Riemannian geometry: a note for reviewing

2024 autumn

Some good references are [Wal09, Pet06, Jos08, DCFF92].

1 Basic concepts and computations 3
1.1 Connections and curvatures 3
1.2 Hessian and scalar Laplacian 4
1.3 Pull-back operation 5
1.4 The 2nd fundamental form 6
1.5 Parallel transports, geodesics and exponential maps 7
1.6 Completeness 9
1.7 Normal coordinates 11
1.8 Hodge star operator and Hodge decomposition 12
1.9 Tensor calculus 15
1.10 Miscellany 18

2 The Bochner technique 18
2.1 Killing vector fields 18
2.2 Harmonic 1-forms 20
2.3 Harmonic maps 20

3 Variation formulae and Jacobi fields 20

4 Curvature and topology 21
4.1 Non-positive sectional curvature 21
4.2 Negative sectional curvature 22
4.3 Non-negative curvature 23
4.4 Constant sectional curvature 23

5 Comparison theorems and splitting theorem 24
5.1 Rauch 24
5.2 Hessian and Laplacian 24
5.3 Volume 24
5.4 Splitting theorem 25

6 Gathering important results 26

A Local isometry and isometry 27

B Covering maps and transformations 28

C Axes, rays and lines 28

1. Basic concepts and computations

1.1. Connections and curvatures

Definition 1 (connection)**.** *$\nabla : TM \times E \to E$, which is linear on TM, a derivation for E, where $E \to M$ is a bundle.*

Definition 2 (Christoffel symbol)**.** $\nabla_{\frac{\partial}{\partial x^i}} e_A = \Gamma^B_{iA} e_B$.

Definition 3 (curvature tensor)**.** $R : TM \otimes TM \otimes E \otimes \to E$,

$$R(X, Y)e := \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e$$

As for a Riemannian manifold (M, g), we consider usually Levi-Civita connection, and several special curvature tensors.

Definition 4 (Levi-Civita connection)**.** *$\nabla : TM \times TM \to TM$, a connection s.t.*

(1) $X(Y, Z) = (X\nabla_Y, Z) + (Y, \nabla_X Z)$;

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$.

Definition 5 (curvature tensors and operator)**.**

(1) $R(X, Y, Z, W) := (R(X, Y)Z, W)$, $R = R_{ijkl} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$;

(2) sectional curvature: $K_\sigma (= \sec(X, Y)) = \frac{R(X,Y,Y,X)}{|X \wedge Y|^2}$, $\sigma = \mathrm{span}\{X, Y\}$;

(3) Ricci curvature: $\mathrm{Ric}_{ij} = g^{kl} R_{iklj}$;

(4) Scalar curvature: $S = g^{ij} \mathrm{Ric}_{ij}$.

(5) curvature operator: $\mathfrak{R} : \wedge^2 TM \to \wedge^2 TM$, *such that* $g(\mathfrak{R}(X \wedge Y), Z \wedge W) = R(X, Y, Z, W)$.

List of properties:

- symmetry of R and first Bianchi;
- independence of basis for K_σ;
- independence of planes for K_σ iff being flat;
- for 3-dim manifolds, CRC implies CSC.

Definition 6 (trace definition of Ricci)**.** $\mathrm{Ric}(v, w) = \mathrm{tr}(x \mapsto R(x, v)w)$. *Taking an ONB of TM,*

(1) $\mathrm{Ric}(v) := \sum R(v, e_i) e_i$;

(2) $\mathrm{Ric}(v,w) = g(\mathrm{Ric}(v), w)$*;*

(3) for $v = e_1$*,* $\mathrm{Ric}(v,v) = \sum R(v, e_i, e_i, v) = \sum_{i=2}^{n} \sec(v, e_i)$*.*

Exercise 7. *(1) show the Koszul formula;*

(2) calculate $\Gamma_{ij}^{k}, R_{ijkl}$*;*

(3) [Y]*show that* $R_{ijkl} =$

$$\frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}\right) + g_{pq}(\Gamma_{ik}^{p}\Gamma_{jl}^{q} - \Gamma_{il}^{q}\Gamma_{kj}^{p}).$$

(4) compute the curvatures of S^n, H^2*;*

(5) [Y]*compute the curvatures of*

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{K^2 - \sum (x^i)^2}, K^2 - \sum (x^i)^2 > 0;$$

(6) [Y]*compute the curvatures of* $(\mathbb{R}^2, e^{a(x^2+y^2)}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y))$*.*

Exercise 8. *(1) what's the relation of curvatures between* g *and* $k \cdot g$*;*

(2) [Y]*prove the integral formulae for* Ric *and* S*:*

(a) for unit vector v*, and* $S_v^{\perp}$ *the set of unit vectors orthogonal to* v*,*

$$\mathrm{Ric}_p(v,v) = \frac{n-1}{\mathrm{Vol}(S^{n-2})} \int_{w \in S_v^{\perp}} \sec(v,w)\, \mathrm{d}V_{\widehat{g}}\,.$$

(b) for $UT_pM \cong S^{n-1}$*,*

$$S_p = \frac{n}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(v,v)\, \mathrm{d}S\,.$$

(3) [Y]*let* (M^3, g) *be Einstein, show that* (M,g) *is of CSC.*

(4) [Y]*(hard, warped product) consider* (N^{n-1}, g_N)*,* $\mathrm{Ric} = \frac{n-2}{n-1}\lambda g_N, \lambda < 0$*, find a function* $\rho : \mathbb{R} \to (0, \infty)$*, such that* $(M^n, g) = (\mathbb{R} \times N, \mathrm{d}r^2 + \rho^2 g_N)$ *becomes an Einstein metric with* $\mathrm{Ric} = \lambda g$*.*

1.2. Hessian and scalar Laplacian

Consider smooth function $f : (M,g) \to \mathbb{R}$.

Definition 9 (Hessian and saclar Laplacian)**.**

(1) $\operatorname{Hess} f := \nabla^2 f = \nabla \mathrm{d} f$, *i.e.*

$$\operatorname{Hess} f(X,Y) = g(\nabla_X \nabla f, Y) = (\nabla_X \,\mathrm{d} f) = XYf - \nabla_X Y f.$$

the Hessian operator is given by $\operatorname{Hess} f(X,Y) = (\mathcal{H}_f(X), Y)$.

(2) $\Delta_g f := \operatorname{tr} \operatorname{Hess} f = g^{ij} \operatorname{Hess} f_{ij}$.

Locally, $\operatorname{Hess} f_{ij} = \operatorname{Hess} f_{ji}$, thus $\operatorname{Hess} f$ is a symmetric 2-form.

Theorem 10 (volume expression of the Laplacian)**.**

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right)$$

Exercise 11. *(1)* [Y]*for* $\mathrm{d}\operatorname{Vol}_g = \sqrt{\det g}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, *compute* $\frac{\partial \det g}{\partial x^i}$, $\frac{\partial \log \det g}{\partial x^i}$ *and* $\frac{\partial \sqrt{\det g}}{\partial x^i}$, *show*

$$\frac{\partial}{\partial x^i} \mathrm{d}\operatorname{Vol}_g = \frac{1}{2} \frac{\partial \log \det g}{\partial x^j} \mathrm{d}\operatorname{Vol}_g .$$

(2) [Y]*prove Theorem 10.*

1.3. Pull-back operation

$f : M \to N$ induces $f_* : TM \to f^*TN$, for immersion, $f^*TN \subset TN$.

$$\begin{tikzcd}
TM \arrow[r, "f_*"] \arrow[dr, "\pi'"'] & f^*TN \arrow[r, "\xi"] \arrow[d, "\widehat{\pi}"] & TN \arrow[d, "\pi"] \\
 & M \arrow[r, "f"] & (N,h)
\end{tikzcd}$$

Theorem 12 (definition of pull-back connection and metric)**.** *There exists compatible pull-back connection and metric defined by*

(1) $\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_A = f_* \left(\frac{\partial f^\alpha}{\partial x^i} \nabla_{\frac{\partial}{\partial y^\alpha}} e_A \right) = f_* \left(\frac{\partial f^\alpha}{\partial x^i} \Gamma^B_{\alpha A}(f) e_B \right)$;

(2) $\widehat{g} = f^* h$, *i.e.* $\widehat{g}(\widehat{e}_A, \widehat{e}_B) = h(e_A, e_B)$.

