Geometry and topology: a problem set

2025 spring

Here are some problems for training on geometry and topology, many of the topological problems are accompanied by solutions (at least hints). Some good references on the same purpose are (Harvard; CUNY; UCLA).

1. Some problems

1.1. Calculations

1. (Harvard24f, Top) Calculate the homology groups of $\mathbb{RP}^2 \times \mathbb{RP}^3$,

   (1) with coefficients in $\mathbb{Z}$;

   (2) with coefficients in $\mathbb{Z}/2\mathbb{Z}$;

   (3) with coefficients in $\mathbb{Z}/3\mathbb{Z}$.

2. (Harvard24s, DG) Show that the orthogonal group $O(n)$ is a smooth manifold.

3. (Harvard23f, Top) Let $A = \mathbb{RP}^2, B$ be the Klein bottle, $C = A \times B$

   (1) Compute $H_*(A; \mathbb{Z})$;

   (2) Compute $H_*(B; \mathbb{Z}/2\mathbb{Z})$;

   (3) Compute $H_*(C; \mathbb{Z}/2\mathbb{Z})$.

4. (Harvard23f, Top) Let T_1, T_2, T_3 be three tori $\mathbb{T}^2$, p, q be two distinct points on S^1. Let X be obtained from the union of T_1, T_2, T_3 by gluing $p \times S^1 \subset T_1$ to $p \times S^1 \subset T_2$ and gluing $q \times S^1 \subset T_3$ to $q \times S^1 \subset T_2$. Compute $\pi_1(X)$.

5. (Harvard22f, Top) Let S^n be the standard n-sphere and let $S^k \subset S^n$ be the locus defined by the vanishing of the last $n-k$ coordinates $x_{k+1}, \cdots, x_n$. Assume $n-1 > k > 0$.

   (1) Find the homology groups of the complement $S^n \backslash S^k$;

   (2) Suppose now that $T \subset S^n$ is the sphere defined by the vanishing of the first $k+1$ coordinates; that is,
   $$T = \left\{(0, \cdots, x_{k+1}, \cdots, x_n) \,\middle|\, \sum x_i^2 = 1\right\}$$
   What is the fundamental class of T in $H_{n-k-1}(S^n \backslash S^k)$?

6. (Harvard08f, DG) Let n be a positive integer, A a symmetric $n \times n$ matrix and Q the quadratic form
   $$Q(x) = \sum A_{ij} x^i x^j.$$
   Define a metric on $\mathbb{R}^n$ using
   $$\mathrm{d}s^2 = e^{Q(x)} \sum \mathrm{d}x^i \otimes \mathrm{d}x^i.$$
   (1) Write down the geodesic equation;

   (2) Write down the curvature tensor.

7. (Harvard08f, Top) Find the universal covering of $\mathbb{RP}^2 \vee S^2$.

8. (Harvard08f, Top) Let X and Y be compact, connected, oriented 3-manifolds, with
   $$\pi_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \pi_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$
   Compute the integral homology of X, Y and the rational homology of $X \times Y$.

   (In fact, the possible fundamental groups of compact connected oriented 3-manifolds are $\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^{\oplus 3}, (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, see Theorem 9.13)

9. (UCLA23f, Top) Let D^2 be the unit disk in $\mathbb{C}$, and $S^1 = \partial D^2$. Let $X = D^2 \times S^1 \times \{0, 1\}/\sim$, where
   $$(x, y, 0) \sim (xy^5, y, 1)$$
   for all $x, y \in S^1$. Compute the homology groups of X.

   (what about $[0, 1]$?)

1.2. Theoretical problems

1. (Harvard24f, Top) X finite connected CW complex, $\pi_1(X) \neq 0$ and is finite. Prove that the universal covering $\widetilde{X}$ can not be contractible.

   (Lefschetz fixed point, universal covering; Euler characteristic)

2. (Harvard24f, Top) Let Σ_g be the closed oriented surface of genus g. Prove that there is a degree 1 map $\Sigma_g \to \Sigma_h$ if and only if $g \geqslant h$. (degree, homology)

3. (Harvard24s, Top) Prove that every closed, connected compacted C^∞ manifold of odd dimension has Euler characteristic 0.

   (UCT, Poincaré duality)

4. (Harvard23f, Top)

   (1) Is there a continuous map from $\mathbb{CP}^2$ to itself with no fixed points?

   (2) Is there a continuous map from $\mathbb{CP}^3$ to itself with no fixed points?

   (cup product, Lefschetz fixed point, special construction)

5. (Harvard22f, Top) What are the possible degrees of continuous maps $\mathbb{CP}^4 \to \mathbb{CP}^4$? Justify your answer.

   (special construction)

6. (Harvard21f, Top) Let M be a connected closed 4-manifold such that $\pi_1(M)$ is perfect; that is, does not have any non-trivial abelian quotients. Determine the possible cohomology groups $H^*(M;\mathbb{Z})$.

   (orientation covering, UCT, Poincaré duality, special construction)

7. (Harvard21s, Top) Suppose f is an orientation-preserving self homeomorphism of $\mathbb{CP}^n$ such that the graph $\Gamma_f \subset \mathbb{CP}^n \times \mathbb{CP}^n$ intersects the diagonal transversely . Compute all possibilities for the number of its fixed points.

   (Lefschetz fixed point/Euler characteristic, intersection, special construction)

8. (Harvard20f, Top) Let $S^1 = \mathbb{R}/\mathbb{Z}$ be a circle, and let S^2 be a two-dimensional sphere. Consider involutions on both, with an involution on S^1 defined by $x \to -x$ and with $j : S^2 \to S^2$ defined by reflection about an equator. Let M be the space of maps that respects these involutions, i.e.

$$M = \{f : S^1 \to S^2 \,|\, f(-x) = j(f(x))\}.$$

Show M is connected but not simply-connected.

(special construction, fibration)

9. (Harvard20s, Top) Determine whether $X = S^2 \vee S^3 \vee S^5$ is homotopy equivalent to

   (1) a manifold;

   (2) a compact manifold;

   (3) a compact orientable manifold.

