14

OPEN PROBLEMS

In the final chapter of this book we discuss some open questions and conjectures that either have served as guiding lights or have emerged in the study of symplectic topology over the last quarter of a century. Since symplectic topology has grown into a vast area of research, it is impossible to be complete. Often, we mention only the most recent papers, since earlier relevant work can be discovered through their references. The choice of topics necessarily reflects our tastes and preferences: we have highlighted some of the open problems that seem to us to be both appealing in their own right and central to symplectic topology. Readers should be aware that this list will inevitably become out of date and hence at best can provide a snapshot of where the field is at the time of writing.

14.1 Symplectic structures

An in depth discussion of the existence and uniqueness problem for symplectic structures is contained in Section 13.1. Sections 13.2 and 13.4 explain what is known about these problems in various examples. Here we highlight some of the open questions that arise from this discussion. We begin with the fundamental existence problem.

Problem 1 (Existence of symplectic structures)
Is there an example of a closed oriented manifold M of dimension $2n \geq 6$, a nondegenerate 2-form ρ on M compatible with the orientation, and a cohomology class $a \in H^2(M;\mathbb{R})$ satisfying $a^n > 0$, such that ρ is not homotopic to a symplectic form representing the class a?

There are many such examples in dimension four (see Sections 13.3 and 13.4). However, the question is open in higher dimensions. In contrast, the uniqueness problem up to isotopy for cohomologous symplectic forms is completely open in dimension four, while counterexamples are known in higher dimensions (see Example 13.2.9). Recall the notation $\mathscr{S}_a$ for the space of symplectic forms representing the cohomology class $a \in H^2(M;\mathbb{R})$.

Problem 2 (Uniqueness in dimension four)
(a) *Is there a closed four-manifold M and a cohomology class $a \in H^2(M;\mathbb{R})$ such that $\mathscr{S}_a$ is disconnected?*
(b) *Is there a closed four-manifold M and a cohomology class $a \in H^2(M;\mathbb{R})$ such that $\mathscr{S}_a$ is nonempty and connected?*

Introduction to Symplectic Topology. Dusa McDuff, Dietmar Salamon. ©Oxford University Press 2017. Published 2017 by Oxford University Press.

Problem 3 (Projective plane uniqueness conjecture)
The space of symplectic forms on the complex projective plane representing a fixed cohomology class is contractible.

Problem 4 (HyperKähler surface uniqueness conjecture)
If M is a closed hyperKähler surface (i.e. a 4-torus or a K3-surface) and the cohomology class $a \in H^2(M;\mathbb{R})$ satisfies $a^2 > 0$, then $\mathscr{S}_a$ is connected.

The uniqueness conjecture for the four-torus is a longstanding open question. By Seiberg–Witten theory, every symplectic form on the four-torus or a K3-surface has first Chern class zero (see Example 13.4.7). In the case of a K3-surface it even follows that any two symplectic forms are homotopic as nondegenerate 2-forms. The analogous statement for two symplectic forms on the four-torus that induce the same orientation is an open problem. The **Donaldson geometric flow** as outlined in [149] is a conjectural approach to settle Problem 4. This geometric flow equation is valid, in principle, for all symplectic four-manifolds, and on $\mathbb{C}\mathrm{P}^2$ it has only one critical point [376]. This motivates the conjecture in Problem 3 which, if true, would imply that $\mathrm{Diff}_h(\mathbb{C}\mathrm{P}^2)$, the group of diffeomorphisms that act trivially on homology, retracts onto $\mathrm{PU}(3)$. However, one might expect that $\mathscr{S}_a$ is disconnected in some examples and in those cases something to go wrong with the analysis (such as failure of long time existence or lack of convergence).

Problem 5 (Donaldson four-six question)
Let (X,ω_X), (Y,ω_Y) be closed symplectic four-manifolds such that X and Y are homeomorphic. Are X and Y diffeomorphic if and only if the product manifolds

$$(X \times S^2, \omega_X \oplus \sigma), \qquad (Y \times S^2, \omega_Y \oplus \sigma)$$

are symplectically deformation equivalent?

Examples of nondeformation equivalent symplectic structures on the same symplectic four-manifold X are discussed in Example 13.4.6. In particular, the examples of Ivan Smith [592, 593] are distinguished by the divisibility properties of their first Chern classes and hence continue to be nondeformation equivalent on $X \times \mathbb{T}^2$. Hence the 2-sphere in Problem 5 cannot be replaced by the 2-torus.

Problem 5 is nontrivial in either direction. When X and Y *are not* diffeomorphic, but are homeomorphic so $X \times S^2$ and $Y \times S^2$ *are* diffeomorphic, the question suggests that the two symplectic structures on these six-manifolds should still *remember* the differences in the smooth structures on X and Y. At the time of writing the only known methods for distinguishing smooth structures on four-manifolds are the Donaldson invariants and the Seiberg–Witten invariants. By Taubes–Seiberg–Witten theory (see Section 13.3) the Seiberg–Witten invariants can also be interpreted as invariants of the symplectic structure, and it is plausible that they give rise to a method for distinguishing the symplectic structures on the products with the two-sphere. (See the related work of Herrera [308].) However, this observation does not get to the heart of the problem:

the Seiberg–Witten invariants may well not be strong enough to distinguish all pairs of smooth structures, and also we have very little understanding of six-dimensional symplectic manifolds and their symplectomorphisms to help us in going in the other direction. An example where X and Y *are not* diffeomorphic, $X \times S^2$ and $Y \times S^2$ *are* diffeomorphic, and the symplectic forms on $X \times S^2$ and $Y \times S^2$ can be distinguished by their Gromov–Witten invariants, was found by Ruan [546] (see Example 13.2.10). More examples along these lines were found by Ruan–Tian [550] and Ionel–Parker [346]. Also the examples of Vidussi [641, 642] furnish symplectic structures on homotopy K3's whose products with the two-sphere are not symplectically deformation equivalent. (See Example 13.2.11.)

Problem 6 (Donaldson almost complex structure question)
Let (M, ω) be a closed symplectic four-manifold and let J be an ω-tame almost complex structure on M. Does there exist a symplectic form that is compatible with J (and represents the same cohomology class as ω in the case $b^+ = 1$)?

This question has a positive answer for $M = \mathbb{C}\mathrm{P}^2$ by Taubes [618] and for $M = S^2 \times S^2$ by Li-Zhang [425] (see Remark 4.1.3). The next question is related to both to the work of Fintushel–Stern mentioned on page 547, and to Problem 5.

Problem 7 (Symplectic knot surgery)
Let X_K be obtained from a K3-surface X by surgery along a knot $K \subset S^3$.
(a) *If X_K admits a symplectic structure, does it follow that K is a fibred knot?*
(b) *If K and K' are fibred knots with the same Alexander polynomial, are $X_K \times S^2$ and $X_{K'} \times S^2$ symplectically deformation equivalent?*

The next question is related to work of Friedl–Vidussi, Ozsvath, Taubes.

Problem 8 (Symplectic mapping torus)
If M is a closed 3-manifold such that $M \times S^1$ admits a symplectic form, does it follow that M is a mapping torus?

Tian-Jun Li has extended the notion of the Kodaira dimension from Kähler surfaces to symplectic 4-manifolds (see page 548). In [420] he has formulated various conjectures about the behaviour of this invariant. Here is one of his conjectures related to Kähler surfaces of general type. It is motivated by an observation of Witten, which asserts that in this case the canonical class and its negative are the only basic classes (see Example 13.4.10).

Problem 9 (Tian-Jun Li's general type existence conjecture)
Let (M, ω, J) be a minimal Kähler surface of general type with first Chern class $c := c_1(\omega)$, so that

$$b^+ > 1, \qquad [\omega] \cdot c < 0, \qquad c \cdot c > 0.$$

Let $a \in H^2(M; \mathbb{R})$ be such that

$$a^2 > 0, \qquad a \cdot c < 0.$$

Then there exists a symplectic form on M in the class a.

The next question concerns the extent to which results known about symplectic forms on the blowup in four dimensions extend to higher dimensions.

Problem 10 (Blowup uniqueness)
Let $\pi_M : \widetilde{M} \to M$ be the one-point blowup of a symplectic $2n$-manifold (M, ω). Define $a := [\omega] \in H^2(M)$, $c := c_1(\omega)$, $\widetilde{a} := \pi_M^* a$, $\widetilde{c} := \pi_M^* c$, and let $e \in H^2(\widetilde{M})$ be the Poincaré dual of the class of the exceptional divisor. Fix a constant $r > 0$.
(a) *If $\widetilde{M}$ admits a symplectic form in the cohomology class $\widetilde{a} - \pi r^2 e$ and with first Chern class $\widetilde{c} - (n-1)e$, does it follow that there exists a symplectic embedding of the closed ball $B^{2n}(r)$ of radius r into M?*
(b) *Is every pair of blowup symplectic forms on $\widetilde{M}$ that arises from two normalized symplectic embeddings of the same ball into M isotopic?*

The answer to question (a) is positive for blowups of rational and ruled 4-manifolds by Example 13.4.5, while the answer to (b) is positive for all symplectic 4-manifolds with $b^+ = 1$ by Remark 7.1.28.

