

Lie groups: a cheat sheet

We consider only examples of matrix Lie groups.

- (1) $\lim_{m \rightarrow \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} \right)^m = e^{X+Y}$, for $X, Y \in M_n(\mathbb{C})$.
- (2) Another definition: $GL(n, \mathbb{C}) = \{A \in GL(2n, \mathbb{R}) \mid AJ_0 = J_0A\}$.
- (3) $U(n) = Sp(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) \cap O(2n)$.
- (4) $SL(2, \mathbb{R}) \cong S^1 \times \mathbb{R}^2$.
- (5) $Sp(n) := Sp(2n, \mathbb{C}) \cap U(2n)$.
- (6) $Spin(n) :=$ universal covering of $SO(n)$.
- (7) $Spin(5) \cong Sp(2)$.
- (8) Lie algebras
Method: $\mathfrak{g} = \{X \in M_n(\cdot) \mid e^{tX} \in G, \forall t \in \mathbb{R}\}$.
 - $\mathfrak{sl}(n, \mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \text{tr} X = 0\}$, $\dim = n^2 - 1$;
 - $\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \text{tr} X = 0\}$, $\dim = 2n^2 - 2$;
 - $\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}$, $\dim = \frac{n(n-1)}{2}$;
 - $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^H = -X\}$, $\dim = n^2$;
 - $\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X^H = -X, \text{tr} X = 0\}$, $\dim = n^2 - 1$;
 - $\mathfrak{sp}(2n, k) = \{X \in M_{2n}(k) \mid M^T J + JM = 0\}$, $\dim = 2n^2 + n$
- (9) Low-dim examples: $SO(3) \cong \mathbb{RP}^3, Sp(1) \cong SU(2) \cong S^3, U(1) \cong S^1, SO(4) \cong S^3 \times SO(3)$ (Hatcher P.294).
- (10) Homogeneous spaces
 - $S^n \cong O(n)/O(n-1) \cong SO(n)/SO(n-1)$;
 - $S^{2n+1} \cong U(n+1)/U(n) \cong SU(n+1)/SU(n)$;
 - $\mathbb{CP}^n \cong SU(n+1)/U(n)$
 - $S^2 \cong S^3/S^1 = SU(2)/U(1)$
- (11) Fundamental groups
Method: For $F \hookrightarrow X \rightarrow B$, $\pi_2(B) \rightarrow \pi_1(F) \rightarrow \pi_1(X) \rightarrow \pi_1(B)$.
 - $\pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2\mathbb{Z}, & n \geq 3 \end{cases}$
 - $\pi_1(U(n)) = \mathbb{Z}, n \geq 1$;

- $\pi_1(SU(n)) = 0, n \geq 2$;
- $\pi_1(GL(n, \mathbb{C})) = \pi_1(U(n)) = \mathbb{Z}, n \geq 1$; (deformation retract, see “matrix exponential” of Hall’s book)
- $\pi_1(SL(n, \mathbb{C})) = \pi_1(SU(n)) = 0, n \geq 2$;
- $\pi_1(SL(n, \mathbb{R})) = \pi_1(SO(n)) = \begin{cases} \mathbb{Z}, & n = 2 \\ \mathbb{Z}/2\mathbb{Z}, & n \geq 3 \end{cases}$;
- $\pi_1(Sp(n)) = 0, n \geq 1$;

(12) Compactness

- non-compact groups: $GL(n, k), SL(n, k), Sp(2n, k)$;
- compact groups: $O(n), SO(n), U(n), SU(n), Sp(n)$.