Topics in differential geometry: Reading report

1 Introduction

The main material of this project is [MR91]. The idea of that paper is to adopt the method in [HK78] and give a new proof of the Alexandrov type theorem for the r-th mean curvature. Moreover, this approach can also be used for hypersurfaces in hyperbolic space and upper semi-sphere, after some modifications. The main work of the report besides typing is filling in some ommitted details of [MR91], especially the proof of the spherical case. I sincerely thank professor for in-depth teaching and those well-arranged notes. It is very lucky for me to take this course.

2 Preliminaries

Mean curvature

Let $x : M^n \to R^{n+1}(c)$ be an immersed compact orientable hypersurface, $k_1, \cdots, k_n$ the principal curvatures.

Definition 1 (r-th mean curvature)**.** *The r-th mean curvature H_r is defined by*

$$P_n(t) = (1 + tk_1) \cdots (1 + tk_n) = 1 + \binom{n}{1} H_1 t + \cdots + \binom{n}{n} H_n t^n. \quad (1)$$

For example, H_1 is the mean curvature, H_2, up to a constant, is the scalar curvature, and H_n is the Gauss curvature.

Here is an important lemma from the inequalities in [Går59].

Lemma 2. *Suppose k_i are all positive at some point in M.*

(1) If $H_r > 0$ everywhere on M, then so is for H_k, $1 \leqslant k \leqslant r-1$.

(2) We have

$$H_k^{\frac{k-1}{k}} \leqslant H_{k-1}, \quad H_k^{\frac{1}{k}} \leqslant H_1. \quad (2)$$

Moreover, for $k \geqslant 2$, the equality holds only at umbilical points.

The existence of convex point

Lemma 3. *Let $x : M^n \to R^{n+1}(c)$ be an immersed compact orientable hypersurface,* *(1) For $c = 0$, there is a point in M, where all the principal curvature are positive.*

(2) For $c = 1$, suppose $\operatorname{im} x \subset S^{n+1}_+$, there is a point in M, where all the principal curvature are positive.

(3) For $c = -1$, there is a point in M, where all the principal curvature are greater than 1.

Remark 4. *For example,*

- *if $c = 0$, since M is compact, we can take a sphere tangent to M;*
- *if $c = 1$, take the point where the height $\langle x, a\rangle$ attains maximum;*
- *if $c = -1$, take the point where distance of H^{n+1} attains maximum.*

An integration formula

Let $x : M^n \to R^{n+1}(c)$ be an embedded compact hypersurface, N the inner unit normal vector field, Ω the compact domain with $\partial\Omega = M$.

Lemma 5 ([Cha95])**.** *Suppose the volume element of $\mathbb{R}^{n+1}$ has the expression*

$$\mathrm{dvol} = \exp_{x(p)}(tN(p)) = F(p,t)\,\mathrm{d}t\,\mathrm{d}A,$$

then we have an integration formula

$$\int_\Omega f\,\mathrm{dvol} = \int_M \int_0^{c(p)} f(\exp_{x(p)}(tN(p)))F(p,t)\,\mathrm{d}t\,\mathrm{d}A \tag{3}$$

where c is the cut function of M.

3 The Euclidean case

Let $x : M^n \to \mathbb{R}^{n+1}$ be an immersed compact orientable hypersurface, N the inner unit normal vector field. By direct calculation,

$$\begin{aligned}\Delta\langle x, x\rangle &= 2(D\langle x, e_i\rangle)_i \\ &= 2\sum_i \delta_{ij}(\langle e_i, e_j\rangle + \langle x, e_\alpha\rangle h^\alpha_{ij}) = 2n(1 + H\langle x, N\rangle).\end{aligned}$$

Using divergence theorem,

$$\int_M (1 + H_1\langle x, N\rangle)\,\mathrm{d}A = 0. \tag{4}$$

Lemma 6 (Minkowski formulae)**.** *Let $x : M^n \to \mathbb{R}^{n+1}$ be an immersed compact orientable hypersurface, N the inner unit normal vector field, then for $1 \leqslant r \leqslant n$, we have*

$$\int_M (H_{r-1} + H_r\langle x, N\rangle)\,\mathrm{d}A = 0. \tag{5}$$

Proof. (From [Hsi56]) For small number t, consider hypersurface

$$x_t(p) = \exp_{x(p)}(-tN(p)) = x(p) - tN(p). \tag{6}$$

Since t is small, N is also a unit normal vector field for x_t, and the principal directions are given by

$$x_{t,*}e_i = (1 + tk_i)e_i, 1 \leqslant i \leqslant n. \tag{7}$$

where e_i are the principal directions for x. Thus $(1 + tk_i)k_i(t) = k_i$. For the area element,

$$\mathrm{d}A_t = (1 + tk_1)\cdots(1 + tk_n)\,\mathrm{d}A = P_n(t)\,\mathrm{d}A. \tag{8}$$

