Research Statement
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Summary

My research applies stable homotopy theory to geometry, topology, and algebra. I am particularly interested in the stable homotopy groups of spheres, K-theory, and their algebro-geometric and equivariant analogs.

1. Introduction

The *k-th homotopy group* of a topological space X, denoted $\pi_k(X)$, is the group of homotopy classes of basepoint-preserving maps from the k-sphere S^k to X. The homotopy groups of spheres are especially important in algebraic topology, since every topological space can be modeled by spaces built out of spheres (CW complexes) up to weak homotopy equivalence. These groups are astonishingly hard to compute, but they eventually stabilize: there is an isomorphism $\pi_n(S^k) \cong \pi_{n+1}(S^{k+1})$ for $n \gg k$, so these groups eventually depend only on $n-k$. The *$(n-k)$-th stable stem*, π^{st}_{n-k}, is any group in this stable range.

Kervaire and Milnor [KM63] linked the stable stems to the classification of high-dimensional manifolds and smooth structures on spheres. For instance, they showed that there exists an exotic $4k$-dimensional sphere[1] if and only if a particular quotient of π^{st}_{4k} is nontrivial. One amazing consequence of the recent work of Hill–Hopkins–Ravenel [HHR16] is that exotic spheres exist in all odd dimensions, except for dimensions 1, 5, 13, 29, 61, and possibly 125. There are many remaining open questions in this area which continue to motivate the development of powerful new techniques in stable homotopy theory, including recent work of Li–Shi–Wang–Xu [LSWX19], Behrens–Hill–Hopkins–Mahowald [BHHM20], Isaksen–Wang–Xu [IWX20], and my own work [BMQ20, Qui21a].

My research applies stable homotopy theory outside of classical topology. In algebraic geometry, this idea extends back over half a century: for example, Grothendieck's proof of the Weil Conjectures relied on étale cohomology, a 'cohomology theory' for algebraic varieties [Gro60]. More recently, Voevodsky and Rost's [Voe03, Voe10, Voe11] celebrated proof of the Bloch–Kato Conjecture, which identified étale cohomology with an arithmetic invariant called Milnor K-theory, used *motivic stable homotopy theory*, the stable homotopy theory of algebraic varieties.

The motivic analogs of stable homotopy groups play a crucial role in the study of algebraic cohomology theories: they are the 'initial' motivic invariant, so there is a comparison map from the motivic stable homotopy groups of an algebraic variety to any of its other cohomology groups. Morel [Mor12] and Röndigs, Spitzweck, and Østvær [RSØ19] have shown that the motivic stable stems sit in exact sequences with important arithmetic invariants, including motivic cohomology, Milnor–Witt K-groups, Milnor K-groups, and Hermitian K-groups. Significant recent advances on the motivic stable stems have been made in the past decade, such as in [DI10, Mor12, DI17, WØ17, RSØ19, BGI21] and some of my projects [Qui19, Qui21b, Qui21c, CQ21].

To illustrate these ideas in my own work, I will briefly describe my results on the *motivic J-homomorphism* (details appear in Section 3). Classically, the J-homomorphism $J^{\mathrm{cl}} : \pi_* SO \to \pi^{\mathrm{st}}_*$ relates the well-known homotopy groups of the infinite special orthogonal group[2] to the stable homotopy groups of spheres [Whi42]. Adams [Ada66] showed that the J-homomorphism detects

[1] A sphere which is homeomorphic, but not diffeomorphic, to the sphere with its standard smooth structure inherited from Euclidean space.

[2] Bott periodicity implies $\pi_* SO$ is 8-periodic: starting with $* = 0$, the groups repeat the pattern $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0, $\mathbb{Z}$.

interesting periodic families of elements in the stable stems. He also showed that the size of its image coincides with the reciprocals of certain special values of the Riemann zeta function.

The motivic J-homomorphism [HKO11, BH21], J^{mot}, relates algebraic K-groups to the motivic stable stems. Unlike its classical counterpart, the image of J^{mot} is largely unknown. This disparity stems from a fundamental difference between topological spaces and algebraic varities: algebraic vector bundles are more complicated than topological vector bundles, and consequently, the source of J^{mot} is much more complicated that the source of J^{cl}.