Problem 11 (Auroux uniqueness question)
Let (M_1, ω_1) and (M_2, ω_2) be closed symplectic four-manifolds such that the cohomology classes $[\omega_1]$ and $[\omega_2]$ have integral lifts. Suppose M_1 and M_2 have the same Euler characteristic, signature, and volume, and that $c_1(\omega_1) \cdot [\omega_1] = c_1(\omega_2) \cdot [\omega_2]$. Does there exist a collection of disjoint Lagrangian tori in M_1 such that Luttinger surgery along them gives rise to a manifold symplectomorphic to (M_2, ω_2)?

For Luttinger surgery see Example 3.4.16. A related result by Auroux [37] asserts that under the assumptions of Problem 11 there exist isomorphic Donaldson hypersurfaces $\Sigma_i \subset M_i$ (see Section 7.4) and a symplectic 4-manifold X with an embedded hypersurface $\Sigma \subset X$ diffeomorphic to Σ_i and with opposite normal bundle such that the fibre sums (see Section 7.2) $M_1 \#_\Sigma X$ and $M_2 \#_\Sigma X$ are symplectomorphic. Another related result by Auroux asserts that if (W_1, ω_1) and (W_2, ω_2) are 4-dimensional Weinstein manifolds (see Definition 7.4.5) with isomorphic contact boundaries and the same Euler characteristic and signature, then they are related by attaching Weinstein handles along the same Legendrian knots in $\partial W_1 \cong \partial W_2$ and subsequent deformation (see [41, Theorem 6.1]).

Problem 12 (Exotic projective plane)
Does there exist a closed symplectic four-manifold that is homeomorphic, but not diffeomorphic, to the complex projective plane, the one-point blowup of the complex projective plane, or the product of two 2-spheres?

The quest for exotic smooth structures on four-manifolds has a long history. For blowups of the complex projective plane it was shown by Donaldson [141], long before the advent of the Seiberg–Witten invariants, that there is an exotic smooth structure on the 9-point blowup of $\mathbb{CP}^2$ (that supports a symplectic form). This was gradually extended by several authors, including Stipsicz–Szabó [601] for the 6-point blowup, Park–Stipsicz–Szabó [524] and Fintushel–Stern [220] for the 5-point blowup, Baldridge–Kirk [47], Akhmedov–Park [15], Fintushel–Park–Stern [218], and Fintushel–Stern [224] for the 3-point blowup, and Akhmedov–Park [16] and Fintushel–Stern [223] for the 2-point blowup of the complex projective plane. The holy grail in this quest would be a positive answer to Problem 12 for the complex projective plane. This would imply a negative answer to part (b) of the next question.

Problem 13 (Miyaoka–Yau inequality)
(a) *Is there any simply connected closed symplectic four-manifold whose Euler characteristic and signature satisfy* $3\sigma > \chi$*?*
(b) *Is every simply connected closed symplectic four-manifold whose Euler characteristic and signature satisfy* $3\sigma = \chi$ *diffeomorphic to* $\mathbb{C}\mathrm{P}^2$*?*

For a brief discussion of the Miyaoka–Yau inequality and some known results, see page 547. A more general question is which Chern numbers c_1^2 and c_2 are realized by simply connected closed symplectic four-manifolds and what can be deduced about the symplectic manifolds from the Chern numbers. This is the so-called *symplectic geography problem* and is briefly discussed on page 547.

14.2 Symplectomorphisms

Fix a closed symplectic manifold (M, ω) and consider the homomorphism

$$\pi_0(\mathrm{Symp}(M, \omega)) \to \pi_0(\mathrm{Diff}(M)). \tag{14.2.1}$$

A fundamental problem in symplectic topology is to understand the kernel and image of this homomorphism. The kernel is trivial if and only if every symplectomorphism that is smoothly isotopic to the identity is symplectically isotopic to the identity. One can also strengthen the question and ask whether the group $\mathrm{Symp}_h(M, \omega)$, of symplectomorphisms that act trivially on homology, is connected.[44]

There are known examples where this group is connected, even though it is an open question whether or not $\mathrm{Diff}_h(M)$ is connected, such as $\mathbb{C}\mathrm{P}^2$, the one-point blowup of $\mathbb{C}\mathrm{P}^2$, $S^2 \times S^2$ (Example 10.4.2), and the monotone k-point blowups of the projective plane for $2 \leq k \leq 4$ (see Evans [210] and Example 10.4.3). A natural conjecture is that the homomorphism (14.2.1) is injective for all ruled surfaces. This is known under some assumptions (Example 13.4.3) but is an open problem in general.

Problem 14 (Symplectic isotopy conjecture for ruled surfaces)
A symplectomorphism of a ruled surface is smoothly isotopic to the identity if and only if it is symplectically isotopic to the identity.

Here is another specific instance of this problem (see Example 10.4.1).

[44] For simplicity, we will restrict this discussion to simply connected manifolds. If $\pi_1(M) \neq 0$, the problem should be refined; one could ask, for example, whether every symplectomorphism that acts trivially on homology and on $\pi_1(M)$ is symplectically isotopic to the identity, or whether every symplectomorphism that is homotopic to the identity is symplectically isotopic to the identity.

Problem 15 (Symplectic isotopy conjecture for tori)
A symplectomorphism of $\mathbb{T}^{2n}$ with a translation invariant symplectic form induces the identity on homology if and only if it is symplectically isotopic to the identity. Equivalently, every exact symplectomorphism of $\mathbb{T}^{2n}$ is Hamiltonian.

There are many examples of simply connected symplectic 4-manifolds where the homomorphism (14.2.1) is not injective. This includes the monotone k-point blowups of the projective plane for $5 \le k \le 8$ by the work of Seidel (Example 10.4.3). The related question of whether $\mathrm{Symp}_h(M, \omega)$ is connected can also have negative answers in higher dimensions, as shown by unpublished work of Michael Callahan from the early 1990s. He proved that the symplectomorphism of the 6-dimensional moduli space of flat SO(3)-connections over a genus-2 Riemann surface that is induced by a Dehn twist along a separating loop is not symplectically isotopic to the identity. Some other explicit examples were found by Dimitroglou Rizell–Evans [138]. So far, all known constructions of nontrivial elements in the kernel of the homomorphism (14.2.1) are based on the use of iterated Dehn twists, around (families of) Lagrangian spheres or other manifolds with periodic geodesic flow. However, the full symplectic mapping class group $\pi_0(\mathrm{Symp}(M, \omega))$ can be very rich for some closed symplectic manifolds that do not contain any Lagrangian spheres. Examples are products of K3-surfaces with suitable symplectic forms, as explained to us by Ivan Smith.

Even if $\mathrm{Symp}_h(M, \omega)$ is not connected it is sometimes interesting to understand whether it is a subgroup of $\mathrm{Diff}_0(M)$. A specific instance of this is the following question, motivated by the discussion in Note 2 on page 537.

Problem 16 (Smooth isotopy for blowups of $\mathbb{C}\mathrm{P}^2$)
Let (M, ω) be the k-point blowup of the complex projective plane with $k \ge 5$ and any symplectic form. Is $\mathrm{Symp}_h(M, \omega) \subset \mathrm{Diff}_0(M)$?

Now consider the image of the homomorphism (14.2.1). The isotopy class of a diffeomorphism $\phi : M \to M$ belongs to this image if and only if ϕ is isotopic to a symplectomorphism. Necessary conditions are the following.

(S) *$\phi^*\omega - \omega$ is exact and $\omega, \phi^*\omega$ can be joined by a path of nondegenerate 2-forms.*

One can ask whether every diffeomorphism that satisfies (S) is smoothly isotopic to a symplectomorphism. For simply connected symplectic four-manifolds the second condition in (S) is redundant by Proposition 13.3.12. For $M = \mathbb{C}\mathrm{P}^2$ a positive answer is equivalent to connectivity of the space of cohomologous symplectic forms in Problem 3. For the standard symplectic form on the 6-manifold $M = \mathbb{T}^2 \times S^2 \times S^2$ there exists a diffeomorphism ϕ that satisfies (S) but is not isotopic to a symplectomorphism, by a theorem of McDuff (Example 13.2.9). In this example, ω and $\phi^*\omega$ can even be joined by a path of symplectic forms, with necessarily varying cohomology classes.

Similar questions are interesting for compactly supported symplectomorphisms of noncompact symplectic manifolds. For example, a theorem of Gromov [287] asserts that the group $\mathrm{Symp}_c(\mathbb{R}^4, \omega_0)$ is contractible (see also [470, Theorem 9.5.2]). In fact, this continues to hold for the group of symplectomorphisms of $(\mathbb{R}^4, \omega_0)$ with support contained in the open four-ball. In higher dimensions even the question of connectivity is open. A first test case is the group of compactly supported diffeomorphisms of $\mathbb{R}^6$. Its connected components form a cyclic group of order 28 (corresponding to the 28 diffeomorphism types of homotopy 7-spheres). This leads to the following question.