For the mean curvature,

$$H_1(t) = \frac{1}{n}\sum\frac{k_i}{1 + tk_i} = \frac{P_n'(t)}{nP_n(t)}. \tag{9}$$

So we have by (4), (8), (9)

$$\begin{aligned}
0 &= \int_M n(1 + H_1(t)\langle x, N\rangle)\,\mathrm{d}A_t \\
&= \int_M (nP_n(t) + P_n'(t)\langle x - tN, N\rangle)\,\mathrm{d}A \\
&= \sum_i \int_M n\binom{n}{i}H_i t^i + \binom{n}{i} iH_i t^{i-1}(\langle x, N\rangle - t)\,\mathrm{d}A.
\end{aligned} \tag{10}$$

Regarding both sides as polynomials of t, we can solve

$$\int_M (H_{r-1} + H_r\langle x, N\rangle)\,\mathrm{d}A = 0 \tag{11}$$

for $1 \leqslant r \leqslant n$. □

Theorem 7 (Heintze-Karcher inequality [HK78, MR91])**.** *Let $x : M^n \to \mathbb{R}^{n+1}$ be an embedded compact hypersurface. If $H_1 > 0$ everywhere on M, then we have*

$$\int_M \frac{1}{H_1}\,\mathrm{d}A \geqslant (n + 1)\mathrm{vol}(\Omega) \tag{12}$$

where Ω is the compact domain with $\partial\Omega = M$. Moreover, the equality holds if and only if M^n is a round sphere.

Proof. Recall that $x_t = \exp_{x(p)}(tN(p)) = x(p) + tN(p)$ here, we have

$$\mathrm{d}V(x+tN) = (1-tk_1)\cdots(1-tk_n)\,\mathrm{d}t\,\mathrm{d}A. \tag{13}$$

Using (3) for $f(x) \equiv 1$,

$$\mathrm{vol}(\Omega) = \int_M \int_0^{c(p)} (1-tk_1)\cdots(1-tk_n)\,\mathrm{d}t\,\mathrm{d}A. \tag{14}$$

Note that $c(p) \leqslant \frac{1}{k_{\max}} \leqslant \frac{1}{H_1(p)}$ since the normal geodesic is well-defined before reaching the focal point. And as an algebraic inequality,

$$(1-tk_1)\cdots(1-tk_n) \leqslant (1-tH_1)^n \tag{15}$$

Then from (14),

$$\mathrm{vol}(\Omega) \leqslant \int_M \int_0^{\frac{1}{H_1}} (1-tH_1)^n\,\mathrm{d}t\,\mathrm{d}A = \frac{1}{n+1}\int_M \frac{1}{H_1}\,\mathrm{d}A. \tag{16}$$

The equality holds if (15) holds, which means M is totally umbilical. □

Theorem 8 ([MR91])**.** *Let $x : M^n \to \mathbb{R}^{n+1}$ be an embedded compact hypersurface. If H_r is constant for some $1 \leqslant r \leqslant n$, then M is a round sphere.*

Proof. From Lemma 3, there is a convex point in M, thus H_r is a positive constant. Let Ω be the compact domain with $\partial\Omega = M$. Using Lemma 2, we have $H_r^{\frac{1}{r}} \leqslant H_1, H_{r-1} \geqslant H_r^{\frac{r-1}{r}}$. Together with Lemma 6, we get

$$\begin{aligned} 0 &= \int_M (H_{r-1} + H_r\langle x, N\rangle)\,\mathrm{d}A \\ &\geqslant \int_M (H_r^{\frac{r-1}{r}} + H_r\langle x, N\rangle)\,\mathrm{d}A \\ &= H_r^{\frac{r-1}{r}} \int_M (1 + H_r^{\frac{1}{r}}\langle x, N\rangle)\,\mathrm{d}A. \end{aligned} \tag{17}$$