In [CQ21], D. Culver and I circumvented this issue with algebraic K-groups by introducing a new tool, the *kq-resolution*, which 'resolves' the motivic stable stems by tensor products of algebraic K-groups. This makes it possible to express the image of J^{mot} in terms of subquotients of algebraic K-groups without actually computing algebraic K-groups explicitly. The *kq*-resolution was inspired by Mahowald's '*bo*-resolution' [Mah81], a resolution of the classical stable stems by topological K-groups, which was used by Davis and Mahowald [DM89] to compute the image of J^{cl} three decades ago. Culver and I applied our *kq*-resolution to compute the image of J^{mot} over algebraically closed fields of characteristic zero. We also produced new infinite periodic families in the motivic stable stems and established connections with the Riemann zeta function. Moreover, we recovered computations [GI15, AM17] of certain Witt and Balmer–Witt groups which arise in algebraic number theory.

Many of my other projects follow a similar theme. By extending well-understood ideas from stable homotopy theory beyond their original context, I am able to produce interesting new results in algebra, geometry, and topology.

The remainder of this research statement is broken into two parts. In Section 2, I present a detailed narrative of my work on Hopf elements, an important family of elements in the stable stems with surprisingly broad applications. I recall the classical Hopf elements and the Hopf invariant one problem, discuss motivic Hopf elements and their connections to number theory, and explain my work on the motivic Hopf invariant one problem and its implications in motivic stable homotopy theory and algebraic geometry. In the remaining sections, I briefly summarize my research on five other topics around stable homotopy theory and its generalizations:

- Kervaire–Milnor sequences, which connect the stable stems, the geometry of high-dimensional manifolds, and smooth structures on spheres (Section 3).
- Mahowald invariants, which can be used to construct interesting maps between high-dimensional spheres (Section 4).
- Trace methods, an approach to computing algebraic K-groups using stable homotopy theory (Section 5).
- Real algebraic K-theory, a generalization of algebraic K-theory with applications in number theory and geometry (Section 6).
- Equivariant algebra, a generalization of commutative and homological algebra with long-term applications to equivariant algebraic K-theory (Section 7).

2. Hopf elements

Let $f : S^{2n-1} \to S^n$. Since $S^{2n-1} \cong \partial D^{2n}$, we can form a $2n$-dimensional CW complex

$$C(f) = e^{2n} \cup_f S^n$$

using f as an attaching map. The integral cohomology of $C(f)$ is then

$$H^*(C(f);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, n, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Let α be a generator for $H^n(C(f);\mathbb{Z})$ and β a generator for $H^{2n}(C(f);\mathbb{Z})$. The *Hopf invariant of* f, $\mathrm{HI}(f)$, is the integer (unique up to a sign) such that

$$\alpha^2 = \alpha \cup \alpha = \mathrm{HI}(f) \cdot \beta.$$

The integral cohomology of a space is a graded commutative ring, so $\alpha^2 = 0$ and $\mathrm{HI}(f) = 0$ when n is odd. On the other hand, there exists a map $f : S^{2n-1} \to S^n$ of Hopf invariant two whenever n is even.[3] The most interesting question is when there exists a map $f : S^{2n-1} \to S^n$ of Hopf invariant one.

Existence and nonexistence are both interesting. If there is *not* a map $f : S^{2n-1} \to S^n$ of Hopf invariant one, then S^{n-1} is not an H-space, S^{n-1} is not parallelizable, and $\mathbb{R}^n$ is not a normed division algebra over the reals. On the other hand, for $n = 2$, 4, and 8, elements of Hopf invariant one can be constructed quite explicitly. For instance, when $n = 2$, the composite

$$\eta : S^3 \cong S(\mathbb{C}^2) \xrightarrow{\pi} \mathbb{CP}^1 \cong S^2 \in \pi_3(S^2) \tag{1}$$

has Hopf invariant one. Similar composites for the quaternions and octonions give Hopf invariant one elements $\nu \in \pi_7(S^4)$ and $\sigma \in \pi_{15}(S^8)$.

