Algebraic geometry(rv)

(24fall)quiddite

This is a very very brief note based on a course lectured by Prof. Zhang, which covers roughly the first 2 chapters of [1], with more examples. Good refenences are [1, 2, 3, 4, 5, 6, 7](the first three books are used frequently) & [8]. I type it in order to review & tide up my mind, so don't blame me for the abundant typos & mis-useages of symbols, terms blabla.

目录

1 Varieties

2 Schemes

2.1 Schemes

Definition 2.1 (spectrum of a ring)**.** *Let A be a ring, the spectrum is a ringed space* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *given by*

(1) $\operatorname{Spec} A$ *with the Zariski topology;*

(2) $\mathcal{O}_{\operatorname{Spec} A}(U) = \{s : u \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}}\} | s(p) \in A_{\mathfrak{p}}\}$.

Luckily, the tidious construction above is used not that often. We always simply use the properties suggested by the following proposition.

Proposition 2.2 (❄)**.** *Let A be a ring,*

(1) for and $p \in \operatorname{Spec} A$, $\mathcal{O}_{\mathfrak{p}} \cong A_{\mathfrak{p}}$;

(2) for any $f \in A$,$\mathcal{O}_{\operatorname{Spec} A}(D(f)) \cong A_f$;

(3) as a result of (2), $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \cong A$.

Definition 2.3 (ringed spaces and morphisms)**.**

(1) A ringed space is a pair $(X, \mathcal{O}_X)$;

(2) A locally ringed space is a r.s. whose stalks $\mathcal{O}_{X,P}$ are local rings $\forall P \in X$;

(3) A morphism between r.s.'s $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$, where $f : X \xrightarrow{\text{conti}} Y$ & $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$;

*(4) A morphism between l.r.s.'s is a morphism $X \xrightarrow{f} Y$ between r.s.'s, which induces **local** homomorphisms $f_p^{\#} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$, i.e. $(f_p^{\#})^{-1}$ preserves the maximal ideal.*

Proposition 2.4.

(1) $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is a l.r.'s.;*

(2) The set of morphisms $(f, f^{\#})$ *between l.r.s.'s* $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ *&* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *consists exactly of the morphisms induced by some* $\varphi : A \xrightarrow{\text{homo}} B$.

Now we can define schemes.

Definition 2.5 (schemes)**.**

(1) An affine scheme is a l.r.s. $(X, \mathcal{O}_X)$ *which is isomorphic to some* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$*;*

(2) A scheme is a l.r.s. $(X, \mathcal{O}_X)$ *which is locally affine, i.e.* $\exists$ *an open cover* $\{U\}$ *s.t. each* $(U, \mathcal{O}_X|_U)$ *is an affine scheme;*

(3) A morphism of schemes is a morphisms of l.r.s.'s.

Example 2.6 (schemes)**.** *In these examples,* $k = \operatorname{alg.cl} k$.

(1) If R *is a d.v.r., then* $\operatorname{Spec} R = \{\circ, \bullet\}$*, where* $\circ$ *is a generic point and* $\bullet$ *is a closed point(see [1]P.74 for detailed explanation);*

(2) $\mathbb{A}^1_k = \operatorname{Spec} k[x] = \{\circ\} \cup k$*, where* $\{\circ\}$ *is a generic point and points in* k *are all closed;*

(3) $\mathbb{A}^2_k = \operatorname{Spec} k[x, y] = \{(0)\} \cup \{f \in k[x, y] \mid f \textit{ is irreducible}\} = \{\circ\} \cup k^2 \cup \{f \in k[x, y] \mid f \textit{ is irreducible}, \deg f \geqslant 2\}$*. The first part is the generic point, the second part consists of closed points, and the third part consists of generic points of such curves* $f(x, y) = 0$.

(4) (❋affine line with a doubled point) Let $X_1 = X_2 = \mathbb{A}^1_k, U_1 = U_2 = \mathbb{A}^1_k \setminus \{0\}$*. Glueing* X_1 *&* X_2 *along* U_1 *&* U_2 *via the identity map* $U_1 \to U_2$*, nothing is done except for* $\{0\}$*.*

this gives a non-affine scheme.

Proposition 2.7 (generic points). *Let X be a scheme, then every non-empty irreducible closed subset Y of X has a unique generic point, i.e. a point $p \in Y$ s.t. $\overline{\{p\}} = Y$*

Let $U = \operatorname{Spec} A$ be an affine open subset of X s.t. $U \cap Y \neq \emptyset$, then $U \cap Y$ is an irreducible closed subset of U(i.e. "reduced" to affine case), thus $U \cap Y = V(p)$ for some $p \in \operatorname{Spec} A$. Obviously, $U \cap Y = \overline{\{p\}}^U = \overline{\{p\}} \cap U$. At the same time, $U \cap Y \neq \emptyset$ is open in Y, from the irreducibility, $\overline{U \cap Y}^Y = Y$, so $Y \subset \overline{\{p\}}$, i.e. $\overline{\{p\}} = Y$. For the uniqueness, if $y = \overline{\{p\}} = \overline{\{p'\}}$, then $V(p) = U \cap Y = V(')$, thus $p = p'$.