Recall that from divergence theorem,

$$(n+1)\mathrm{vol}(\Omega) + \int_M \langle x, N\rangle\,\mathrm{d}A = 0. \tag{18}$$

So

$$0 \geqslant \mathrm{area}(M) - (n+1)H_r^{\frac{1}{r}}\mathrm{vol}(\Omega). \tag{19}$$

Using Theorem 7,

$$(n+1)H_r^{\frac{1}{r}}\mathrm{vol}(\Omega) \leqslant \mathrm{area}(M). \tag{20}$$

Thus the equality in (20) holds, and hence M is totally umbilical by the rigidity of Theorem 7. □

4 The hyperbolic case

Let $\mathbb{R}_1^{n+2}$ be the real vector space $\mathbb{R}^{n+2}$ endowed with the Lozrentzain metric

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_{n+1}y_{n+1}.$$

The hyperbolic space $R^{n+1}(-1)$ can be regarded as $H^{n+1} = \{x \in R_1^{n+2} \,|\, |x|^2 = -1, x_0 \geqslant 1\}$, with the induced positive-definite metric. Then an immersed compact orientable hypersurface $x : M \to H^{n+1}$ can be viewed as $x : M \to R_1^{n+2}$ with $|x|^2 = -1, x_0 \geqslant 1$. Let N be the inner unit normal vector field, $a \in R_1^{n+2}$. By direct calculation,

$$\begin{aligned}\Delta\langle x, a\rangle &= (D\langle e_i, a\rangle)_i \\ &= \sum_i \delta_{ij}(\langle e_i, a\rangle + \langle e_\alpha, a\rangle h_{ij}^\alpha) = n(\langle x, a\rangle + H_1\langle N, a\rangle).\end{aligned}$$

Using divergence theorem,

$$\int_M (\langle x, a\rangle + H_1\langle N, a\rangle)\,\mathrm{d}A = 0. \tag{21}$$

Lemma 9 ([MR91]). *Let $x : M^n \to H^{n+1}$ be an immersed compact orientable hypersurface, N the inner unit normal vector field, then for $1 \leqslant r \leqslant n$ and arbitrary $a \in \mathbb{R}_1^{n+2}$, we have*

$$\int_M (H_{r-1}\langle x, a\rangle + H_r\langle N, a\rangle)\,\mathrm{d}A = 0. \tag{22}$$

Proof. For small number t, consider hypersurface

$$x_t(p) = \exp_{x(p)}(-tN(p)) = x(p)\cosh t - N(p)\sinh t. \tag{23}$$

By solving Jacobi field equation, the unit normal vector field is given by $N_t = -x\sinh t + N\cosh t$, and the principal directions are given by

$$x_{t,*}e_i = (\cosh t + k_i \sinh t)e_i, 1 \leqslant i \leqslant n \tag{24}$$

where e_i are the principal directions for x. Thus $(\cosh t + k_i \sinh t)k_i(t) = \sinh t + k_i \cosh t$. For the area element,

$$\begin{aligned}\mathrm{d}A_t &= (\cosh t + k_1 \sinh t)\cdots(\cosh t + k_n \sinh t)\,\mathrm{d}A \\ &= \cosh^n t \cdot P_n(\tanh t)\,\mathrm{d}A.\end{aligned} \tag{25}$$

For the mean curvature,

$$\begin{aligned}H_1(t) &= \frac{1}{n}\sum \frac{\sinh t + k_i \cosh t}{\cosh t + k_i \sinh t} = \frac{1}{n}\sum \frac{\tanh t + k_i}{1 + k_i \tanh t} \\ &= \frac{n\cosh t \sinh t \cdot P_n(\tanh t) + P_n'(\tanh t)}{n\cosh^2 t \cdot P_n(\tanh t)}.\end{aligned} \tag{26}$$

So we have by (21), (25), (26),

$$
\begin{aligned}
0 &= \int_M n(\langle x_t, a\rangle + H_1(t)\langle N_t, a\rangle)\,\mathrm{d}A_t \\
&= \int_M n\cosh^2 t\cdot P_n(\tanh t)\langle x\cosh t - N\sinh t, a\rangle\,\mathrm{d}A \\
&+ \int_M (n\cosh t\sinh t\cdot P_n(\tanh t) + P_n'(\tanh t))\langle -x\sinh t + N\cosh t, a\rangle\,\mathrm{d}A \\
&= \int_M (nP_n(\tanh t)\langle x, a\rangle + P_n'(\tanh t)\langle -x\tanh t + N, a\rangle)\,\mathrm{d}A.
\end{aligned}
\tag{27}
$$