Adams [Ada60] proved that there exists an element of Hopf invariant one in $\pi_{2n-1}(S^n)$ if *and only if* $n = 2$, 4, or 8 using the *Adams spectral sequence*, a tool for computing the stable homotopy groups of spheres. There is an element $f : S^{2n-1} \to S^n$ with $\mathrm{HI}(f) = 1$ if and only if a particular *potential Hopf element* survives in the spectral sequence. The potential Hopf elements form a submodule in the E_2-term of the Adams spectral sequence

$$(\text{submodule of potential Hopf elements}) = \mathbb{F}_2\{h_0, h_1, h_2, \ldots\}, \tag{2}$$

where h_n detects a map of Hopf invariant one $f : S^{2^{n+1}-1} \to S^{2^n}$ if it survives. Adams showed $d_2(h_n)$ is nontrivial for $n \geq 4$, so the only surviving elements are h_0, h_1, h_2, and h_3, where h_0 detects the degree two self-map of the sphere and h_1, h_2, and h_3 detect η, ν, and σ, respectively.

2.1. **The first motivic Hopf map.** Morel and Voevodsky [MV99] introduced *motivic homotopy theory*, or the homotopy theory of algebraic varieties, to apply homotopical techniques to problems in algebraic geometry and number theory. In motivic homotopy theory, topological spaces are replaced by *motivic spaces*, or presheaves of spaces on the Nisnevich site of smooth schemes over a base field F. In simpler terms, a motivic space assigns an ordinary topological space to each smooth scheme over F, subject to some technical constraints. A *homotopy* between motivic spaces is the same as a classical homotopy, except that the unit interval is replaced by the affine line. In motivic homotopy theory, there are two different circles:

(1) The *simplicial circle* $S^{1,0}$ is the motivic space which assigns an ordinary topological circle to each smooth F-scheme.
(2) The *geometric circle* $S^{1,1}$ is the motivic space represented by the multiplicative group scheme $\mathbb{G}_m = \mathrm{Spec}(F[x, x^{-1}])$.

These are not homotopy equivalent[4], so smashing them together produces *motivic spheres* $S^{m,n}$ which are bigraded.[5] Some of these are homotopy equivalent to familiar varities: for instance, $S^{2,1} = S^{1,0} \wedge S^{1,1}$ is homotopy equivalent to $\mathbb{P}^1$. Motivic homotopy groups are defined by considering homotopy classes of maps out of these motivic spheres, and motivic stable homotopy

[3] For those familiar with unstable homotopy theory, the Whitehead product of the identity map on S^{2n} with itself has Hopf invariant two.
[4] For instance, their motivic cohomology groups are different.
[5] The first coordinate in the bigrading, m, is the dimension of the motivic sphere and the second coordinate, n, is the motivic weight.

groups are defined by stabilizing with respect to suspension in both coordinates. The motivic stable stems are usually significantly more complicated than the classical stable stems.

The motivic stable stems form a graded commutative ring. Many classical procedures for studying commutative rings, such as completion, localization, or inverting an element, are quite useful for studying the motivic stable stems. The first Hopf map η (1) has a motivic lift: the composite

$$\eta_F : S^{3,2} \simeq \mathbb{A}^2_F \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^1_F \simeq S^{2,1} \tag{3}$$

represents a nontrivial element in $\pi^F_{3,2}(S^{2,1})$. Morel [Mor12] showed that η_F is not nilpotent, in contrast with the classical Hopf map which satisfies $\eta^4 = 0$. Most elements in the motivic stable stems are η-torsion, so inverting η_F produces a simpler ring we can hope to compute.

Surprisingly, a great deal of arithmetic information is preserved after inverting η_F. Morel [Mor12] showed that the η_F-periodic motivic stable stems are isomorphic to the Witt ring of symmetric bilinear forms, and Bachmann and Hopkins [BH20] connected them to connective Balmer–Witt groups. The 2-completed η_F-periodic stable stems have been computed explicitly over many base fields F: Guillou–Isaksen [GI15] and Andrews–Miller [AM17] for $F = \bar{F}$ algebraically closed of characteristic zero; Guillou–Isaksen [GI16] for $F = \mathbb{R}$; and Wilson [Wil18] over finite fields, the p-adic rationals, and the rationals.

