# About CAT($-1$) surfaces

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# The main result

The goal is to present the following theorem, the main reference is [SJ24].

- Let $\Sigma$ be a non-simply connected closed surface, we denote by $\mathcal{A}_\Sigma$ the collection of CAT$(-1)$ surface $M$, which is homeomorphic to $\Sigma$.

## Theorem

*Let $\Sigma$ be a non-simply connected closed surface, $U \subset \mathcal{A}_\Sigma$ be an open subset, then there exists a hyperbolic surface $M \in U$, s.t.*

$$\operatorname{sys}(M) = \sup_{M' \in U} \operatorname{sys}(M')$$

## Corollary

*Let $\Sigma$ be a non-simply connected closed surface, then the maximal systole of the CAT$(-1)$ metrics on $\Sigma$ is attained by a hyperbolic metric.*

# Definitions and results

Throughout, we consider only closed surfaces.

**Definition (conical singularity)**

Let $(M, g)$ be a closed surface with metric $g$. We say $g$ has a *conical singularity* of order $\beta$ at $p$, if $g$ can be written as

$$g = e^{2u(z)}|z|^{2\beta}|\mathrm{d}z|^2, \beta > -1 \in \mathbb{R}$$

around $p$, where $u : \mathbb{C} \to \mathbb{R}$ is continuous. The *total angle* is defined as $2\pi(\beta + 1)$.

- Every closed *Alexandrov surface* can be approximated by a piecewise hyperbolic surface with conical singularities[SJ24]. We consider mainly this kind of surfaces.
- ***CAT($-1$) surfaces*** are Alexandrov surfaces of *Alexandrov curvature* at most $-1$.
- For closed piecewise hyperbolic surface with conical singularities,

  $$\text{it is a CAT}(-1)\text{ surface} \iff \text{all the total angles are not less than } 2\pi.$$
- Fix $N \geqslant 0$ and $s > 0$, let $\Sigma$ be a non-simply connected closed surface. Then the space of piecewise hyperbolic surfaces $M \cong \Sigma$ of CAT($-1$) surface with **at most** $N$ **conical singularities** and **systoles at least** $s$ is *compact*.

# Systolic decomposition and kite excision I

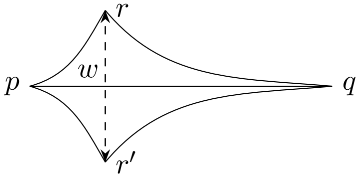
- The intersecting systolic loops of $M$ meet at **one** or **two points**, or **along a line**.
- The *systolic decomposition* of $M$ is the collection of the connected components of the complementary set in $M$ of the systolic loops.
  - the *vertices* are the intersection points and the endpoints of intersecting lines;
  - the *edges* are the separated geodesic arcs by the vertices.
- The number of domains, edges and vertices in the systolic decomposition of $M$ have an upper bound which depends only on the topology of $M$.

The vertices(both conical singularities) $p, q$ are called the *main vertices* of the kite $K$, and the length $w$ of diagonal $[r, r']$ is called the *width* of $K$.

- We say $K$ is *exact* at $p$, if

*(1)* $p$ is a small singularity, i.e. $\theta_p \in [2\pi, 3\pi)$;

*(2)* $\measuredangle rpr' = \theta_p - 2\pi < \pi, \measuredangle rqr' \leqslant \min\{\theta_q - 2\pi, \pi\}$.




$M_w := (M \backslash K_w)/\sim$, $\sim$ means identifying $[p, r], [q, r]$ with $[p, r'], [q, r']$ respectively.

# Systolic decomposition and kite excision II

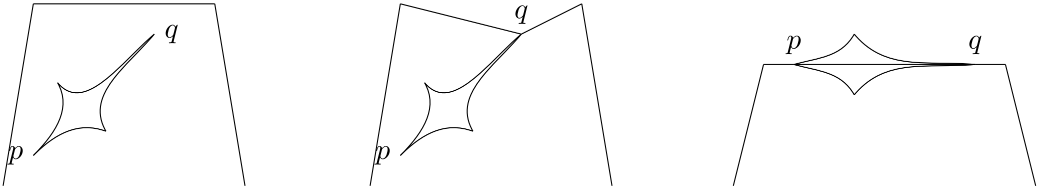
**Proposition**

*Let $K_w$ be an exact kite, then*

*(1) $M_w$ is a CAT$(-1)$ surface, with the same number of conical singularities as $M$;*

*(2) $M_w$ converges to $M$ w.r.t. the bilipschitz distance as $w$ tends to zero.*

We now consider systolic decomposition with kite excision. There are three typical cases of position for a kite with main diagonal $[p, q]$ within a systolic domain $D$.