Locally, drop the hats,

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \frac{\partial f^\alpha}{\partial x^i} \Gamma^k_{j\alpha}(f) \frac{\partial}{\partial y^k};$$

$$\widehat{g}_{ij} = h\left(f_* \frac{\partial}{\partial x^i}, f_* \frac{\partial}{\partial x^j} \right) = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta}.$$

Exercise 13. *(1) show the well-defined-ness and compatibility.*

(2) [Y]*show that* $\widehat{R}_{ij\gamma\delta} = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} R_{\alpha\beta\gamma\delta}$.

1.4. The 2nd fundamental form

The 2nd fundamental form, which generalize the Hessian, is defined to indicate the deviation under pull-back.

General case

Definition 14 (2nd fundamental form)**.** $B \in \Gamma(M, T^*M \otimes T^*M \otimes f^*TN)$, $B(X,Y) := \widehat{\nabla}_X f_* Y - f_* \nabla_X Y$.

Locally, $B_{ij}^{\alpha} = B_{ji}^{\alpha}$, thus B is a symmetric (2,1)-tensor, as a result,

$$\widehat{\nabla}_X f_* Y - \widehat{\nabla}_Y f_* X = f_* \nabla_X Y - f_* \nabla_Y X = f_*[X,Y].$$

Exercise 15. *(1) compute the local expression of B.*

(2) [Y]$f : (M,g) \to (N,h)$, and $\widetilde{\nabla}$ is the affine connection on $T^*M \otimes f^*TN$ induced by ∇^M, ∇^N, then $B = \widetilde{\nabla} \mathrm{d}f$, where $\mathrm{d}f$ is regarded as a smooth section in $\Gamma(M, T^*M \otimes f^*TN)$.

The case of Riemannian immersion

Given an immersion $f : M \to (\overline{M}, \overline{g}, \overline{\nabla})$, $f^*T\overline{M} \subset T\overline{M} = f^*T\overline{M} \oplus T^{\perp}M$. We write $(\widehat{g}, \widehat{\nabla}), (g, \nabla)$ for the induced structures on f^*TN, TM.

List of properties:

- $g_{ij} = \frac{\partial f^{\alpha}}{\partial x^i} \frac{\partial f^{\beta}}{\partial x^j} \overline{g}_{\alpha\beta}$;
- $B \in \Gamma(M, T^*M \otimes T^*M \otimes T^{\perp}M)$, i.e. $\widehat{g}(B(X,Y), f_*Z) = 0$ for any $X, Y, Y \in \Gamma(M, TM)$. Equivalently (*drop of push-forward*),
$$\widehat{g}(\widehat{\nabla}_X f_* Y, f_* Z) = \widehat{g}(f_* \nabla_X Y, f_* Z) = g(\nabla_X Y, Z).$$
- for any $X, Y, Z, W \in \Gamma(M, TM)$, $R(X,Y,Z,W) - \overline{R}(X,Y,f_*Z,f_*W)$
$$= \widehat{g}(B(X,W), B(Y,Z)) - \widehat{g}(B(X,Z), B(Y,W)).$$

Definition 16 (Weingarten map)**.** $X, Y \in \Gamma(M, TM), \eta \in \Gamma(M, T^{\perp}M)$, $g(W_{\eta}(X), Y) := B_{\eta}(X,Y) := g(B(X,Y), \eta)$.

Remark 17. *Take $(\widehat{M}, \widehat{g}) = (\mathbb{R}^N, g_{\mathbb{R}^N})$, we shall get Gauss' Theorema Egregium, especially for the immersion of a surface into $\mathbb{R}^3$.*

Exercise 18. *(1) [Y]show the orthogonal relation with(out) the rank theorem.*

(2) [Y] *consider immersion of a surface into* $\mathbb{R}^3$*, with unit normal vector* n*, write the expression of first and second fundamental form,* B_n*, and Gauss' Theorema Egregium:*

$$K = \frac{\det II}{\det I} = \sec(X, Y) = \frac{R(X, Y, Y, X)}{g_D(X, X)g_D(Y, Y) - g_D(X, Y)^2}.$$

(3) [Y] *show that* $\operatorname{Ric} g_D = K g_D, S = 2K$*.*

(4) [Y] *consider* $S^n \to \mathbb{R}^{n+1}$ *and the local parametrization*

$$\gamma : D \to U^+_{n+1} \subset \mathbb{R}^{n+1}, \gamma(u) = \left(u^1, \cdots, u^n, \sqrt{1 - |u|^2}\right)$$

where $D = \{u \mid |u| < 1\}$*.*

(a) compute $g_D = \gamma^* g_{\text{can}}$*;*

(b) compute the second fundamental form;

(c) compute the mean curvature $H = \frac{1}{n} \operatorname{tr}_{g_D} B$*.*

1.5. Parallel transports, geodesics and exponential maps

Parallel transport

Let $\gamma : I \to (M, g)$ be a smooth curve.

Proposition 19 (definition of parallel transport)**.** *For any* $v \in T_{\gamma(t_0)}M$*, there exists a unique vector field* $V \in \Gamma(I, \gamma^*TM)$ *(along* γ*) with*

(1) $V(t_0) = v$*;* *(2)* $\widehat{\nabla} V = 0$*.*

Define the parallel transport along γ *by* $P_{t_0,t,\gamma} = V(t)$*, for any* $t_0, t \in I$*.*

List of properties: the gist is a take a *parallel frame*.

- $P_{t_2,t_3,\gamma} \circ P_{t_1,t_2,\gamma} = P_{t_1,t_3,\gamma}, P_{t,t,\gamma} = \text{id}$.
- $P_{s,t,\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is a linear isometry for any $s, t \in I$;
- $F(t, (s, v)) := (t, P_{s,t,\gamma}(v))$ is a smooth function;
- $\frac{\mathrm{d}}{\mathrm{d}t} P_{t,t_0,\gamma}(V(t)) = P_{t,t_0,\gamma}(\widehat{\nabla} V(t))$, for any vector field V along γ.

Exercise 20. *prove the properties above.*

Geodesic and exponential map

Proposition 21 (definition of geodesic)**.** *For any $p \in M, v \in T_pM, t_0 \in \mathbb{R}$, there is an open interval $I \ni t_0$ and a smooth curve $\gamma : I \to M$ with*

(1) $\gamma(t_0) = p, \gamma'(t_0) := (\gamma_* \frac{\mathrm{d}}{\mathrm{d}t})|_{t_0} = v$;

(2) $\widehat{\nabla}\gamma' = 0$ along I.

The curve satisfying (2), i.e.

$$\widehat{\nabla}\gamma' = \widehat{\nabla}\gamma_* \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}^2\gamma^i}{\mathrm{d}t^2}\frac{\partial}{\partial x^i} + \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma_{ij}^k(\gamma)\frac{\partial}{\partial x^k} = 0,$$

is called a geodesic along I. Up to a shift of position, we suppose $\gamma(0) = p, \gamma'(0) = v$ and write $I_{p,v}$ for the maximal existence interval of γ.

List of properties:

- $|\gamma'|$ is a constant for the geodesic γ;
- $\gamma_{cv}(t) = \gamma_v(ct)$, i.e. invariant under rescaling.
- $P_{0,t,\gamma_v}(v) = \gamma_v'(t)$.

Definition 22 (exponential map)**.** *Write $\mathcal{E}_p = \{v \,|\, 1 \in I_{p,v}\}$, the exponential map $\exp_p : \mathcal{E}_p \to M$ is defined by*

$$\exp_p(v) = \gamma_v(1),$$

where γ_v is the geodesic with $\gamma(0) = p, \gamma'(0) = v$.

List of properties:

- $\exp_p(tv) = \gamma_v(t)$, for $t \in I_{p,v}$;
- $\exp$ is smooth on $\mathcal{E} = \{(p, v) \,|\, v \in \mathcal{E}_p\}$;
- $\exp$ is a local diffeomorphism, since the differential

  $$\exp_{*,0} : T_0(T_pM) \to T_pM$$

  is the identity map.
- set $B_r(p) = \{\exp_p(v) \,|\, |v| < r\}$, then $\exp|_{B_r(p)}$ is a diffeomorphism. The injectivity radius of p is

  $$\mathrm{inj}_p(M) := \sup\{r \,|\, \exp|_{B_r(p)} \text{ is diffeomorphic}\},$$

  and $\mathrm{inj}(M) := \inf_p \mathrm{inj}_p(M)$.