   (Poincaré duality, oriention covering, special construction)

10. (Harvard20s, Top) Suppose that X is a space written as a union of two simply connected open subsets U_1, U_2.

    (1) Show that $H_1(X)$ is a free abelian group.

    (2) Find an example in which $\pi_1(X)$ is a non-trivial group. Why does this not contradict the the Seifert-van Kampen theorem?

    (3) Find an example in which $\pi_1(X)$ is non-abelian.

    (special construction)

11. (Harvard19f, Top) Let $\mathrm{Sym}^n(X)$ denote the nth symmetric power of a CW complex X, i.e. X^n/S_n, where the symmetric group S_n acts by permuting coordinates. Show that for all $n \geqslant 2$, the fundamental group of $\mathrm{Sym}^n(X)$ is abelian.

    (fundamental group, special construction)

12. (Harvard19f, Top) Suppose that m is odd. Show that if nis odd there is a fixed point free action of $\mathbb{Z}/m$ on S^n. What happens if n is even?

    (special construction, Lefschetz fixed point)

13. (CUNY24f, Top) Compute the fundamental group of $\mathbb{R}^3$ minus a union of the Z-axis and the unit circle in the XY-plane.

    (deformation retract)

14. (CUNY24f, Top) Prove that every map $\mathbb{CP}^3 \to S^1 \times S^1 \times S^1$ is nullhomotopic.

15. (CUNY24s, Top) Prove that there is no orientation reversing homeomorphism $\mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$.

    (intersection form)

16. (CUNY23s, Top) Let M and N be surfaces of genus 2, X the space obtained as a quotient of the disjoint union of M and N by identifying a separating curve on M with a non-separating curve on N.

    (1) Compute $\pi_1(X)$;

    (2) Is there a retraction $X \to N$?

17. (CUNY24s, DG) Consider the manifold $S^n \times \mathbb{T}^m$, where $n \geqslant 2, m \geqslant 1$

    (1) Does it admit a Riemannian metric with $\mathrm{Ric} > 0$? Justify.

    (2) Does it admit a Riemannian metric with $\sec \geqslant 0$? Justify.

    (3) Does it admit a Riemannian metric with $\sec \leqslant 0$? Justify.

    (CUNY24f, DG) $\mathbb{RP}^n \times \mathbb{RP}^n, n \geqslant 2$......

    (curvature and topology)

2. Past qualifying exams

2.1. Problems on topology

Remarks: MV sequence is a must.

1. [23f] Let M be the quotient space

$$([0,1] \times \mathbb{CP}^2)/(0, [z_0, z_1, z_2]) \sim (1, [\overline{z}_0, \overline{z}_1, \overline{z}_2])$$

   compute the homology groups.

2. [23f] Consider a map $f : \mathbb{T}^2 \to \mathbb{RP}^2$ from the torus to the projective plane. Suppose the induced map $f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{RP}^2)$ is trivial. Is f always null-homotopic?

3. [23f] Let M be a closed smooth manifold. Show that M is a smooth fiber bundle over S^1 if and only if there exists a closed, nowhere vanishing differential 1-form on M.

4. [23f]

    (1) Let M be a compact, orientable $(2n+1)$-dimensional manifold with boundary N. Consider the map

$$i^* : H^n(M; \mathbb{R}) \to H^n(N; \mathbb{R})$$

    induced by the inclusion $i : N \to M$. Show that $\dim \operatorname{im} i^* = \frac{1}{2} \dim H^n(N; \mathbb{R})$.

(2) Show that for any $n \geqslant 1$, $\#_n\mathbb{CP}^2$ is not the boundary of a compact, orientable 5-dimensional topological manifold.

5. [24s] Let a, b denote the two standard generators of $\pi_1(X) = F_2$

   (1) Draw a picture of the directed graph of the covering space of X that corresponds to the subgroup $H = \langle a^2, b^2, ab^2a, ba^2b, (ab)^2 \rangle$.

   (2) Is the covering above a normal covering?

6. [24s] Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Prove that the complement $\mathbb{R}^n \backslash M$ is connected and simply-connected.

7. [24s] Let M be a 2-dimensional manifold without boundary such that $\pi_1(M)$ is an infinite group. Compute $\pi_n(M)$ for all $n \geqslant 2$.

8. [24s] Let M, N be closed connected orientable manifolds of the same dimension n. Suppose that $f : M \to N$ induces a non-zero map $f_* : H_n(M; \mathbb{Z}) \to H_n(N; \mathbb{Z})$. Are the following statements true?

   (1) $\dim_{\mathbb{Q}} H_k(M; \mathbb{Q}) \geqslant \dim_{\mathbb{Q}} H_k(N; \mathbb{Q})$ for all k;

   (2) $\operatorname{rank} H_k(M; \mathbb{Z}) \geqslant \operatorname{rank} H_k(N; \mathbb{Z})$ for all k. (here the rank is defined to be the minimal number of generators)

9. [24s] Prove that $\mathbb{RP}^n \times S^2$ does not have an open covering consisting of k contractible open subsets if $k \leqslant n+1$.

10. [24f] What is the fundamental group of $SO(3)$?

11. [24f] Solve the following problems

    (1) Is it true that every continuous map $f : S^{2024} \to \mathbb{RP}^{2024}$ is null-homotopic?

    (2) Is it true that every continuous map $g : S^{2024} \to \mathbb{CP}^{2012}$ is null-homotopic?

12. [24f] Consider the quotient space $X = ([0,1] \times S^1 \times S^1)/\sim$, where $\sim$ is generated by

    $$(0, x, y) \sim (0, z, w) \text{ if } xy = zw, (1, x, y) \sim (1, z, w) \text{ if } x^2y^6 = z^2w^6$$

    Compute the homology groups.

13. [24f] Show that $\mathbb{CP}^n$ is not an H-space for any $n \geqslant 1$.

