Topological Approaches to Epistemic Logic: HW collection
2025 summer

1. Topological Approaches to Epistemic Logic: HW1

1. (30') Let (X, τ) be a topological space and $A \subseteq X$. Show that A is closed in (X, τ) iff A contains all of its limit points.

   Solution. "$\Rightarrow$": Suppose by contradiction that there is some x with $x \in d(A)$ and $x \notin A$. Since $X \backslash A$ is open, there is a neighborhood $x \in U \subseteq X \backslash A$, i.e. $U \cap A = \emptyset$, this is impossible since $x \in d(A)$.
   "$\Leftarrow$": For any $x \in X \backslash A$, we know $x \notin d(A)$, so there is some neighborhood $x \in U$ with $U \cap A = \emptyset$, i.e. $U \subseteq X \backslash A$. Since x is arbitrary, $X \backslash A$ is open, which means A is closed.

2. Given a topological space (X, τ) and $A \subseteq X$, A is called an open domain if $A = Int(Cl(A))$. Show that

   (1) (20') The interior of a closed set is an open domain.

   (2) (20') The intersection of two open domains is an open domain.

   Solution. *(1)* Take $A = Int(B)$, where B is closed, then we have $Int(Cl(Int(B))) \subseteq Int(Cl(B)) = Int(B)$, on the other side, $Int(Cl(Int(B))) \supset Int(Int(B)) = Int(B)$, so $A = Int(Cl(A))$.

   (2) Suppose A, B are two open domains, we have

   $$\begin{aligned} Int(Cl(A \cap B)) &= Int(Cl(Int(Cl(A)) \cap Int(Cl(B)))) \\ &= Int(Cl(Int(Cl(A) \cap Cl(B)))) \\ &\overset{*}{=} Int(Cl(A) \cap Cl(B)) \\ &= Int(Cl(A)) \cap Int(Cl(B)) = A \cap B. \end{aligned}$$

   Here that $\overset{*}{=}$ follows from *(1)*, since $Int(Cl(A) \cap Cl(B))$ is the interior of a closed set.

3. Prove that $p \to \Box\Diamond p$ is valid on a topological space (X, τ) iff every closed set of (X, τ) is open.

   Solution. $p \to \Box\Diamond p$ means that for any $p \subseteq X$, we have $p \subseteq Int(Cl(p))$. "$\Rightarrow$": If U is closed, then $U \subseteq Int(Cl(U)) = Int(U) \subseteq U$, thus U is open. "$\Leftarrow$": For any $p \subseteq X$, $Cl(p)$ is closed, thus is open from the assumption, then $p \subseteq Cl(p) = Int(Cl(p))$.

4. (20', bonus) Prove Proposition 7 on slide 25 of Lecture 2:
   For all $\varphi \in \mathcal{L}_K$,

(1) for any S4*-Kripke model* $\mathcal{M} = (X, R, V)$ *and* $x \in X$,

$$\mathcal{M}, x \models \varphi \iff B(\mathcal{M}), x \models \varphi;$$

(2) for any Alexandroff model $\mathcal{X} = (X, \tau, V)$ *and* $x \in X$,

$$\mathcal{X}, x \models \varphi \iff A(\mathcal{X}), x \models \varphi.$$

Solution. *(1)* $K\varphi$ is true in the S4-Kripke model $\mathcal{M}$ iff $y \in V(\varphi)$ for all y with xRy. From the definition of τ_R, this happens iff for $x \in U = \{y|xRy\} \in \tau_R$, we have $y \in V(\varphi)$ for all $y \in U$, i.e. $K\varphi$ is true in the topo-model $B(\mathcal{M})$.

(2) $K\varphi$ is true in the Alexandroff model $\mathcal{X}$ iff there is a neighborhood $x \in U \in \tau$ with $y \in V(\varphi)$ for all $y \in U$. Since τ is Alexandroff, this happens iff $y \in V(\varphi)$ for all $y \in \cap_{x \in U \in \tau} U$, i.e. $x \sqsubseteq_\tau y$ by definition. Thus it is equivalent that $K\varphi$ is true in the S4-Kripke model $A(\mathcal{X})$.

2. Topological Approaches to Epistemic Logic: HW2

1. Given a topological space (X, τ) and any two subsets $U_1, U_2 \subseteq X$,

   (1) (15') if U_1 is open and dense and U_2 is dense, then $U_1 \cap U_2$ is dense;

   (2) (15') if U_1 and U_2 are nowhere-dense, then $U_1 \cup U_2$ are nowhere-dense.