Exercise 23. *prove the following Gauss' lemma: fix $p \in M, r < \mathrm{inj}_p(M)$ and I an open interval. suppose*

(1) $w(s): I \to T_pM$ *satisfies* $|w(s)| = r$ *and*

(2) $\alpha(t,s) := \exp_p(tw(s))$ *for* $(t,s) \in \mathbb{R} \times I, tw(s) \in \mathcal{E}_p$.

then

$$\left\langle \alpha_* \frac{\partial}{\partial s}, \alpha_* \frac{\partial}{\partial t} \right\rangle = 0.$$

Exercise 24. *(1)* [Y]*let* M *be a smooth manifold and* ∇ *any connection on* TM. *We define the curvature endomorphism by*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

then ∇ *is said to be flat if* $R(X,Y)Z \equiv 0$. *show that the followings are euqivalent.*

(a) ∇ *is flat;*

(b) for every point $p \in M$, *there exists a parallel local frame defined on a neighborhood of* p;

(c) for all $p, q \in M$, *parallel transport along an admissible curve segment from* p *to* q *depends only on the path-homotopy class.*

(d) parallel transport around any sufficiently small closed curve is the identity, i.e. for every $p \in M$, *there exists a neighborhood* U *of* p *such that if* $\gamma : [a,b] \to U$ *is an admissible curve in* U *starting and ending at* p, *then* $P_{ab} : T_pM \to T_pM$ *is the identity map.*

(2) [Y]*a vector field* X *is said to be parallel if* $\nabla X \equiv 0$.

(a) let $p \in \mathbb{R}^n, v \in T_p\mathbb{R}^n$, *show that there is a unique parallel vector field* Y *on* $\mathbb{R}^n$ *such that* $Y_p = v$.

(b) let $X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ *be the spherical coordinate of an open subset* $U \subset S^2$, *let* $X_\varphi = X_* \frac{\partial}{\partial \varphi}, X_\theta = X_* \frac{\partial}{\partial \theta}$. *compute* $\nabla_{X_\theta} X_\varphi, \nabla_{X_\varphi} X_\varphi$, *and conclude that* X_φ *is parallel along the equator and along each meridian* $\theta = \theta_0$.

(c) let $p = (1,0,0) \in S^2$, *show that there is no parallel vector field* W *on any neighborhood of* p *in* S^2 *such that* $W_p = X_\varphi|_p$.

(d) conclude that no neighborhood of p *in* (S^2, g) *is isometric to an open subset of* $(\mathbb{R}^2, g_{\text{can}})$.

1.6. Completeness

COMPLETENESS OF MANIFOLDS AND VECTOR FIELDS

A riemannian manifold is naturally a metric space under

$$d_g(p,q) = \inf_{\gamma \in \mathcal{L}} \operatorname{length}(\gamma) = \inf_{\gamma \in \mathcal{L}} \int |\gamma'|$$

where $\mathcal{L}$ is the collection of piecewise smooth curves joining p, q.

Using Gauss' lemma (Exercise 23), one can show

Proposition 25. *Fix $p \in M, r < \operatorname{inj}_p(M)$, then for any v with $|v| < r$,*

$$d_g(p, \exp_p(v)) = |v|.$$

Thus the shortest curve joining p, q must be a geodesic.

Definition 26 (completeness of a manifold)**.** *(M, g) is (geodesically) complete if $\exp_p(v)$ is well-defined for all $p \in M, v \in T_pM$. Or equivalently, all the geodesics are well-defined on $\mathbb{R}$.*

Definition 27 (completeness of a vector field)**.** *X is complete if it has a global flow, i.e. the integral curve extends to $\mathbb{R}$.*

Exercise 28. *(1) let (M, g) be complete, V a smooth vector field with $|V| \leqslant C$, show that V is complete.*

(2) let (M, g) be complete, show that every Killing vector field is complete.

Hopf-Rinow theorem

Theorem 29 (Hopf-Rinow)**.** *The followings are euqivalent*

(1) (M, g) is geodesically complete;

(2) there exists some $p \in M$ such that $\exp_p$ is well-defined on T_pM;

(3) every closed and bounded subset of M is compact.

(4) (M, d_g) is metrically complete.

Exercise 30. *(1) every compact manifold is complete;*

(2) [P]if $(M, g_1), (M, g_2)$ satisfies $g_1 \geqslant g_2$ and (M, g_2) is complete, then (M, g_1) is also complete.

(3) [P]a riemannian manifold is said to be homogeneous if the isometry group acts transitively. show that the homogeneous manifolds are complete.

(4) [P]let $O \subset (M, g)$ be an open subset, show that if (O, g) is complete, then $O = M$.

(5) [P]*let* $(M,g) = (\mathbb{R}\times N, \mathrm{d}r^2 + \rho^2 g_N)$ *where* $\rho : \mathbb{R} \to (0,\infty)$, (N, g_N) *is complete. show that* (M,g) *is complete.*

(6) [P]*show that any Riemannian manifold* (M,g) *admts a conformal change* $(M, \lambda^2 g)$ *that is complete.*

1.7. Normal coordinates

Definition 31 (normal coordinates)**.** *Take an ONB of* T_pM*, and define* $B : \mathbb{R}^n \to T_pM, r \mapsto r^i e_i$*, which is an isometry. The (reversed) map*

$$\varphi = B^{-1} \circ \exp_p^{-1} : U \to T_pM \to \mathbb{R}^n$$

gives $(x^i) = (r^i \circ \varphi)$*, the normal coordinates centered at* p*.*

List of properties:

- $\varphi_* \frac{\partial}{\partial x^i}|_p = \frac{\partial}{\partial r^i}$ and $\varphi_*(e_i) = B^{-1}e_i = \frac{\partial}{\partial r^i}$, so $\frac{\partial}{\partial x^i}|_p = e_i$;
- $g_{ij}(p) = \delta_{ij}$;
- for $v = v^i \frac{\partial}{\partial x^i}|_p, \gamma_v^i(t) = tv^i$;
- $\Gamma_{ij}^k|_p = 0$, thus $\frac{\partial}{\partial x^k} g_{ij}|_p = 0$.

Theorem 32 (local expansion of metric)**.** *Under any normal coordinates,*

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj}|_p x^k x^l + O(|x|^3), \quad g^{ij} = \delta_{ij} + \frac{1}{3} R_{iklj}|_p x^k x^l + O(|x|^3),$$

and also,

$$\det g = 1 - \frac{1}{3} \operatorname{Ric}_{ij}|_p x^i x^j + O(|x|^3), \quad \frac{\partial g_{ij}}{\partial x^k x^l} = \frac{1}{3}(R_{iklj}|_p + R_{ilkj}|_p).$$

Exercise 33. *show for small* r *that*

(1) $\mathrm{Vol}(B(p,r)) = \omega_n r^n \left(1 - \frac{S_p}{6(n+2)} r^2 + O(r^3)\right)$;

(2) $\mathrm{Area}(S(p,r)) = n\omega_n r^{n-1} \left(1 - \frac{S_p}{6n} r^2 + O(r^3)\right)$.

Consider the distance function $r(q) := d_g(p,q)$ on $U = M \setminus \mathrm{cut}(p)$.
List of properties:

- r is continuous and is smooth on $U \setminus \{p\}$;
- $r(q) = |\exp_p^{-1}(q)|$;
- $\nabla r = g^{ij} \frac{\partial r}{\partial x^i} \frac{\partial}{\partial x^j}$ is a smooth vector field on $U \setminus \{p\}$.

In normal coordinates, recall that $\gamma_v^i(t) = x^i \circ \gamma_v(t) = tv^i$ for $v = v^i \frac{\partial}{\partial x^i}|_p$, so $r(q) = |\exp_p^{-1}(q)| = |\exp_p^{-1}(\exp_p(x^i(q)\frac{\partial}{\partial x^i}|_p))| = \sqrt{\sum (x^i(q))^2}$.

Definition 34 (radial vector field)**.** $\partial_r := \frac{x^i}{r}\frac{\partial}{\partial x^i} = \sum_i \frac{\partial r}{\partial x^i}\frac{\partial}{\partial x^i}$.

Theorem 35. *On* $U \backslash \{p\}$

(1) ∂_r *is nowhere-vanishing and orthogonal to the level set of* r*;*

(2) (Gauss' lemma) $\nabla r = \partial_r, |\partial_r| = 1$.

List of properties: (as corollaries)

- $\mathcal{H}_r(\partial r) = \nabla_{\partial_r}\partial_r = 0$.
- $\sum_j g_{ij}x^j = x^i$, $g_{ij} = \delta_{ij} - \sum_k \frac{\partial g_{ik}}{\partial x^j}x^k$;
- $\sum_j \frac{\partial g_{ij}}{\partial x^k}x^j = \sum_j \frac{\partial g_{kj}}{\partial x^i}x^j$, $\sum_{i,j}\frac{\partial g_{ij}}{\partial x^k}x^i x^j = \sum_{i,j}\frac{\partial g_{jk}}{\partial x^i}x^i x^j = 0$
- $\sum_{i,j}\Gamma_{ij}^k x^i x^j = 0$.