14. [24f] Let M be an oriented connected closed manifold of dimension $n \geqslant 2$. Let $f : S^n \to M$ be a continuous map of mapping degree 1. Show that f must be a homotopy equivalence.

15. [24f] Let M be a smooth closed manifold of dimension $n \geqslant 1$, $f : M \to M$ be a smooth map such that $f^2 = \mathrm{id}$. Show that the set $\{x \in M \mid f(x) = x\}$ can not be a single point.

16. [25s] Let M, N be closed connected manifolds of the same dimension. Suppose $f : M \to N$ is an immersion. Is the following true or false?

    (1) f is injective.

    (2) f is surjective.

    (3) f is a homeomorphism.

    (4) f is a covering map.

17. [25s] Let X denote a closed orientable surface of genus 2. Consider a group homeomorphism $\rho : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$ such that $\rho(e_4) = 1$ and $\rho(e_i) = 0$ for $i = 1, 2, 3$.

    (1) Draw a picture of a covering space Y that corresponds to the subgroup $\ker \rho \subset \pi_1(X)$.

    (2) On the picture, label all curves on Y which are preimages of e_1, e_2, e_3, e_4.

    (3) Let T denote the non-trivial deck transformation of the covering $Y \to X$. Compute the signature $\sigma(T_*)$ of $T_* : H_1(Y) \to H_1(Y)$.

    (4) Prove that there exists a homeomorphism $\phi : X \to X$ such that $\phi(e_1) = e_4$.

18. [25s] Compute the cohomology ring of $(S^2 \times S^8)\#(S^4 \times S^6)$.

19. [25s] Suppose that M is a compact connected non-orientable manifold of dimension 3. Prove that $\pi_1(M)$ is infinite.

    (universal covering, Hurewicz theorem, Whitehead theorem, Lefschetz fixed point)

20. [25s] Let $SL_3(\mathbb{C})$ be the space of 3×3 complex matrices with determinant 1. Compute $\pi_3(SL_3(\mathbb{C}))$.

2.2. Problems on geometry

1. [23f] On a Riemannian manifold (M, g), suppose that f is a smooth function such that $|\operatorname{grad} f| = 1$. Show that the integral curves of $\operatorname{grad} f$ are geodesics.

2. [23f] Let $F : (M, g) \to (M, g)$ be an isometry that fixes $p \in M$. Show that $F_{*,p} = -\operatorname{id}$ on T_pM iff $F^2 = \operatorname{id}_M$ and p is an isolated fixed point.

3. [23f] Consider an Einstein metric (N^{n-1}, g_N) with $\operatorname{Ric} = \frac{n-2}{n-1}\lambda g_N, \lambda < 0$. Find a function $\rho : \mathbb{R} \to (0, \infty)$ such that $(M^n, g) = (\mathbb{R} \times N, \mathrm{d}r^2 + \rho^2 g_N)$ becomes an Einstein metric with $\operatorname{Ric} = \lambda g$.

4. [23f] Suppose (M, g) is a Riemannian manifold. There is a unique bundle endomorphism $\mathfrak{R} : \wedge^2 TM \to \wedge^2 TM$ called the curvature operator of g, satisfying

$$g(\mathfrak{R}(X \wedge Y), Y \wedge Z) = -R(X, Y, Z, W)$$

for all tangent vectors X, Y, Z, W at a point of M. A Riemannian metric is said to have positive curvature operator if $\mathfrak{R}$ is positive definite.

(1) Show that positive curvature operator implies positive sectional curvature.

(2) Let g be the Fubini-Study metric on $\mathbb{CP}^2$. Does it have positive sectional curvature? Does it have positive curvature operator?

5. [24s] Let (M, g) be a complete Riemannian manifold. Is every Killing vector field X on M complete?

6. [24s] Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be self-dual if $*\omega = \omega$ and anti-self-dual if $*\omega = -\omega$.

(1) Show that every 2-form ω on M can be written uniquely as a sum of self-dual form and an anti-self-dual form.

(2) On $M = \mathbb{R}^4$ with the Euclidean metric, determine the self-dual and anti-self-dual forms in standard coordinates.

7. [24s] Classifying all 2024-dimensional complete Riemannian manifolds with constant sectional curvature $K = 1$.

8. [24s] Compute the Ricci curvature and scalar curvature of $(\mathbb{R}^2, g = e^{-\pi(x^2+y^2)}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y))$. Is it a complete Riemannian manifold?

9. [24s] Can $M = S^1 \times S^2$ admit a smooth Riemannian metric g with $\mathrm{Ric} = f \cdot g$ for some smooth function f on M?

10. [24f] Does $SO(3)$ admit a smooth Riemannian metric with constant Ricci curvature?

11. [24f] Let (M, g) be a connected Riemannian manifold of dimension n. Suppose that there exists some $f \in C^\infty(M, \mathbb{R})$ such that $\mathrm{Ric} = (n-1)fg$.

    (1) If $n = 2$, is f necessarily a constant?

    (2) If $n = 2024$, is f necessarily a constant?

12. [24f] Let (M, g) be a Cartan-Hadamard manifold. Given $p \in M$, let $f : M \to [0, \infty)$ be the function $f(x) = \frac{1}{2}d(x,p)^2$. Show that f is strictly geodesically convex, i.e. for any non-trivial geodesic $\gamma : [0,1] \to M$, the following inequality holds for all $t \in (0,1)$

$$f(\gamma(t)) < (1-t)f(\gamma(t)) + tf(\gamma(1)).$$

13. [24f] Let (M, g) be a smooth Riemannian manifold of dimension n and $p \in M$. Show that when r is small enough,

$$\mathrm{Vol}(B(p,r)) = \omega_n r^n \left(1 - \frac{S_p}{6(n+2)} r^2 + O(r^3)\right).$$

14. [24f] Solve the following problems.

    (1) Does $S^1 \times S^1$ admit a Riemannian metric with conjugate radius $= \infty$?

    (2) Does $S^1 \times S^1$ admit a Riemannian metric with conjugate radius $< \infty$?