   Solution. *(1)* We write $\bullet^c := X \backslash \bullet$ for simplicity, then

$$\begin{aligned} X = Cl(U_2) &= Cl((U_2 \cap U_1) \cup (U_2 \cap U_1^c)) \\ &= Cl(U_1 \cap U_2) \cup Cl(U_2 \cap U_1^c) \\ &\overset{*}{\subseteq} Cl(U_1 \cap U_2) \cup U_1^c, \end{aligned}$$

   so $U_1 \subseteq Cl(U_1 \cap U_2)$, as a result, $X = Cl(U_1) \subseteq Cl(U_1 \cap U_2)$. Here $\overset{*}{\subseteq}$ follows since U_1 is open, $Cl(U_2 \cap U_1^c) \subseteq Cl(U_1^c) = U_1^c$.

   (2) By direct calculation,

$$\begin{aligned} Int(Cl(U_1 \cup U_2)) &= Int(Cl(U_1) \cup Cl(U_2)) \\ &\overset{*}{=} (Cl((Cl(U_1) \cup Cl(U_2))^c))^c \\ &\overset{**}{=} (Cl(Int(U_1^c) \cap Int(U_2^c)))^c \\ &\overset{***}{=} X^c = \emptyset, \end{aligned}$$

   so $U_1 \cup U_2$ is nowhere-dense. Here for $\overset{*}{=}, \overset{**}{=}$, we simply use the duality: $Int(\bullet^c) = (Cl(\bullet))^c$, and for $\overset{***}{=}$, we use *(1)* since $Int(U_1^c)$ is open and dense:

$$Cl(Int(U_1^c)) = Cl((Cl(U_1))^c = (Int(Cl(U_1)))^c = X.$$

2. Let (X, τ) be a topological space and define τ_{dense} on X as follows:

$$\tau_{dense} = \{U \in \tau \mid Cl(U) = X\} \cup \{\emptyset\}.$$

   Prove that

   (1) (10') for all non-empty $U, V \in \tau_{dense}$, we have $U \cap V \neq \emptyset$;

   (2) (20') τ_{dense} is a topology;

   (3) (20') τ_{dense} is extremally disconnected.

   Solution. *(1)* From Problem 1, $Cl(U \cap V)$ is dense, so $U \cap V \neq \emptyset$.

(2) We verify all the axioms:

- $Cl(X) = X$, so by definition, $\emptyset, X \in \tau_{dense}$;
- For any $\mathcal{A} \neq \{\emptyset\} \subseteq \tau_{dense}$,

$$Cl(\cup\mathcal{A}) = Cl(\cup_{U\neq\emptyset\in\mathcal{A}}U) = \cup_{U\neq\emptyset\in\mathcal{A}}Cl(U) = X,$$

  so τ_{dense} is closed under unions.
- For $U, V \in \tau_{dense}$, $U \cap V$ is empty if one of them is empty, and is dense from Problem 1 if both are non-empty. In both cases, $U \cap V \in \tau_{dense}$.

(3) We write $Cl_d(\bullet)$ for the closure in τ_{dense}. $Cl_d(\emptyset) = \emptyset$ is open. For $U \neq \emptyset \in \tau_{dense}$, we **claim** $X = Cl(U) \subseteq Cl_d(U)$, then of course $Cl_d(U) \in \tau_{dense}$. So it remains to show the **claim** above: By definition, for any $x \in Cl(U)$, and any $x \in V \in \tau$, we have $U \cap V \neq \emptyset$, especially, for any $x \in V \in \tau_{dense} \subseteq \tau$, we have $U \cap V \neq \emptyset$, thus $x \in Cl_d(U)$.

Remark: In general topology, τ_{dense} is coined as an irreducible space.

3. (20') As stated on slide 50 of Lecture 2, the logic $\mathsf{KD45}_B$ is sound and complete with respect to the class of extremally disconnected spaces (under the closure of interior semantics). Which axioms of $\mathsf{KD45}_B$ are invalidated by some topo-models (that are not extremally disconnected) when we interpret $B\varphi$ as $[\![B\varphi]\!]^{\mathcal{X}} = Cl(Int([\![\varphi]\!]^{\mathcal{X}}))$? Justify your answers by giving counterexamples that invalidate these axioms and explaining why the axioms are false in these models.