1.8. Hodge star operator and Hodge decomposition

Inner product

Definition 36 (musical operators)**.**

(1) $X^\flat := g_{ij}X^i \mathrm{d}x^j$*;* *(2)* $\omega^\sharp := g^{ij}\omega_i \frac{\partial}{\partial x^j}$

A natural way to extend g is $g(\mathrm{d}x^i, \mathrm{d}x^j)(= g((\mathrm{d}x^i)^\sharp, (\mathrm{d}x^j)^\sharp)) = g^{ij}$, or

$$g(\mathrm{d}x^I, \mathrm{d}x^J) = k! \det \begin{pmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \vdots & \ddots & \vdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{pmatrix} =: k! g^{IJ}$$

for $\wedge^k T^*M$. For $\varphi = \sum f_{i_1 \cdots i_k}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, we write

$$\varphi_{i_1 \cdots i_k} = \sum_{\sigma \in S_k} (-1)^{|\sigma|} f_{i_{\sigma(1)} \cdots i_{\sigma(k)}}$$

where $\varphi_{i_1 \cdots i_k}$ is skew-symmetric.

Definition 37 (inner product for k-forms)**.** *(1)* $\langle \varphi, \psi \rangle := \frac{1}{k!}g(\varphi, \psi)$*;*

(2) $(\varphi, \psi) := \int \langle \varphi, \psi \rangle \, \mathrm{d}\,\mathrm{Vol} = \frac{1}{k!}\int g(\varphi, \psi) \, \mathrm{d}\,\mathrm{Vol}$.

List of properties:

- $\varphi = \frac{1}{k!} \sum \varphi_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{i_1 < \cdots < i_k} \varphi_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$;
- $\langle \varphi, \psi \rangle = g^{IJ} \varphi_I \psi_J = \frac{1}{k!} \sum g^{i_1 j_1} \cdots g^{i_k j_k} \varphi_{i_1 \cdots i_k} \psi_{j_1 \cdots j_k}$;
- $\langle \mathrm{d}\,\mathrm{Vol}, \mathrm{d}\,\mathrm{Vol} \rangle = 1$.

Exercise 38. *prove the properties above.*

Hodge star operator

Definition 39 (Hodge star operator)**.** *Take an ONB of T^*M, $\xi^1 \wedge \cdots \wedge \xi^n = \mathrm{d}\,\mathrm{Vol}_g$. Define the linear operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ by*

$$*(v_I \xi^I) = v_I \operatorname{sgn}(I, I^c) \xi^{I^c}$$

where $I = (i_1 \cdots i_k), I^c = (j_1 \cdots j_{n-k}), i_1 < \cdots < i_k, j_1 < \cdots < j_{n-k}$.

List of properties:

- $*1 = \mathrm{d}\,\mathrm{Vol}_g, * \mathrm{d}\,\mathrm{Vol}_g = 1$, and $** \, v = (-1)^{k(n-k)} v$, for $v \in \Omega^k(M)$;
- $*(u \wedge v) = \langle *u, v \rangle = (-1)^{k(n-k)} \langle u, *v \rangle$, for $u \in \Omega^k(M), v \in \Omega^{n-k}(M)$;
- $u \wedge *v = v \wedge *u = \langle u, v \rangle \, \mathrm{d}\,\mathrm{Vol}_g, \langle *u, *v \rangle = \langle u, v \rangle$, for $u, v \in \Omega^k(M)$. Thus $(u, v) = \int u \wedge *v$.

Definition 40 (adjoint operator of d)**.** $(\mathrm{d}\varphi, \psi) =: (\varphi, \mathrm{d}^* \psi)$.

Theorem 41 (expression of d^*)**.** *On $\Omega^k(M)$, $\mathrm{d}^* = (-1)^{nk+n+1} * \mathrm{d} *$.*

Proof. For $u \in \Omega^{k-1}(M), v \in \Omega^k(M)$,

$$\begin{aligned}
\int \langle u, * \, \mathrm{d} * v \rangle \, \mathrm{d}\,\mathrm{Vol}_g &= \int u \wedge ** \, \mathrm{d} * v \\
&= (-1)^{(k-1)(n-k+1)} \int u \wedge \mathrm{d} * v \\
&\overset{*}{=} (-1) \cdot (-1)^{k-1} \cdot (-1)^{(k-1)(n-k+1)} \int \mathrm{d}u \wedge *v \\
&= (-1)^{nk+n+1} \int \langle \mathrm{d}u \, , v \rangle \, \mathrm{d}\,\mathrm{Vol}_g \, .
\end{aligned}$$

Here we use Stokes' formula for $\overset{*}{=}$. $\square$

Exercise 42. *for $\omega \in \Omega^p(M)$, show that*

$$(\mathrm{d}\omega)(X_0, \cdots, X_p) = \sum (-1)^i (\nabla_{X_i} \omega)(X_0, \cdots, \widehat{X_i}, \cdots, X_p).$$

Exercise 43. *for 1-form ω, show that*

$$\mathrm{d}^*\omega = -g^{ij}\left(\frac{\partial \omega_i}{\partial x^j} - \Gamma_{ij}^k \omega_k\right) =: -\nabla^i \omega_i.$$

Divergence

Definition 44 (divergence)**.** *The divergence of X is defined by*

$$\operatorname{div} X \cdot \mathrm{d}\,\mathrm{Vol}_g = \mathcal{L}_X \mathrm{d}\,\mathrm{Vol}_g\,.$$

List of properties:

- $\operatorname{div} X = \frac{\partial X^i}{\partial x^i} + \Gamma_{is}^s X^i = \nabla_i X^i$ (regrad $\nabla_i X^j$ as coefficient of $\nabla_i X$);
- divergence theorem: if X is of compact support, then
$$\int \operatorname{div} X \,\mathrm{d}\,\mathrm{Vol}_g = 0.$$
- for 1-form ω with compact support, $\mathrm{d}^*\omega = \operatorname{div}\omega^\sharp$, so
$$\int \mathrm{d}^*\omega \,\mathrm{d}\,\mathrm{Vol}_g = 0.$$
- for $f_0, f_1 \in C_0^\infty(M)$, $\operatorname{div} f_1 \nabla f_2 = g(\nabla f_1, \nabla f_2) + f_1 \Delta f_2$, so
$$\int f_1 \Delta f_2 = -\int g(\nabla f_1, \nabla f_2) = \int f_2 \Delta f_1.$$

Exercise 45. *(1) solve Exercise 43 with the divergence theorem;*

(2) regard ∇X as $\nabla X^\flat$, then $\operatorname{div} X = \operatorname{tr}_g(\nabla X)$, this is a more general definition of divergence. for any smooth k-tensor field, define

$$\operatorname{div} F = \operatorname{tr}_g(\nabla F),$$

where the trace is taken on the first two indices. For smooth covariant k-tensor field F and $(k+1)$-tensor field on a compact manifold (M, g) with boundary, show that

$$\int_M \langle \nabla F, G\rangle \,\mathrm{d}\,\mathrm{Vol}_g = \int_{\partial M} \left\langle F \otimes N^\flat, G\right\rangle \mathrm{d}\,\mathrm{Vol}_{\widehat{g}} - \int_M \langle F, \operatorname{div} G\rangle \,\mathrm{d}\,\mathrm{Vol}_g$$

where $\widehat{g}$ is the induce metric of ∂M.

Hodge decomposition

Definition 46 (Beltrami-Laplace operator (a.k.a. Hodge laplacian))**.**

$$\Delta := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$$

A k-form u is called harmonic if $\Delta u = 0$, denote by $\mathcal{H}^k(M)$ the set of harmonic k-forms.

