Lie groups: a cheat sheet

Consider only matrix Lie groups, and real dimensions.

(1) $\lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m = e^{X+Y}$, for $X, Y \in M_n(\mathbb{C})$.

(2) Another definition: $GL(n,\mathbb{C}) = \{A \in GL(2n,\mathbb{R}) \mid AJ_0 = J_0A\}$.

(3) $U(n) = Sp(2n,\mathbb{R}) \cap GL(n,\mathbb{C}) \cap O(2n)$.

(4) $SL(2,\mathbb{R}) \cong S^1 \times \mathbb{R}^2$.

(5) $Sp(n) := Sp(2n,\mathbb{C}) \cap U(2n)$.

(6) $Spin(n) :=$ universal covering of $SO(n)$.

(7) $Spin(5) \cong Sp(2)$.

(8) Lie algebras and dimensions (Petersen P.366)
Method: $\mathfrak{g} = \{X \in M_n(\cdot) \mid e^{tX} \in G, \forall t \in \mathbb{R}\}$.

- $\mathfrak{sl}(n,\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{tr}X = 0\}$, $\dim = n^2 - 1$;
- $\mathfrak{sl}(n,\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}X = 0\}$, $\dim = 2n^2 - 2$;
- $\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}$, $\dim = \frac{n(n-1)}{2}$;
- $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^H = -X\}$, $\dim = n^2$;
- $\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X^H = -X, \mathrm{tr}X = 0\}$, $\dim = n^2 - 1$;
- $\mathfrak{sp}(2n,k) = \{X \in M_{2n}(k) \mid M^TJ + JM = 0\}$, $\dim = 2n^2 + n$

(9) Low-dim examples (Hatcher P.294)
$SO(3) \cong \mathbb{RP}^3, Sp(1) \cong SU(2) \cong S^3, U(1) \cong S^1, SO(4) \cong S^3 \times SO(3)$.

(10) Homogeneous spaces

- $S^n \cong O(n)/O(n-1) \cong SO(n)/SO(n-1)$;
- $S^{2n+1} \cong U(n+1)/U(n) \cong SU(n+1)/SU(n)$;
- $\mathbb{CP}^n \cong SU(n+1)/U(n)$
- $S^2 \cong S^3/S^1 = SU(2)/U(1)$

(11) Fundamental groups
Fibration: $F \hookrightarrow X \to B$, $\pi_{k+1}(B) \to \pi_k(F) \to \pi_k(X) \to \pi_k(B)$.

- $\pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2\mathbb{Z}, & n \geqslant 3 \end{cases}$
- $\pi_1(U(n)) = \mathbb{Z}, n \geqslant 1$;

- $\pi_1(SU(n)) = 0, n \geqslant 2$;
- $\pi_1(GL(n,\mathbb{C})) = \pi_1(U(n)) = \mathbb{Z}, n \geqslant 1$;(deformation retract, see "matrix exponential" of Hall's book)
- $\pi_1(SL(n,\mathbb{C})) = \pi_1(SU(n)) = 0, n \geqslant 2$;
- $\pi_1(SL(n,\mathbb{R})) = \pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2\mathbb{Z}, & n \geqslant 3 \end{cases}$;
- $\pi_1(Sp(n)) = 0, n \geqslant 1$;

(12) Compactness

- non-compact groups: $GL(n,\mathbb{k}), SL(n,\mathbb{k}), Sp(2n,\mathbb{k})$;
- compact groups: $O(n), SO(n), U(n), SU(n), Sp(n)$.