   Solution. (Hinted by [Baltag et al., 2019], appendix A.1, and [Stalnaker., 2006]) The following axioms are not valid if the topo-model is not extremally disconnected:

   - $B(\varphi \to \psi) \to (B\varphi \to B\psi)$, equivalently, $Cl(Int([\![\varphi]\!]^c \cup [\![\psi]\!])) \subseteq (Cl(Int([\![\varphi]\!])))^c \cup Cl(Int([\![\psi]\!]))$, or for any $A, B \subseteq X$, we have $Cl(Int(A \cup B)) \subseteq Int(Cl(A)) \cup Cl(Int(B))$.
   - $B\varphi \to \neg B\neg\varphi$, equivalently, $Cl(Int([\![\varphi]\!])) \subseteq Int(Cl([\![\varphi]\!]))$.

   For example, take $\mathbb{R}$ with the standard topology τ, then

$$Cl(Int((0,1) \cup \emptyset)) \not\subseteq Int(Cl((0,1))) \cup Cl(Int(\emptyset)),$$
$$Cl(Int((0,1))) \not\subseteq Int(Cl((0,1))),$$

   The other two axioms translate into tautologies for topological spaces.

4. (20', bonus) Let (X, R) be a total preordered set. Show that the Alexandroff topology τ_R, defined as

$$\tau_R = \{U \subseteq X \mid U \text{ is an up-set of } (X, R)\},$$

is hereditarily extremally disconnected.

Solution. For any subspace (P, τ_P) of (X, τ_R), we write $Cl_P(\bullet)$ for the closure in τ_P. Suppose $U \in \tau_P$, i.e. $U = P \cap V$ for some $V \in \tau_R$. So

$$\begin{aligned} Cl_P(U) &\overset{*}{=} P \cap Cl(P \cap V) \\ &\overset{**}{=} P \cap \{y \in X \mid \exists x \in P \cap V, xRy\} \\ &= P \cap (\cup_{x \in P \cap V} \{y \in X \mid xRy\}) \end{aligned}$$

is open in τ_P. Hence τ_R is hereditarily extremally disconnected. Here $\overset{*}{=}$ follows, since in general topology, we have $Cl_P(\bullet) = P \cap Cl(\bullet)$. And $\overset{**}{=}$ follows from the fact that, in general, $Cl(A) = \{y \in X \mid \exists x \in A, xRy\}$, as long as R is total.

3. Topological Approaches to Epistemic Logic: HW3

1. For this exercise we work with the topological subset space semantics introduced in Lecture 4. Show that

   (1) (20') Every φ in the language of classical propositional logic (elements of $\mathcal{L}_{K\blacksquare}$ that do not have occurrences of K or $\blacksquare$) is persisent with respect to the subset space semantics.

   (2) (20') Show that not every formula in $\mathcal{L}_{K\blacksquare}$ is persisent. Justify your answer by finding a counterexample and showing that it is not persisent.

   (3) (20') Are there any persistent formulae with respect to topological subset space models in $\mathcal{L}_{K\blacksquare}$ that have occurrences of K or $\blacksquare$? That is, is there a formula $\psi \in \mathcal{L}_{K\blacksquare}$ that has some occurrences of K or $\blacksquare$ such that for all topological models $\mathcal{X} = (X, \tau, V)$ and for all epistemic scenarios (x, U) and (x, O) of $\mathcal{X}$, we have $(x, U) \models \psi$ iff$(x, O) \models \psi$? Justify your answer either by given an example of such a persistent formula and proving your claim; or by proving that only the formulae in the language of classical propositional logic are persistent.

Solution. *(1)* This is obvious, since in classical propositional logic

- $\mathcal{X}, (x, U) \models p$ iff $x \in V(p)$,
- $\mathcal{X}, (x, U) \models \neg\varphi$ iff $\mathcal{X}, (x, U) \not\models \varphi$,
- $\mathcal{X}, (x, U) \models \varphi \wedge \psi$ iff $\mathcal{X}, (x, U) \models \varphi$ and $\mathcal{X}, (x, U) \models \psi$

all these expressions do not concern U.

(2) In topological subset space semantics

- $\mathcal{X}, (x, U) \models K\varphi$ iff $(\forall y \in U)(\mathcal{X}, (y, U) \models \varphi)$, i.e. $U \subseteq [\![\varphi]\!]_U$,
- $\mathcal{X}, (x, U) \models \blacksquare\varphi$ iff $\forall O \in \mathcal{O}(x \in O \subseteq U \Rightarrow \mathcal{X}, (x, O) \models \varphi)$, i.e. $x \in \bigcap_{x \in O \subseteq U} [\![\varphi]\!]_O$.