In classical stable homotopy theory, Mahowald's *bo-resolution* [Mah81] is a powerful tool for studying stable homotopy groups after inverting certain non-nilpotent elements.[6] Note that any element which is not torsion with respect to a non-nilpotent element gives rise to an infinite family of nontrivial elements: if v is non-nilpotent and x is not v-torsion, then the family of elements $\{x, vx, v^2x, \ldots\}$ is an infinite family of period $|v|$. As mentioned in the introduction, the *bo*-resolution 'resolves' the stable stems by the topological K-groups of a point. Since topological K-groups are 8-periodic, this resolution naturally isolates the 8-periodic elements from the more complicated elements in the stable stems.

To understand periodic phenomena in the motivic stable stems, D. Culver and I introduced the *kq-resolution*, a motivic analog of the *bo*-resolution. According to work of Ananyevskiy, Röndigs, and Østvær [ARØ20], the motivic cohomology theory *kq*, the very effective cover of Hermitian K-theory, is the motivic analog of *bo*. There are two interesting periodicities in *kq*: an $(8, 4)$-fold periodicity called *v_1-periodicity* and a $(1, 1)$-fold periodicity called η_F-periodicity. Culver and I leveraged the first kind of periodicity to completely classify $(8, 4)$-periodic elements in the motivic stable stems over algebraically closed fields, and we leveraged the second kind of periodicity to recover the η_F-periodic computations of Guillou–Isaksen and Andrews–Miller. Our computation of the $(8, 4)$-periodic elements is closely related to the motivic J-homomorphism discussed in the introduction.

Theorem 2.1 (Culver–Q., [CQ21, Thm. C])**.** Let $F = \bar{F}$ be an algebraically closed field of characteristic zero. The v_1-periodic 2-complete motivic stable stems are isomorphic to

$$\mathbb{F}_2[\tau, h_0, h_1, v_1^4]/(h_0h_1, h_0v_1^4, \tau h_1^3) \oplus \bigoplus_{k\geq 0} \Sigma^{4k-1,2k}\mathbb{Z}/2^{\rho(k)}[\tau] \oplus \mathbb{F}_2[\tau, h_1, v_1^4]/(\tau h_1^3),$$

where $|h_0| = (0, 0)$, $|h_1| = (1, 1)$, $|v_1^4| = (8, 4)$, $|\tau| = (0, -1)$, and $\rho(k)$ is the 2-adic valuation of $8k$.

Theorem 2.2 (Guillou–Isaksen; Andrews–Miller; Culver–Q., [CQ21, Thm. E])**.** Let $F = \bar{F}$ be an algebraically closed field of characteristic zero. The η-periodic 2-complete F-motivic stable

[6] For those familiar with chromatic homotopy theory, Mahowald used the *bo*-resolution to prove the 2-primary Telescope Conjecture at height one.

stems are given by
$$\pi^F_{**}(S^{0,0})^\wedge_2[\eta^{-1}] \cong \mathbb{F}_2[\eta^{\pm 1}, v_1^4]\{x, y\}$$
where $|v_1^4| = (8,4)$, $|x| = (0,0)$, and $|y| = (8,5)$.

Preliminary computations over more general base fields, such as finite fields, the p-adic rationals, and the real numbers, suggest that the kq-resolution can be used to prove analogous results more generally.

Goal 2.3. Describe the η-periodic motivic stable stems over more general base fields using the kq-resolution.

Culver and I were surprised to find that the kq-resolution also has applications beyond understanding periodicity in the stable stems. In particular, we showed that the kq-resolution can be used to understand "low-dimensional Milnor–Witt stems", or motivic stable homotopy groups where the gap between topological dimension and motivic weight is small.

Theorem 2.4 (Culver–Q., [CQ21, Thm. B])**.** The n-th Milnor–Witt stem of a field is detected by elements in filtration at most n in the kq-resolution.