Problem 17 (Symplectomorphism group of $\mathbb{R}^{2n}$)
(a) *Is every compactly supported symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$ smoothly isotopic to the identity (through an isotopy with uniform compact support)?*
(b) *Is every compactly supported symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$ symplectically isotopic to the identity (through an isotopy with uniform compact support)?*

Since the answer to question (b) in Problem 17 is positive for $2n = 4$, it follows from symplectic rigidity that the group

$$\mathrm{Symp}_{\mathrm{c},0}(\mathbb{R}^4, \omega_0) = \mathrm{Symp}_{\mathrm{c}}(\mathbb{R}^4, \omega_0)$$

is a C^0-closed subset of $\mathrm{Diff}_{\mathrm{c}}(\mathbb{R}^4)$. This is an open question for $\mathbb{R}^{2n}$ when $2n \geq 6$.

Problem 18 (C^0 closure of the identity component)
Let (M, ω) be a symplectic manifold without boundary. Is the identity component $\mathrm{Symp}_{\mathrm{c},0}(M, \omega)$ of the group $\mathrm{Symp}_{\mathrm{c}}(M, \omega)$ of compactly supported symplectomorphisms a closed subset of $\mathrm{Symp}_{\mathrm{c}}(M, \omega)$ with respect to the C^0-topology?

It is also interesting to investigate the more general question of what is known about the topology of the symplectomorphism group $\mathrm{Symp}(M, \omega)$ versus the topology of the diffeomorphism group of M. This question is only understood in very few cases (Examples 10.4.2 and 10.4.3). Via Moser isotopy it is closely related to the topology of the space of symplectic forms (Example 13.4.11).

The next conjecture is a restatement of Conjecture 10.2.18. For a discussion of this conjecture and some known results see Section 10.2.

Problem 19 (C^0 flux conjecture)
The group of compactly supported Hamiltonian symplectomorphisms is a C^0-closed subgroup of $\mathrm{Symp}_{\mathrm{c},0}(M, \omega)$.

Hofer geometry

There are various important questions that can be posed about the Hofer metric on the group of Hamiltonian symplectomorphisms. One central problem is whether the Hofer diameter is always infinite.

Problem 20 (Hofer diameter conjecture)
The group of Hamiltonian symplectomorphisms has infinite diameter in the Hofer metric for every nonempty closed symplectic manifold of positive dimension.

The Hofer diameter conjecture has been confirmed for $\mathbb{C}\mathrm{P}^n$ and for symplectically aspherical manifolds; cf. page 480. In general it is an open problem. Here we discuss the proof in Entov–Polterovich [198] for $M = S^2$, which is closely related to a question posed by Kapovich and Polterovich in 2006. Their question was posed for the 2-sphere but it makes sense for any closed manifold and is based on the following concept. Let $\mathcal{G}$ be a topological group equipped with a bi-invariant metric $\rho : \mathcal{G} \times \mathcal{G} \to [0, \infty)$. A group homomorphism $\mathbb{R} \to \mathcal{G} : t \mapsto \phi_t$ is called a **quasi-geodesic one parameter group** if there exist constants $\delta > 0$ and $c > 0$ such that

$$\delta|t| \le \rho(\mathbb{1}, \phi_t) \le c|t| \qquad \text{for all } t \in \mathbb{R}. \tag{14.2.2}$$

A quasi-geodesic one parameter group is called **quasi-dense** in $\mathcal{G}$ if

$$A(\{\phi_t\}) := \sup_{\psi \in \mathcal{G}} \inf_{t \in \mathbb{R}} \rho(\psi, \phi_t) < \infty. \tag{14.2.3}$$

One can prove that the finiteness of the number $A(\{\phi_t\})$ is independent of the choice of the quasi-geodesic one parameter group in $\mathcal{G}$. (Here is an argument explained to us by Polterovich: Let ϕ_t and ϕ'_t be two quasi-geodesic one parameter groups satisfying (14.2.2) and assume $A(\{\phi_t\}) < \infty$. Then there exists a constant A and a map $\mathbb{Z} \to \mathbb{Z} : j \mapsto k_j$ such that $\rho(\phi'_k, \phi_{j_k}) \le A$ for all $k \in \mathbb{Z}$. It follows from the quasi-geodesic property that $\lim_{|k|\to\infty} |j_k| = \infty$. Moreover, it follows from (14.2.2) and the bi-invariance of the metric that, for all $k \in \mathbb{N}$, we have $\delta|j_{-k} + j_k| \le \rho(\mathbb{1}, \phi_{j_k + j_{-k}}) \le \rho(\phi_{-j_k}, \phi'_{-k}) + \rho(\phi'_{-k}, \phi_{j_{-k}}) \le 2A$. Thus, reversing the orientation of $t \mapsto \phi'_t$ if necessary, we obtain $\lim_{k\to\infty} j_k = \infty$ and $\lim_{k\to\infty} j_{-k} = -\infty$. It then follows from (14.2.2) and the bi-invariance of the metric that $|j_{k+1} - j_k| \le \rho(\phi_{j_k}, \phi_{j_{k+1}})/\delta \le (2A + \rho(\mathbb{1}, \phi'_1))/\delta =: B$. This implies that every ϕ_j has distance at most $A + cB$ to some ϕ'_k and hence $A(\{\phi'_t\}) < \infty$.)

Now let (M, ω) be a closed symplectic manifold and consider the Hofer metric ρ on the group $\mathcal{G} = \mathrm{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms. In the case $M = S^2$ it was shown by Entov–Polterovich in [198, Section 5.5] that the one parameter subgroup generated by a Hamiltonian function $H : S^2 \to \mathbb{R}$ is a quasi-geodesic whenever $\int_{S^2} H\sigma = 0$, H is a Morse function, zero is a regular value of H, and no connected component of $H^{-1}(0)$ divides the 2-sphere into two connected components of equal area. The existence of a quasi-geodesic one parameter subgroup shows that $\mathrm{Ham}(S^2)$ has infinite Hofer diameter.

Problem 21 (Kapovich–Polterovich)
Let $\mathbb{R} \to \mathrm{Ham}(S^2) : t \mapsto \phi_t$ be a 1-parameter subgroup generated by $H \in C^\infty(S^2)$ and assume that there is a constant $\delta > 0$ such that $\rho(\phi_t, \mathrm{id}) \ge \delta|t|$ for all $t \in \mathbb{R}$. Is the number

$$A(H) := \sup_{\psi \in \mathrm{Ham}(S^2)} \inf_{t \in \mathbb{R}} \rho(\psi, \phi_t)$$

finite or infinite?

As noted above, the finiteness of the number $A(H)$ in Problem 21 is independent of H. The question asks if some, and hence every, quasi-geodesic one parameter group in $\mathrm{Ham}(S^2)$ is quasi-dense with respect to the Hofer metric. A positive answer (i.e. finiteness of $A(H)$), would assert that $\mathrm{Ham}(S^2)$ is contained in a tube of finite Hofer distance around the one parameter subgroup $\{\phi_t \,|\, t \in \mathbb{R}\}$. The expected answer is that $A(H)$ is infinite. This is formulated as a much more general conjecture by Polterovich–Shelukhin [533]. They introduce the invariant

$$\mathrm{aut}(M,\omega) := \sup_{\psi \in \mathrm{Ham}(M,\omega)} \inf_{H \in C^\infty(M)} \rho(\psi, \phi_H). \tag{14.2.4}$$

Thus, $\mathrm{aut}(M,\omega)$ is the supremum over all $\psi \in \mathrm{Ham}(M,\omega)$ of the Hofer distance of ψ to the set of all Hamiltonian symplectomorphisms that are generated by time-independent Hamiltonian functions. Polterovich and Shelukhin conjecture that this invariant is always infinite. This would imply the Hofer diameter conjecture and show that the number $A(H)$ in Problem 21 is always infinite.

Problem 22 (Autonomous Hamiltonian conjecture)
The number $\mathrm{aut}(M,\omega)$ *is infinite for every closed symplectic manifold* (M,ω).

Polterovich–Shelukhin [533] verify this conjecture for Riemann surfaces of genus at least four and their products with symplectically aspherical manifolds. They also derive consequences of this result in Hamiltonian dynamics.