Now we are going to a criterion for affine-ness(see [1]P.81 or [3]P.28).

Procedure 2.8 (Construction of X_f). *Let X be a scheme, $f \in \mathcal{O}_X(X)$*

1. $X_f = \{p \in X \mid f_p \notin \mathfrak{m}_p = \mathfrak{m}_{\mathcal{O}_p}$*(equivalently,* f_p *is invertible in* $\mathcal{O}_p$)$\}$;
2. *properties of* X_f:
   (a) X_f ***is open in*** X***;***
   (b) $X_f \cap X_g = X_{fg}$***;***
   (c) if X ***has a finite cover*** $\{U_i\}$***, s.t. each*** $U_i \cap U_j$ ***is q.c., then*** $\mathcal{O}_X(X_f) = (\mathcal{O}_X(X))_f$***.***

Proposition 2.9 (❋criterion for affine-ness). *Let X be a scheme, then X is affine $\iff$ $\exists$ finitely many $\{f_i\}$ s.t.*

(1) X_{f_i} *are affine;*

(2) $\{f_i\}$ *generates* $\mathcal{O}_X(X)$.

Definition 2.10 (residue field). *Let X be a scheme, $(\mathcal{O}_x, \mathfrak{m}_x)$ be the local ring at $x \in X$. $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is called the residue field of x.*

Remark 2.11. *In order to define a morphism $f : \operatorname{Spec} K \to X$, where K is a field, it suffices to identify a point $x \in X$ & an inclusion $k(x) \hookrightarrow K$. e.g. $k(x) \xrightarrow{\text{id}} k(x) \leftrightsquigarrow \operatorname{Spec} k(x) \hookrightarrow X$.*

2.2 Properties of schemes & morphisms I

Let's begin with an annoying table of definitions.

Definition 2.12 (some special schemes)**.** *A scheme X is called*

(1) quasi-compact, if $\mathrm{sp}(X)$ *is q.c.;*

(2) connected, if $\mathrm{sp}(X)$ *is connected;*

(3) irreducible, if $\mathrm{sp}(X)$ *is irreducible;*

(4) reduced, if $\forall U \overset{open}{\subset} X$, $\mathcal{O}_X(U)$ *is reduced, i.e.* $\mathrm{nil}(\mathcal{O}_X(U)) = \{0\}$*;*

(5) integral, if $\forall U \overset{open}{\subset} X$, $\mathcal{O}_X(U)$ *is a domain;*

(6) locally noetherian, if $\forall U = \operatorname{Spec} A \overset{open}{\subset} X$, A *is noetherian;*

(7) noetherian, if X is l.n. & q.c.;

Remark 2.13. *The condition is (6) can be replaced with "$\exists$ a cover $\{U_i\}$ of X, where $U_i = \operatorname{Spec} A_i \overset{open}{\subset} X$, each A_i is noetherian". The equivalence between "$\forall U$" & "$\exists$ a cover $\{U_i\}$" also holds for (1)(2)(3)(5).*

Here's some connections between these definitions.

Proposition 2.14. *Let X be a scheme,*

(1) X is integral $\iff$ X is reduced & irreducible;

(2) if $X = \operatorname{Spec} A$ is affine, then X is noetherian $\iff$ A is noetherian;

Let's continue with an annoying table of definitions.

Definition 2.15 (some special morphisms)**.** *Let $f : X \to Y$ be a morphism between schemes, f is (called)*

*(1) locally of finite type, if $\forall V = \operatorname{Spec} B \overset{open}{\subset} Y, \exists$ a cover $\{U_j\}$ of $f^{-1}(V)$, where $U_j = \operatorname{Spec} A_j \overset{open}{\subset} X$, each A_j is a f.g. B-**algebra**;*

(2) of finite type, if $\forall V = \operatorname{Spec} B \overset{open}{\subset} Y, \exists$ *a* ***finite*** *cover* $\{U_j\}$ *of* $f^{-1}(V)$*, where* $U_j = \operatorname{Spec} A_j \overset{open}{\subset} X$*, each* A_j *is a f.g.* B-***algebra****;;*

(3) finite, if $\forall V = \operatorname{Spec} B \overset{open}{\subset} Y, f^{-1}(V) = \operatorname{Spec} A \overset{open}{\subset} X$*, where* A *is a f.g.* B-***module****;*

(4) quasi-finite, if $\forall y \in Y, f^{-1}(y)$ *is a finite set;*

(5) quasi-compact, if $\forall V = \operatorname{Spec} B \overset{open}{\subset} Y, f^{-1}(V)$ *is q.c..*

Here's a famous & useful trick.

