Symplectic geometry: Final report
2024 autumn

1. Basic definitions

Definition 1.1 (symplectic form & symplectic manifold)**.** *Let M be a C^∞ manifold,*

(1) $\omega \in \Omega^2(M)$ is called a symplectic form if

- *(a) ω is closed;*
- *(b) ω is non-degenerate;*

(2) a pair (M, ω) is called a symplectic manifold.

Remark 1.2. *(1)a means $\mathrm{d}\omega = 0$, and (1)b means the matrix of ω w.r.t. some basis of $T^*M \otimes T^*M$ is invertible, or equivalently, $\iota_X \omega \neq 0 \in \Omega^1(M)$ for $X \neq 0$.*

There are 2 quick observations:

(1) the existence of such ω implies that $\dim M = 2n$ is even(since the matrix of M is anti-symmetric & invertible);

(2) $\omega^n \neq 0 \in \Omega^{2n}(M)$(by taking a basis of $\Omega^2(M)$), and is thus a non-zero top form.

Example 1.3 (2 canonical models for symplectic manifolds)**.**

(1) Take $M = \mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, then $\omega_{std} = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i$ is a symplectic form(since the matrix w.r.t. such basis is $J_0 = \begin{pmatrix} & I \\ -I & \end{pmatrix} \in GL(2n, \mathbb{R})$, which is invertible);

(2) Take $M = T^*X$, for $\dim X = n$, with coordinates $(q_1, \cdots, q_n)$ on X and $(p_1, \cdots, p_n)$ on $T^*_q X$. Then $\omega_{can} = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$ is a symplectic form. (ω_{can} is well-defined since $\lambda_{can} = \sum_i p_i \wedge \mathrm{d}q_i$ is globally defined and $\omega_{can} = \mathrm{d}\lambda_{can}$).

Remark 1.4. *(1) The example (2) is related to classical mechanics, where q means position and p means momentum.*

(2) The Darboux's theorem shows that a symplectic manifold is locally diffeomorphic to $(\mathbb{R}^{2n}, \omega_{std})$. And the Weinstein's neighborhood theorem shows for a compact Lagrangian submanifold L, there exists

some neighborhood which is diffeomorphic to (T^*L, ω_{can}).
Thus we say: ***Symplectic*** *geometry does not have any local deformations, which is in sharp contrast to* ***Riemannian*** *geometry. (These results can be proved with Moser's trick, which we ommit)*

(3) There are ***contact*** *analogous of these theorems. A contact manifold(roughly speaking, the boundary of a symplectic manifold)is locally diffeomorphic to $(\mathbb{R}^{2n+1}, \xi = \ker(\mathrm{d}z - \sum y_i \mathrm{d}x_i))$, and for a Legendrian submanifold Λ, there exists some neighborhood which is diffeomorphic to $(J^1\Lambda = T^*\Lambda \times \mathbb{R}_z, \xi = \ker(\mathrm{d}z - \lambda_{can}))$. There are abundant materials on Legendrian submanifold, especially Legendrian knots.*

2. Symplectic linear algebra

Definition 2.1. $Sp(2n) := \{A \in GL(2n, \mathbb{R}) \mid A^T J A = J\}$.

Lemma 2.2. *$U(n) = Sp(2n) \cap GL(n, \mathbb{C}) \cap O(2n)$, and the intersection of any 2 terms is $U(n)$.*

($A \in GL(n, \mathbb{C}) \iff AJ_0 = J_0 A, A \in Sp(2n) \iff A^T J_0 A = J_0, A \in O(2n) \iff A^T A = I$, so 2 of them implies the other. By calcultion, we can see $A \in U(n) \iff A \in Sp(2n) \& A \in O(2n)$).

Lemma 2.3. *$\exists$a homotopy equivalence between $Sp(2n) \& U(n)$.*

(The map $f : [0,1] \times Sp(2n) \to Sp(2n), (t, A) \to A(A^T A)^{-t/2}$ gives a such equivalence. Now that we have $f_1 : Sp(2n) \to U(n)$).

Lemma 2.4. *$U(n)$ is a maximal compact subgroup of $Sp(2n)$.*

(The idea is: for $A \in Sp(2n) \backslash U(n), P = (A^T A) \in Sp(2n)$ and has an eigenvalue $\lambda \neq 1$, thus the iteration $\{P^i\}_{i=1}^{\infty}$ is divergent).