Regarding both sides as polynomials of $\tanh t$, we can solve

$$
\int_M (H_{r-1}\langle x, a\rangle + H_r\langle N, a\rangle)\,\mathrm{d}A = 0 \tag{28}
$$

for $1 \leqslant r \leqslant n$. □

Definition 10. *We define a positive function* $\rho_n : (1,\infty) \to (0,\infty)$ *with parameter* $n \in \mathbb{N}$ *by*

$$
\rho_n(u) = \int_0^{\coth^{-1} u} (\cosh t - u\sinh t)^n \cosh t\,\mathrm{d}t. \tag{29}
$$

We have the following Heintze-Karcher type inequality.

Theorem 11 ([MR91])**.** *Let* $x : M^n \to H^{n+1}$ *be an embedded campact hypersurface. If* $H_r > 1$ *everywhere on* M*, then we have*

$$
\int_M (\langle x, a\rangle + H_r^{\frac{1}{r}}\langle N, a\rangle)\rho_n(H_r^{\frac{1}{r}})\,\mathrm{d}A \geqslant 0 \tag{30}
$$

for $a \in \mathbb{R}^{n+2}_1$ *with* $|a|^2 = -1$*. Moreover, the equality holds if and only if* M *is a geodesic sphere.*

Proof. Recall that $x_t = \exp_{x(p)}(tN(p)) = x(p)\cosh t + N(p)\sinh t$ here,

$$
\mathrm{d}V(x\cosh t + N\sinh t) = (\cosh t - k_1\sinh t)\cdots(\cosh t - k_n\sinh t)\,\mathrm{d}t\,\mathrm{d}A. \tag{31}
$$

Note $\overline{\Delta}\langle x, a\rangle = (n+1)\langle x, a\rangle$, and $\overline{\nabla}\langle x, a\rangle = a$, from divergence theorem,

$$
(n+1)\int_\Omega \langle x, a\rangle\,\mathrm{d}V + \int_M \langle N, a\rangle\,\mathrm{d}A = 0. \tag{32}
$$

Using (3) for $f(x) = (n+1)\langle x, a\rangle$,

$$
-\int_M \langle N, a\rangle\,\mathrm{d}A = (n+1)\int_M\int_0^{c(p)} \langle x_t, a\rangle \prod_i (\cosh t - k_i\sinh t)\,\mathrm{d}t\,\mathrm{d}A. \tag{33}
$$

From Lemma 3, there is a point in M, where all the principal curvatures are greater than 1. Using Lemma 2, we have $1 < H_r^{\frac{1}{r}} \leqslant H_1$. Note that $c(p) \leqslant \coth^{-1} k_{\max} \leqslant \coth^{-1} H_1(p) \leqslant \coth^{-1} H_r^{\frac{1}{r}}(p)$. And as an algebraic inequality,

$$\prod_i (\cosh t - k_i \sinh t) \leqslant (\cosh t - H_1 \sinh t)^n \leqslant (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n. \quad (34)$$

Then from (33),

$$\begin{aligned} & -\frac{1}{n+1}\int_M \langle N, a\rangle \,\mathrm{d}A \\ \leqslant & \int_M \int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n \langle x_t, a\rangle \,\mathrm{d}t\,\mathrm{d}A. \end{aligned} \quad (35)$$

On the other hand, by taking $w = \cosh t - H_r^{\frac{1}{r}} \sinh t$, we can show

$$\begin{aligned} & (n+1)\int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n (\sinh t - H_r^{\frac{1}{r}} \cosh t)\,\mathrm{d}t \\ = & \int_1^0 \mathrm{d}w^{n+1} = -1. \end{aligned} \quad (36)$$

So multiplying by $\langle N, a\rangle$ and integrating over M, we have

$$\begin{aligned} & -\frac{1}{n+1}\int_M \langle N, a\rangle \,\mathrm{d}A \\ = & \int_M \langle N, a\rangle \int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n (\sinh t - H_r^{\frac{1}{r}} \cosh t)\,\mathrm{d}t\,\mathrm{d}A. \end{aligned} \quad (37)$$