Proposition 2.16 (Nike's trick)**.** *Let* X *be a scheme,* $\operatorname{Spec} A, \operatorname{Spec} B \overset{open}{\subset} X$*, then* $\operatorname{Spec} A \cap \operatorname{Spec} B$ *is covered by (principle) open* $\{\operatorname{Spec} C\}$*, which is open both in* $\operatorname{Spec} A \,\&\, \operatorname{Spec} B$*.*

$\forall p \in \operatorname{Spec} A \cap \operatorname{Spec} B$, take $f \in A, g \in B$ s.t. $p \in D_{\operatorname{Spec} B}(g) \subset D_{\operatorname{Spec} A}(f) \subset \operatorname{Spec} A \cap \operatorname{Spec} B$.
Let $g' = g|_{D_{\operatorname{Spec} A}(f)} \in \mathcal{O}_{\operatorname{Spec} A}(D_{\operatorname{Spec} A}(f)) = A_f$(since $D_{\operatorname{Spec} A}(f) \subset \operatorname{Spec} B$, this can be done). Then we write $g' = \frac{h}{f^n}$, where $h \in A, n \in \mathbb{N}$.

$$D_{\operatorname{Spec} B}(g) = D_{\operatorname{Spec} A_f}(g') = \operatorname{Spec}(A_f)_{g'} = \operatorname{Spec} A_{fh},$$

where "$=$" holds on the "set" level. Thus $D_{\operatorname{Spec} B}(g)$ is open in $\operatorname{Spec} B$.

Remark 2.17. *As for intersections of the form* $U \cap \operatorname{Spec} A$*, where* U *is an arbitrary open set, the result is easier(since openness is "weaker"):* $U \cap \operatorname{Spec} A$ *is covered by open* $\{D_{\operatorname{Spec} A}(f)\}$*.*

Proposition 2.18 (closed points)**.** *Let* X *be a scheme which is of finite type over a field* k*, then the set of closed points is dense in* X*.*

Accoding to the condition, we have a finite cover $\{U_i\}$ of X(Spec k is a singleton), where $U_i = \operatorname{Spec} A_i \overset{open}{\subset} X$, each A_i is a f.g. k-algebra.
Only need to prove that, $\forall U = \operatorname{Spec} B \overset{open}{\subset} X$, it contains a closed point of X. Let p be a closed point in U, and consider $U_i \ni p$. Using 2.16, take a principle open set $p \in D(f) \neq \emptyset$ in $U_i \cap U$. The inclusion $i : D(f) \to U_i$ induces $i^\# : A_i \to (A_i)_f$ between Jacobson rings, so $p = i(p)$ is closed in U_i. Thus p is closed in X. The existence of such $p \in U$ follows the existence of maximal ideals(reduce to affine case).

Remark 2.19. *2.18 fails generally, e.g. (1)$X = \{\circ, \bullet\}$.*

problem 3.3 & 3.13

Definition 2.20 (open & closed immersions)**.** *Let $f : X \to Y$ be a morphism of schemes, f is called a*

(1) open immersion, if $(X, \mathcal{O}_X) \overset{f}{\cong} (Z, \mathcal{O}_Z)$, for some open subscheme $(Z, \mathcal{O}_Z)$ of Y;

(2) closed immersion, if $\operatorname{sp}(X) \overset{f}{\cong} \operatorname{sp}(Z)$ & $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective;

(3) immersion, if f can be factorized as $h \circ g : X \to U \to Y$, where $g : X \to U$ is a closed imm. & $h : U \to Y$ is an open imm..

(4) 2 closed imm.'s $f_1 : X_1 \to Y, f_2 : X_2 \to Y$ are equivalent if $\exists$ an isom. $g : X_1 \to X_2$ s.t. the following diagram commutes.

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\sim"'] & Y \\ X_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

The following proposition characterizes closed immersions in affine case.

Proposition 2.21. *Let A be a ring, X be a scheme. $X \to \operatorname{Spec} A$ is a closed imm. $\iff$ $(X, \mathcal{O}_X) \cong (\operatorname{Spec} A/\mathfrak{a}, \mathcal{O}_{\operatorname{Spec} A})$ for some ideal $\mathfrak{a}$ of A.*

Definition 2.22 (fiber product & fiber)**.**

(1) Let X, Y be schemes over S, the fiber product $X \times_S Y$ is defined by the following diagram of morphisms:

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

(2) the fiber of $f : X \to Y$ at y is defined by $X_y = X \times_Y \operatorname{Spec} k(y)$

$$\begin{tikzcd}
X \times_Y \operatorname{Spec} k(y) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d] \\
\operatorname{Spec} k(y) \arrow[r, hook] & Y
\end{tikzcd}$$

where "$\hookrightarrow$" exists in the sense of 2.11.