    (3) Does $S^2 \times S^1$ admit a Riemannian metric with conjugate radius $= \infty$?

15. [25s] Let $B^n(r) = \{x \in \mathbb{R}^n \mid |x| < r\}$. Compute the sectional curvature of

$$g = \frac{r^4}{(r^2 - |x|^2)} \sum_i \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

16. [25s] Let (M, g) be a Riemannian manifold and X be a Killing field. If γ is a geodesic, show that $J(t) = X \circ \gamma(t)$ is a Jacobi field along γ.

17. [25s] A Riemannian metric h on a Lie group G is said to be left-invariant if $L_g^* h = h$ for all $g \in G$. A right-invariant metric is defined similarly. A metric that is both left- and right-invariant is said to be bi-invariant. Let (M, g) be a Riemannian manifold with a bi-invariant Riemannian metric g.

   (1) Show that for all left-invariant vector fields X, Y, Z,
$$h([X,Y],Z) = h(X,[Y,Z]).$$

   (2) Show that for any left-invariant vector fields X, Y,
$$\nabla_X Y = \frac{1}{2}[X,Y].$$

18. [25s] Let (M, g) be a simply-connected complete Riemannian manifold. If the differential of each exponential map is length increasing, i.e.
$$|(\exp_p)_{*,v}(\widetilde{v}| \geqslant |\widetilde{v}|$$
for all $p \in M, v, \widetilde{v} \in T_pM$. Show that (M, g) has non-positive sectional curvature.

19. [25s] Let (M, g) be an n-dimensional compact oriented Riemannian manifold with positive sectional curvature. Given an isometry $F : M \to M$ such that F preserves the orientation when n is enen, F changes the orientation when n is odd. Show that F has a fixed point.

3. Some problems on topology

3.1. Homology and cohomology

Homology group and cohomology ring

1. Show that the fundamental group of an H-space is abelian and that S^{even} is not an H-space.
   (variant) Show that $\mathbb{CP}^n$ is not an H-space if $n \geqslant 1$.

   Answer:

(cup product, H-space)

2. (Harvard95, Top) Show that for $m > n$ there does not exist an antipodal map $f : S^m \to S^n$, that is a continuous map carrying antipodal points to antipodal points.

   Answer:

   (cup product, covering space)

3. Let $\overline{\mathbb{CP}^2}$ be the complex projective plane with the opposite orientation.

   (1) Show that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ have the same cohomology groups but different cohomology rings.

   (2) Show that $(S^2 \times S^2) \# \overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ have the same cohomology rings.

   Answer:

   (connected sum, cohomology ring)

4. Show that there is no degree one map from $S^2 \times S^2$ to $\mathbb{CP}^2$.

   Answer:

   (cup product, pairing)

5. Show that $\mathbb{CP}^{2n}$ does not cover any manifold except itself.

   Answer:

   (Lefschetz fixed point, cup product)

6. If $f : M \to N$ is a degree one map between compact connected oriented manifolds without boundary, show that the induced map on $\pi_1(-)$ is surjective. Show the induced map on $H_k(-)$ is surjective.

   Answer:

   (covering-subgroup correspondence, Poincaré duality, pairing)

7. Consider the quotient space

$$X = ([0,1] \times S^1 \times S^1)/\sim,$$

   where $\sim$ is generated by

$$(0, x, y) \sim (0, z, w), \text{if } xy = zw$$

   and

$$(1, x, y) \sim (1, z, w), \text{if } x^2y^6 = z^2w^6.$$

   Compute $H_n(X; \mathbb{Z})$ for all n.

   Answer:

(computation, MV sequence, deformation)

8. Let M be the quotient space

$$([0,1] \times \mathbb{CP}^2)/(0, [z_0, z_1, z_2]) \sim (1, [\overline{z_0}, \overline{z_1}, \overline{z_2}]).$$

Compute the homology group $H_k(M; \mathbb{Z})$ for all $k \geqslant 0$.

Answer:

(computation, MV sequence, deformation)

9. Let $M_h' \subset M_g$ be a compact subsurface of genus h with one boundary circle. Show that there is no retraction $M_g \to M_h'$ if $h > g/2$.

Answer:

(homology, degree)

10. *(1)* Let M be the boundary of an orientable $2n+1$-dimensional manifold N, show that

$$\dim(\ker(i_* : H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q}))) = \frac{1}{2}\dim H_n(M;\mathbb{Q}).$$

(2) Show that for any $n \neq 0$, the connected sum of n copies of $\mathbb{CP}^2$ is not a boundary of any orientable 5-manifold.

Answer:

(Lefschetz duality, cup product, boundary)

11. Show that there does not exists a closed 3-manifold M such that $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Answer:

(cup product, special construction)

12. Compute the degree of the self-map $g \to g^q$ defined on the group $SU(2)$, where q is an arbitrary integer.

13. Fine the minimal number k such that $\mathbb{CP}^n$ can be covered by open contractible subsets $U_1, \cdots, U_k$.

Answer:

(cup product, relative cohomology)

14. Consider the abelianization $\mathbb{Z}/2 * \mathbb{Z}/3 \to \mathbb{Z}/2 \oplus \mathbb{Z}/3$, compute its kernel.

15. Let M be a 4-dimensional manifold such that $\partial M = \mathbb{RP}^3$. Consider the map $i_* : H_1(\mathbb{RP}^3; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ induced by the inclusion $\partial M \to M$, show that i_* must be trivial.

    Answer:

(cup product, duality, relative homology, E-M space)

16. Let M be a smooth closed manifold of dimension 4. Show that there does not exist smooth $\mathbb{Z}/2\mathbb{Z}$ action on M with a single fixed point.

    Answer:

(metric, classification of coverings, question...)

Orientation, fundamental class and Poincaré duality

1. Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.
2. Let M be a closed smooth n-manifold.
   (1) Does there always exist a smooth map $f : M \to S^n$ from M to the n-sphere, such that f is essential (i.e. not homotopic to a constant map)?
   (2) What about $\mathbb{T}^n$?