For example, Kp is not persistent for some interpretation. Consider $X = \{a, b, c\}$, $\mathcal{O}$ the discrete topology, $A = \{a\}, B = \{a, b\}$ and $V(p) = A$.

- $(a, A) \models Kp$, since $A \subseteq A$.
- $(a, B) \not\models Kp$, since $B \not\subseteq A$.

(3) As mention in [Moss and Parikh, 1992], it is shown in [Georgatos, 1994] that, $\blacklozenge Kp, \blacksquare\hat{K}p$ are persistent (called "stable" in the context), where $\blacklozenge\varphi := \neg\blacksquare\neg\varphi, \hat{K}\varphi := \neg K\neg\varphi$.

We consider $\neg\blacksquare\neg Kp$ here. Suppose $(x, U) \models \neg\blacksquare\neg Kp$, i.e. $\exists O \in \tau(x \in O \subseteq U, \forall y \in O((y, O) \models p))$. Now for any $x \in V \in \tau$, write $W = V \cap O \in \tau$, we have

$$x \in W \subseteq U, \forall y \in W((y, W) \models p)$$

since $W \subseteq V$. Thus $(x, V) \models \neg\blacksquare\neg Kp$.

2. For this exercise we work with the topological subset space semantics introduced inn Lecture 4. Given a topo-model $\mathcal{X} = (X, \tau, V)$ and $p \in Prop$, prove the following:

   (1) (20') $V(p)$ is an open set of (X, τ) iff $p \to \blacklozenge Kp$ is valid in $\mathcal{X}$.

   (2) (20') $V(p)$ is a closed set of (X, τ) iff $\blacksquare\hat{K}p \to p$ is valid in $\mathcal{X}$.

   Solution. *(1)* "$\Leftarrow$": For any $x \in V(p)$, and any $x \in U \in \tau$, we have $(x, U) \models p$, and thus $(x, U) \models \blacklozenge Kp$, i.e. there is some $O \in \tau$ with $x \in O \subseteq U$, and $O \subseteq V(p)$. Hence $V(p)$ is open.

   "$\Rightarrow$": Suppose $x \in U \in \tau$. If $x \notin V(p)$, then $(x, U) \not\models p$, thus $(x, U) \models (p \to \blacklozenge Kp)$. If $x \in V(p)$, then there is some $x \in W \in \tau$ with $W \subseteq V(p)$. Let $O = W \cap U \in \tau$, then $x \in O \subseteq U, \forall y \in O((y, O) \models p)$. In both cases, $(x, U) \models (p \to \blacklozenge Kp)$.

   (2) "$\Rightarrow$": Suppose $x \in U \in \tau$, and assume $(x, U) \models \blacksquare\hat{K}p$, i.e. for all $O \in \tau$ with $x \in O \subseteq U$, there is some $y \in O$ with $y \in X \backslash V(\neg p) = V(p)$. Thus $x \in Cl(V(p)) = V(p)$, which means $(x, U) \models p$. Hence $(x, U) \models (\blacksquare\hat{K}p \to p)$.

   "$\Leftarrow$": For any $x \in U \in \tau$. If $(x, U) \models \blacksquare\hat{K}p$ and $(x, U) \models p$, then $x \in Cl(V(p)) \cap V(p)$. If $(x, U) \not\models \blacksquare\hat{K}p$, then $(x, U) \models \neg\blacksquare\hat{K}p = \blacklozenge K\neg p$, i.e. there is some $O \in \tau$ with $x \in O \subseteq U$, and $O \subseteq V(\neg p) = (V(p))^c$, which means $x \in Int((V(p))^c) = (Cl(V(p)))^c$. In combine, $X = (Cl(V(p)) \cap V(p)) \cup (Cl(V(p)))^c$, thus $V(p) = Cl(V(p))$.

3. (20', bonus) For this exercise, we work with topo-e-models and the corresponding semantics introduced in the slides of Lecture 3.

   By adding a piece of evidence $P \subseteq X$ to a topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$, we can create another topo-e-model $\mathfrak{M}^{+P}$. Let us define $\mathfrak{M}^{+P} = (X, \mathcal{E}^{+P}, \tau^{+P}, V)$, where $\mathcal{E}_0^{+P} = \mathcal{E}_0 \cup \{P\}$ and τ^{+P} is the topology generated by $\mathcal{E}_0^{+P}$.