2.3 Properties of schemes & morphisms II

Appendix

A Category theory

A.1 colimit & limit

Definition A.1 (direct system)**.** *Let I be a directed set, a direct system $\{X_i, f_{ij}\}$ over I consists of a family of objects $\{X_i\}_{i \in I}$ & morphisms $f_{ij} : X_i \to X_j$ s.t.*

(1) $f_{ii} = \mathrm{id}_{X_i}, \forall i$;

(2) $f_{ik} = f_{jk} \circ f_{ij}, \forall i \leqslant j \leqslant k$.

"colimit" has many names, including "direct limit", "inductive limit".

Definition A.2 (colimit)**.** *Let $\{X_i, f_{ij}\}$ be a direct system, then colimit $\varinjlim X_i$ is defined by the following diagram.*

$$
\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr] \arrow[ddr, bend right] & & X_j \arrow[dl] \arrow[ddl, bend left] \\
& \varinjlim X_i \arrow[d, dashed] & \\
& Y &
\end{tikzcd}
$$

Definition A.3 (inverse system)**.** *Let I be a directed set, a inverse system $\{X_i, g_{ij}\}$ over I consists of a family of objects $\{X_i\}_{i \in I}$ & morphisms $g_{ij} : X_j \to X_i$ s.t.*

(1) $g_{ii} = \mathrm{id}_{X_i}, \forall i$;

(2) $g_{ik} = g_{ij} \circ g_{jk}, \forall i \leqslant j \leqslant k$.

"limit" has many names, including "inverse limit", "projective limit".

Definition A.4 (limit)**.** *Let $\{X_i, g_{ij}\}$ be an inverse system, then limit $\varprojlim X_i$ is defined by the following diagram.*

$$\begin{tikzcd}
& Y \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& \varprojlim X_i \arrow[dl] \arrow[dr] & \\
X_j & & X_i \arrow[ll, "g_{ij}"']
\end{tikzcd}$$

Example A.5 (colimit & limit)**.**

(1) Let I be equiped with the discrete order($i \leqslant j \iff i = j$), $\{X_i\}$ be a family of objects, then

(a) $\varinjlim X_i = \coprod X_i$, it's called the sum or coproduct;

(b) $\varprojlim X_i = \prod X_i$, it's called the product.

(2) Let $I = \emptyset$,

(a) the colimit coincides with the initial object;

(b) the limit coincides with the terminal object.

(3) In the category of R-algebras, $A \coprod B = A \otimes_R B$;

(4) Let $I = \{a, b, c\}$, where $a \leqslant b, c$, $\{X_i\}$ be a family of objects, then

$$\varprojlim X_i = X_b \times_{X_a} X_c$$

Proposition A.6 (with adjoint functors)**.** *Let $\mathcal{C}, \mathcal{D}$ be 2 categories, F, G be a pair of adjoint functors, i.e.*

(1) $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$; *(2) $\mathrm{Hom}_{\mathcal{C}}(G(-), \star) \cong \mathrm{Hom}_{\mathcal{D}}(-, F(\star))$.*

Let I be a directed set,

(1) $\{Y_i\} \subset \mathcal{D}$ be a direct system, then $G(\varinjlim Y_i) = \varinjlim G(Y_i)$;

(2) $\{X_i\} \subset \mathcal{C}$ be an inverse system, then $F(\varprojlim X_i) = \varprojlim F(X_i)$.

B Commutative algebra

B.1 Valuation rings

Definition B.1 (valuation rings)**.** *Let k be a field, A be a subring(thus a domain) of k. We say A is a valuation ring of k if $\forall x \neq 0 \in k$, either $x \in A$ or $\frac{1}{x} \in k$.*

Proposition B.2 (properties of v.r.)**.** *Let A be a v.r. of k.*

(1) A is a local ring, and $\mathfrak{m}_A = \{x \in A \,|\, x \text{ is not invertible}\} = \{x \neq 0 \in A \,|\, \frac{1}{x} \notin A\} \cup \{0\}$;

(2) A is integrally closed in k;

(3) if B is a ring s.t. $A \subset B \subset k$, the B is also a v.r. of k. Moreover,

(a) $\mathfrak{m}_B \subset A$;

(b) $\mathfrak{m}_B$ is a prime ideal of A;

(c) $B = A_{\mathfrak{m}_B}$, i.e. B is a local ring of A

(4) $\forall 2$ ideals $\mathfrak{a}, \mathfrak{b}$ of A, either $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{a} \supset \mathfrak{b}$. Moreover, if any subring B of k with this compariable properties, must be a v.r..

Now we are going to construct v.r.'s of a field k.

Procedure B.3. *Fix a field k and an algebraically closed filed Ω.*

1. $\Sigma = \{(A, f) \,|\, A \subset k,\ f : A \xrightarrow{\text{homo}} \Omega\}$;

2. define a partial order on Σ:

$$(A, f) \leqslant (B, g) \iff A \subset B \& g|_A = f,$$

then Σ has at least one maximal element(Zorn's lemma);

3. let (B, g) be a maximal element of Σ, then

(a)** (B, g) **is a local ring &** $\mathfrak{m}_B = \ker g$**;

(b) (B, g) ***is a v.r. of*** k.