Lemma 2.5. *$\pi_1(U(n)) = \mathbb{Z}$.*

(Using the fibration $SU(n) \to U(n) \xrightarrow{\det_{\mathbb{C}}} S^1$, we get an exact sequence $\pi_1(SU(n) \to \pi_1(U(n)) \to \pi_1(S^1) \to \pi_0(SU(n))$, since $\pi_1(SU(n)) = 0$, we have $\pi_1(U(n)) = \mathbb{Z}$).

We can thus define the Maslov index.

Definition 2.6 (Maslov index)**.** *The maslov index is a functin $\mu : \pi_1(Sp(2n)) \to \mathbb{Z}$, i.e. $\mu : \Gamma : S^1 \to Sp(2n) \to \mu(\Gamma) \in \mathbb{Z}$, s.t.*

(1) if $\Gamma_1 \xrightarrow{homotopy} \Gamma_2$, then $\mu(\Gamma_1) = \mu(\Gamma_2)$;

(2) $\mu(\Gamma_1 \cdot \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2)$*;*

(3) for $n = n' + n''$*, consider* $Sp(2n') \oplus Sp(2n'') = Sp(2n)$*, then* $\mu(\Gamma_1 \oplus \Gamma_2) = \mu(\Gamma_1) + \mu(\Gamma_2)$*;*

(4) for $\Gamma = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$, $\mu(\Gamma) = 1$.

Proposition 2.7. *There exists uniquely a such function.*

(μ can be defined from $S^1 \xrightarrow{\Gamma} Sp(2n) \xrightarrow{f_1} U(n) \xrightarrow{\det_{\mathbb{C}}} S^1$, by $\mu = \deg(\det_{\mathbb{C}} \circ f_1 \circ \Gamma)$. It's well defined and satisfies all the conditions. The uniqueness follows from *(4)*).

Remark 2.8. *(1) We can define Maslov index for* $LGr(n)$ *(the Lagrangian Grassmannian, i.e. the space of Lagrangians in* $\mathbb{R}^{2n}$*) through a similar procedure;*

(2) The Maslov index is used in the construction of Floer homology.

3. Symplectic manifolds and its relation to Kälher manifolds

Definition 3.1 (exact symplectic manifold)**.** *A symplectic manifold* (M, ω) *is called exact if* $\omega = \mathrm{d}\lambda$ *for some* $\lambda \in \Omega^1(M)$*.*

Example 3.2. *(1) The examples in 1.3 are both exact.*

(2) For some closed surface $\Sigma \subset \mathbb{R}^3$*, take* $\omega_p(v_1, v_2) = \langle N_p, v_1 \times v_2 \rangle$*, then* (Σ, ω) *is a non-exact symplectic manifold. This follows from the following proposition.*

Proposition 3.3. *If* (M^{2n}, ω) *is closed ,then* M *can not be exact.*

(Using Poincaré duality, $H^{2n}(M) \neq 0$, thus ω^n can not be exact). The example *(2)* is critical, since we have the following result.

Proposition 3.4. S^{2n} *is not symplectic for* $n > 1$*.*

(Note that for closed (M^{2n}, ω), ω^k is not exact for $k = 1, \cdots, n$).

Definition 3.5 (Kähler manifold)**.** *A Kähler manifold is a manifold with a symplectic form* ω *and an almost complex structure* J*, s.t.*

(1) $\omega(Ju, Jv) = \omega(u, v)$*;*

(2) $\omega(v, Jv) > 0$ *for* $v \neq 0$*.*

Remark 3.6. *(1) The conditions in 3.5 can be replaced by the bilinear form $g(u,v)=\omega(u,Jv)$ is symmetric and positive definite.*

(2) There are some direct results from the definition.

(a) ω is a $(1,1)$-form and $\partial\omega=0, \bar{\partial}\omega=0$;

(b) write $\omega=\frac{i}{2}\sum h_{jk}\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_k$, then $(h_{jk})(p)$ is an Hermitian and positive definite(thus invertible) matrix at any point $p\in M$.

There are some examples of complex yet not symplectic manifolds, symplectic yet not complex manifolds, and symplectic & complex yet not Kähler manifolds. These gaps are of their own interests.