We now introduced a new modality, $[+\varphi]\psi$, into our language $\mathcal{L}$ (on slide 51 of Lecture 3) and interpret it in a given topo-e-model $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ at $x \in X$ as follows:

$$\mathfrak{M}, x \in [\![[+\varphi]\psi]\!]^{\mathfrak{M}} \iff [\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset \text{ implies } x \in [\![\psi]\!]^{\mathfrak{M}^{+\varphi}}.$$

Show that the following formula is valid in all topo-e-models:

$$[+\varphi]\Box\psi \leftrightarrow ([\exists]\varphi \to (\Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi)))).$$

Solution. *(1)* "$\to$": Suppose $\mathfrak{M}, x \models [+\varphi]\Box\psi$. If $[\![\varphi]\!]^{\mathfrak{M}} = \emptyset$, then φ is not valid and $[\exists]\varphi$ is not valid, thus "$\to$"holds. We only need to show that $\mathfrak{M}, x \models \Box(\varphi \to [+\varphi]\psi)$, in the case that $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$. Note by definition, $\mathfrak{M}^{+\varphi}, x \models \Box\psi$, i.e. there is some $U \in \tau^{+\varphi}$ with $x \in U$, and for all $y \in U$, $\mathfrak{M}^{+\varphi}, y \models \psi$. This means, for all $y \in U$, $\mathfrak{M}, y \in [\![[+\varphi]\psi]\!]^{\mathfrak{M}}$, i.e. $U \subseteq [\![[+\varphi]\psi]\!]^{\mathfrak{M}}$, since $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$. Now we have $x \in U \subseteq ([\![\varphi]\!]^{\mathfrak{M}})^c \cup [\![[+\varphi]\psi]\!]^{\mathfrak{M}}$, so $\mathfrak{M}, x \models \Box(\varphi \to [+\varphi]\psi)$.

(2) "$\leftarrow$": Suppose $\mathfrak{M}, x \models [\exists]\varphi \to (\Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi)))$, and $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$, the goal is to show that $\mathfrak{M}^{+\varphi}, x \models \Box\psi$. The conditions infer $\mathfrak{M}, x \models \Box[+\varphi]\psi \vee (\varphi \wedge \Box(\varphi \to [+\varphi]\psi))$.

- If $\mathfrak{M}, x \models \Box[+\varphi]\psi$, then there is some $U \in \tau$ with $x \in U$, and for all $y \in U$, $\mathfrak{M}, y \models [+\varphi]\psi$, i.e. $\mathfrak{M}^{+\varphi}, y \models \psi$, since $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$. Note that $U \in \tau \subseteq \tau^{+\varphi}$, so $\mathfrak{M}^{+\varphi}, x \models \Box\psi$.
- If $\mathfrak{M}, x \models (\varphi \wedge \Box(\varphi \to [+\varphi]\psi))$, then $\mathfrak{M}, x \models \Box(\varphi \to [+\varphi]\psi)$. This means there is some $U \in \tau$ with $x \in U$, and for all $y \in U$, $\mathfrak{M}, y \models (\varphi \to [+\varphi]\psi)$, i.e. $U \subseteq ([\![\varphi]\!]^{\mathfrak{M}})^c \cup [\![[+\varphi]\psi]\!]^{\mathfrak{M}}$. Note that $x \in [\![\varphi]\!]^{\mathfrak{M}}$, so we may suppose (up to an intersection) $U \subseteq [\![[+\varphi]\psi]\!]^{\mathfrak{M}}$, i.e. for all $y \in U$, $\mathfrak{M}^{+\varphi}, y \models \psi$, since $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$. Note that $U \in \tau \subseteq \tau^{+\varphi}$, so $\mathfrak{M}^{+\varphi}, x \models \Box\psi$.

In both cases, $\mathfrak{M}, x \models \Box(\varphi \to [+\varphi]\psi)$.

4. Topological Approaches to Epistemic Logic: Final Exam

1. Let $\mathfrak{M} = (X, \mathcal{E}_0, \tau, V)$ be a topo-e-model and τ_{dense} be the corresponding dense topology defined as

$$\tau_{dense} = \{U \in \tau \,|\, Cl(U) = X\} \cup \{\emptyset\}.$$

Check our definitions of justified belief B and (fallible) knowledge K given in Lecture 3: for any $P \subseteq X$, we have

$$x \in KP \iff x \in Int(P) \text{ and } Cl(Int(P)) = X,$$
$$x \in BP \iff Cl(Int(P)) = X.$$

Show that

(1) (10') $KP = Int_{dense}(P)$, and

(2) (10') $BP = Cl_{dense}(Int_{dense}(P))$.

