, q)) will not be systolic under the insertion of a kite. Thus $\#_s(M_\alpha) < \#_s(M)$.

Thus the result follows. □

1.5. **Proof of the main result.** Now we can show Theorem 1.1.

Proof. According to Theorem 1.14, the supremum of the systole on U is attained by some piecewise hyperbolic surfaces in U with conical singularities. Among these surfaces, take M to be with a minimal $\#_s(M)$.Our goal is to show that M has no conical singularity, thus M is a hyperbolic surface.

Suppose by contradiction that m is a conical singularity of M. By Theorem 1.20, we can deform M by a kite insertion into some $M_\alpha \in U$, for $\alpha > 0$ small enough (Proposition 1.17) with one of the following properties

(1) $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$;
(2) $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ and $\#_s(M_\alpha) < \#_s(M)$.

But M attains the maximal of the systoles, so (1) is impossible. Since M has a minimal $\#_s(M)$, (2) is also impossible. □

References

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