Calabi quasimorphisms

An important development since the mid 1990s, pursued by Entov, Polterovich, Py, and many others, was to bring the theory of quasimorphisms to bear on questions in symplectic topology and Hamiltonian dynamics. A map $\mu : \mathcal{G} \to \mathbb{R}$ on a group $\mathcal{G}$ is called a **quasimorphism** if there is a constant $c > 0$ such that

$$|\mu(\phi \circ \psi) - \mu(\phi) - \mu(\psi)| \le c \tag{14.2.5}$$

for all $\phi, \psi \in \mathcal{G}$. A quasimorphisms $\mu : \mathcal{G} \to \mathbb{R}$ is called **homogeneous** if

$$\mu(\phi^k) = k\mu(\phi) \tag{14.2.6}$$

for all $\phi \in \mathcal{G}$ and all $k \in \mathbb{Z}$. The relevant case for the discussion at hand is when $\mathcal{G} = \mathrm{Ham}(M,\omega)$ is the group of Hamiltonian symplectomorphisms on a closed symplectic manifold. A homogeneous quasimorphism $\mu : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ is called a **Calabi quasimorphism** if, for every displaceable open set $U \subset M$ and every compactly supported time-dependent Hamiltonian $\{H_t\}_{0 \le t \le 1}$, we have

$$\overline{\bigcup_{0 \le t \le 1} \mathrm{supp}(H_t)} \subset U \qquad \Longrightarrow \qquad \mu(\phi_H) = \int_0^1 \int_M H_t \omega^n \, dt. \tag{14.2.7}$$

In other words, if ϕ is generated by a time-dependent Hamiltonian function with compact support in a displaceable open set, then $\mu(\phi)$ agrees with the Calabi invariant of ϕ (see page 411). Although Calabi quasimorphisms are more naturally defined on the universal cover of $\mathrm{Ham}(M)$, they sometimes descend to

Ham(M). The existence of Calabi quasimorphisms on Ham(M) was established by Entov–Polterovich [198] for $M = \mathbb{C}\mathrm{P}^n$ and by Pierre Py [534,535] for all closed symplectic 2-manifolds of positive genus. For an extensive discussion of Calabi quasimorphisms and their applications in symplectic topology, see the papers [73, 198,199,200,201,202,203,204] by Entov, Polterovich, and their collaborators, and the book by Polterovich–Rosen [532]. In [203, Section 5.2] Entov, Polterovich, and Py posed the following questions.

Problem 23 (Quasimorphism question)
(a) *Does there exist a nonzero homogeneous quasimorphism* $\mu : \mathrm{Ham}(S^2) \to \mathbb{R}$ *that is continuous with respect to the* C^0*-topology on* $\mathrm{Ham}(S^2)$*?*
(b) *If yes, can it be made Lipschitz with respect to the Hofer metric?*

They show in [203] that the difference of two Calabi quasimorphisms on $\mathrm{Ham}(S^2)$ is C^0-continuous, although the Calabi homomorphism, and hence all Calabi quasimorphisms, are not C^0-continuous. They conclude that a negative answer to question (a) in Problem 23 would imply that the Calabi quasimorphism on $\mathrm{Ham}(S^2)$ constructed by Entov–Polterovich [198] is unique, while a positive answer to question (b) in Problem 23 would imply that $A(H) = \infty$ in Problem 21. For a full exposition of this circle of ideas, including a discussion of related developments and open questions, see the book by Polterovich–Rosen [532].

Finite subgroups of $\mathrm{Symp}(M, \omega)$

In [638], Polterovich observed that the group of Hamiltonian symplectomorphisms of a closed symplectically aspherical manifold has no nontrivial finite subgroups because there is a measure of the size of the iterates ϕ^k of an element $\phi \in \mathrm{Ham}(M) \setminus \{\mathrm{id}\}$ that grows at least linearly with k. This is not true for a manifold such as S^2 that supports an action of S^1. Now manifolds with a Hamiltonian circle action are symplectically uniruled, by a theorem of McDuff [462]. This is a condition on the genus zero Gromov–Witten invariants (see the discussion on page 563 below). Thus one might investigate connections between the properties of the Gromov–Witten invariants and the geometry of the group of Hamiltonian symplectomorphisms. One such question is the following.

Problem 24 (Finite groups of Hamiltonian symplectomorphisms)
Does there exist a closed symplectic manifold (M, ω) *with vanishing genus zero Gromov–Witten invariants such that* $\mathrm{Ham}(M, \omega)$ *has a nontrivial finite subgroup?*

For other interesting questions about the structure of finite subgroups of $\mathrm{Symp}(M, \omega)$, see Mundet-i-Riera [499] and the references therein.

14.3 Lagrangian submanifolds and cotangent bundles

Lagrangians are an important class of submanifolds of symplectic manifolds, and we concentrate here on questions about their existence and properties. Much of the current interest in understanding them is motivated by mirror symmetry, because Lagrangian manifolds (perhaps decorated with additional structure) form the objects of the Fukaya category, one side of the mirror symmetry correspondence. The influence of this powerful ansatz can be seen even within the rather narrowly focused questions discussed here: for example, the work of Vianna [640] discussed after Problem 25 was motivated by ideas from the mirror side. However, to discuss mirror symmetry itself is beyond the scope of this book.

Lagrangian tori

A longstanding open question which has attracted the attention of many researchers has been the *Audin conjecture*, which asserts that every Lagrangian torus in $\mathbb{R}^{2n}$ with the standard symplectic structure has minimal Maslov number two. After various partial results by Viterbo [645] (for $n = 2$), Buhovski [81], and Fukaya–Oh–Ohta–Ono [249, Theorem 6.4.35] (in the monotone case, cf. Definition 3.4.4), Damian [131], and others, the Audin conjecture was settled in 2014 by Cieliebak–Mohnke [117] (see Remark 3.4.6). Another proof of the Audin conjecture was outlined by Fukaya [247]. The standard torus $S^1 \times S^1$, also called the Clifford torus, and the Chekanov torus in Example 3.4.7 are, up to Hamiltonian isotopy, the only known examples of monotone Lagrangian tori with factor $\pi/2$ in $(\mathbb{R}^4, \omega_0)$. This leads to the following natural question.

Problem 25 (Monotone Lagrangian tori in $\mathbb{R}^4$)
Is every monotone Lagrangian torus in $(\mathbb{R}^4, \omega_0)$ *Hamiltonian isotopic to either the Clifford torus or the Chekanov torus?*

In [42], Auroux constructs infinitely many monotone Lagrangian tori in $\mathbb{R}^6$ (with factor $\pi/2$) that are pairwise not Hamiltonian isotopic and even not ambiently symplectomorphic, i.e. there is no symplectomorphism of $\mathbb{R}^6$ taking one to another. By taking products with S^1, he obtains infinitely many pairwise nonsymplectomorphic monotone Lagrangian tori in $(\mathbb{R}^{2n}, \omega_0)$ for every $n \geq 3$. After arbitrarily small nonmonotone Lagrangian perturbations, the tori constructed by Auroux become Hamiltonian isotopic to standard product tori. For many other interesting results about monotone Lagrangian tori in $\mathbb{R}^{2n}$ for $n \geq 4$, see the work of Dimitroglou Rizell–Evans [137].

After appropriate rescaling, the Clifford and Chekanov tori in $\mathbb{C}^2$ can also be embedded as monotone Lagrangian tori into $\mathbb{CP}^2$. In [639], Vianna found a third monotone Lagrangian torus in $\mathbb{CP}^2$, that is not Hamiltonian isotopic to the Clifford and Chekanov tori. The three monotone Lagrangian tori in $\mathbb{CP}^2$ are distinguished by their relative Gromov–Witten invariants (the count of pseudoholomorphic discs with Maslov index two). In a subsequent paper [640], Vianna constructed infinitely many monotone Lagrangian tori in $\mathbb{CP}^2$ that are pairwise not Hamiltonian isotopic. As explained in [639, §6.1], these tori can be considered as subsets of suitable affine charts in $\mathbb{CP}^2$ and hence can also be embedded as monotone Lagrangian tori into $(\mathbb{C}^2, \omega_0)$. It is as yet unknown whether the resulting tori in $\mathbb{C}^2$ are Hamiltonian isotopic to the appropriately scaled Clifford or Chekanov torus.

In Problem 25 it makes sense to drop the monotonicity hypothesis and replace Hamiltonian isotopy by Lagrangian isotopy. In particular, the Chekanov torus is Lagrangian isotopic to the Clifford torus. This leads to the following question.

Problem 26 (Exotic nonmonotone Lagrangian tori in $\mathbb{R}^4$)
Does there exist a Lagrangian 2-torus in $(\mathbb{R}^4, \omega_0)$ that is not Lagrangian isotopic to the Clifford torus?

The next question concerns Lagrangian tori in symplectic 4-tori. Call a symplectic 4-torus $(\mathbb{T}^4, \omega)$ with a translation invariant symplectic form **irrational** if the integrals of ω over the six coordinate 2-tori are rationally independent.

Problem 27 (Irrational four-torus question)
Is every Lagrangian 2-torus in an irrational symplectic 4-torus displaceable by a Hamiltonian isotopy?

The above questions concentrate on the four-dimensional case, and have obvious analogues in higher dimensions. In particular, one might ask if every monotone torus in $\mathbb{C}\mathrm{P}^n$ (or more generally a product of projective spaces) is 'local', i.e. Hamiltonian isotopic to a torus in the affine part of the manifold, a result that is known in dimension four by [682].

Lagrangian knots

Another question concerns the existence of local Lagrangian knots. It goes back to a theorem by Eliashberg and Polterovich [195] which asserts that every Lagrangian embedding $\iota : \mathbb{R}^2 \to \mathbb{R}^4$ that is standard at infinity, is isotopic to the standard linear embedding by a compactly supported Hamiltonian isotopy of $\mathbb{R}^4$. The analogous question in higher dimensions is open.

Problem 28 (Local Lagrangian knots)
Fix an integer $n \geq 3$, let $\iota_0 : \mathbb{R}^n \to \mathbb{R}^{2n}$ be the standard Lagrangian embedding given by

$$\iota_0(x) := (x, 0) \qquad \text{for } x \in \mathbb{R}^n,$$

and let $\iota : \mathbb{R}^n \to \mathbb{R}^{2n}$ be a Lagrangian embedding that agrees with ι_0 outside of a compact set. Does it follow that ι is isotopic to ι_0 by a compactly supported (Hamiltonian) isotopy?