We completely analyzed the kq-resolution in filtrations zero and one over algebraically closed fields, so the first two Milnor–Witt stems of algebraically closed fields can be extracted from our computations. More generally, these low-dimensional computations are related to *Morel's* π_1 *Conjecture* [Mor12], recently proven by Röndigs, Spitzweck, and Østvær [RSØ19], which relates three invariants of the base field: Milnor K-theory [Mil70], the first motivic stable stem, and Hermitian K-theory. Röndigs, Spitzweck, and Østvær [RSØ21] recently related the second motivic stable stem to other arithmetic invariants, such as motivic cohomology.

Although Culver and I only analyzed the kq-resolution in filtrations zero and one, we expect that higher filtrations will also be accessible by extending classical analyses of the bo-resolution to the motivic setting. For instance, motivic analogs of results of Lellmann–Mahowald [LM87] and Beaudry–Behrens–Bhattacharya–Culver–Xu [BBB+20] on the bo-resolution would allow us to compute the first nine Milnor–Witt stems.

Goal 2.5. Describe low-dimensional Milnor–Witt stems over general base fields using the kq-resolution.

To approach these more complicated computations, Culver and I are currently working on one additional piece of motivic technology. Classically, the bo-resolution is most easily understood using certain cohomology theories called *Brown–Gitler spectra*. These were originally introduced to study immersions of manifolds [BJG73], but they were applied quite effectively by Mahowald [Mah81], Lellmann–Mahowald [LM87], and others [BBB+20] to understand the bo-resolution in a large range.

Goal 2.6. Define motivic Brown–Gitler spectra using the *motivic lambda algebra* described in the next section.

2.2. **Additional motivic Hopf elements.** Classically, Brown–Gitler spectra are defined using the *lambda algebra*, a differential graded associative algebra whose homology is the E_2-term of the Adams spectral sequence. As a starting point for defining motivic Brown–Gitler spectra, Culver and I worked with W. Balderrama to define the *motivic lambda algebra* [BCQ21, Thm. A]. Our main goal with the motivic lambda algebra is to define motivic Brown–Gitler spectra, but we have already explored other interesting applications in motivic homotopy theory and algebraic geometry.

The element $\eta_F \in \pi^F_{3,2}(S^{2,1})$ is not the only element with Hopf invariant one in the motivic stable stems. In [DI13], Dugger and Isaksen produced additional motivic Hopf maps $\nu_F \in$

$\pi^F_{7,4}(S^{4,2})$ and $\sigma_F \in \pi^F_{15,8}(S^{8,4})$. Balderrama, Culver, and I have investigated the existence of additional Hopf invariant one elements in the motivic stable stems using the motivic lambda algebra.

We began by identifying the submodules of potential Hopf elements over various base fields (cf. (2) for the classical result). We then computed all of the d_2-differentials on these potential Hopf elements. In both steps, we used the motivic lambda algebra to compute an additive basis for the E_2-term of the motivic Adams spectral sequence in low filtrations and to determine key multiplicative relations which were inaccessible using other techniques. Unlike in the classical Hopf invariant one problem, the potential motivic Hopf elements sometimes supported differentials of length greater than two.[7] Our most interesting computations were over the real numbers:

Theorem 2.7 (Balderrama–Culver–Q., special case of [BCQ21, Thms. E and H])**.** The submodule of potential Hopf elements over the reals is isomorphic to

$$\mathbb{F}_2[\rho]\{h_1, h_2, h_3, \ldots\} \oplus \mathbb{F}_2[\tau^2]\{h_0\} \oplus \bigoplus_{n\geq 1} \mathbb{F}_2[\tau^{2^{n+1}}, \rho]/(\rho^{2^n})\{\tau^{2^{n-1}} h_n\}.$$