   Answer:

   (special construction)
3. (Harvard90, Top) Let $\Sigma \hookrightarrow \mathbb{R}^3$ be an embedded, closed surface of genus 3. Define the Gauss map $\phi : \Sigma \to S^2$ by associating to each point $x \in \Sigma$ the outward pointing unit normal vector. Let ω be the volume form on S^2, compute $\int_\Sigma \phi^*\omega$.
   (variant) Let Σ be a closed surface of genus 2 embedded in $\mathbb{R}^3$, consider the map $\Sigma \to S^2$ by assigning to p the unit normal X_p to Σ at p translating to the origin. What is the degree of this map?

   Answer:

   (pairing, Euler class)
4. (Harvard92, Top) Let X be a compact orientable surface of genus 2, and let $\phi : X \to X$ be a fixed point-free homeomorphism of finite order.
   (1) Show that ϕ is of order 2 and orientation-reversing.
   (2) Show that such homeomorphism of surface of genus 2 do exist.

Answer:

(covering space, Euler characteristic)

5. Let M be a smooth compact connected n-manifold without boundary, and $\phi : M \to M$ be a smooth map which is smoothly homotopic to the identity. Show that ϕ must be surjective.

   Answer:

   (factoring through, $\mathbb{Z}/2\mathbb{Z}$-coefficient)

6. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L be the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection. Find a differential form on M which represents the Poincaré dual of S.

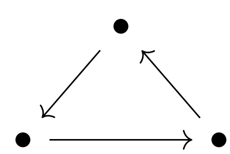
   Answer:

   (Poincaré duality)

7. *(1)* Show that $\mathbb{CP}^{2n}$ cannot be the boundary of a compact manifold.

   (2) Show that $\mathbb{CP}^3$ is the boundary of some compact manifold.

   (3) (Harvard93, Top) Let X be the topological space obtained by identifying all the three sides of a triangle as shown in the diagram.

   Compute the homology groups of X with coefficients in $\mathbb{Z}$ and in $\mathbb{Z}/3\mathbb{Z}$. Is X a closed manifold?

   Answer:

(Euler characteristic, special construction, fibration)

8. Let M, N be closed connected oriented 3-manifolds with fundamental groups $\pi_1(M) = \mathbb{Z}_3 \oplus \mathbb{Z}_2$ and $\pi_1(N) = \mathbb{Z} \oplus \mathbb{Z}_2$. Find all integral homology groups of M, N and rational homology of $M \times N$.

   Answer:

   (Poincaré duality, UCT, Künneth formula)

9. Let M be a compact orientable manifold of dimension $4n+2$, show that $\dim H_{2n+1}(M; \mathbb{R})$ is even.

   Answer:

   (intersection form)

10. Let $f : M \to N$ be continuous map between two closed oriented connected manifolds. Suppose $\deg f \neq 0$, show that $b_i(M) \geqslant b_i(N)$ for all i.

    Answer:

    (Poincaré duality, pairing)

11. Let M be a closed connected $n-1$-dimensional manifold with a smooth embedding $M \hookrightarrow S^n$. Show that M is orientable and $S^n \backslash M$ has exactly two components. Does this result hold if we allow M to have a boundary?

    Answer:

    (tubular neighborhood, MV sequence, or Alexander duality)

Cohomology with compact support, product and transverse intersection

1. (Harvard96, Top) Let X be a compact n-dimensional manifold, and $Y \subset X$ a closed submanifold of dimension m. Show that the Euler characteristic $\chi(X\backslash Y)$ of the complement of Y in X is given by

$$\chi(X\backslash Y) = \chi(X) + (-1)^{n-m-1}\chi(Y).$$

Does the same result hold if we do not assume that X is compact, but only that the Euler characteristic of X and Y are finite?

Answer:

(hard, cohomology with compact support, or tubular neighborhood)

2. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $f : S^n \to S^n$ a continuous map. Assume that the degree of f is odd. Show that there exists $x_0 \in S^n$ such that $f(-x_0) = -f(x_0)$.

Answer:

3. Classify all vector bundles over S^1 up to isomorphism.

4. Consider a map $\pi : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ given by $l \mapsto l \otimes_{\mathbb{R}} \mathbb{C}$. Compute the induced map on cohomology mod 2.

   Answer:

   (intersection)

5. Consider a map $\pi : \mathbb{RP}^n \to \mathbb{CP}^n$ induced by the inclusion $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$. Compute the induced map on cohomology mod 2.

   Answer:

6. Let M be a closed orientable 4-dimensional manifold. Given any homology class $\alpha \in H_2(M; \mathbb{Z})$, show that there exists an embedded surface $F \hookrightarrow M$ such that $[F] = \alpha$.

   Answer:

(intersection, Thom transverse theorem, E-M space)

3.2. Homotopy

Whitehead theorem, Hurewicz theorem and CW approximation

1. Let M be a simply-connected, closed 3-manifold. Show that M is homotopy equivalent to S^3.

   Answer:

   (Whitehead theorem)

2. Let M be a connected, orientable manifold of dimension $n \geqslant 2$ and $f : S^n \to M$ be a map of degree 1. Show that f must be a homotopy equivalence.

   Answer:

   (Poincaré duality, Whitehead theorem)

3. Let $X = \mathbb{R}^3 \backslash (\{(0,0,z) | z \in \mathbb{R}\} \cup \{(1,0,0)\})$, compute $\pi_2(X)$.

   Answer:

   (deformation, universal covering, Hurewicz theorem)

4. For $k \geqslant 2$, show that $\pi_k(S^k \vee S^k) \cong \mathbb{Z} \oplus \mathbb{Z}$.

   Answer:

(attaching cells)

Sharpness: $\pi_3(S^2 \vee S^2) = \mathbb{Z}^{\oplus 3}$, $S^3 \to S^2 \vee S^2$ gives the third $\mathbb{Z}$.