In a similar vein, Dimitroglou Rizell–Evans [137] proved that if $n \geq 5$ is an odd integer, then any two embedded Lagrangian tori in $(\mathbb{R}^{2n}, \omega_0)$ are smoothly isotopic. The questions discussed here can of course be extended to general symplectic $2n$-manifolds (M, ω) and one can ask whether any two homologous Lagrangian embeddings of a given n-manifold into (M, ω) are smoothly (or Lagrangian or Hamiltonian) isotopic. In this connection, Borman–Li–Wu [76] asserts that certain rational projective surfaces contain pairs of embedded Lagrangian real projective planes that are homologous but not smoothly isotopic.

Lagrangian spheres

Lagrangian spheres in smooth projective varieties (i.e. symplectic manifolds that are embedded as complex submanifolds in some projective space and inherit their symplectic structure from the Fubini–Study form) arise naturally as vanishing cycles when the algebraic variety in question is the fibre of a Lefschetz fibration. Thus the study of Lagrangian spheres plays a central role in both symplectic and algebraic geometry. For various interesting results about Lagrangian spheres in Milnor fibres, see the work of Seidel [577, 578] and Keating [361]. The following problem is a fundamental question in the subject and was pointed out to us by Ivan Smith.

Problem 29 (Donaldson's Lagrangian sphere question)
Is every Lagrangian sphere in a smooth algebraic variety Hamiltonian isotopic to a vanishing cycle of an algebraic degeneration?

Cotangent bundles

Problem 30 (Nearby Lagrangian conjecture)
Let N be a closed smooth manifold. Then every exact Lagrangian submanifold $L \subset T^*N$ is Hamiltonian isotopic to the zero section.

The nearby Lagrangian conjecture is a longstanding open problem in symplectic topology and there are quite a few partial results towards it. In particular the projection $L \to N$ along the fibres of T^*N is a homotopy equivalence by a result of Abouzaid–Kragh [5]. Constraints on exact Lagrangian submanifolds of cotangent bundles were found by Fukaya–Seidel–Smith [253, 254]. However, the nearby Lagrangian conjecture has only been confirmed for $N = S^1$, where it can be proved with elementary methods, and for $N = S^2$ by Richard Hind [309]. It is open for all other manifolds.

Problem 31 (Eliashberg cotangent bundle question)
Let N_0, N_1 be closed smooth manifolds that are homeomorphic. Are they diffeomorphic if and only if their cotangent bundles T^*N_0 and T^*N_1 (with their standard symplectic structures) are symplectomorphic?

The first result in this direction is due to Abouzaid, who showed in [37] that if Σ is an exotic $(4k+1)$-dimensional sphere that does not bound a parallelizable manifold (and these exist), then $T^*\Sigma$ and T^*S^{4k+1} are *not* symplectomorphic. He does this by showing that every homotopy sphere that embeds as a Lagrangian in T^*S^{4k+1} must bound a parallelizable manifold, which he constructs directly out of certain perturbed J-holomorphic curves. (See Example 13.2.12.) This result has now been extended by Ekholm–Kragh–Smith [173].

Lagrangian Hofer geometry

A circle in a symplectic 2-sphere is called an **equator** if it divides the 2-sphere into two connected components of equal area. Any two equators are Hamiltonian isotopic (see part (v) of Exercise 3.4.5).

Problem 32 (Equator conjecture)
The space of equators on (S^2, σ) has infinite Hofer diameter.

This problem has been known to Lalonde and Polterovich since 1991; a positive answer would imply that the quantity $A(H)$ in Problem 21 is infinite. The analogous conjecture for *diameters of the disc* was confirmed by Khanevsky [368] in 2009. This question is, of course, a Lagrangian version of Problem 20, and can be asked for the orbit of any Lagrangian submanifold under the Hamiltonian group. Besides Ostover's observation in [518] and Usher's work in [635], very little is known about this question. In particular, to our knowledge there is no monotone Lagrangian in a closed manifold for which this orbit has been shown to have finite diameter.

14.4 Fano manifolds

A closed complex n-manifold M is called a **Fano manifold** if the top exterior power $L := \Lambda^{n,0}TM$ of its tangent bundle is **ample**, i.e. there is a $k \in \mathbb{N}$ such that $L^{\otimes k}$ is **very ample**, i.e. the holomorphic sections of $L^{\otimes k}$ determine an immersion into projective space. By pulling back the Fubini–Study form on projective space one obtains after scaling a Kähler form in the de Rham cohomology class $c_1(TM) \in H^2(M; \mathbb{R})$. Conversely, if (M, ω, J) is a closed Kähler manifold such that $[\omega] = c_1(\omega)$ then (M, J) is Fano by the Kodaira embedding theorem.

A theorem of Mori asserts that every Fano manifold is **uniruled**, i.e. admits a nonconstant holomorphic sphere (also called a *rational curve*) through every point. In fact he proved that every Fano manifold is **rationally connected**, i.e. for any two points in M there is a finite connected sequence of rational curves (also called a *genus zero stable map*) containing both points. This can be used to prove that every Fano manifold is simply connected. Mori's proofs of these theorems involve intrinsic algebraic geometric methods in characteristic p, and it would be interesting to find analytic proofs based on the theory of holomorphic curves. So far no one has succeeded in this endeavour.

Problem 33 (Mori theory question)
Is there an analytic proof of Mori's theorem that every closed Fano manifold is rationally connected?

In another direction one can ask if Mori's results carry over to nonKähler symplectic manifolds. The natural analogue of a Fano manifold in symplectic topology is a **monotone symplectic manifold**, i.e. a closed symplectic manifold (M, ω) such that $c_1(\omega) \in H^2(M; \mathbb{Z})$ is an integral lift of $[\omega] \in H^2(M; \mathbb{R})$. The above results do not all extend. Indeed, Fine–Panov [216] developed some ideas in Reznikov [538] to construct infinitely many examples in each dimension greater than or equal to twelve of nonsimply connected monotone symplectic manifolds. These manifolds cannot support any Kähler structure at all since their fundamental groups are hyperbolic. Further, since they fibre over hyperbolic manifolds, they are not rationally connected. However, one can still hope that a symplectic analogue of the uniruled condition might hold. This condition needs to be translated with some care. A first basic question would be whether every closed monotone symplectic manifold (M, ω) contains a symplectically embedded 2-sphere. For every symplectic manifold of dimension at least six, monotone or not, this question has a positive answer if and only if the cohomology class of ω does not vanish on $\pi_2(M)$. This follows from an h-principle for symplectic embeddings due to Gromov [288] (see also Eliashberg–Mishachev [190] and the discussion on page 327). Once such a symplectically embedded 2-sphere has been found, there is one through every point in M because the symplectomorphism group of (M, ω) acts transitively on M. Therefore a more interesting question is whether there is a nonconstant J-holomorphic sphere through every point for every ω-compatible almost complex structure. This question can be recast and strengthened in terms of the Gromov–Witten invariants. A closed symplectic manifold (M, ω) is called **symplectically uniruled** if there is a nonzero Gromov–Witten invariant with a point constraint. This means that there exists a nonzero class $A \in H_2(M; \mathbb{Z})$ and an integer k such that, for one and hence every regular ω-compatible almost complex structure J, the homology class in $H_*(M^k)$ represented by the evaluation map $\mathrm{ev}_{A,k,J}$ in (4.5.5) has a nonzero intersection pairing with a product homology class of the form $\mathrm{pt} \times c_2 \times \cdots \times c_k$. A class A with this property is called a **uniruling class**. With this definition it was proved by Hu–Li–Ruan [333] that every uniruled Kähler manifold is symplectically uniruled.

Problem 34 (Rational curves in monotone symplectic manifolds)
Is every closed monotone symplectic manifold (M, ω) symplectically uniruled?

As we discuss in more detail below, one way to approach such a problem might be to use the decomposition in Section 14.5.

There are many other questions one can ask about monotone symplectic manifolds. Here is a question pointed out to us by Paul Biran and Slava Kharlamov. If (M, ω) is a closed monotone symplectic manifold and $\tau : M \to M$ is an anti-symplectic involution so that its fixed point set $L := \mathrm{Fix}(\tau)$ is a Lagrangian submanifold, does there exist an upper bound, independent of τ, on the number of connected components of L that are tori? Apparently a reasonable guess would be that for dim $M = 2n \geq 4$ an upper bound is given by $n+1-N$, where $N \in \mathbb{N}$ is the **minimal Chern number**, defined by $N\mathbb{Z} = \langle c_1(TX), H_2(X; \mathbb{Z}) \rangle$. At least in complex dimensions $n = 2, 3$, there seems to be no known example with more than $n + 1 - N$ toral components.