(a) Case1 (b) Case2 (c) Case3

# Systolic decomposition and kite excision III

## Proposition

*Let $M$ be a closed piecewise hyperbolic CAT($-1$) surfaces. Consider a kite $K_w \subset M$ exact at $p$ and satisfying one of the three cases. Then for sufficiently small width $w$, we have*

$$\operatorname{sys}(M_w) \geqslant \operatorname{sys}(M).$$

## Theorem (piecewise hyperbolic)

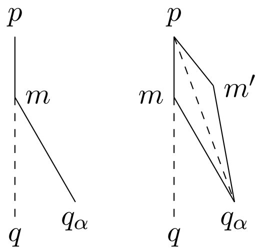
*Let $\Sigma$ be a non-simply connected closed surface. Let $U \subset \mathcal{A}_\Sigma$ be an open set. Then there exists a const $N$ which depends only on the topology of $\Sigma$ and a piecewise hyperbolic surface $M \in U$ with at most $N$ conical singularities, s.t.*

$$\operatorname{sys}(M) = \sup_{M' \in U} \operatorname{sys}(M').$$

# Kite insertion and deformation I

Let $m \in M$ be a conical singularity, $(p, q)$ be a geodesic arc passing through $m$, s.t. $m$ is the only conical singularity on $[p, q]$.

- For $0 < \alpha < \frac{1}{2}(\theta_m - 2\pi)$, take $q_\alpha \in M$ with $|mq_\alpha| = |mq|$ and $\measuredangle qmq_\alpha = \alpha$. Denote by $M'$ the surface by cutting along $[p, m], [m, q_\alpha]$.
- Let $K_\alpha = \bar{p}\bar{m}\bar{q}_\alpha\bar{m}'$ be a kite in $\mathbb{H}^2$, with $|\bar{p}\bar{m}| = |pm|, |\bar{q}_\alpha\bar{m}| = |q_\alpha m|$, $|\bar{p}\bar{m}'| = |pm'|, |\bar{q}_\alpha\bar{m}'| = |q_\alpha m'|$ and $\measuredangle\bar{p}\bar{m}\bar{q}_\alpha = \measuredangle\bar{p}\bar{m}\bar{q}_\alpha = \pi - \alpha$. Attach $K_\alpha$ to $M'$ along the corresponding arcs, to get $M_\alpha$.




## Proposition (approximation)

*(1) $M_\alpha$ is also a CAT$(-1)$ surface, with more conical singularities than $M$;*

*(2) The surface $M_\alpha$ converges to $M$ w.r.t. the bilipschitz distance as $\alpha$ tends to zero.*

Now we consider the deformation of systolic loop with kite insertion.

# Kite insertion and deformation II

Let $M$ be a closed CAT$(-1)$ surface, $C$ be a loop free homotopy class.

*(1)* $L_M(C) := \min\{\ell(S) \,|\, S \in C\}$. *(2)* $\#_s(M) := \#\{$systolic loops of $M\}$.

Let $m \in M$ be a conical singularity. we can choose a geodesic arc $(p, q)$ passing through $m$, s.t. **at least one systolic loop of $M$ transversely intersects** $(p, q)$ .

## Proposition (loop length)

*Let $C$ be the free homotopy class of a systolic loop $\gamma$ of $M$, for $\alpha > 0$ small enough*

*(1) if $\gamma$ does not transversely intersect $[p, q]$, then $L_{M_\alpha}(C) = L_M(C) = \mathrm{sys}(M)$;*

*(2) if $\gamma$ transversely intersects $[p, q]$, then $L_{M_\alpha}(C) > L_M(C) = \mathrm{sys}(M)$.*

## Theorem (deformation via kite insertion)

*Let (M) be a closed piecewise hyperbolic CAT$(-1)$ surface, with a conical singularity $m$. Then $M$ can be deformed into a closed piecewise hyperbolic CAT$(-1)$ surface $M_\alpha$, s.t. for $\alpha > 0$ small enough, one of the following statements holds*

*(1)* $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$; *(2)* $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ *and* $\#_s(M_\alpha) < \#_s(M)$.

# Proof of the main result

Now we can show the proof.

**Proof.**

According to Theorem(piecewise hyperbolic), the supremum of the systole on $U$ is attained by some piecewise hyperbolic surfaces in $U$ with conical singularities.
Among these surfaces, take $M$ to be of minimal $\#_s(M)$.
**Goal**: $M$ has no conical singularity, thus $M$ is a hyperbolic surface.
Suppose by contradiction that $m$ is a conical singularity. By Theorem(deformation via kite insertion), we can deform $M$ by a kite insertion into some $M_\alpha \in U$, for $\alpha > 0$ small enough (approximation) with one of the following properties

*(1)* $\mathrm{sys}(M_\alpha) > \mathrm{sys}(M)$;

*(2)* $\mathrm{sys}(M_\alpha) = \mathrm{sys}(M)$ and $\#_s(M_\alpha) < \#_s(M)$.

But $M$ attains the maximal of the systoles, so (1) is impossible.
Since $M$ has a minimal $\#_s(M)$, (2) is also impossible. □

Thank you for your attention!