4. Morse homology and Floer homology

The Morse theory is about the critical point of smooth functions. A classical example is the flat torus with the height function.

Definition 4.1 (Morse function & index, pseudo-gradient vector field)**.** *Let $f:M\to\mathbb{R}$ be a smooth function,*

(1) f is a Morse function if all its critical points are non-degenerate.

(2) By a lemma of Morse, we can write $f(x_1,\cdots,x_n)=f(p)-(x_1)^2-\cdots-(x_i)^2+(x_{i+1})^2+\cdots+(x_n)^2$ for some chart $(x_1,\cdots,x_n)$ around a critical point p, the number i is invariant under coordinate change. So we define the Morse index $\operatorname{ind}(p)=i$.

(3) A vector field X is pseudo-gradient for some Morse function f if

(a) $\mathrm{d}f(X)\leqslant 0$ on M, and the equality holds for critical points;

(b) $X=\operatorname{grad}f$ in the charts described as in (2).

Given a pseudo-gradient vector field, we can define the stable & unstable submanifolds for a critical point p, as we shall do for dynamic systems.

Definition 4.2. $M(p,q):=\{x\in M\mid \lim\limits_{t\to\infty}\varphi^t(x)=q, \lim\limits_{t\to-\infty}\varphi^t(x)=q\}$, *where φ^t is the flow of X, and p,q are critical points.*

We assume Smale's condition, i.e. $W^u(p)\pitchfork W^s(q)$ for any 2 critical points p,q, which implies that $\dim(W^u(p)\cap W^s(q))=\operatorname{ind}p-\operatorname{ind}q$. So directly, we have $\dim M(p,q)=\operatorname{ind}p-\operatorname{ind}q$. Let $\widetilde{M}(p,q)=M(p,q)/\mathbb{R}$.

Thus we can define the Morse complex and ($\mathbb{Z}_2$-)homology:

$$CM.(f) := \operatorname{span}_{\mathbb{Z}_2}\{\operatorname{crit}(f)\}$$
$$\partial(p) := \sum_{\substack{q\in\operatorname{crit}(f),\\ \operatorname{ind}p-\operatorname{ind}q=1}} |\widetilde{M}(p,q)|q$$
$$HM_k(f) := H_k(CM.(f),\partial)$$

Remark 4.3. *(1)* $\partial^2=0$, *since*

$$\partial^2 p = \sum_q |\widetilde{M}(p,q)|\partial q = \sum_q\sum_r |\widetilde{M}(p,q)||\widetilde{M}(q,r)|r$$

where $\operatorname{ind}p = \operatorname{ind}q+1, \operatorname{ind}q = \operatorname{ind}r+1$. *In this case* $\dim\widetilde{M}(p,r)=1$ *and* $\partial\widetilde{M}(p,r)$ *consists of a even number of points. Thus* $\partial^2 p = \sum_{\substack{r\in\operatorname{crit}(f),\\ \operatorname{ind}p=\operatorname{ind}r+2}} |\partial\widetilde{M}(p,r)|r = 0 \mod 2$.

(2) Similar to mapping degree, if we take an orientation for $M(p,q)$, *then we can elevate this complex to be over* $\mathbb{Z}$.

Roughly speaking, Floer homology is infinite analogous of Morse homology, where the Morse index is replaced by Maslov index.

$$CF.(L_0,L_1) := \operatorname{span}_{\mathbb{Z}_2}\{\varphi^1(L_0)\pitchfork L_1\}$$
$$d(p) := \sum_q |\widetilde{M}(p,q)|q$$
$$HF_k(f) := H_k(CF.(L_0,L_1),d)$$

The function $M(p,q)$ here is obtained from the Floer's equations.

Example 4.4 (easy example, cylinder)**.** *Let* $M = T^*S^1, \omega=\omega_{can} = \mathrm{d}\theta\wedge\mathrm{d}\xi$, *take* $L_0 = L_1 = T^*_0S^1$, *and* $H(\theta,\xi) = \frac{\xi^2}{2}$. *Then the flow of* H *is* $\varphi^t(\theta,\xi) = (\theta+t\xi,\xi)$, *so* $\varphi^1(L_0)\pitchfork L_1 = \{(0,n)\,|\,n\in\mathbb{Z}\}$. *As a result,* $CF(L_0,L_1) = \mathbb{Z}_2[x,\frac{1}{x}]$*(consider the degree). In this case there's no complicate intersection relaton, in fact* $d=0$, *and* $HF(L_0,L_1) = \mathbb{Z}_2[x,\frac{1}{x}]$.