14.5 Donaldson hypersurfaces

In the mid nineties Donaldson proved that for every closed symplectic manifold (M, ω) with an integral symplectic form the cohomology class $k[\omega]$ is Poincaré dual to a symplectic submanifold $Z \subset M$ for every sufficiently large integer k (see Section 7.4). The symplectic submanifolds constructed by Donaldson have very special properties that are not necessarily shared by all symplectic submanifolds in the same homology class. For example they are connected when M is connected and has dimension at least four, and they are simply connected when M is simply connected and has dimension at least six. Moreover, by an as yet unpublished theorem of Giroux, for k sufficiently large they can be chosen such that the complement $W := M \setminus Z$ is a Weinstein manifold (see Definition 7.4.5). On the other hand a theorem of McLean (also not yet published) asserts that certain closed symplectic manifolds contain symplectic submanifolds that are Poincaré dual to $k[\omega]$, but yet do not have a Weinstein complement. These submanifolds must therefore be different from those constructed by Donaldson. This leads to the question of how the term *Donaldson hypersurface* should be defined. As a temporary working definition we call a codimension two symplectic submanifold $Z \subset M$ of a closed integral symplectic manifold (M, ω) a **Donaldson hypersurface** if it is Poincaré dual to the cohomology class $k[\omega]$ for some integer $k > 0$.

By the theorems of Donaldson and Giroux, every closed integral symplectic manifold admits a decomposition into a Donaldson hypersurface $Z \subset M$ Poincaré dual to $k[\omega]$ and a Weinstein manifold $W := M \setminus Z$ for every sufficiently large integer $k > 0$. This decomposition is so far rather little understood but its properties seem likely to affect the symplectic topology of (M, ω) in very interesting ways. For example, (M, ω) is called **subcritical** if, for some k, it admits such a decomposition in which the Weinstein manifold W has the homotopy type of a cell complex with cells of dimension at most $n - 1$. Moreover, there is the notion of a *flexible Weinstein manifold* introduced by Cieliebak–Eliashberg [111]. One can then compare the properties of closed integral symplectic manifolds that admit a Donaldson–Giroux decomposition with subcritical Weinstein part with those where the Weinstein part is flexible.

One can also ask whether these decompositions have consequences for the Gromov–Witten invariants, for example whether symplectic manifolds with flexible decompositions are uniruled. The work of Jian He [306] explains a new way to understand the Gromov–Witten invariants of subcritical manifolds in the Kähler case, giving a new proof that such a manifold is uniruled.

It is also important to gain a better understanding of the difference between Liouville and Weinstein domains. The symplectic form ω on a Liouville domain (W, ω, X) is exact, i.e. $\omega = \mathcal{L}_X \omega = d\iota(X)\omega$, and the corresponding Liouville vector field X points out along the boundary ∂W (see Definition 3.5.32), but a Weinstein domain also carries a function f that satisfies the stringent requirements in Definition 7.4.5: f must be a generalized Morse function and also such that the flow of X is gradient like, i.e. $df \cdot X \geq 0$. Neither of these conditions is well understood. Here is a sample question.

Problem 35 (Eliashberg's Weinstein manifold question)
Let (W, ω) be a noncompact connected symplectic manifold without boundary, let X be a global complete Liouville vector field on W, and let $f : W \to \mathbb{R}$ be a smooth function that is proper and bounded below such that $df \cdot X \geq 0$. Can the pair (X, f) be perturbed to a Weinstein structure as in Definition 7.4.5 (i.e. f is a generalized Morse function that satisfies (a) and (b))?

The next question was posed by Donaldson and seconded by Eliashberg.

Problem 36 (Hypersurface stability question)
Do Donaldson hypersurfaces 'stabilize' in dimension six? More precisely, is there a family of simply connected smooth four-manifolds $X_{b^+,b^-,\varepsilon}$ (parametrized by pairs of nonnegative integers $b^{\pm}$ and elements $\varepsilon \in \{0,1\}$, with Euler characteristic $2+b^++b^-$, signature b^+-b^-, and even/odd intersection form when $\varepsilon = 0/1$) having the following significance: If (M,ω) is a simply connected closed symplectic 6-manifold with integral symplectic form then, for k sufficiently large, the Donaldson hypersurface $X \subset M$ Poincaré dual to $k[\omega]$ can be constructed to be diffeomorphic to one of the manifolds $X_{b^+,b^-,\varepsilon}$?

This question is backed up by the observation that hyperplane sections of simply connected algebraic 3-folds are minimal Kähler surfaces of general type whenever they have sufficiently large degree. Hence they have very simple Seiberg–Witten invariants, namely, their only basic classes are $\pm K$ (see Example 13.4.10). Thus, when they have isomorphic intersection forms, their diffeomorphism types cannot be distinguished by any currently known method.

14.6 Contact geometry

The existence problem for contact structures was settled in 2014 by Borman–Eliashberg–Murphy [75]. A necessary condition for the existence of a co-oriented contact structure on a $(2n+1)$-dimensional manifold M is that the tangent bundle admits a rank-$2n$ subbundle $\xi \subset TM$, equipped with a nondegenerate 2-form ω, such that the rank-1 quotient bundle TM/ξ admits a trivialization, i.e. there exists a 1-form $\alpha \in \Omega^1(M)$ such that $\xi = \ker\alpha$. In other words, there exists a pair (α,ω) consisting of a nonvanishing 1-form $\alpha \in \Omega^1(M)$ and a nondegenerate 2-form ω on $\xi := \ker\alpha$. Such a pair (α,ω) is called an **almost contact structure**. They also introduced the notion of an *overtwisted contact structure* in higher dimensions (see Definition 3.5.4 for dimension three) and proved that the map $\alpha \mapsto (\alpha, d\alpha|_{\ker\alpha})$ from the space of overtwisted contact forms to the space of almost contact structures is a weak homotopy equivalence. In particular, the existence of an almost contact structure is equivalent to the existence of an overtwisted contact structure, and every homotopy class of almost contact structures contains a unique isotopy class of overtwisted contact structures. (See also the discussion on page 137 in Section 3.5.) Their theorem does not settle the existence and uniqueness problem for **tight** (i.e. not overtwisted) contact structures. An important problem is which homotopy classes of almost contact structures contain an isotopy class consisting of tight contact structures and, if so, how many such isotopy classes it contains. There are many results about this problem in dimension three, some of which are discussed in Section 3.5. For spheres of dimension $4m+1$ the problem was studied by Ustilovsky in the late 1990s. In [637] he found infinitely many isotopy classes of contact structures on the 5-sphere, that are all homotopic as contact structures and are tight because they are symplectically fillable. However the technical background for this result was not put in place until Gutt's thesis [302]; see also Bourgeois–Oancea [79]. A general question in higher dimensions is which closed contact manifolds are fillable by Liouville domains, respectively Weinstein domains (see Definitions 3.5.32, 3.5.38, and 7.4.5). Borman–Eliashberg–Murphy [75] proved that overtwisted contact structures are not symplectically fillable. Further, in [192] Eliashberg–Murphy construct many symplectic cobordisms, showing in particular that in all dimensions ≥ 5 there is a symplectic cobordism whose inner (i.e. concave) boundary is overtwisted while the outer (convex) boundary is tight.

Another central problem that has inspired many developments in symplectic and contact geometry is the Weinstein conjecture which has already been discussed in Section 1.2 for hypersurfaces of $(\mathbb{R}^{2n}, \omega_0)$.

Problem 37 (Weinstein conjecture)
The Reeb vector field of every contact form on a closed contact manifold has a closed orbit.

For contact hypersurfaces of Euclidean space the Weinstein conjecture was settled by Viterbo [643] in the late 1980s (see Theorem 12.4.6). It was confirmed by Hofer [317] for the 3-sphere and by Taubes [612] for all closed contact 3-manifolds. In [130], Cristofaro-Gardiner and Hutchings used embedded contact homology [337,340] to establish the existence of at least two closed Reeb orbits in dimension three. In [342], Hutchings and Taubes extended the Weinstein conjecture to **stable Hamiltonian structures** on 3-manifolds. In higher dimensions the general Weinstein conjecture is open.

Problem 38 (Arnold chord conjecture)
If M is a closed contact manifold and $L \subset M$ is a Legendrian submanifold then, for every contact form on M, there exists a Reeb orbit with endpoints on L.

In [343, 344] Hutchings and Taubes settled the Arnold chord conjecture in dimension three. Their work is based on Hutchings' embedded contact homology and a theorem of Taubes [613, 614, 615, 616, 617] which asserts that embedded contact homology is isomorphic to Seiberg–Witten Floer homology. The Arnold chord conjecture is open in higher dimensions.

An important notion in contact topology is that of a **positive loop of contactomorphisms**, i.e. a loop of contactmorphisms $\psi_t = \psi_{t+1}$ of a contact manifold (M, ξ) that is generated by a time-dependent family of positive Hamiltonian functions $H_t = H_{t+1} = \alpha(X_t) : M \to (0, \infty)$ (Lemma 3.5.14) for some, and hence every, contact form α. By a theorem of Eliashberg–Kim–Polterovich [189], the nonexistence of a positive loop is equivalent to the orderability of the identity component $\mathrm{Cont}_0(M, \xi)$ of the group of contactomorphisms, while the nonexistence of a contractible positive loop is equivalent to the orderability of the universal cover of $\mathrm{Cont}_0(M, \xi)$.