Corollary B.4. *Let* A *be a subring of* k*, then* $\operatorname{int.cl} A = \cap B$*, where the intersection is taken over* $\{B \mid A \subset B \subset k \,\&\, B$ *is a v.r. of* $k\}$.

- Obviously $\operatorname{int.cl} A \subset \cap B$;
- Conversely, if $x \in \operatorname{int.cl} A$ but $x \notin A$, let $B = A[\frac{1}{x}]$, then $\frac{1}{x}$ is not invertible in B. Let $\mathfrak{m}$ be a maximal ideal of B s.t. $\frac{1}{x} \in \mathfrak{m}$, and let $\Omega = \operatorname{alg.cl} B/\mathfrak{m}$. The quotient gives a map $f : B \to \Omega$. From B.3, (B, f) can be extended to some valuation ring (C, g). But $f(\frac{1}{x}) = 0$, thus $\frac{1}{x} \in \ker C$, i.e $x \notin C$.

There's another construction which happens to be equivalent to B.3.

Procedure B.5. *Fix a field* k.

1. $\Sigma = \{(A, \mathfrak{m}) \mid A \subset k$ *is a local ring with maximal ideal* $\mathfrak{m}\}$;
2. *define a partial order(called dominance) on* Σ:

$$(A, \mathfrak{m}) \leqslant (B, \mathfrak{n}) \iff A \subset B \& \mathfrak{m} \subset \mathfrak{n},$$

   then Σ *has at least one maximal element;*

3. $(A, \mathfrak{m})$ ***is a maximal element of*** $\Sigma \iff A$ ***is a v.r. of*** k.

Proposition B.6. *Let* $A \subset B$ *be 2 domains,* B *f.g. over* A. $\forall x \neq 0 \in B, \exists u \neq 0 \in A$ *s.t. any* $f : A \to \Omega = \operatorname{alg.cl} \Omega, f(u) \neq 0$ *can be extended to* $g : B \to \Omega$ *with* $g(v) \neq 0$.

Using B.6, we can prove one form of Hilbert's Nullstellensatz.

Corollary B.7. *Let* k *be a field and* B *a f.g.* k*-algebra. If* B *is a field, the* B/k *is a finite algebraic extension.*

Take $A = k, v = 1, \Omega = \text{alg.cl}\, k$, then we get some $g : B \to \Omega$, which is non-trivial thus injective.

Explanation: Consider only the case when $B = k[x]$. Take some $\xi \neq 0 \in \Omega = \text{alg.cl}\, k$, we get a homomorphism by sending x to ξ.

Finally, we explore the relartion between v.r.'s & valuations of a field.

Definition B.8 (valuations)**.** *Let k be a field, G be a totally ordered abelian group. A valuation of k with values in G is a mapping $v : k^* \to G$ s.t.*

(1) $v(xy) = v(x)v(y)$*;*

(2) $v(x + y) \geqslant \min\{v(x), v(y)\}$*, if* $x + y \neq 0$*.*

Procedure B.9.

1. *From a v.r A of k to a valuation*

   (a) $U = \{\textit{units of } A\}, G = k^*/U$*;*

   (b) define a partial order on G:

$$[x] \leqslant [y] \iff \frac{y}{x} \in A,$$

   *then G becomes **a totally ordered group**, moreover, the quotient $v : k \to G$ is **a valuation with values in** G.*

2. *From a valuation $v : k^* \to G$ of k to a v.r.*

   (a) $A = \{x \in k^* \mid v(x) \geqslant 0\} \cup \{0\}$*;*

   *(b) A **is a v.r. of** k, which is called the v.r. of v.*

B.2 Jacobson rings

Definition B.10 (Jacobson rings)**.** *We say a ring A is a Jacobson ring if $\forall \mathfrak{p} \in \operatorname{Spec} A, \mathfrak{p} = \cap \mathfrak{m}$, where the intersection is taken over $\{\mathfrak{m} \in \operatorname{Spm} \mid \mathfrak{p} \subset \mathfrak{m}\}$.*

Remark B.11. *In non-commatative cases, Jacobson rings are defined via primitive ideals.*

Example B.12 (Jacobson rings)**.** *The following rings are Jacobson.*

(1) A field k;

(2) A polynomial ring $k[x_1, \cdots, x_n]$;

(3) A p.i.d. A with $\mathrm{Jac}(A) = 0$;

(4) A ring of Krull dimension 0, e.g. a ring with only one prime ideal.

Here's an interesting example.