5. Let X be a CW complex such that the reduced homology $\widetilde{H}_*(X;\mathbb{Z}) = 0$. Show that the suspension ΣX is contractible.

   Answer:

   (van Kampen theorem, Whitehead theorem)

6. Let $f : X \to Y$ be a ma between connected CW complexes. Prove the following two results:

   (1) Suppose f induces isomorphism on $\pi_1(-)$ and isomorphism on $H_k(-;\mathbb{Z})$ for all k, then f is a homotopy equivalence.

   (2) Suppose X, Y are both n-dimensional, and f induces isomorphism on $\pi_k(-)$ for all $k \leqslant n$, then f is a homotopy equivalence. (hint: relative Hurewicz: $\pi_{<n}(X, A) = 0 \implies H_{<n}(X, A) = 0$, and $\mathrm{Ab}(\pi_n(X, A)) \to H_n(X, A)$ is surjective)

   Answer:

   (relative Hurewicz theorem, Whitehead theorem, link)

Eilenberg-Maclane spaces

1. If a finite dimensional CW complex X is a $K(G, 1)$ space, then the group $G = \pi_1(X)$ must be torsion free.

Answer:

(E-M space, lens spaces $K(\mathbb{Z}/m\mathbb{Z}, 1) = S^\infty/(\mathbb{Z}/m\mathbb{Z})$)

2. Consider the Eilenberg-Maclane space $X = K(G, n)$, where G is a finite group. Show that $H^*(X; \mathbb{Q}) = 0$ for $* > 0$.

   Answer:

   (E-M space, spectral sequence, question...)

3. For any integer $n, k \geqslant 2$, show that there does not exists a free action of $G = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ on S^n.

   Answer:

   (E-M space, gluing cells)

4. More problems that matter

There are some problems on topology

1. Calculate $\pi_1(SO(n)), \pi_2(SO(n))$.

   Answer:

(Lie group, fibration)

2. Let $X = S^1 \vee S^1, \pi_1(x) = \langle a, b\rangle$. Find the covering of X corresponding to $\left\langle a^2, b^2, ab^2a, ba^2b, (ab)^2 \right\rangle$. Is this subgroup normal?

3. Let M be a compact manifold with $\partial M = \mathbb{RP}^n$. Show that

   (1) n must be odd,

   (2) the map $H_1(\mathbb{RP}^n) \to H_1(M)$ is trivial.

   (3) Give an example for $n = 3$.

   Answer:

(Euler characteristic, pairing, special construction)

4. Show that any continuous map $f : \mathbb{RP}^2 \to \Sigma_g$ is null-homotopic.

   Answer:

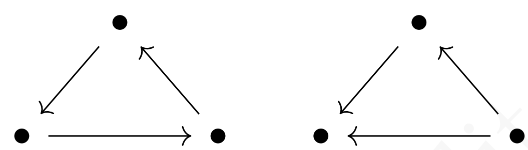
(factoring through)

5. Show that $f : \mathbb{T}^n \to S^n$ is null-homotopic iff $\deg f = 0$.

   Answer:

(CW approximation, factoring through, Hopf theorem)

6. Define X_1, X_2 by identifying sides of a triangle in the following ways:



compute $\pi_2(X_1), \pi_2(X_2)$.

Answer:

(attaching cells, universal covering)

7. Show that $\pi_3(S^2) = \mathbb{Z}\langle \eta \rangle$, where $\eta : S^3 \to S^2, (z_1, z_2) \to z_1/z_2$.
8. Compute $[\mathbb{RP}^n, \mathbb{T}^n]$.

Answer:

(E-M space, representation theorem, UCT)

9. Compute $\pi_2(S^1 \vee S^2)$.
10. Let M be a closed 3-dimensional manifold with $\pi_1(M) = 0$, show that $M \simeq S^3$.
11. Let M be a closed connected orientable n-dimensional manifold, $f : S^n \to M, \deg f = 1$. Show that f is a homotopy equivalence.
12. Are $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ homotopy equivalent?

13. Compute $\pi_i(U(n))$ for $i \leqslant 3, n \geqslant 2$.

    Answer:

    (Lie group, fibration)

14. Let M^m be a closed connected oriented manifold embedded in $\mathbb{R}^n$. Show that $\pi_k(\mathbb{R}^n \backslash M) = \begin{cases} 0 & , k < n-m-1 \\ \mathbb{Z} & , k = n-m-1 \end{cases}$.

    Answer:

    (intersection, factoring through, Alexander duality, Hurewicz theorem)

15. 设 M, N 是 n 维闭流形，M 可定向，N 不可定向。设 $f : M \to N$ 是一个连续映射，证明 $f_* : H_n(M; \mathbb{Z}/2\mathbb{Z}) \to H_n(N; \mathbb{Z}/2\mathbb{Z})$ 是零。

    (degree mod 2, orientable 2-covering)

16. 不可定向的 n 维流形不能嵌入 $\mathbb{R}^{n+1}$。

    (Alexander duality, change of coefficient; SW classes)

17. 考虑商空间 $X = S^3 \times [0,1]/\sim$，其中等价关系 $\sim$ 由

    $$(v,0) \sim (-v,0), (v,1) \sim (-v,1)$$

    生成，计算 X 的各阶整系数同调群。

    (computation, MV sequence)

18. $\mathbb{CP}^3 \# \mathbb{CP}^3$ 与 $S^2 \times \mathbb{CP}^2$ 是否同伦等价？

    (cohomology)

19. 证明任意不可定向闭曲面无法嵌入 $\mathbb{R}^3$。

20. 令 M 是一个可定向 $4k+2$ 维闭流形，证明 M 的欧拉数一定是偶数。

    (Euler characteristic)

21. 考虑一个 3 维 CW 复形 X，它的二维骨架是 $S^2 \vee S^2$，并且有唯一的 3 维胞腔，粘贴映射为

$$f : \partial D^3 = S^2 \to S^2 \vee S^2$$

    考虑诱导映射 $f_* : \mathbb{Z} \cong H_2(S^2; \mathbb{Z}) \to H_2(S^2 \vee S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$，假设 $f_*(1) = (m, n)$。当 m, n 满足什么条件时 X 与 S^2 同伦等价?