5. A_∞ algebra

Definition 5.1 (A_∞ algebra)**.** *An* A_∞ *algebra over a field* k *is a graded vector space* $V=\oplus_{p\in\mathbb{Z}}V^p$ *with homogeneous* k*-linear maps:* $\mu^n : V^{\otimes n}\to$

$V^{2-n}, n \geqslant 1$, *s.t.*

$$\sum_{\substack{r+s+t=n \\ r+1+t=d}} (-1)^{r+st} \mu^d (\mathrm{id}^{\otimes r} \otimes \mu^s \otimes \mathrm{id}^{\otimes t}) = 0$$

Remark 5.2. *(1) If* $\mathrm{char} k = 2$, *we can ignore the sign* $(-1)^{r+st}$*;*

(2) Take $n = 1$, $(\mu^1)^2 = 0$, *so* (V, μ^1) *becomes a complex;*

(3) Take $n = 2$, $\mu^1 \circ \mu^2 = \mu^2 \circ (\mathrm{id} \otimes \mu^1) + \mu^2 \circ (\mu^1 \otimes \mathrm{id})$, *so* μ^2 *can be understood as a multiplication, with* μ^1 *as a derivation;*

(4) Take $n = 3$, $\mu^2 \circ (\mathrm{id} \otimes \mu^2) - \mu^2 \circ (\mu^2 \otimes \mathrm{id}) = \mu^1 \circ \mu^3 + \mu^3 \circ (\mathrm{id}^{\otimes 2} \otimes \mu^1 + \mathrm{id} \otimes \mu^1 \otimes \mathrm{id} + \mu^1 \otimes \mathrm{id}^{\otimes 2})$, *so* μ^3 *compensates for the associativity;*

(5) If $V^p = 0$ *for* $p \neq 0$, *then it reduces to an associate algebra.*

For an exact symplectic manifold (M, ω), the Fukaya category is an A_∞-category, where the objects are exact Lagrangian submanifolds, and the morphisms are given by Floer complex via intersection.

6. Legendrian knots

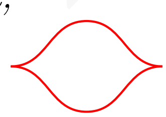
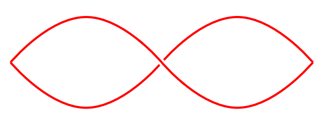
Take a Legendrian $\Lambda \subset (\mathbb{R}^3_{xyz}, \alpha = \mathrm{d}z - y\mathrm{d}x)$, we can take two different projections

(1) front projection: $\pi_F : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xz}$;

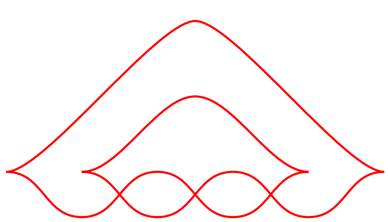
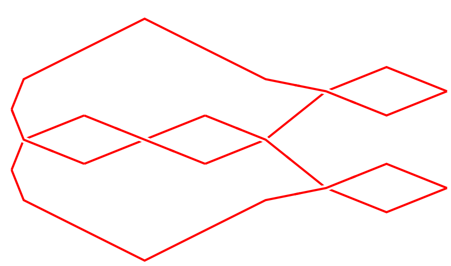
(2) Lagrangian projection: $\pi_L : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$.

Example 6.1 (2 kinds of projections)**.**

(1) unknot;

π_F :  π_L : 

(2) right trefoil.

π_F :  π_L : 

Reidemeister's theorem tells us, the isotopy of Legendrian knots are given by Reidemeister's moves of knots.

There are several invariants for the Legendrian knots, like Thurston-Bennequin number, rotiaion number and Chekanov's dg algebra$(\mathcal{A}(\Lambda), \partial)$. $\mathcal{A}(\Lambda)$ is generated by $\langle a_1, \cdots, a_n, t^{\pm 1}\rangle$, where a_i represent the Reeb chords for the knot, t represent the base point we choose, and t^{-1} is the formal inverse of t. The grading is obtained from rotation number, and it varies as the crossings  ,  changes.