Example B.13 (❋Jacobson yet not noetherian)**.** *Let k be a field, $R = k[x_1, x_2, \cdots]/(x_1^2, x_2^2, \cdots)$. The only prime ideal of R is $(x_1, x_2, \cdots)$, which is not f.g..*

Proposition B.14 (properties of Jacobson rings)**.**

(1) A ring A is Jacobson $\iff$ $A[x]$ is Jacobson([9]P.18);

(2) As a result of (1), a f.g. algebra over a Jacobson ring is also Jacobson;

(3) Let A, B be Jacobson, $f : A \to B$, then $f^{-1}(\mathfrak{m})$ is a maximal ideal of $A, \forall$ maximal ideal $\mathfrak{m}$ of B;

(4) As a result of (3), $f^{\#} : \operatorname{Spec} B \to \operatorname{Spec} A$ maps closed points in $\operatorname{Spec} B$ to closed points in $\operatorname{Spec} A$.

B.3 Nakayama's lemma

Theorem B.15 (Nakayama's lemma)**.** *Let M be a f.g. A-module, $\mathfrak{a}$ be an ideal. If $\mathfrak{a}M = M$, then $\exists x \in A$ s.t.*

(1) $x \equiv 1 \mod \mathfrak{a}$;

(2) $xM = 0$.

Corollary B.16. *Let M be a f.g A-module,([4]P.556)*

(1) if $u : M \xrightarrow{\text{homo}} M$ is surjective, then u is bijective;

(2) if A is local with $\mathfrak{m}$, then a subset $\{m_1, \cdots, m_r\}$ generates $M \iff \{m_1, \cdots, m_r\}$ generates $M/\mathfrak{m}M$ over $A/\mathfrak{m}$.

(1) Consider M as $A[x]$-module, where $x \cdot m = u(m)$([10]P.9);

(2) "$\Rightarrow$" is obvious. As for "$\Leftarrow$", let $N = (x_1, \cdots, x_r)$, then $N \hookrightarrow M \twoheadrightarrow M/\mathfrak{m}M$ is exact, i.e. $M = N + \mathfrak{m}M$.

Remark B.17. *Essentially, B.15 generalizes the existence of annihilating polynomial in linear algebra. Thus the idea in (1) is natural.*

C Sheaves

We consider only sheaves of Abelian groups, thus rings & modules are treated as special cases.

Definition C.1 (presheaves & sections & morphisms)**.** *Let X be a topo. space,*

(1) a presheaf $\mathcal{F}$ on X is a contra. functor from $\mathrm{Open}(X) \to \mathbf{Ab}$*;*

(2) a morphism between 2 presheaves $\mathcal{F}, \mathcal{G}$ is a natural transformation from $\mathcal{F} \to \mathcal{G}$.

Or we can adopt human's language,

(1) a presheaf $\mathcal{F}$ on X consists of

(a) $\mathcal{F}(U) \in \mathbf{Ab}, \forall U \in \mathrm{Open}(X)$;

(b) restriction $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V), \forall V \subset U \in \mathrm{Open}(X)$;

s.t.

(a) $\mathcal{F}(\emptyset) = 0$;

(b) $\rho_{UU} = \mathrm{id}$*;*

(c) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$*.*

(2) any element $s \in \mathcal{F}(U)$ *is called a section of* $\mathcal{F}$ *on* U*, we sometimes write* $\Gamma(U, \mathcal{F})$ *for* $\mathcal{F}(U)$*;*

(3) a morphism $f : \mathcal{F} \to \mathcal{G}$ *consists of morphisms* $f(U) : \mathcal{F}(U) \to \mathcal{G}(U), \forall U \in \mathrm{Open}(X)$*, s.t. the following diagram commutes* $\forall V \subset U \in \mathrm{Open}(X)$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "f(V)"] & \mathcal{G}(V) \end{tikzcd}$$

Definition C.2 (sheaves)**.** *Let* X *be a topo. space. A sheaf* $\mathcal{F}$ *on* X *is a presheaf, s.t. if* $U \in \mathrm{Open}(X), \{V_i\} \subset \mathrm{Open}(X)$ *is a cover of* U*,*

(1) (factorizing)if $s \in \mathcal{F}(U)$ *s.t.* $s|_{V_i} = 0, \forall i$*, then* $s = 0$*;*

(2) (glueing)if $s_i \in \mathcal{F}(V_i)$ *s.t.* $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}, \forall i, j$*, then* $\exists s \in \mathcal{F}(U)$ *s.t.* $s|_{V_i} = s_i, \forall i$*.*

Definition C.3 (stalks & germs)**.** *Let* X *be a topo. space* $p \in X$*,* $\mathcal{F}$ *be a presheaf on* X*.*

(1) We define the stalk $\mathcal{F}_p$ *at* p *by*

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U),$$

where the colimit is taken over $\{U \overset{open}{\subset} X \mid p \in U\}$*;*

(2) Any element $s_p \in \mathcal{F}_p$ *is called a germ of* $\mathcal{F}$ *at* p*.*

Remark C.4. *To be more concrete,* $\mathcal{F}_p = \{(U, s) \mid p \in U \overset{open}{\subset} X, s \in \mathcal{F}(U)\}/\sim$*, where* $(U, s) \sim (V, t) \iff \exists W \overset{open}{\subset} X$ *s.t.*

(1) $p \in W \subset U \cap V$*;*

(2) $s|_W = t|_W$.