    (definition of cellular homology; the sufficiency is hard/not required)

22. 对于所有 $k \leqslant 2025$，计算 $\pi_k(\mathbb{CP}^{1012})$。

    (fibration)

23. 考虑 $\mathbb{T}^2$ 到 $\mathbb{RP}^3$ 映射同伦类构成的集合 $[\mathbb{T}^2, \mathbb{RP}^3]$，请问这个集合中有几个元素?

    (CW approximation/attaching cells, representation theorem)

24. 令 M 是闭流形，满足 $H_1(M; \mathbb{Z}) = 0$。考虑连续映射 $f : M \to S^1$，证明 f 不可能是纤维化。

    (factoring through, abelianization; or SSS)

5. Some problems on geometry

Differential geometry

1. Let ω be the $(n-1)$-form on $\mathbb{R}^n \backslash \{0\}$ defined by

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n$$

   (a) Show that $\iota^*_{S^{n-1}}\omega$ is the Riemannian volume form of S^{n-1} w.r.t the round metric and the standard orientation.

   (b) Show that ω is closed but not exact on $\mathbb{R}^n \backslash \{0\}$.

2. (s) Let M, N be n-dimensional smooth compact connected manifolds, and $f : M \to N$ a smooth map with rank equals n everywhere. Show that f is a covering map.

3. Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be self-dual if $*\omega = \omega$ and anti-self-dual if $*\omega = -\omega$.

   (1) Show that every 2-form ω on M can be written uniquely as a sum of self-dual form and an anti-self-dual form.

   (2) On $M = \mathbb{R}^4$ with the Euclidean metric, determine the self-dual and anti-self-dual forms in standard coordinates.

Riemannian geometry

1. Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature.

2. Let (M, g) be a Riemannian manifold, $X \in \Gamma(M, TM)$ a nowhere-vanishing vector field. For $f \in C^\infty(M)$, show that $X = \operatorname{grad} f$ iff $Xf = |X|^2$ and X is orthogonal to the level sets of f at all regular points of f.

3. Suppose that K is a Killing vector field on a closed and oriented Riemannian manifold (M, g), and ω is a harmonic form. Show that $L_K\omega = 0$. (equivalently, $\omega(K)$ is a constant)

4. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$.

   (1) Find a 6-form on $\mathbb{R}^7\backslash\{0\}$ such that $\mathrm{d}\alpha = 0, \int_{S^6} \alpha = 1$.

   (2) For any smooth map $f : S^{11} \to S^6$, show that there exists a 5-form φ on S^{11} with $f^*\alpha = \mathrm{d}\varphi$.

   (3) Let $H(f) = \int_{S^{11}} \varphi \wedge \mathrm{d}\varphi$, show that $H(f)$ is independent of the choice of φ satisfying the condition in *(2)*.

   (4) Show that $H(f)$ is an even integer for any such smooth map f.

5. Let (M, g) be a complete Riemannian manifold. Suppose a smooth function f on M satisfies

$$|\nabla f| = 1, \operatorname{Hess} f = 0.$$

   Set $N = f^{-1}(0)$ and $h = g|_N$. Show that (N, h) is a totally geodesic submanifold of (M, g).

6. Is there a compact manifold that admits both a metric of positive definite Ricci curvature and a metric of non-positive sectional curvature?

7. (s) Prove that a Killing vector field on an odd-dimensional Riemannian manifold cannot have an isolated zero.

8. Consider the $(0, 2)$-tensor

$$T = \operatorname{Ric} + bS \cdot g + cg, b, c \in \mathbb{R}$$

(1) Show that $\nabla^* T = 0$ if $b = -\frac{1}{2}$. $G = T|_{b=-\frac{1}{2}}$ is the Einstein tensor.

(2) Show that if $c = 0$, then $G = 0$ in dimension 2.

(3) When $n > 2$, show that if $G = 0$, the metric is Einstein.

(4) When $n > 2$, show that if $G = 0, c = 0$, the metric is Ricci flat.

9. Let X be a vector field on a complete Riemannian manifold and consider the map $F_t : M \to M$ given by $F_t(p) = \exp_p(tX_p)$.

(1) For $v \in T_pM$, show that $J(t) = (F_t)_{*,p}(v)$ is a Jacobi field along $\gamma(t) = \exp_p(tX_p)$ with initial conditions $J(0) = v$ and $J'(0) = \nabla_v X$.

(2) Select an ONB for T_pM and let $J_i(t) = (F_t)_{*,p}(e_i)$. Show that

$$(\det(F_t)_*)^2 = \det(g(J_i(t), j_j(t))).$$

(3) Suppose $(\nabla X)_p = 0$, show that as long as $\det(F_t)_* \neq 0$ it holds

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\det(F_t)_*)^{\frac{1}{n}} \leqslant -\frac{(\det(F_t)_*)^{\frac{1}{n}}}{n} \operatorname{Ric}(\gamma', \gamma').$$

10. (s) Let X be a vector field on a Riemannian manifold.

(1) Show that

$$|L_X g|^2 = 2|\nabla X|^2 + 2\operatorname{tr}(\nabla X \circ \nabla X).$$

(2) Establish the following integral formulae on a closed oriented Riemannian manifold:

$$\int_M \operatorname{Ric}(X, Y) = \int_M \operatorname{div} X \cdot \operatorname{div} Y - \int_M \operatorname{tr}(\nabla Y \circ \nabla X),$$

$$\int_M \left(\operatorname{Ric}(X, X) + g(\operatorname{tr} \nabla^2 X, X) + \frac{1}{2}|L_X g|^2 - (\operatorname{div} X)^2 \right) = 0.$$

(3) Finally, show that X is a Killing field iff

$$\operatorname{div} X = 0, \operatorname{tr} \nabla^2 X = -\operatorname{Ric}(X).$$

11. Show that all geodesics on the sphere are precisely the great circles.

12. Let (M, g) be a Riemannian manifold, and f be a smooth function such that $|\operatorname{grad} f| = 1$. Show that the integral curves of $\operatorname{grad} f$ are geodesics.