The submodule of these elements which survive to the E_∞-term of the $\mathbb{R}$-motivic Adams spectral sequence, i.e. the submodule which could detect a motivic Hopf invariant one element, is contained in

$$\begin{aligned}\mathbb{F}_2[\rho]\{h_1, h_2, h_3, \rho h_4\} \oplus \mathbb{F}_2[\tau^2]\{h_0\} \oplus \bigoplus_{n=1}^{3} \mathbb{F}_2[\tau^{2^{n+1}}, \rho]/(\rho^{2^n})\{\tau^{2^{n-1}} h_n\} \\ \oplus \bigoplus_{n\geq 4} \mathbb{F}_2[\tau^{2^{n+1}}, \rho]\{\rho^{2^{n-1}-1}\tau^{2^{n-1}} h_n\}/(\rho^{2^n}).\end{aligned}$$

The second expression above is exciting because it suggests an unexpected connection between $\mathbb{R}$-motivic Hopf invariant one elements and Mahowald's η_j-family [Mah77], a famous family of elements in the classical stable stems. The generators in the last summand of the Hopf elements above have precisely the same dimensions as the η_j elements.

Goal 2.8. Reconstruct Mahowald's η_j-family using $\mathbb{R}$-motivic stable homotopy theory.

Our computations also had applications in unstable motivic homotopy theory, algebraic geometry, and abstract algebra. We proved that the nonexistence of maps of Hopf invariant one between motivic spheres implies the nonexistence of H-space structures on motivic spheres in certain dimensions [BCQ21, Thm. F]. Consequently, our results imply that many affine varieties which model motivic spheres do not admit group structures, or equivalently, that many rings do not admit commutative Hopf algebra structures.

For example, Asok, Doran, and Fasel [ADF17] have shown that the affine quadric hypersurfaces

$$Q_{2n-1} = \operatorname{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle \sum x_i y_i - 1\rangle),$$
$$Q_{2n} = \operatorname{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/\langle \sum x_i y_i - z(1+z)\rangle)$$

are models for the motivic sphere $S^{2n-1,n}$ and $S^{2n,n}$, respectively. Our computations imply that Q_n rarely admits a group structure, or equivalently, the ring defining Q_n rarely admits a commutative Hopf algebra structure over the base field F. For example:

Theorem 2.9 (Balderrama–Culver–Q., special case of [BCQ21, Thm. H])**.** Let $F = \mathbb{F}_q$ be a finite field of odd order. The variety Q_n admits an H-space structure only if $n = 0,\ 1,\ 3,\ 7$.

[7] These longer differentials are forced to exist by comparison with known Milnor–Witt K-groups.

Finally, the motivic lambda algebra opens the door to potentially groundbreaking new machine-assisted computations in motivic stable homotopy theory. Using simple programs written in Lisp and Python, Balderrama and I have recomputed the E_2-term of the motivic Adams spectral sequence [DI10, DI17] in a large range. We are working to improve our algorithms, which would lead to substantial advances.

Goal 2.10. With machine assistance, compute the E_2-term of the motivic Adams spectral sequence beyond the known range using the motivic lambda algebra.

3. Kervaire–Milnor sequences

In [Mil56], Milnor produced the first *exotic sphere*: a 7-sphere which is homeomorphic, but not diffeomorphic, to S^7 with its standard smooth structure. One of the classical applications of stable homotopy theory is understanding in which dimensions there exists an exotic sphere.

Kervaire and Milnor [KM63] introduced the group Θ_n of *homotopy n-spheres*, i.e. the group of h-cobordism classes of smooth n-manifolds which are homotopy equivalent to S^n, and showed that for $n \geq 5$, $\Theta_n = 0$ if and only if there is a unique differentiable structure on S^n. They showed $\Theta_{4k} \cong \operatorname{coker} J_{4k}$ and that there are exact sequences

$$0 \to \Theta^{bp}_{2k+1} \to \Theta_{2k+1} \to \operatorname{coker} J_{2k+1} \to 0,$$

$$0 \to \Theta_{4k+2} \to \operatorname{coker} J_{4k+2} \xrightarrow{\Phi_K} \mathbb{Z}/2 \to \Theta^{bp}_{4k+1} \to 0.$$

Here, $\operatorname{coker} J$ is the cokernel of the J-homomorphism, $\Theta^{bp}_n \subseteq \Theta_n$ is the subgroup of homotopy n-spheres which bound a parallelizable manifold, and Φ_K is the Kervaire invariant. The groups Θ^{bp}_{2k+1} are well-understood, but $\operatorname{coker} J$ and Φ_K are more mysterious.