Problem 39 (Positive loop question)
Which closed contact manifolds (M, ξ) admit (contractible) positive loops of contactomorphisms?

Positive loops of contactomorphisms obviously exist on contact manifolds with periodic Reeb flows, such as pre-quantization circle bundles over symplectic manifolds (Example 3.5.11) and unit cotangent bundles of closed Riemannian manifolds with periodic geodesic flows (Example 3.5.7). These loops are all noncontractible [189]. Eliashberg–Kim–Polterovich [189] also found contractible positive loops of contactomorphisms for the standard contact structures on spheres (Example 3.5.9). The Eliashberg–Kim–Polterovich construction can be adapted to the regular level sets of plurisubharmonic Morse functions on symplectic $2n$-manifolds with critical points of indices at most $n-2$. These are apparently the only known existence results. Nonexistence is known in various cases such as the standard contact structure on $\mathbb{R}\mathrm{P}^{2n-1}$.

Another fundamental question in contact topology is whether contact manifolds are determined by their symplectizations (Definition 3.5.24); see Cieliebak–Eliashberg [111, p. 239].

Problem 40 (Symplectization question)
Do there exist closed contact 3-manifolds (M, ξ) and (M', ξ') that are not contactomorphic but have symplectomorphic symplectizations?

Examples of such contact manifolds in dimensions at least five were found by Sylvain Courte [129]. In dimension three this problem is open.

14.7 Continuous symplectic topology

Symplectic rigidity asserts that the group of symplectomorphisms is a closed subset of the group of all diffeomorphisms with respect to the C^0-topology (Theorem 12.2.1). This has prompted the development of continuous symplectic topology into an important and very active research area, that studies questions such as the relation between the C^0-topology and that given by the Hofer metric, and the persistence of symplectic phenomena under C^0 limits.

A rigidity theorem by Cardin–Viterbo [93] asserts that if F and G are smooth functions on a symplectic manifold that can be approximated in the C^0-topology by sequences of smooth functions F_n and G_n with vanishing Poisson bracket, then the Poisson bracket of F and G must also vanish. Their result was refined by Entov–Polterovich [201] who showed that the supremum norm of the Poisson bracket $\{F, G\}$ is a lower bound for the limit inferior of the supremum norms of $\{F_n, G_n\}$. Their result was in turn refined by Buhovski [82] when he established his *'2/3-law'* for the convergence rate. These results also led to the introduction of new invariants by Buhovsky–Entov–Polterovich [86] and they are closely related to the study of Calabi quasimorphisms on the universal cover of the group of Hamiltonian symplectomorphisms (see Section 14.2) and to the study of *symplectic quasi-states* on the space of functions on a symplectic manifold as well as to Floer homology (see Section 11.4).

Symplectic rigidity suggests that there is a notion of a symplectic homeomorphism. For example, one can define a **symplectic homeomorphism** between two open subsets $U, V \subset \mathbb{R}^{2n}$ as a homeomorphism $\phi : U \to V$ such that every point in U has an open neighbourhood such that the restriction of ϕ to this neighbourhood preserves a given symplectic capacity of all open sets (e.g. the Hofer–Zehnder capacity or the Gromov width). Another possible definition is that each point in U has a neighbourhood such that the restriction of ϕ to this neighbourhood can be approximated in the C^0-topology by a sequence of symplectomorphisms, and the same holds for ϕ^{-1}. As mentioned in the discussion at the end of Section 12.2, there are many open questions about the precise relation between the different definitions. However, they have in common that the symplectic homeomorphisms are the morphisms of a category (i.e. compositions and inverses of symplectic homeomorphisms are again symplectic homeomorphisms) and that the symplectic form is preserved by every smooth symplectic homeomorphism that is defined on a symplectic manifold. Given such choice one can define a **topological symplectic manifold** as a topological manifold equipped with an atlas whose transition maps are symplectic homeomorphisms. The following question was pointed out to us by Yasha Eliashberg and Leonid Polterovich. It has been known to them since the early 1990s.

Problem 41 (Topological symplectic four-sphere)
Does the 4*-sphere admit the structure of a topological symplectic manifold?*

A related and somewhat vaguely worded question is which symplectic properties are preserved by symplectic homeomorphisms. The notion of a symplectic homeomorphism seems to be considerably more flexible than that of a symplectomorphism. For example Buhovsky–Opshtein [84] construct symplectic homeomorphisms of Euclidean space $\mathbb{C}^3$ (in the sense of C^0 approximation by symplectomorphisms) that restrict to the map $(z, 0, 0) \mapsto (\frac{1}{2}z, 0, 0)$ for z in the open unit disc. On the other hand, Humilière–Leclercq–Seyfaddini [334] show that symplectic homeomorphisms are compatible with the construction in Proposition 5.4.5 of the symplectic reduction of a coisotropic manifold.

In dimension two a well-known folk theorem states that every area-preserving homeomorphism can be C^0-approximated by area-preserving diffeomorphisms. (For a short proof see Sikorav [587].) Nevertheless, there are still many open questions. The following question about area-preserving homeomorphisms has attracted much interest, partly because of its immediacy and partly because it can be tackled from many different points of view.

Problem 42 (Simplicity conjecture)
The group $\mathrm{Homeo}_c(\mathbb{D})$ *of compactly supported area preserving homeomorphisms of the open two-disc is not simple.*

There have been various attempts to settle this conjecture by constructing normal subgroups of $\mathrm{Homeo}_c(\mathbb{D})$; the difficulty has always been to show that these subgroups are proper. For example, in [413], Le Roux reduces this problem to a question about properties of *fragmentations* of elements $\phi \in \mathrm{Ham}_c(\mathbb{D})$ into products of symplectomorphisms with support in discs of small area. Oh–Müller [510] suggest a different approach in which the normal subgroup is the group of hameomorphisms. These are limits of symplectomorphisms under a combination of the Hofer metric topology and the C^0-topology (see Definition 12.3.17 in Section 12.3). The following problem describes an elementary example of an area-preserving homeomorphism that might not belong to this group. Note that every continuous function $f : (0, 1] \to \mathbb{R}$ that equals zero near 1 determines a compactly supported area-preserving homeomorphism $\phi_f : \mathbb{D} \to \mathbb{D}$, defined by $\phi_f(0) := 0$ and

$$\phi_f(re^{i\theta}) := re^{i(\theta+f(r))} \tag{14.7.1}$$

for $0 < r \leq 1$ and $\theta \in \mathbb{R}$.

Problem 43 (Infinite twist conjecture)
If $\lim_{r\to 0} f(r) = \infty$ *then* ϕ_f *is not a hameomorphism.*

A positive answer to the infinite twist conjecture would also settle the simplicity conjecture in Problem 42. For further background see Le Roux [414].

14.8 Symplectic embeddings

The Gromov nonsqueezing Theorem 12.1.1 was the the first striking result about a nontrivial obstruction to the existence of a symplectic embedding and has led to many important developments in the subject. One can extend this question and ask in general when one symplectic manifold can be symplectically embedded into another of the same dimension. For example, one can ask when one $2n$-dimensional open ellipsoid can be embedded into another.

Problem 44 (Symplectic ellipsoid embeddings)
For $n \in \mathbb{N}$ *and* $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ *with* $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ *define*

$$E(a) := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} < 1 \right\}$$

Under what conditions on a_i *and* b_i *does there exist a symplectic embedding of* $E(a)$ *into* $E(b)$ *(with respect to the standard symplectic structure* ω_0*)?*

For $n = 1$ the obvious answer is if and only if $a_1 \leq b_1$. For $n = 2$ an answer to the symplectic ellipsoid embedding problem was given by McDuff [463] in 2009. This answer was later reformulated by Hutchings [338, 339] and McDuff [467]. Define the ordered sequence

$$0 < c_1(a) \leq c_2(a) \leq c_3(a) \leq \cdots ,$$

by ordering the set $\{n_1 a_1 + n_2 a_2 \,|\, n_i \in \mathbb{Z},\, n_i \geq 0,\, n_1 + n_2 > 0\}$ with multiplicities. Hutchings and McDuff proved that there is a symplectic embedding of $E(a)$ into $E(b)$ if and only if $c_k(a) \leq c_k(b)$ for all $k \in \mathbb{N}$. In Hutchings' work these numbers appear as invariants of embedded contact homology and he shows that the condition is necessary for the existence of a symplectic embedding. McDuff proved that the condition is sufficient. In dimensions $2n \geq 6$ the symplectic ellipsoid embedding problem is open in this general form. In [301] Guth constructed an embedding that shows that the obvious extension of the above condition to $n > 2$ is no longer satisfied by all ellipsoidal embeddings; however, so far no other substitute condition has been proposed. Of course, when $a_1 = a_2 = \cdots = a_n$ this becomes a question about the Gromov width of the ellipsoid $E(b)$ (see equation (12.1.2)), and the answer follows from Gromov's nonsqueezing theorem.