Theorem 47 (Hodge decomposition)**.** *There is an orthogonal decomposition*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathrm{d}^*(\Omega^{k+1}(M)).$$

Moreover, $\dim_{\mathbb{R}} \mathcal{H}^k(M) < \infty$*.*

Theorem 48. $\mathcal{H}^k(M) \cong H^k_{dR}(M;\mathbb{R})$*.*

Exercise 49. *(1) show that* $\Delta u = 0$ *iff* $\mathrm{d}u = 0, \mathrm{d}^*u = 0$*;*

(2) prove Theorem 48;

(3) show that $H^1_{dR}(\mathbb{R}^2\backslash\{0\};\mathbb{R}) \neq 0$*.*

(4) suppose that M *is connected, show that* $H_{dR}(M,\mathbb{R}) \cong \mathbb{R}$*.*

1.9. Tensor calculus

Covraiant derivatives

A seemingly natural way to extend ∇ is using musical operators, i.e.

$$\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j = \left(\nabla_{\frac{\partial}{\partial x^i}} (\mathrm{d}x^j)^\sharp\right)^\flat = \left(\nabla_{\frac{\partial}{\partial x^i}} g^{jk} \frac{\partial}{\partial x^k}\right)^\flat = -\Gamma^j_{ik} \mathrm{d}x^k.$$

But Leibniz rule simplifies the calculations greatly:

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^j\right) \frac{\partial}{\partial x^k} = \frac{\partial}{\partial x^i} \left\langle \mathrm{d}x^j, \frac{\partial}{\partial x^k} \right\rangle - \left\langle \mathrm{d}x^j, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \right\rangle = -\Gamma^s_{ik}\delta_{js} = -\Gamma^j_{ik}.$$

Definition 50 (covraiant derivative)**.** *For* $T \in \Gamma(M, \otimes^r T^*M \otimes \otimes^s TM)$*, the covariant derivative* $\nabla T \in \Gamma(M, \otimes^{r+1} T^*M \otimes \otimes^s TM)$ *is defined by*

$$(\nabla T)(X, X_1, \cdots, \omega_s) = (\nabla_X T)(X_1, \cdots, \omega_s).$$

For $T = T^{j_1\cdots j_s}_{i_1\cdots i_r} \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}$, $\nabla T = W^{j_1\cdots j_s}_{ii_1\cdots i_r} \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} =$

$$\left(\frac{\partial}{\partial x^i} T^{j_1\cdots j_s}_{i_1\cdots i_r} - \sum_{l=1}^r \Gamma^p_{ii_l} T^{j_1\cdots j_s}_{i_1\cdots p\cdots i_r} + \sum_{m=1}^s \Gamma^{j_m}_{iq} T^{j_1\cdots q\cdots j_s}_{i_1\cdots i_r}\right) \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}}.$$

We ususlly write $T^{j_1\cdots j_s}_{i_1\cdots i_r}$, i.e. the coefficient, instead of the whole tensor.

Definition 51 (2nd covariant derivative)**.** $\nabla^2 T := \nabla(\nabla T)$*, or locally*

$$\nabla_k \nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \nabla_k (W^{j_1 \cdots j_s}_{i i_1 \cdots i_r}).$$

Remark 52. *Caution!* $(\nabla_k(\nabla_i T))^{j_1 \cdots j_s}_{i_1 \cdots i_r} \neq \nabla_k \nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r}$*, in fact, the first one is not a tensor.*

Lemma 53. $\nabla^2_{X,Y} T = \nabla_X \nabla_Y T - \nabla_{\nabla_X Y} T$*, or locally*

$$\nabla_k \nabla_i T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = (\nabla_k(\nabla_i T))^{j_1 \cdots j_s}_{i_1 \cdots i_r} - (\Gamma^j_{ki} \nabla_j T)^{j_1 \cdots j_s}_{i_1 \cdots i_r}.$$

Proof.

$$\begin{aligned}
\nabla_k (W^{j_1 \cdots j_s}_{i i_1 \cdots i_r}) &= \frac{\partial}{\partial x^k} W^{j_1 \cdots j_s}_{i i_1 \cdots i_r} + \sum_m \Gamma^{j_m}_{kq} W^{j_1 \cdots q \cdots j_s}_{i i_1 \cdots i_r} - \sum_l \Gamma^p_{k i_l} W^{j_1 \cdots j_s}_{i i_1 \cdots p \cdots i_r} \\
&\quad - \Gamma^j_{ki} W^{j_1 \cdots j_s}_{j i_1 \cdots p \cdots i_r} \\
&= \frac{\partial}{\partial x^k} (\nabla_i T)^{j_1 \cdots j_s}_{i_1 \cdots i_r} + \sum_m \Gamma^{j_m}_{kq} (\nabla_i T)^{j_1 \cdots q \cdots j_s}_{i_1 \cdots i_r} \\
&\quad - \sum_l \Gamma^p_{k i_l} (\nabla_i T)^{j_1 \cdots j_s}_{i_1 \cdots p \cdots i_r} - \Gamma^j_{ki} W^{j_1 \cdots j_s}_{j i_1 \cdots p \cdots i_r} \\
&= (\nabla_k(\nabla_i T))^{j_1 \cdots j_s}_{i_1 \cdots i_r} - (\Gamma^j_{ki} \nabla_j T)^{j_1 \cdots j_s}_{i_1 \cdots i_r}.
\end{aligned}$$

$\square$

Ricci identity

From the definition of curvature tensor,

$$\begin{aligned}
R(X,Y)T &= \nabla_X \nabla_Y T - \nabla_{\nabla_X Y} T - \nabla_Y \nabla_X T + \nabla_{\nabla_Y X} T \\
&= \nabla^2_{X,Y} T - \nabla^2_{Y,X} T.
\end{aligned}$$

$$\begin{aligned}
\nabla_k \nabla_l T^{j_1 \cdots j_s}_{i_1 \cdots i_r} - \nabla_l \nabla_k T^{j_1 \cdots j_s}_{i_1 \cdots i_r} &= \left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) T \right) \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \mathrm{d}x^{j_s} \right) \\
&= \left(R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) T \right) T^{j_1 \cdots j_s}_{i_1 \cdots i_r} \\
&\quad + \sum_m R^{j_m}_{klq} T^{j_1 \cdots q \cdots j_s}_{i_1 \cdots i_r} - \sum_t R^p_{kl i_t} T^{j_1 \cdots j_s}_{i_1 \cdots p \cdots i_r}
\end{aligned}$$

Since $R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) f = 0$ for smooth function f, we obtain the following:

Theorem 54 (Ricci identity)**.**

$$\nabla_k \nabla_l T^{j_1 \cdots j_s}_{i_1 \cdots i_r} - \nabla_l \nabla_k T^{j_1 \cdots j_s}_{i_1 \cdots i_r} = \sum_m R^{j_m}_{klq} T^{j_1 \cdots q \cdots j_s}_{i_1 \cdots i_r} - \sum_t R^p_{kl i_t} T^{j_1 \cdots j_s}_{i_1 \cdots p \cdots i_r}.$$

In particular, for vector fields and 1-forms,

$$\nabla_k\nabla_l X^i - \nabla_l\nabla_k X^i = R^i_{klq}X^q,$$

$$\nabla_k\nabla_l\omega_j - \nabla_l\nabla_k\omega_j = -R^p_{klj}\omega_p.$$

Exercise 55. *prove the Ricci identity in (normal) local coordinates.*

Contraction and 2nd Bianchi identity

Using Leibniz rule for 2-tensor T,

$$Xg(g,T) = g(\nabla_X g, T) + g(g, \nabla_X T) = g(g, \nabla_X T),$$

this works similarly for 4-tensor S,

$$Xg(g\otimes g, S) = g(\nabla_X g\otimes g, S) + g(g\otimes g, \nabla_X T) = g(g\otimes g, \nabla_X T).$$

Proposition 56 (magic formulae for 2- and 4-tensors)**.**

$$\nabla_k g^{ij}T_{ij} = g^{ij}\nabla_k T_{ij},$$

$$\nabla_s g^{ij}g^{kl}S_{ijkl} = g^{ij}g^{kl}\nabla_s S_{ijkl}.$$

Theorem 57 (2nd Bianchi identity)**.**

$$\nabla_i R_{jkpq} + \nabla_j R_{kipq} + \nabla_k R_{ijpq} = 0.$$

As a corollary,

$$\begin{aligned} 0 &= g^{jp}g^{kq}\left(\nabla_i R_{jkpq} + \nabla_j R_{kipq} + \nabla_k R_{ijpq}\right) \\ &= -\nabla_i g^{jp}g^{kq}R_{kjpq} + g^{jp}\nabla_j g^{kq}R_{ikqp} + g^{kq}\nabla_k g^{jp}R_{ijpq} \\ &= -\nabla_i S + g^{jp}\nabla_j \operatorname{Ric}_{ip} + g^{kq}\nabla_k \operatorname{Ric}_{iq}, \end{aligned}$$

i.e. $\nabla_i S = 2g^{jk}\nabla_j \operatorname{Ric}_{ik}$, this is the contracted Bianchi identity.

Theorem 58 (Schur's lemma)**.** *Let (M,g) be a connected Riemannian manifold with $\dim M \geqslant 3$. If $f \in C^\infty(M)$, and one of the followings hold*

(1) $K = f$, i.e. $R(X,Y,Y,X) = |X\wedge Y|^2 f$ for $X,Y \in TM$;

(2) $\operatorname{Ric} = (n-1)fg$

then f is a constant.