3.1. **The J-homomorphism and its motivic analog.**

The J-homomorphism $J : \pi_n SO \to \pi_n^{\mathrm{st}}$ is a homomorphism from the homotopy groups of the infinite special orthogonal group to the stable homotopy groups of spheres. Thinking of an element of $SO(q)$ as a transformation $S^{q-1} \to S^{q-1}$, we see that an element of $\pi_n SO(q)$ is represented by a map $S^n \times S^{q-1} \to S^{q-1}$. The Hopf construction then produces a map $S^{n+q} \cong S^n * S^{q-1} \to S(S^{q-1}) \cong S^q$ which represents an element in $\pi_{n+q}(S^q)$. Letting q tend to infinity yields the J-homomorphism.

The homotopy groups $\pi_* SO$ are known by Bott periodicity [Bot59]. In [Ada66], Adams determined the image of J.[8] Most interestingly, he showed $J_{4k-1} \cong \mathbb{Z}/2^{\rho(k)}$, where $\rho(k)$ is the 2-adic valuation of $8k$. Equivalently, $\rho(k)$ is the denominator of $\zeta(1-2k)$.

Roughly speaking, the *motivic J-homomorphism* [BH21, HKO11] is obtained by replacing $SO(q)$ by $GL_q(F)$ (or another matrix group) in the definition of the classical J-homomorphism. Bachmann and Hopkins [BH20] defined a motivic cohomology theory j_o. The image of the motivic J-homomorphism is expected to be a direct summand of $j_{o*}(F)$.

Culver and I computed the 2-completed motivic stable homotopy groups of j_o over algebraically closed fields of characteristic zero using the kq-resolution [CQ21, Thm. D]. The homotopy groups of j_o are detected in the 0- and 1-lines of the kq-resolution. Our preliminary computations over other base fields suggest these can often be computed explicitly:

Goal 3.1. Compute $\pi^F_{**}(j_o)$ over more general base fields using the kq-resolution.

[8]Technically, Adams determined the image up to a factor of two in degrees $n \equiv 3 \mod 4$. The factor was removed in subsequent work of Quillen [Qui71], Sullivan [Sul74], and Becker–Gottlieb [BG75].

3.2. **Nontriviality of Θ_n.** Returning to the classical Kervaire–Milnor sequence, the analysis of Θ_n can be divided into two cases: n odd and n even. When n is odd, work of Browder [Bro69], Hill–Hopkins–Ravenel [HHR16], and Wang–Xu [WX17] implies that $\Theta_n = 0$ if and only if $n = 1, 5, 13, 29, 61$, and perhaps 125. When n is even, the situation is less clear.

To show that $\Theta_n \neq 0$ for n even, it suffices to find a nontrivial element of Kervaire invariant zero in coker J_n. Building on work of Behrens–Hill–Hopkins–Mahowald [BHHM20], M. Behrens, M. Mahowald, and I proved the following theorem:

Theorem 3.2 (Behrens–Mahowald–Q., [BMQ20, Cor. 1.4])**.** There exist exotic spheres in over half of the even dimensions. More precisely, $\Theta_n \neq 0$ for more than half of the even congruence classes of n mod 192. These congruence classes can be listed explicitly.

Our result used *topological modular forms*, a cohomology theory combining information from the stable homotopy groups of spheres and the ring of weakly holomorphic integral modular forms. The *Hurewicz homomorphism* is a ring homomorphism

$$\pi_*^{\mathrm{st}} \to \mathit{tmf}_*.$$

The target is 192-periodic, so it detects infinite families of long periodicity in the stable stems. We completely computed the image of the 2-primary Hurewicz homomorphism [BMQ20, Thm. 1.2] using the *tmf-resolution*, an analog of the *bo*-resolution. Our analysis used several new spectral sequences to access this resolution. I plan to use these to construct new elements in the stable stems using Mahowald invariants, as discussed in Section 4.