Solution. Before proving the statements, we shall show

- $Int_{dense}(P) \subseteq Int(P)$, (the equality holds if $Int(P) \in \tau_{dense}$) since $LHS = \cup_{P \subseteq U \in \tau_{dense}} U \subseteq \cup_{P \subseteq U \in \tau} U = RHS$;
- $Cl_{dense}(P) \supseteq Cl(P)$, since $LHS = (Int_{dense}(P^c))^c \supseteq (Int(P^c))^c = RHS$.

(1) If $Cl(Int(P)) \neq X$, then $KP = \emptyset$. If $Int(P) = \emptyset$, then $Int_{dense}(P) = \emptyset = KP$. If $Int(P) \neq \emptyset$, then $Int(P) \notin \tau_{dense}$, and for any $\emptyset \neq U \in \tau_{dense}$, with $U \subseteq P$, we have $Cl(U) \subseteq Cl(Int(P)) \neq X$, thus $U \notin \tau_{dense}$. This tells us $Int_{dense}(P) = \emptyset = KP$.
Now suppose $Cl(Int(P)) = X$, in which case $KP = Int(P) \neq \emptyset$. Then $Int_{dense}(P) = Int(P)$, since $Int(P) \in \tau_{dense}$ and $Int_{dense}(P) \subseteq Int(P)$. In both cases $Int_{dense}(P) = KP$.

(2) If $Cl(Int(P)) \neq X$, we have $BP = \emptyset$. From the proof of *(1)*, $Int_{dense}(P) = \emptyset = BP$. Now suppose $Cl(Int(P)) = X$, in which case $BP = X$. Note that

$$Cl_{dense}(Int_{dense}(P)) \supseteq Cl(Int_{dense}(P)) = Cl(Int(P)) = X,$$

so $Cl_{dense}(Int_{dense}(P)) = BP$.

2. For this exercise, we work with the unimodal language $\mathcal{L}_B$ and define an alternative topological semantics for B.

Given a topo-model $\mathcal{X} = (X, \tau, V)$, the semantics of $B\varphi$ is given as

$$[\![B\varphi]\!]^{\mathcal{X}} = Int(Cl(Int([\![\varphi]\!]^{\mathcal{X}}))).$$

Definition 1. *Given a Kripke frame (X, R)*

	$\mathsf{wKD45}$ axioms & rules	
(K_B)	$B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi)$	Closure under conjunction
(D_B)	$B\varphi \to \neg B \neg \varphi$	Consistency
$(\mathsf{4}_B)$	$B\varphi \to BB\varphi$	Positive Introspection
$(\mathsf{w5}_B)$	$B\hat{B}B\varphi \to B\varphi$	weak Negative Introspection
(Nec)	from φ, infer $B\varphi$	

- *a nonempty $C \subseteq X$ is called a cluster if for all $x, y \in C, xRy$.*
- *$C \subseteq X$ is called a maximal cluster if it is a cluster and for all $x, y \in X$, if $x \in C$ and xRy, then $y \in C$.*
- *(X, R) is called a rooted weak pin if there is a unique $x \in X$ and a family of disjoint maximal clusters $\{C_i\}_{i\in I}$ such that for all $y \in \cup_{i\in I} C_i, (x, y) \in R$, $(y, x) \notin R$, and $X = \cup_{i\in I} C_i \cup \{x\}$.*

Lemma 1. $\mathsf{wKD45}$ *is sound and complete with respect to the class of finite rooted weak pins.*

In this exercise, you are asked to show all the items below for the $\mathsf{wKD45}_B$ given in the table.

(1) (20') Axioms K_B and D_B are valid in all topo-models under the new semantics.

(2) (10') Also show that the negative introspection axiom for belief, namely $\neg B\varphi \to B\neg B\varphi$, is not valid in all topo-models under the new semantics.

(3) (30') Prove that $\mathsf{wKD45}$ is complete with respect to the class of all topo-models.

Solution. *(1)* Let (X, τ) be a topological space, write for convenience $P(U) = Int(Cl(Int(U)))$, for any $U \subseteq X$.