There are many open questions about the Gromov width, and more generally about symplectic packing in the sense of Remark 7.1.31. For example, it would be interesting to understand which manifolds are fully fillable by one ball, in the sense that they contain a symplectically embedded ball that occupies an arbitrarily large fraction of its volume. Another question concerns monotone manifolds with symplectic form normalized so that $[\omega] = c_1(\omega)$. In the four-dimensional case these manifolds are $S^2 \times S^2$ and k-point blowups of $\mathbb{CP}^2$ for $k \leq 8$, and they have integral Gromov width (the supremum of the numbers πr^2 such that there exists a symplectic embedding of the ball of radius r; see (12.1.2)). For example, the Gromov width of $\mathbb{CP}^2$ with the appropriately rescaled Fubini–Study form is 3, and for $S^2 \times S^2$ with the product form it is 2.

Problem 45 (Gromov width of monotone manifolds)
If (M, ω) is a closed symplectic manifold such that

$$[\omega] = c_1(\omega),$$

is its Gromov width at least one?

Here is a question of a different flavour.

Problem 46 (Gromov width of cohomologous symplectic forms)
Is there a closed manifold M with cohomologous symplectic forms ω_0, ω_1 such that (M, ω_0) and (M, ω_1) have different Gromov widths?

Related to the existence question is also a uniqueness problem, in this case whether two symplectic embeddings of a ball into a symplectic manifold are symplectically isotopic. By a theorem of McDuff [450], the space of symplectic embeddings of the closed ball $\overline{B}^4(r)$ into the open ball $B^4(1)$ is connected for every radius $0 < r < 1$. It is a remarkable fact that the analogous question is open in higher dimensions even for arbitrarily small radii.

Problem 47 (Ball isotopy question)
Let $n \geq 3$. Is there a constant $0 < \varepsilon < 1$ such that the space of symplectic embeddings of $\overline{B}^{2n}(\varepsilon)$ into $B^{2n}(1)$ is connected?

Problem 48 (Symplectic camel problem)
Is there any closed $2n$-dimensional symplectic manifold (M, ω) and a real number $r > 0$ such that the space of symplectic embeddings $\overline{B}^{2n}(r) \to M$ is disconnected?

For open manifolds the answer to Problem 48 is positive by Gromov's camel obstruction. Gromov's counterexamples are convex at infinity. (See the discussion on page 512 and Definition 3.5.39.) Returning to the existence problem, the following question about the existence of several disjoint symplectically embedded balls (the *packing problem*) was discussed in Remark 7.1.31.

Problem 49 (Ball packing)
For which integers $n \geq 3$ and $k \geq 2$ is

$$v_k(B^{2n}) := \sup \left\{ \frac{k\mathrm{Vol}(B^{2n}(\varepsilon))}{\mathrm{Vol}(B^{2n}(1))} \;\middle|\; \begin{array}{l} \textit{there exist symplectic embeddings} \\ \iota_i : \overline{B}^{2n}(\varepsilon) \to B^{2n}(1),\ i = 1, \dots, k, \\ \textit{with pairwise disjoint images} \end{array} \right\} < 1?$$

When $n = 2$ it is known that $v_k(B^4) < 1$ for $k = 2, 3, 5$ by a result of Gromov [287], that $v_k(B^4) < 1$ for $k = 6, 7, 8$ by a result of McDuff–Polterovich [469], and that $v_k(B^4) = 1$ for $k \geq 9$ by a result of Biran [66]. When $n \geq 3$ it is known that $v_k(B^{2n}) = 1$ for k sufficiently large by a result of Buse–Hind [89]. However, Gromov showed in [287] that the disjoint union of two open balls of radii r_1, r_2 embeds into $B^{2n}(1)$ if and only if $r_1^2 + r_2^2 \leq 1$. Hence

$$v_2(B^{2n}) = \frac{1}{2^{n-1}}$$

for all $n \geq 1$.

There are many interesting open problems about embedding other shapes. For example, one can consider the analogue of Problem 49 with domain (and perhaps also target) the polydisc

$$P(a_1, \dots, a_n) := B^2(r_1) \times \cdots \times B^2(r_n), \qquad \pi r_i^2 = a_i.$$

In general, polydiscs are harder to understand than ellipsoids, at least as domains. There are now two proofs that the closed polydisc $\overline{P(1,2)}$ embeds in the open ball $B^4(r)$ only if $\pi r^2 > 3$, one by Hind–Lisi [310] using finite energy foliations and one by Hutchings [341] that uses refined information about the chain complex of embedded contact homology.

14.9 Symplectic topology of Euclidean space

In the final section of this book we return to the symplectic topology of Euclidean space, which is where we started in Chapter 1. One open problem is whether every symplectic form on $\mathbb{R}^{2n}$ which is *standard at infinity* is diffeomorphic to the standard symplectic form.

Problem 50 (Standard-at-infinity)
Let $n \geq 3$ and let ω be a symplectic form on $\mathbb{R}^{2n}$ that agrees with ω_0 on the complement of a compact set. Is $(\mathbb{R}^{2n}, \omega)$ symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$?

The same question in dimension four has a positive answer by a celebrated theorem of Gromov [287]. A theorem of Floer–Eliashberg–McDuff [451] asserts that if (M,ω) is a symplectic manifold with $\pi_2(M) = 0$ that is symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ outside of a compact set, then M is necessarily diffeomorphic to $\mathbb{R}^{2n}$. Mark McLean and others constructed many examples of symplectic structures on $\mathbb{R}^{2n}$ that are convex at infinity, but are not symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ (see McLean [474, 475], Abouzaid–Seidel [6], and Seidel [576]). Problem 50 can be rephrased as the question of which symplectic $2n$-manifolds can have the standard $(2n-1)$-sphere as a contact boundary.

Problem 51 (Symplectic Hadamard question)
Let (M,ω) be a connected, simply connected, symplectic $2n$-manifold and let J be an ω-compatible almost complex structure such that the Riemannian metric $\langle\cdot,\cdot\rangle := \omega(\cdot, J\cdot)$ is complete and has nonpositive sectional curvature. Is (M,ω) symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$?

By Hadamard's theorem the manifold M in Problem 51 is diffeomorphic to $\mathbb{R}^{2n}$. By a theorem of McDuff [446] the question has a positive answer whenever J is integrable.

In [136], Dimitroglou Rizell constructed Lagrangian submanifolds $L \subset \mathbb{R}^{4k+2}$, diffeomorphic to $S^1 \times S^{2k}$, with infinite Gromov width (the supremum over all numbers πr^2 such that there is a symplectic embedding of $B^{2n}(r)$ into $\mathbb{R}^{2n}$ which sends the intersection of $B^{2n}(r)$ with a Lagrangian plane to L). His work builds on the ideas of Murphy [500] on *loose Legendrean knots* and of Eliashberg–Murphy [191] on *Lagrangian caps*. Such examples exist also in $\mathbb{R}^{2n}$ for any $n \geq 3$ but the question is open in $\mathbb{R}^4$.

Problem 52 (Lagrangian infinite width)
Is there a compact Lagrangian submanifold in $\mathbb{R}^4$ with infinite Gromov width?

This question brings us back to symplectic capacities as discussed in Section 12.1. Recall that a symplectic capacity is called *normalized* if the unit ball and the unit cylinder in $\mathbb{R}^{2n}$ have capacity π. A longstanding conjecture about symplectic capacities is the following.

Problem 53 (Convex capacity conjecture)
All normalized capacities agree on convex subsets of Euclidean space.

This conjecture, if true, would imply the following conjecture by Viterbo which relates the capacity of a general convex subset of Euclidean space to the capacity of the ball.

Problem 54 (Viterbo's symplectic isoperimetric conjecture)
All normalized capacities satisfy the inequality

$$\frac{c(\Sigma)}{\mathrm{Vol}(\Sigma)^{1/n}} \leq \frac{c(B)}{\mathrm{Vol}(B)^{1/n}} \tag{14.9.1}$$

for every compact convex set $\Sigma \subset \mathbb{R}^{2n}$ with nonempty interior.

It is known that the inequality (14.9.1) holds up to a constant factor which is independent of the dimension. An equivalent formulation is the inequality

$$\mathrm{Vol}(\Sigma) \geq \frac{c(\Sigma)^n}{n!}. \tag{14.9.2}$$

for every compact convex set $\Sigma \subset \mathbb{R}^{2n}$ with nonempty interior. In [46], Artstein, Karasev, and Ostrover proved that the Hofer–Zehnder capacity satisfies the inequality $c_{\mathrm{HZ}}(K \times K^*) \geq 4$ for every symmetric (i.e. $v \in K$ implies $-v \in K$) convex set $K \subset \mathbb{R}^n$ with nonempty interior. Assuming Viterbo's isoperimetric inequality for the Hofer–Zehnder capacity, one would then get

$$\mathrm{Vol}(K \times K^*) \geq \frac{4^n}{n!} \tag{14.9.3}$$

for every symmetric convex set $K \subset \mathbb{R}^n$ with nonempty interior. (Here K is the unit ball of a norm on $\mathbb{R}^n$ and $K^* \subset (\mathbb{R}^n)^*$ is the unit ball of the dual norm. It is known from Kuperberg [383] that $\mathrm{Vol}(K \times K^*) \geq \pi^n/n!$.) The inequality (14.9.3) is known as the **Mahler conjecture**. Its appearance in this context shows yet again the tight connection between the symplectic world and other ideas in geometry.