Putting together (35) and (37),

$$
\begin{aligned}
0 \leqslant & \int_M \int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n \langle x_t, a\rangle \,\mathrm{d}t\,\mathrm{d}A \\
& - \int_M \langle N, a\rangle \int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n (\sinh t - H_r^{\frac{1}{r}} \cosh t)\,\mathrm{d}t\,\mathrm{d}A \\
& = \int_M \int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n \\
& \quad \cdot (\langle x \cosh t + N \sinh t, a\rangle - (\sinh t - H_r^{\frac{1}{r}} \cosh t)\langle N, a\rangle)\,\mathrm{d}t\,\mathrm{d}A \\
& = \int_M (\langle x, a\rangle + H_r^{\frac{1}{r}} \langle N, a\rangle) \int_0^{\coth^{-1} H_r^{\frac{1}{r}}} (\cosh t - H_r^{\frac{1}{r}} \sinh t)^n \,\mathrm{d}t\,\mathrm{d}A \\
& = \int_M (\langle x, a\rangle + H_r^{\frac{1}{r}} \langle N, a\rangle)\rho_n(H_r^{\frac{1}{r}})\,\mathrm{d}A.
\end{aligned}
\tag{38}
$$

The equality holds if (34) holds, which means M is totally umbilical. $\square$

Theorem 12 ([MR91])**.** *Let $x : M^n \to H^{n+1}$ be an embedded campact hypersurface. If H_r is constant for some $1 \leqslant r \leqslant n$, then M is a geodesic hypersphere.*

Proof. From Lemma 3, there is a point in M, where all the principal curvatures are greater than 1, thus H_r is a constant greater than 1. Then $\rho_n(H_r^{\frac{1}{r}})$ is a positive constant. Using Lemma 2, $H_{r-1} \geqslant H_r^{\frac{r-1}{r}}$. Together with Lemma 9, we get

$$
\begin{aligned}
0 &= \int_M (H_{r-1}\langle x, a\rangle + H_r\langle N, a\rangle)\,\mathrm{d}A \\
&\geqslant \int_M (H_r^{\frac{r-1}{r}} \langle x, a\rangle + H_r\langle N, a\rangle)\,\mathrm{d}A \\
&= H_r^{\frac{r-1}{r}} \int_M (\langle x, a\rangle + H_r^{\frac{1}{r}} \langle N, a\rangle)\,\mathrm{d}A.
\end{aligned}
\tag{39}
$$

Using Theorem 11,

$$
0 \leqslant \int_M (\langle x, a\rangle + H_r^{\frac{1}{r}} \langle N, a\rangle)\,\mathrm{d}A. \tag{40}
$$

Thus the equality in (40) holds, and hence M is totally umbilical by the rigidity of Theorem 11. $\square$

5 The spherical case

Let $x : M^n \to S^{n+1}$ be an immersed compact orientable hypersurface, N the inner unit normal vector field, $a \in \mathbb{R}^{n+2}$. By direct calculation,

$$\begin{aligned}\Delta\langle x, a\rangle &= (D\langle e_i, a\rangle)_i \\ &= \sum_i \delta_{ij}(\langle e_i, a\rangle + \langle e_\alpha, a\rangle h_{ij}^\alpha) = n(\langle x, a\rangle - H_1\langle N, a\rangle).\end{aligned}$$

Using divergence theorem,

$$\int_M (\langle x, a\rangle - H_1\langle N, a\rangle)\,\mathrm{d}A = 0. \tag{41}$$

Lemma 13 ([MR91, Biv83])**.** *Let $x : M^n \to S^{n+1}$ be an immersed compact orientable hypersurface, N the inner unit normal vector field, then for $1 \leqslant r \leqslant n$ and arbitrary $a \in \mathbb{R}^{n+2}$, we have*

$$\int_M (H_{r-1}\langle x, a\rangle - H_r\langle N, a\rangle)\,\mathrm{d}A = 0. \tag{42}$$

Proof. For small number t, consider hypersurface

$$x_t(p) = \exp_{x(p)}(-tN(p)) = x(p)\cos t - N(p)\sin t. \tag{43}$$

By solving Jacobi field equation, the unit normal vector field is given by $N_t = x\sin t + N\cos t$, and the principal directions are given by

$$x_{t,*}e_i = (\cos t - k_i \sin t)e_i, 1 \leqslant i \leqslant n \tag{44}$$

where e_i are the principal directions of x. Thus $(\cos t - k_i\sin t)k_i(t) = \sin t - k_i\cos t$. For the area element,

$$\begin{aligned}\mathrm{d}A_t &= (\cos t - k_1\sin t)\cdots(\cos t - k_n\sin t)\,\mathrm{d}A \\ &= \cos^n t P_n(-\tan t)\,\mathrm{d}A.\end{aligned} \tag{45}$$