K_B: Only need to show $P(U \cap V) = P(U) \cap P(V)$ for any $U, V \subseteq X$. We shall prove a stronger result:

$$Int(Cl(U \cap V)) = Int(Cl(U)) \cap Int(Cl(V))$$

for any $U, V \in \tau$. On the one hand, $Int(Cl(U \cap V)) \subseteq Int(Cl(U))$, similarly, $Int(Cl(U \cap V)) \subseteq Int(Cl(V))$, so $LHS \subseteq RHS$. Now consider the other part when $RHS \neq \emptyset$.

- if $U \cap V = \emptyset$, suppose by contradiction that there is some $x \in Int(Cl(U)) \cap Int(Cl(V))$, then there are $U', V' \in \tau$ with $x \in U' \subseteq Cl(U), x \in V' \subseteq Cl(V)$. Let $W = U' \cap V' \in \tau$, note that by definition of closure, $\emptyset \neq W \cap U \subseteq V' \subseteq Cl(V)$, so $\emptyset \neq (W \cap U) \cap V$, that is impossible. As a result, $Int(Cl(U \cap V)) = \emptyset = Int(Cl(U)) \cap Int(Cl(V))$.
- If $U \cap V \neq \emptyset$, for $x \in Int(Cl(U)) \cap Int(Cl(V))$, there are $U', V' \in \tau$ with $x \in U' \subseteq Cl(U), x \in V' \subseteq Cl(V)$. Let $W = U' \cap V' \in \tau$, by definition of closure, $\emptyset \neq W \cap U \subseteq Cl(V)$, and thus $\emptyset \neq W \cap U \cap V \subseteq Cl(U \cap V)$. Hence $x \in Int(Cl(U \cap V))$, as a result, $RHS \subseteq LHS$.

D_B: Only need to show $P(U) \subseteq (P(U^c))^c$, i.e. $Int(Cl(Int(U))) \subseteq Cl(Int(Cl(U)))$ for any $U \subseteq X$. This holds since

$$Int(Cl(Int(U))) \subseteq Int(Cl(U)) \subseteq Cl(Int(Cl(U))).$$

(2) For example, consider $\mathbb{R}$ with the standard topology, take $U = (-\infty, 0) \cup (1, \infty)$, then

$$(P(U))^c = U^c = [0, 1],$$

$$P((P(U))^c) = P([0, 1]) = (0, 1).$$

Then $(P(U))^c \nsubseteq P((P(U))^c)$, i.e. $\neg B\varphi \to B\neg B\varphi$ fails.

(3) We shall mimick the proof of Theorem 7 in [Baltag et al, 2019].

Lemma 2. *For all $\varphi \in \mathcal{L}_B$, and any Kripke model $\mathcal{M} = (X, R, V)$ based on a weak rooted pin with root denoted by $\bullet$,*

$$\|\varphi\|^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{M}_{\tau_R}},$$

where τ_R is the topology generated by $\{C_i\}_{i \in I}$.

Proof. There are some (easily-verified) facts about τ_R

- The opens sets of τ_R is of the forms: $\emptyset, X$ or $\cup_{j \in J} C_j$ for $J \subseteq I$.
- $R(x) = \begin{cases} X & \text{, if } x \text{ is the root } \bullet \\ C_i & \text{, if } x \in C_i \end{cases}$.
- $Cl(\cup_{j \in J} C_j) = \cup_{j \in J} C_j \cup \{\bullet\}$, since $RHS = (\cup_{i \notin J} C_i)^c$ is closed.

The proof follows by induction on φ for sub-formula, we only need to consider the case $\varphi = B\psi$, where the result holds for ψ.
"$\subseteq$": If $\|\psi\|^{\mathcal{M}} = \emptyset$, then $[\![\varphi]\!]^{\mathcal{M}_{\tau_R}} = Int(Cl(Int(\emptyset)) = \emptyset$. Now suppose $\|\psi\|^{\mathcal{M}} \neq \emptyset$ and $\mathcal{M}, x \models B\psi$, i.e. $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. If

$x = \bullet$, then $R(x) = X$, from the hypothesis,

$$X = Int(Cl(Int(R(x)))) \subseteq Int(Cl(Int([\![\psi]\!]^{\mathcal{M}_{\tau_R}}))).$$

If $x \in C_i$ for some i, then $R(x) = C_i$, similarly,

$$C_i = Int(Cl(Int(R(x)))) \subseteq Int(Cl(Int([\![\psi]\!]^{\mathcal{M}_{\tau_R}}))).$$