Thus any germ s_p comes from some sections.

Proposition C.5 (sheaves are determined by stalks)**.** *Let X be a topo. space, $\mathcal{F}, \mathcal{G}$ be 2 sheaves on X, $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism. Then*

(1) φ *is injective* $\iff$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ *is injective* $\forall p \in X$*;*

(2) φ *is surjective* $\iff$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ *is surjective* $\forall p \in X$*;*

(3) Thus φ *is an isom.* $\iff$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ *is an isom.* $\forall p \in X$*;*

Remark C.6. *This result applies* ***only*** *to sheaves.*

Procedure C.7 (sheafification)**.** *Let X be a topo. space, $\mathcal{F}$ be a presheaf on X.*

1. $\forall U \overset{open}{\subset} X$*, define* $\mathcal{F}^{\ddagger}(U) = \{s : U \to \cup_{p \in U} \mathcal{F}_p \,|\, s \text{ satisfies } 1a\&1b\}$*:*
   (a) $\forall p \in U, s(p) \in \mathcal{F}_p$
   (b) $\forall p \in U, \exists p \in V \subset U, t \in \mathcal{F}(V)$ *s.t.* $\forall q \in V, t_q = s(q)$.
2. *properties of* $\mathcal{F}^{\ddagger}$*:*
   (a) $\mathcal{F}^{\ddagger}$ ***is a sheaf;***
   (b) $\exists$ ***a natural morphism*** $\theta : \mathcal{F} \to \mathcal{F}^{\ddagger}$***, which satisfies the following universal property.***

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[d] & \mathcal{F}^{\ddagger} \arrow[dl, dashed, "\exists!"] \\ \mathcal{G}\,(sheaf) & \end{tikzcd}$$

   Moreover, the pair $(\mathcal{F}^{\ddagger}, \theta)$ ***is unqiue in this sense.***
   (c) $\mathcal{F}_p = \mathcal{F}^{\ddagger}_p, \forall p \in X$.

Definition C.8 (kernel, image & cokernel)**.** *Let X be a topo. space, $\mathcal{F}, \mathcal{G}$ be 2 sheaves on X, $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism.*

(1) $\ker\varphi(U) := \ker(\varphi(U))$*;*

(2) $\mathrm{p.im}\,\varphi(U) := \mathrm{im}(\varphi(U))$*;*

(3) $\mathrm{p.coker}\,\varphi(U) := \mathrm{p.coker}(\varphi(U))$

In general, $\mathrm{p.im}\,\varphi, \mathrm{p.coker}\,\varphi$ *fail to be sheaves, so we define*

(1) $\mathrm{im}\,\varphi = (\mathrm{p.im}\,\varphi)^{\ddagger}$*;*

(2) $\mathrm{coker}\,\varphi = (\mathrm{p.coker}\,\varphi)^{\ddagger}$*.*

Definition C.9 (injective & surjective morphisms, exact sequences)**.** *Let X be a topo. space,*

(1) a sequence $\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \to \cdots$ of sheaves on X is called exact if $\mathrm{im}\,\varphi^{i-1} = \ker\varphi^{i}, \forall i$*;*

(2) $\varphi : \mathcal{F} \to \mathcal{G}$ is called injective if $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact;

(3) $\varphi : \mathcal{F} \to \mathcal{G}$ is called surjective if $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ is exact.

Proposition C.10 (exactness on the stalk level)**.** *Let X be a topo. space, a sequence $\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \to \cdots$ of sheaves on X is exact $\iff$ $\cdots \to \mathcal{F}^{i-1}_p \xrightarrow{\varphi^{i-1}_p} \mathcal{F}^{i}_p \xrightarrow{\varphi^{i}_p} \mathcal{F}^{i+1}_p \to \cdots$ is exact $\forall p \in X$.*

Proposition C.11 (left-exactness of restriction)**.** *Let X be a topo. space, $U \overset{open}{\subset} X$. Then the functor $\Gamma(U, \star)$ is left exact, i.e. if $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ is an exact sequence of sheaves on X, then $0 \to \mathcal{F}'(U) \xrightarrow{\varphi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U)$ is an exact sequence in* **Ab***.*

Remark C.12. *The problem occurs since* $(\mathrm{im}\,\varphi)(U) \neq \mathrm{im}(\varphi(U)) = (\mathrm{p.im}\,\varphi)(U)$ *in general. Roughly speaking, the left-exactness comes from (3) of the following proposition, since* $\ker(\psi(U)) = (\ker\psi)(U) = (\mathrm{im}\,\varphi)(U) \cong \mathcal{F}'(U) \cong (\mathrm{p.im}\,\varphi)(U) = \mathrm{im}(\varphi(U))$*.*