For the mean curvature,

$$\begin{aligned}H_1(t) &= \frac{1}{n}\sum\frac{\sin t - k_i\cos t}{\cos t - k_i\sin t} = \frac{1}{n}\sum\frac{\tan t - k_i}{1 - k_i\tan t} \\ &= -\frac{n\cos t\sin t\cdot P_n(-\tan t) + P_n'(-\tan t)}{n\cos^2 t\cdot P_n(-\tan t)}.\end{aligned} \tag{46}$$

So we have by (41), (45), (46),

$$\begin{aligned}
0 &= \int_M n(\langle x_t, a\rangle + H_1\langle N_t, a\rangle)\,\mathrm{d}A_t \\
&= \int_M n\cos^2 t \cdot P_n(-\tan t)\langle x\cos t - N\sin t, a\rangle\,\mathrm{d}A \\
&+ \int_M (-n\cos t\sin t \cdot P_n(-\tan t) - P_n'(-\tan t))\langle x\sin t + N\cos t\rangle\,\mathrm{d}A \\
&= \int_M (nP_n(-\tan t)\langle x, a\rangle - P_n'(-\tan t)\langle x\tan t + N, a\rangle)\,\mathrm{d}A.
\end{aligned} \tag{47}$$

Regarding both sides as polynomials of $-\tan t$, we can solve

$$\int_M (H_{r-1}\langle x, a\rangle - H_r\langle N, a\rangle)\,\mathrm{d}A = 0 \tag{48}$$

for $1 \leqslant r \leqslant n$. □

Definition 14. *We define a positive functin $\tau_n : (0, \infty) \to (0, \infty)$ with parameter $n \in \mathbb{N}$ by*

$$\tau_n(u) = \int_0^{\cot^{-1} u} (\cos t - u\sin t)^n \cos t\,\mathrm{d}t. \tag{49}$$

We have the following Heintze-Karcher type inequality.

Theorem 15 ([MR91])**.** *Let $x : M^n \to S^{n+1}_+$ be an embedded campact hypersurface lying in the upper semi-sphere. If $H_r > 0$ everywhere on M, then we have*

$$\int_M (\langle x, a\rangle - H_r^{\frac{1}{r}}\langle N, a\rangle)\tau_n(H_r^{\frac{1}{r}})\,\mathrm{d}A \geqslant 0 \tag{50}$$

where a is the north pole of S^{n+1}. Moreover, the equality holds if and only if M is umbilical.

Proof. Recall that $x_t = \exp_{x(p)}(tN(p)) = x(p)\cos t + N(p)\sin t$ here,

$$\mathrm{d}V(x\cos t + N\sin t) = (\cos t - k_1\sin t)\cdots(\cos t - k_n\sin t)\,\mathrm{d}t\,\mathrm{d}A. \tag{51}$$

Note $\overline{\Delta}\langle x, a\rangle = -(n+1)\langle x, a\rangle$, and $\overline{\nabla}\langle x, a\rangle = a$, from divergence theorem,

$$(n+1)\int_\Omega \langle x, a\rangle\,\mathrm{d}V = \int_M \langle N, a\rangle\,\mathrm{d}A. \tag{52}$$

Using (3) for $f(x) = (n+1)\langle x, a\rangle$,

$$\int_M \langle N, a\rangle\,\mathrm{d}A = (n+1)\int_M\int_0^{c(p)} \langle x_t, a\rangle \prod_i (\cos t - k_i\sin t)\,\mathrm{d}t\,\mathrm{d}A. \tag{53}$$

From Lemma 3, there is a convex point in M. Using Lemma 2, we have $0 < H_r^{\frac{1}{r}} \leqslant H_1$. Note that $c(p) \leqslant \cot^{-1} k_{\max} \leqslant \cot^{-1} H_1(p) \leqslant \cot^{-1} H_r^{\frac{1}{r}}(p)$. And as an algebraic inequality

$$\prod_i (\cos t - k_i \sin t) \leqslant (\cos t - H_1 \sin t)^n \leqslant (\cos t - H_r^{\frac{1}{r}} \sin t)^n. \tag{54}$$