In both cases, we have $x \in [\![\varphi]\!]^{\mathcal{M}_{\tau_R}}$.
"$\supseteq$": Consider only the case when $\|\psi\|^{\mathcal{M}} \neq \emptyset$. Suppose by contradiction that there is some $x \in Int(Cl(Int(\|\psi\|^{\mathcal{M}})))$ with $x \notin \|\varphi\|^{\mathcal{M}}$, i.e. $R(x) \nsubseteq \|\psi\|^{\mathcal{M}}$. If $x = \bullet$, then $R(x) = X$, and thus $Int(\|\psi\|)^{\mathcal{M}} = \cup_{j\in J} C_j$ for some $J \subseteq I$. But

$$x \notin \cup_{j\in J} C_j = Int((\cup_{j\in J} C_j) \cup \{\bullet\}) = Int(Cl(\cup_{j\in J} C_j)).$$

So it is impossible. If $x \in C_i$ for some i, then $R(x) = C_i \nsubseteq \|\psi\|^{\mathcal{M}}$. Thus $C_i \cap Int(\|\psi\|^{\mathcal{M}}) = \emptyset$, since there are no other open sets contained in C_i. Then $Int(\|\psi\|^{\mathcal{M}}) \subseteq \cup_{j\neq i} C_j$,

$$Int(Cl(Int(\|\psi\|^{\mathcal{M}}))) \subseteq Int(Cl(Int(\cup_{j\neq i} C_j))) = \cup_{j\neq i} C_j \not\ni x.$$

This is also impossible. In conclusion, $\|\psi\|^{\mathcal{M}} \supseteq [\![\varphi]\!]^{\mathcal{M}_{\tau_R}}$. $\square$

Now back to the original problem. Let $\varphi \in \mathcal{L}_B$ such that $\varphi \notin$ $\mathsf{wKD45}_B$. Then by Lemma 1, there exists a relational model $\mathcal{M} = (X, R, V)$ where (X, R) is a weak rooted pin, and $x \in X$ such that $\mathcal{M}, x \not\models \varphi$. Therefore, by Lemma 2, we obtain $\mathcal{M}_{\tau_R}, x \not\models \varphi$, so φ is not valid. This shows the completeness.

3. For this exercise we work with the subset space semantics introduced in Lecture 4. Recall the axiom (AP) on slide 23 of Lecture 4:

$$\text{(AP)} \quad (p \to \blacksquare p) \wedge (\neg p \to \blacksquare \neg p) \text{ for all } p \in Prop.$$

(1) (10') Show that for every φ in the language of classical propositional logic (elements of $\mathcal{L}_{K\blacksquare}$ that do not have occurrences of K or $\blacksquare$),

$$(\varphi \to \blacksquare\varphi) \wedge (\neg\varphi \to \blacksquare\neg\varphi)$$

is valid in all subset space models.

(2) (10') Show that the validity in *(1)* does not hold for all $\psi \in \mathcal{L}_{K\blacksquare}$. That is, find a $\psi \in \mathcal{L}_{K\blacksquare}$ such that $(\psi \to \blacksquare\psi) \wedge (\neg\psi \to \blacksquare\neg\psi)$ is not valid and justify your answer.

Solution. *(1)* For any subset space model $\mathcal{X} = (X, \mathcal{O}, V)$, and epistemic scenario (x, U), we only need to show $(x, U) \models (\varphi \to \blacksquare\varphi)$, if there is no occurrence of $K, \blacksquare$ in φ (thus also, $\neg\varphi$). Suppose $(x, U) \models \varphi$, i.e. $x \in V(\varphi)$. Then for any $O \in \mathcal{O}$ with $x \in O \subseteq U$, we have $x \in V(\varphi)$, so $(x, O) \models \varphi$. Thus $(x, U) \models \blacksquare\varphi$.

(2) For example, consider $X = \{a, b\}$, $\mathcal{O}$ the discrete topology, p with $V(p) = \{a\}$. Then since $\{a, b\} \nsubseteq V(p)$, we have $(a, \{a, b\}) \not\models Kp$, i.e. $(a, \{a, b\}) \models \neg Kp$. However, for $O = \{a\} \in \mathcal{O}$, we have $a \in O \subseteq \{a, b\}$, and $(x, O) \models Kp$, i.e. $(x, O) \not\models \neg Kp$. By definition, $(a, \{a, b\}) \not\models \blacksquare\neg Kp$. As a result, $(a, \{a, b\}) \not\models (\neg Kp \to \blacksquare\neg Kp)$.