13. Let (M, g) be a complete Riemannian manifold, and V be a smooth vector field with $|V|_g \leqslant C$ for some constant C. Show that V is a complete vector field.

14. Let $f : (M, g) \to (M, g)$ be an isometry that fixes $p \in M$. Show that $F_{*,p} = -\operatorname{id}$ on T_pM iff $F^2 = \operatorname{id}_M$ and p is an isolated fixed point.

15. (s) Let M be an even dimensional compact and oriented Riemannian manifold with positive sectional curvature. Show that M is simply connected.

16. Suppose M is a compact 2-dimensional Riemannian manifold without boundary with positive sectional curvature. Show that any two compact closed geodesics on M must intersect with each other.

    (Gauss-Bonnet, Euler characteristic)

17. A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

18. Let (M, g) be a Riemannian manifold and X be a Killing field. If γ is a geodesic, show that $J(t) = X \circ \gamma(t)$ is a Jacobi field along γ.

19. Let M, g be a Riemannian manifold and $f : M \to \mathbb{R}$ be a Lipschitz function. Then for any smooth φ,

$$-\int_M \langle \nabla\varphi, \nabla f\rangle \,\mathrm{d}\operatorname{Vol} = \int_M \Delta\varphi f \mathrm{d}\operatorname{Vol}$$

20. Suppose (M, g) is a compact connected riemannian manifold. Every non-trivial free homotopy class in M is represented by a closed geodesic that has minimum length among all admissible loops in the given free homotopy class.

21. Let (M, g) be a complete connected Riemannian manifold and $p, q \in M$.

    (1) Show that every path-homotopy class of paths from p to q contains a geodesic segement γ.

(2) Show that if M has non-positive sectional curvature, then γ is the unique geodesic segment in the given path homotopy class.

22. (s) On a Riemannian manifold (M, g), let F be the set of smooth functions f on M with $|\operatorname{grad} f| \leqslant 1$. For any $x, y \in M$, show that
$$d(x,y) = \sup\{|f(x) - f(y)| \mid f \in F\}.$$

23. (s) Let M be an orientable closed and embedded minimal hypersurface in the round sphere S^{n+1}. Denote by λ_1 the first eigenvalue for the Beltrami-Laplace operator on M. Prove that $\lambda_1 \geqslant \frac{n}{2}$.

24. Does there exist a compact Riemannian manifold (M, g) with $\mathrm{Ric} \geqslant 0$, $b_1(M) = 0$ and $|\pi_1(M)| = \infty$?

6. Zhiyuan cup '25

1. Let M be a smooth manifold of dimension $n \geqslant 3$. Denote by $C_k(M)$ the configuration space of k points, that is
$$C_k(M) = \{(x_1, \cdots, x_k) \in M^k : x_i \neq x_j\}$$
Express the fundamental group of $C_k(M)$ in terms of $G = \pi_1(M)$.

2. Compute $\pi_k(\Sigma_2)$ for all $k \geqslant 1$.

3. Consider $\mathbb{R}^2$ with metric $g = \rho^2(\mathrm{d}x^2 + \mathrm{d}y^2)$, where $\rho : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is a smooth function. Compute the sectional curvature K.

4. Let L, M be smooth manifolds, $\omega \in \Omega^k(M)$ be a closed k-form. Let $f : L \times (-1,1) \to M$ be a smooth map, denote by $i_t : L \to L \times (-1,1)$ the inclusion map to $L \times \{t\}$, and write $f_t = f \circ i_t$. Suppose that $f_t^*\omega = 0$ for every $t \in (-1,1)$. Prove that there exists a $(k-1)$-form $\theta \in \Omega^{k-1}(L \times (-1,1))$ such that $f^*\omega = \mathrm{d}t \wedge \theta$ and $i_t^*\theta$ is closed for each t.

5. Let (M, g) be a 3-dimensional Riemannian manifold with $\mathrm{Ric} = \lambda g$ where $\lambda > 0$. Prove that M is orientable.

6. *(1)* Compute $\pi_2(\mathbb{CP}^2)$;

   (2) Compute the map $\pi_2(\mathbb{CP}^1) \to \pi_2(\mathbb{CP}^2)$ induced by the standard inclusion $\mathbb{CP}^1 \to \mathbb{CP}^2$.

7. Let (M, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric} \geqslant (n-1)kg$, ω any harmonic 1-form on M. Prove that $\forall p \in M$ with $\omega(p) \neq 0$, we have
$$\Delta|\omega| \geqslant (n-1)k|\omega|.$$

8. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a continuous map without a fixed point. Prove that there exists a non-zero $x \in H^1(\mathbb{T}^n)$ that satisfies $f^* x = x$.

9. Let (M, g) be a closed Riemannian manifold of dimension $\geqslant 2$ with negative sectional curvature. Let $\widetilde{M}$ be its universal cover, then $\Gamma = \pi_1(M)$ can be identified as a subgroup of $\mathrm{Isom}(\widetilde{M})$ by deck transformations.

   (1) Prove that there are $\gamma_1, \gamma_2 \in \pi_1(M)$ with different axes.

   (2) Prove that the centralizer of $\Gamma \subset \mathrm{Isom}(\widetilde{M})$ is trivial.

10. Let $E = \mathbb{C}^* \times \mathbb{D}^* / \sim$ where $\sim$ is defined by $(t, q) \sim (q^n t, q)$ for all $n \in \mathbb{Z}$. Compute the homology groups $H_i(E; \mathbb{Z})$.

    (hint: E is a fiber bundle over $\mathbb{D}^*$ via the projection, and there is a homeomorphism $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$ where $q = e^{2\pi i \tau}$ for $t \in \mathbb{C}\backslash\mathbb{R}$)

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