Proof. Assuming *(2)*, $S = g^{ij}\operatorname{Ric}_{ij} = n(n-1)f$.

$$\nabla_k S = 2g^{ij}\nabla_i \operatorname{Ric}_{kj} = 2(n-1)g^{ij}\nabla_i f g_{kj} = 2(n-1)\nabla_k f.$$

Thus $n(n-1)\nabla_k f = 2(n-1)\nabla_k f$, which implies that f is constant. $\square$

Exercise 59. *prove the 2nd Bianchi identity in local coordinates.*

1.10. Miscellany

Riemannian submersions

Exercise 60. *let $\pi : (\overline{M}, \overline{g}) \to (M, g)$ be a Riemannian submersion.*

(1) let $H \subset T\overline{M}$ be the subbundle such that $H_p \perp \ker \pi_{*,p}$,

(a) for each $X \in \Gamma(M, TM)$, there exists a unique $\overline{X} \in \Gamma(\overline{M}, H)$ such that $\pi_* \overline{X} = X$;

(b) let $\sigma : [a, b] \to \overline{M}$ be a smooth curve, then for each $p \in \pi^{-1}(\sigma(a))$, there exists $\varepsilon > 0$ and a unique smooth curve $\overline{\sigma} : [a, a + \varepsilon] \to \overline{M}$ such that

$$\overline{\sigma}(a) = p, \pi \circ \overline{\sigma} = \sigma, \overline{\sigma}'(t) \in H_{\overline{\sigma}(t)}.$$

(2) for $X, Y \in \Gamma(M, TM)$, we have

$$\nabla^{g}_{\overline{X}} \overline{Y} = \overline{\nabla^{h}_{X} Y} + \frac{1}{2} [\overline{X}, \overline{Y}]^{v}$$

where Z^v is the orthogonal projection of Z to $\ker \pi_*$.

(3) [P]for $X, Y \in \Gamma(M, TM)$, we have

$$R(X, Y, Y, X) = \overline{R}(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) + \frac{3}{4} \left| [\overline{X}, \overline{Y}]^{v} \right|^2.$$

(4) show that $\pi \circ \exp_p(v) = \exp_{\pi(p)}(\mathrm{d}\pi_p(v))$. in particular, if $\widetilde{\gamma}$ is a geodesic, then $\pi \circ \widetilde{\gamma}$ is a geodesic.

(5) [Y]show that

(a) (M, g) is complete if $(\overline{M}, \overline{g})$ is complete;

(b) π is a fibration if $(\overline{M}, \overline{g})$ is complete.

(c) give a counterexample when $(\overline{M}, \overline{g})$ is not complete.

Warped products

Lie groups

2. The Bochner technique

2.1. Killing vector fields

Bochner formula for smooth functions

Proposition 61. *Let $f : M \to \mathbb{R}$ be a smooth function over (M, g), then*

$$\frac{1}{2}\Delta_g|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla\Delta_g f, \nabla f).$$

Curvature and Killing vector fields

Definition 62 (Killing vector field)**.** $L_X g = 0$ *(the flow is isometric).*

Using Koszul formula, we can show

$$g((L_X\nabla)_Y Z, W) = 0, \text{ i.e. } L_X\nabla = 0.$$

which gives a useful relation

$$R(X,Y)Z + \nabla^2_{Y,Z}X = 0.$$

It can also be stated and proven in terms of coefficients.

$$g_{il}\nabla_j\nabla_k X^i + R_{ijkl}X^i = 0.$$

Theorem 63. *Let X be a Killing vector field, $f = \frac{1}{2}|X|^2$,*

(1) $\nabla f = -\nabla_X X$*;*

(2) For any vector field V,

$$\operatorname{Hess} f(V,V) = g(\nabla_V X, \nabla_V X) - R(V,X,X,V).$$

In particular,

$$\Delta_g f = |\nabla X|^2 - \operatorname{Ric}(X,X).$$

Theorem 64. *Let (M, g) be a compact Riemannian manifold*

(1) if $\operatorname{Ric} < 0$, *then M has no non-trivial Killing vector field.*

(2) (Bochner) if $\operatorname{Ric} \leqslant 0$, then a vector field is parallel iff it is Killing.

The following theorem is proven using "linear algebra".

Theorem 65. *Let (M, g) be a compact Riemannian manifold with positive sectional curvature. If M is of even dimension, then every Killing field has a zero.*

Remark 66. *There are examples of non-vanishing Killing vector fields if M is odd, e.g. $V_x = (x_2, -x_1, \cdots, x_{2n}, -x_{2n-1})$ on S^{2n-1}.*

2.2. Harmonic 1-forms

Bochner formula for harmonic 1-forms

Proposition 67. *Let (M, g) be a compact Riemannian manifold, $\alpha \in \Omega^1(M)$ be a harmonic form, then*

$$\frac{1}{2}\Delta_g|\alpha|^2 = |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).$$

For general 1-form α, the Bochner formula is

$$\frac{1}{2}\Delta_g|\alpha|^2 = -g(\Delta\alpha, \alpha) + |\nabla\alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp).$$

where Δ is the Hodge laplacian.

Curvature and Betti numbers

Theorem 68. *Suppose (M, g) is a compact Riemannian manifold of non-negative Ricci curvature.*

(1) Every harmonic 1-form is parallel. Hence $b_1(M) \leqslant \dim M$.

(2) If $\mathrm{Ric} > 0$, then $b_1(M) = 0$.

2.3. Harmonic maps

Bochner formula for smooth maps

Proposition 69. *Let $f : (M, g) \to (N, h)$ be a smooth map, then*

$$\begin{aligned}\frac{1}{2}\nabla_g|\mathrm{d}f|^2 = (\widehat{\nabla}\Delta f, \mathrm{d}f) + |\widetilde{\nabla}\mathrm{d}f|^2 + g^{ik}g^{jl}h_{\alpha\beta}\,\mathrm{Ric}_{ij}\, f_k^\alpha f_l^\beta \\ - g^{ij}g^{kl}R_{\alpha\beta\gamma\delta}f_i^\alpha f_j^\delta f_k^\beta f_l^\gamma.\end{aligned}$$

Curvature and harmonic maps

3. Variation formulae and Jacobi fields

Variations

proper variation, 1st variation of the energy, 2nd variation of the energy

Jacobi fields

Definition 70 (Jacobi field)**.** *Let $\gamma : [a, b] \to (M, g)$ be a geodesic. A vector field J along γ is called a Jacobi field if*

$$\widehat{\nabla}\widehat{\nabla}J + R(J, \gamma')\gamma' = 0$$

Proposition 71 (local expansion of the length)**.** *Let $g(t) = |J|^2$, where J is a Jacobi field along a geodesic γ, then*

Theorem 72 (characterization of a Jacobi field)**.** *Every Jacobi field is given by some variation along some geodesic. Let (M, g) be a Riemannian manifold, $\gamma : [0, 1] \to M$ be a geodesic, then the Jacobi field along γ with $J(0) = 0$ and $J'(0) = v$ is given by*

$$J = \alpha_* \frac{\partial}{\partial s}\Big|_{s=0}, \quad \alpha = \exp_{\gamma(0)}(t(\gamma'(0) + sv))$$

for s small enough. In particular,

$$J(t) = (\exp_{\gamma(0)})_{*, t\gamma'(0)}(tv).$$

Jacobi fields in CSC space, conjugate point, index theorem, cut locus and topology (maybe Morse theory?)

4. Curvature and topology

4.1. Non-positive sectional curvature

Theorem 73 (Cartan-Hadamard)**.** *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature. For any $p \in M$, $\exp_p : T_pM \to M$ is a covering map. The universal covering $\widetilde{M} \cong \mathbb{R}^n$.*

Corollary 74. *Suppose M, N are compact smooth manifolds. If one of them is simply-connected, then $M \times N$ does not admit a Riemannian metric with non-positive sectional curvature.*

Theorem 75 (characterization of CH manifolds)**.** *Let (M, g) be a simply-connected complete manifold. The followings are euqivalent.*

(1) M has non-positive sectional curvature;

(2) The differential of exponential map is length increasing, i.e.

$$|(\exp_p)_{*,v}(\widetilde{v})| \geqslant |\widetilde{v}|$$

for all $p \in M, v, \widetilde{v} \in T_pM$.