Proposition C.13 (on injectivity)**.** *Let X be a topo. space, $\mathcal{F}, \mathcal{G}$ be 2 presheaves on X, $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism.*

(1) If $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective $\forall U \overset{open}{\subset} (X)$, then the induced morphism $\varphi^{\ddagger} : \mathcal{F}^{\ddagger} \to \mathcal{G}^{\ddagger}$ is injective;

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\theta_1"'] \arrow[d, "\varphi"'] & \mathcal{F}^{\ddagger} \arrow[ddl, dashed, "\varphi^{\ddagger}"] \\
\mathcal{G} \arrow[d, "\theta_2"] & \\
\mathcal{G}^{\ddagger} &
\end{tikzcd}$$

(2) As a result of (1), if φ is a morphism of sheaves, then $\operatorname{im}\varphi$ is naturally a subsheaf of $\mathcal{G}$. Moreover, $\operatorname{im}\varphi \cong \mathcal{F}/\ker\varphi$.

$$\mathcal{F}(U) \longrightarrow \text{p.im}\,\varphi(U) \hookrightarrow \mathcal{G}(U)$$

$$\begin{tikzcd}
\mathcal{F} \arrow[d] \arrow[dr, dashed, two heads] & \\
\text{p.im}\,\varphi \arrow[r] \arrow[d] & \operatorname{im}\varphi \arrow[dl, dashed, hook] \\
\mathcal{G} &
\end{tikzcd}$$

(3) As a result of (2), if $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact, then $\mathcal{F} \cong \operatorname{im}\varphi$.

Definition C.14 (direct image & inverse image)**.** *Let $f : X \to Y$ be a conti. map of topo. spaces, $\mathcal{F}, \mathcal{G}$ are sheaves on X, Y resp.,*

(1) $(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$, this gives the direct image $f_*\mathcal{F}$ on Y;

(2) $(f^{-1}\mathcal{G})(U) := \varinjlim \mathcal{G}(V)$, where the colimit is taken over $\{V \overset{open}{\subset} Y \mid f(U) \subset V\}$, this gives the inverse image $f^{-1}\mathcal{G}$ on X.

Remark C.15.

(1) Calculating $f_*\mathcal{F}$ is always a crucial problem in algebraic geometry;

(2) $f^{-1}\mathcal{G}$ is difficult to define, but easy to use.

Proposition C.16 ($f^{-1}(-)$& $f_*(\star)$ are adjoint)**.** *Let $f : X \to Y$ be a conti. map of topo. spaces, $\mathcal{F}, \mathcal{G}$ are sheaves on X, Y resp.,*

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

More precisely, we have 2 natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ & $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

Let's examine some examples.

Example C.17 (sheaves)**.** *Let X be a topo. space, A be an ab. grp..*

(1) (constant sheaf) Equip A with the disc. topo.. Define $\mathcal{A}(U) = \{f : U \xrightarrow{\text{conti}} A\}, \forall U \overset{open}{\subset} X$, then $\mathcal{A}(U) \cong A$ if U is connected.

(2) (skyscraper sheaf) For $p \in X$, define $i_p(A)(U) = \begin{cases} A, \textit{if } p \in U \\ 0, \textit{otherwise} \end{cases}$.

Note that $(i_p(A))_q = \begin{cases} A, \textit{if } q \in \overline{\{p\}} \\ 0, \textit{otherwise} \end{cases}$. Also, let $\mathcal{A}$ be the const. sheaf on $\{p\}$, $j : \{p\} \to X$, then $i_p(A) = j_*\mathcal{A}$.

Definition C.18 (flasque sheaves)**.** *Let X be a topo. space, $\mathcal{F}$ be a sheaf on X. $\mathcal{F}$ is called flasque if $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective $\forall V \subset U \in \mathrm{Open}(X)$.*

Proposition C.19 (properties of flasque sheaves)**.**

(1) Let X be a topo. space, $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ be an exact sequence of sheaves on X

(a) if $\mathcal{F}'$ is flasque, then $\Gamma(U, \star)$ is exact, i.e. $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is an exact sequence in **Ab***;*

(b) if $\mathcal{F}'$&$\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

(2) Let $f : X \to Y$ be a conti. map of topo. spaces, $\mathcal{F}$ be a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.

(3) Let X be a topo. space, $\mathcal{F}$ be a sheaf on X. Define $\mathcal{F}^{\dagger}(U) = \{s : U \to \cup_{p \in U} \mathcal{F}_p \,|\, \forall p \in U, s(p) \in \mathcal{F}_p\}$. Then

(a) $\mathcal{F}^{\dagger}$ is a flasque sheaf;

(b) $\exists$ a natural injective morphism $\mathcal{F} \to \mathcal{F}^{\dagger}$.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^{\dagger} \arrow[dl, dashed, "\exists!"] \\ \mathcal{G}\,\textit{(fls. sh.)} & \end{tikzcd}$$

0-extension of sheaves.

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