7. Microlocal sheaves

Definition 7.1 (presheaves & sheaves)**.** *Let X be a topological space,*

(1) a presheaf $\mathcal{F}$ on X is a contravariant functor from $\mathrm{Open}(X) \to \mathbf{Ab}$*;*

(2) a sheaf $\mathcal{F}$ on X is a presheaf, s.t. if $U, V_i \in \mathrm{Open}(X), U = \cup_i V_i$*,*

   (a) (factorizing)if $s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = 0, \forall i$, then $s = 0$;

   (b) (glueing)if $s_i \in \mathcal{F}(V_i)$, $s_i = s_j$ on $V_i \cap V_j, \forall i, j$, then $\exists s \in \mathcal{F}(U)$, $s|_{V_i} = s_i, \forall i$.

The definition of micro-support is not easy.

Definition 7.2 (micro-support)**.** *For a sheaf $\mathcal{F}$ on X,*

(1) we say $\mathcal{F}$ propagates at $(x,p) \in T^*X$, if $\forall \varphi \in C^1(X), \varphi(x) = 0, \mathrm{d}\varphi(x) = p$, we have $\varinjlim_{x \in U} H^j(U; \mathcal{F}) \cong \varinjlim_{x \in U} H^j(U \cap \{\varphi < 0\}; \mathcal{F}), \forall j$;

(2) $\mu\mathrm{supp}(\mathcal{F})$ is the closure of all $(x,p) \in T^*X$, at where $\mathcal{F}$ does not propagate;

(3) for a Legendrian Λ, $\mathrm{Sh}_\Lambda(X) := \{\mathcal{F} \in \mathrm{Sh}(X) \mid \mu\mathrm{supp}(\mathcal{F}) \cap \partial_\infty T^*X \subset \Lambda\}$.

Remark 7.3. *(1) $\mu\mathrm{supp}(\mathcal{F})$ is conical for p;*

(2) $(x,p) \in \mu\mathrm{supp}(\mathcal{F})$ means that (x,p) is singular for $\mathcal{F}$ in some sense;

*(3) $\mathrm{Sh}_\Lambda(X)$ is a **Legendrian isotopy invariant**, by Guillermou-Kashiwara-Schapira.*

Example 7.4. *Let* $X = \mathbb{R}, \mathcal{F} = \mathbb{C}_{[0,\infty)}$ *on* X. *Obviously,* $(x, 0) \in \mu\mathrm{supp}(\mathcal{F})$ *only for* $x \geqslant 0$. *Take* $f(x) = x$, *then* $f(0) = 0, \mathrm{d}f_0 = \mathrm{d}x_0$, *then*

$$
\begin{aligned}
R\mathit{\Gamma}(\{f < \delta\}; \mathcal{F}) &= \varinjlim \mathit{\Gamma}((-\varepsilon, \delta), \mathcal{F}) = \mathbb{C}, \\
R\mathit{\Gamma}(\{f < -\delta\}; \mathcal{F}) &= \varinjlim \mathit{\Gamma}((-\varepsilon, -\delta), \mathcal{F}) = 0,
\end{aligned}
$$

where $R(\mathit{\Gamma})$ *is the right derived functor of* $\mathit{\Gamma}$. *So* $(0, \mathrm{d}x) \in \mu\mathrm{supp}(\mathcal{F})$, *similarly,* $(0, -\mathrm{d}x) \notin \mu\mathrm{supp}(\mathcal{F})$. *Thus* $\mu\mathrm{supp}(\mathcal{F})$ *looks like* $\llcorner$. *In the same way, we have*

$$
\mathbb{C}_{[0,\infty)} \to \llcorner \qquad \mathbb{C}_{(0,\infty)} \to \ulcorner \qquad \mathbb{C}_{(-\infty,0]} \to \urcorner \qquad \mathbb{C}_{(-\infty,0)} \to \lrcorner
$$

In general, we consider the constructible objects in the derived category of $\mathrm{Sh}(X)$. A theorem of Nadler-Zaslow gives an equivalence between such category of constructible objects and the derived Fukaya category, which builds the bridge between Lagrangian intersection theory and microlocal sheaf theory.