(3) The exponential map is distance increasing, i.e.

$$d_g(\exp_p(v), \exp_p(\widetilde{v})) \geqslant |v - \widetilde{v}|$$

for all $p \in M, v, \widetilde{v} \in T_pM$.

Moreover, if the conditions are satisfied, then the exponential map is diffeomorphic.

Theorem 76 (Cartan). *Let* (M, g) *be a CH manifold,* G *a compact Lie group acting smoothly and isometrically on* M*, then* G *has a fixed point.*

Theorem 77 (Cartan). *Let* (M, g) *be a complete Riemannian manifold with non-positive sectional curvature, then* $\pi_1(M)$ *is torsion free.*

4.2. Negative sectional curvature

Proposition 78. *Let* (M, g) *be a complete Riemannian manifold with non-positive sectional curvature and* $\pi : \widetilde{M} \to M$ *the universal covering. If* $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ *is a common axis for all elements of* $\mathrm{Aut}_\pi(\widetilde{M})$*, then* M *is not compact.*

Exercise 79. *Let* (M, g) *be a closed Riemannian manifold of dimension* $\geqslant 2$ *with negative sectional curvature. Let* $\widetilde{M}$ *be its universal,* $\Gamma = \pi_1(M)$ *can be identified as a subgroup of* $\mathrm{Isom}(\widetilde{M})$ *by deck transformations.*

(1) Prove that there are $\gamma_1, \gamma_2 \in \pi_1(M)$ *with different axes.*

(2) Prove that the centralizer of $\Gamma \subset \mathrm{Isom}(\widetilde{M})$ *is trivial.*

Theorem 80 (Preissmann). *Let* (M, g) *be a compact Riemannian manifold with negative sectional curvature.*

(1) Any non-trivial abelian subgroup of $\pi_1(M)$ *is isomorphic to* Z.

(2) $\pi_1(M)$ *is not abelian.*

Corollary 81. *Suppose* M, N *are compact cmooth manifolds. Then* $M \times N$ *does not admit a Riemannian metric of negative sectional curvature.*

Theorem 82. *Let* (M, g) *be a compact Riemannian manifold with negative sectional curvature.*

(1) (Byers) Any non-trivial solvable subgroups of $\pi_1(M)$ *is isometric to* $\mathbb{Z}$*. In particular,* $\pi_1(M)$ *is not solvable.*

(2) Any subgroup of $\pi_1(M)$ *which contains a non-trivial abelian normal subgroup is isomorphic to* $\mathbb{Z}$.

growth rate of fundamental group?

4.3. Non-negative curvature

Theorem 83 (Myers)**.** *Let (M^n, g) be a complete manifold. If*

$$\mathrm{Ric} \geqslant \frac{(n-1)g}{R^2}$$

then $\mathrm{diam}(M,g) \leqslant \pi R$. *In particular, M is compact and $\pi_1(M)$ is finite.* (Cheng) If $\mathrm{diam}(M,g) = \pi R$, then M is isometric to (S^n, g_{can}).

Theorem 84 (Synge)**.** *Let (M,g) be a compact Riemannian manifold with positive sectional curvature.*

(1) If $\dim M$ *is even and M is orientable, then M is simply connected;*

(2) If $\dim M$ *is odd, then M is orientable.*

Corollary 85. *Let (M,g) be a compact Riemannian manifold with positive sectional curvature. If* $\dim M$ *is even and M is not orientable, then $\pi_1(M) = \mathbb{Z}/2\mathbb{Z}$.*

Theorem 86 (Weinstein-Synge)**.** *Let (M^n, g) be a compact Riemannian manifold with positive sectional curvature. Given an isometry $F : M \to M$ such that F preserve the orientation if n is even, changes the orientation if n is odd. Then F has a fixed point.*

4.4. Constant sectional curvature

Theorem 87 (Riemann-Hopf-Killing)**.** *Let (M,g) be a complete manifold with constant sectional curvature, then it is isometric to a Riemannian quotient of the form $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is one of the models spaces*

(1) $\mathbb{R}^n$, *(2)* $S^n(r)$, *(3)* $\mathbb{H}^n(r)$

and $\Gamma \subset \mathrm{Isom}(\widetilde{M})$ *is discrete and acts freely.*

A corollary of the Cartan-Ambrose-Hicks theorem.

Theorem 88. *Let (M, g_M) be connected, φ, ψ be two local isometries from M to (N, g_N). If there exists some point $p \in M$ with $\varphi(p) = \psi(p)$ and $\varphi_{*,p} = \psi_{*,p}$, then $\varphi = \psi$.*

Corollary 89. *Let (M,g) be a connected simply-connected complete Riemannian manifold. The followings are equivalent.*

(1) (M,g) is of constant sectional curvature.

(2) *For every pair of points $p, q \in M$ and linear isometry $\Pi : T_pM \to T_pM$, there exists an isometry $\varphi : M \to M$ with $\varphi(p) = q, \varphi_{*,p} = \Phi$.*

Corollary 90. *Let (M, g) be a complete and of constant sectional curvature 1. If $\dim M = 2m$, then (M, g) is isometric to S^{2m} or $\mathbb{RP}^{2m}$.*

Jacobi fields in space forms, function sn_k, warped product, expression of the Hessian:

$$\mathcal{H}_r = \frac{\mathrm{sn}_k' r}{\mathrm{sn}_k r}\pi_r,$$

$$\Delta_g r = (n-1)\frac{\mathrm{sn}_k' r}{\mathrm{sn}_k r},$$

$$\Delta_g r^2 = 2 + 2(n-1)r \cdot \frac{\mathrm{sn}_k' r}{\mathrm{sn}_k r}.$$

5. Comparison theorems and splitting theorem

5.1. Rauch

Rauch comparison, Jacobi field comparison, conjugate comparison, metric comparison, estimate of injective radius.

Corollary 91. *Suppose $0 <\leqslant C_1 \leqslant K_M \leqslant C_2$, let γ be any geodesic in M and l be the distance along γ between two consecutive conjugate points on γ, then*

$$\frac{\pi}{\sqrt{C_2}} \leqslant l \leqslant \frac{\pi}{\sqrt{C_1}}.$$

In particular, $\exp_p$ has no critical points on $B\left(0, \frac{\pi}{\sqrt{C_2}}\right)$.

5.2. Hessian and Laplacian

Hessian comparison, Laplacian comparison

5.3. Volume

volume comparison, proof of Cheng's rigidity, Lichnerowicz's eigenvalue inequality.

Proposition 92 (Gromov)**.** *Let (M, g) be a complete Riemannian manifold of dimension n with $\mathrm{Ric} \geqslant (n-1)kg$ for some constant $k > 0$. Then*

$$\mathrm{Vol}_g(M) \leqslant \mathrm{Vol}_{g_k}\left(S^n(\frac{1}{\sqrt{k}})\right).$$

If the equality holds, then (M,g) is isometric to $S^n\left(\frac{1}{\sqrt{k}}\right)$.

Proposition 93 (Cheng)**.** *Let (M,g) be a complete Riemannian manifold of dimension n with $\operatorname{Ric} \geqslant (n-1)kg$ for some constant $k>0$. If $\operatorname{diam} M = \frac{\pi}{\sqrt{k}}$, then (M,g) is isometric to $S^n\left(\frac{1}{\sqrt{k}}\right)$.*

5.4. Splitting theorem

Cheeger-Gromoll's splitting theorem, corollaries.

Corollary 94. *Let (M,g) be a complete Riemannian manifold with $\operatorname{Ric} \geqslant 0$.*

(1) (M,g) is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$, where N does not contain a geodesic line and $\operatorname{Ric} g_N \geqslant 0$.

(2) The isometry group splits

$$\operatorname{Isom}(M,g) \cong \operatorname{Isom}(\mathbb{R}^k, g_{\mathbb{R}^k}) \times \operatorname{Isom}(N, g_N).$$

Theorem 95 (structure of manifolds with $\operatorname{Ric} \geqslant 0$)**.** *Let (M,g) be a compact Riemannian manifold with $\operatorname{Ric} \geqslant 0$, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ its universal covering with pull-back metric.*

(1) There exists some integer $k \geqslant 0$ and a compact Riemannian manifold (N, g_N) with $\operatorname{Ric} g_N \geqslant 0$ such that $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$.

(2) The isometry group splits

$$\operatorname{Isom}(M,g) \cong \operatorname{Isom}(\mathbb{R}^k, g_{\mathbb{R}^k}) \times \operatorname{Isom}(N, g_N).$$

(3) There exists a finite normal subgroup G of $\operatorname{Isom}(N,h)$, a Bieberbach group B_k and an exact sequence

$$0 \to G \to \pi_1(M) \to B_k \to 0.$$

Corollary 96. *Let (M,g) be a compact Riemannian manifold with $\operatorname{Ric} \geqslant 0$, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ its universal covering with pull-back metric.*

(1) If $\widetilde{M}$ is contractible, then $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ and (M,g) is flat.