Then from (53),

$$\frac{1}{n+1} \int_M \langle N, a \rangle \,\mathrm{d}A \leqslant \int_M \int_0^{\cot^{-1} H_r^{\frac{1}{r}}} (\cos t - H_r^{\frac{1}{r}} \sin t)^n \langle x_t, a \rangle \,\mathrm{d}t \,\mathrm{d}A. \tag{55}$$

On the other hand, by taking $w = \cos t - H_r^{\frac{1}{r}} \sin t$, we can show

$$\begin{aligned} &(n+1) \int_0^{\cot^{-1} H_r^{\frac{1}{r}}} (\cos t - H_r^{\frac{1}{r}} \sin t)^n (-\sin t - H_r^{\frac{1}{r}} \cos t) \,\mathrm{d}t \\ &= \int_1^0 \mathrm{d}w^{n+1} = -1. \end{aligned} \tag{56}$$

So multiplying by $\langle N, a \rangle$ and integrating over M, we have

$$\begin{aligned} &-\frac{1}{n+1} \int_M \langle N, a \rangle \,\mathrm{d}A \\ &= \int_M \langle N, a \rangle \int_0^{\cot^{-1} H_r^{\frac{1}{r}}} (\cos t - H_r^{\frac{1}{r}} \sin t)^n (-\sin t - H_r^{\frac{1}{r}} \cos t) \,\mathrm{d}t \,\mathrm{d}A \end{aligned} \tag{57}$$

Putting together (55) and (57),

$$\begin{aligned} 0 \leqslant & \int_M \int_0^{\cot^{-1} H_r^{\frac{1}{r}}} (\cos t - H_r^{\frac{1}{r}} \sin t)^n \langle x_t, a \rangle \,\mathrm{d}t \,\mathrm{d}A \\ &+ \int_M \langle N, a \rangle \int_0^{\cot^{-1} H_r^{\frac{1}{r}}} (\cos t - H_r^{\frac{1}{r}} \sin t)^n (-\sin t - H_r^{\frac{1}{r}} \cos t) \mathrm{d}t \,\mathrm{d}A \\ &= \int_M \int_0^{\cot^{-1} H_r^{\frac{1}{r}}} (\cos t - H_r^{\frac{1}{r}} \sin t)^n \\ &\quad \cdot (\langle x \cos t + N \sin t, a \rangle + (-\sin t - H_r^{\frac{1}{r}} \cos t) \langle N, a \rangle) \,\mathrm{d}t \,\mathrm{d}A \\ &= \int_M (\langle x, a \rangle - H_r^{\frac{1}{r}} \langle N, a \rangle) \tau_n(H_r^{\frac{1}{r}}) \,\mathrm{d}A. \end{aligned} \tag{58}$$

The equality holds if (54) holds, which means M is totally umbilical. $\square$

Theorem 16 ([MR91])**.** *Let $x : M^n \to S^{n+1}_+$ be an embedded campact hypersurface. If H_r is constant for some $1 \leqslant r \leqslant n$, then M is a geodesic hypersphere.*

Proof. From Lemma 3, there is a convex point in M, thus H_r is a positive constant. Then $\tau_n(H_r^{\frac{1}{r}})$ is a positive constant. Using Lemma 2, $H_{r-1} \geqslant H_r^{\frac{r-1}{r}}$. Together with Lemma 13, we get

$$\begin{aligned} 0 &= \int_M (H_{r-1}\langle x, a\rangle - H_r\langle N, a\rangle)\,\mathrm{d}A \\ &\geqslant \int_M (H_r^{\frac{r-1}{r}}\langle x, a\rangle - H_r\langle N, a\rangle)\,\mathrm{d}A \\ &= H_r^{\frac{r-1}{r}} \int_M (\langle x, a\rangle - H_r^{\frac{1}{r}}\langle N, a\rangle)\,\mathrm{d}A. \end{aligned} \tag{59}$$

Using Theorem 15,

$$0 \leqslant \int_M (\langle x, a\rangle - H_r^{\frac{1}{r}}\langle N, a\rangle)\,\mathrm{d}A. \tag{60}$$

Thus the equality (60) holds, and hence M is totally umbilical by the rigidity of Theorem 15. □

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