(2) If $(\widetilde{M}, \widetilde{g})$ does not contain a line, then $\pi_1(M)$ is finite and $b_1(M) = 0$.

(3) If $\pi_1(M)$ is finite, then $\widetilde{M}$ is compact and $b_1(M) = 0$.

Corollary 97. *Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$. If there exists some point $p \in M$ such that $\mathrm{Ric}_p > 0$, then $\pi_1(M)$ is finite and $b_1(M) = 0$.*

Corollary 98. *Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric} \geqslant 0$, and $\dim M = n$. Then $b_1(M) \leqslant n$. Moreover, $b_1(M) = n$ iff (M, g) is flat.*

Corollary 99. *$S^3 \times S^1$ can not admit Ricci flat metrics.*

6. Gathering important results

(1) Koszul formula

(2) for 3-dim manifolds, Einstein implies CSC.

(3) volume expression of the Laplacian {see 10}

(4) symmetry and orthogonality of the 2nd fundamental form

(5) Gauss' lemma {see 23}

(6) Hopf-Rinow theorem {see 29}

(7) local expansion of metric {see 32}

(8) properties of the radial vector field and corollaries {see 35}

(9) expression of d^* {see 41}

(10) divergence theorem {see 1.8}

(11) Ricci identity {see 54}

(12) 2nd Bianchi identity {see 57}

(13) Schur's lemma {see 58}

(14) Bochner formula for smooth functions {see 61}

(15) Bochner formula for Killing vector fields{see 63}

(16) Bochner formula for harmonic 1-forms {see 67}

(17) *Bochner formula for smooth maps {see 69}

(18) 1st and 2nd variation of the energy

(19) characterization of the Jacobi field {see 72}

(20) index theorem and topology

(21) Cartan-Hadamard theorem {see 73}

(22) characterization of CH manifolds {see 75}

(23) Cartan's fixed point and torsion free theorem {see 76, 77}

(24) Preissmann theorem {see 80}

(25) Byers theorem {see 82}

(26) no product manifold admits a metric of negative sectional curvature

(27) Myers theorem {see 83}

(28) Synge theorem {see 84}

(29) Weinstein-Synge theorem {see 86}

(30) Riemann-Hopf-Killing theorem {see 87}

(31) properties of space of CSC

(32) Rauch comparison and corollaries

(33) Hessian comparison and Laplacian

(34) volume comparison

(35) proof of Cheng's rigidity theorem

(36) Linchnerowicz's eigenvalue inequality and rigidity

(37) Cheeger-Gromoll splitting theorem and corollaries

A. Local isometry and isometry

Definition 100 ((local) isometry)**.** *Let $\varphi : (M, g_M) \to (N, g_N)$ be smooth.*

(1) φ is called a local isometry if $\varphi_{*,p} : T_pM \to T_{\varphi(p)}M$ is a linear isometry for every $p \in M$, or equivalently, $g_M = \varphi^* g_N$.

(2) φ is called an isometry if φ is surjective and preserve the distance.

List of properties:

- if φ is a local isometry, then φ is totally geodesic;

- for smooth curve $\gamma : [a,b] \to M$ and $\widetilde{\gamma} = \varphi \circ \gamma$, γ is a geodesic iff $\widetilde{\gamma}$ is a geodesic.

Theorem 101. *Let $\varphi : (M, g_M) \to (N, g_N)$ be smooth and bijective. The followings are equivalent*

(1) φ is an isometry.

(2) φ is a diffeomorphism and a local isometry.

(3) φ is a diffeomorphism and for every smooth curve $\gamma : [a,b] \to M$,

$$\operatorname{length}(\varphi \circ \gamma) = \operatorname{length}(\gamma).$$

Exercise 102. *prove the theorem above.*

B. Covering maps and transformations

Riemannian covering maps

Definition 103 (Riemannian covering map)**.** *A smooth covering map $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering map if it is a local isometry.*

Theorem 104. *Suppose $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry.*

(1) If $(\widetilde{M}, \widetilde{g})$ is complete, then π is a Riemannian covering map and (M, g) is complete.

(2) If π is a covering map, then (M, g) is complete iff $(\widetilde{M}, \widetilde{g})$ is complete.

Deck transformations

Definition 105 (deck transformation)**.** *Let $\pi : \widetilde{M} \to M$ be the universal covering of M. A deck transformation $F : \widetilde{M} \to \widetilde{M}$ is a homeomorphism such that $\pi \circ F = F$, enote by $\operatorname{Aut}_\pi(\widetilde{M})$ the set of deck transformations*

Theorem 106. *(1) $\pi_1(M) \cong \operatorname{Aut}_\pi(\widetilde{M})$;*

(2) $\operatorname{Aut}_\pi(\widetilde{M})$ acts smoothly freely and properly on $\widetilde{M}$;

(3) $\operatorname{Aut}_\pi(\widetilde{M})$ acts transitively on each fiber of π.

C. Axes, rays and lines

Free homotopy class

Definition 107. *Two loops $\gamma_0, \gamma_1; [0,1] \to M$ are said to be freely homotopic if they are homotopic through closed paths, i.e. there exists a homotopy $H(s,t) : [0,1] \times [0,1] \to M$ such that*

$$H(0,t) = \gamma_0(t), H(1,t) = \gamma_1(t) \text{ and } H(s,0) = h\mathcal{H}(s,1).$$

Axes

Definition 108 (axis of an isometry)**.** *Let (M,g) be complete, $F : M \to M$ be an isometry. A geodesic $\mathbb{R} \to M$ is called an axis of F if $F \circ \gamma$ is a non-trivial translation of γ, i.e.*

$$F(\gamma(t)) = \gamma(t+c)$$

for some constant $c \neq 0$. F is axial if it has an axis.

Lemma 109. *Let (M,g) be complete, F be an isometry. If $\delta_F(p) = d(p, F(p))$ has a positive minimum, then F has an axis.*

Theorem 110. *Let (M,g) be a compact Riemannian manifold, $F : \widetilde{M} \to \widetilde{M}$ be a non-trivial deck transformation of $\pi : \widetilde{M} \to M$.*

(1) δ_F has a positive minumum and $\delta_F \geq 2\,\mathrm{inj}(M)$, thus F is axial.

(2) The axis corresponding to this minimum is mapped under π to a closed geodesic, whose length is minimal in its free homotopy class.

Exercise 111. [Y]*suppose (M,g) is a compact connected riemannian manifold. every non-trivial free homotopy class in M is represented by a closed geodesic that has minimum length among all admissible loops in the given free homotopy class.*

Geodesic rays

Definition 112 (geodesic ray)**.** *A geodesic ray is a unit-speed geodesic $\gamma : [0,\infty) \to M$ such that $d(\gamma(s), \gamma(t)) = |s-t|$ for any $s, t \geqslant 0$.*

Lemma 113. *Let (M,g) be a complete Riemannian manifold. The followings are equivalent.*

(1) M is non-compact.

(2) For any $p \in M$, there is a geodesic ray starting from p.

Proposition 114 (definition of Busemann function)**.** *Let (M,g) be a complete Riemannian manifold, $\gamma:[0,\infty)\to M$ be a geodesic ray starting from a point p. Define*

$$b^t_\gamma(x)=d(x,\gamma(t))-t=d(x,\gamma(t))-d(\gamma(0),\gamma(t))$$

then $b^t_\gamma(x)$ is non-increasing for t. Define the Busemann function by

$$b_\gamma(x)=\lim_{t\to\infty}b^t_\gamma(x).$$

List of properties:

- $|b^t_\gamma(x)|\leqslant d(x,\gamma(0))$;
- $|b^t_\gamma(x)-b^t_\gamma(y)|\leqslant d(x,y)$.

Exercise 115. [Y]*compute the busemann functions on the upper half plane $\mathbb{H}^2$ with canonical metric of constant sectional curvature -1.*

Geodesic lines

Definition 116 (geodesic line)**.** *A geodesic line is a unit-speed geodesic $\gamma:\mathbb{R}\to M$ such that $d(\gamma(s),\gamma(t))=|s-t|$ for any $s,t\in\mathbb{R}$.*

Lemma 117. *Let (M,g) be a connected complete non-compact manifold. If M contains a compact subset K such that $M\backslash K$ has at least two un-bounded components, then there is a geodesic passing through K.*

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