3.3. **Kervaire invariant one.** The Kervaire invariant $\Phi_K : \mathrm{coker}\, J_{4k+2} \to \mathbb{Z}/2$ also figures into the Kervaire–Milnor sequences. Determining the dimensions where there exists a smooth framed manifold of Kervaire invariant one was a longstanding open problem in geometric topology. Hill, Hopkins, and Ravenel [HHR16] proved that this is only possible in dimensions $2, 6, 14, 30, 62$, and possibly 126 using *equivariant stable homotopy theory*, or the stable homotopy theory of spaces with group actions.

One key step in the work of Hill–Hopkins–Ravenel was producing a genuine C_8-equivariant cohomology theory, Ω, whose fixed points detected "potential Kervaire classes". They studied these fixed points using the *slice spectral sequence*, a computational tool introduced by Dugger [Dug99] as an equivariant analog of Voevodsky's [Voe02] motivic slice spectral sequence.

In [CKQ21, Thm. D], Culver, H. Kong, and I reproduced some C_2-equivariant slice differentials appearing in Dugger's work using the *algebraic slice spectral sequence*, a new tool combining the motivic Adams spectral sequence and the motivic slice spectral sequence. This is a motivic version of the algebraic Atiyah–Hirzebruch spectral sequence which appears throughout stable homotopy theory.

Bruner and Greenlees [BG03, BG10] used the algebraic Atiyah–Hirzebruch spectral sequence to compute the connective unitary and orthogonal K-theory of classifying spaces of finite groups. The motivic cohomology of classifying spaces appears throughout algebraic geometry [Tot99].

Goal 3.3. Use the algebraic slice spectral sequence to compute the connective algebraic and Hermitian K-theory of finite groups.

3.4. **Hyperreal oriented cohomology theories.** Let X be a space with involution. A *Real vector bundle* over X is a complex vector bundle $p : E \to X$ where the action of complex conjugation on the total space is compatible with the involution on X. A *Real orientable* cohomology theory is a cohomology theory for spaces with involution which has Thom isomorphisms for Real vector bundles. In [Ati66], Atiyah defined *topological K-theory with reality*, $K\mathbb{R}^{\mathrm{top}}$, using Real vector bundles. $K\mathbb{R}^{\mathrm{top}}$ simultaneously encodes unitary and orthogonal K-theory: unitary K-theory is its underlying nonequivariant theory and orthogonal K-theory is its C_2-fixed points.

$K\mathbb{R}^{\text{top}}$ is the first in a family of Real orientable cohomology theories called *real Johnson–Wilson theories* $E\mathbb{R}(n)$, $n \geq 0$. Li, Shi, Wang, and Xu [LSWX19] showed that as n increases, the Hurewicz images of the fixed points $E\mathbb{R}(n)^{C_2}$ grows larger.

G. Li, V. Lorman, and I [LLQ19, Thm. C] showed that the Tate construction (an analog of Tate cohomology in stable homotopy theory) of $E\mathbb{R}(n)$ splits into infinitely many copies of $E\mathbb{R}(n-1)$. By taking fixed points, this splitting relates the complicated Hurewicz image of $E\mathbb{R}(n)^{C_2}$ to the simpler Hurewicz image of $E\mathbb{R}(n-1)^{C_2}$ (cf. [LLQ19, Thm. A]). Our results generalized earlier work of Greenlees–May [GM95], which was the $n = 1$ case of our result, and Ando–Morava–Sadofsky [AMS98], which was the nonequivariant version of our result. The key idea in our results was *mixed genuineness*, which I discuss further in Section 6.

The C_8-equivariant cohomology theory Ω appearing in the work of Hill–Hopkins–Ravenel is a *hyperreal oriented cohomology theory*. Currently, I am working with H. Chatham, Li, and Lorman to extend our Tate splitting to hyperreal analogs of $E\mathbb{R}(n)$ developed by Beaudry–Hill–Shi–Zeng [BHSZ20]. These hyperreal theories have quite large Hurewicz images, so the splitting we expect to prove should have interesting applications towards the stable stems.

Goal 3.4. Prove that the Tate construction of the fixed points of hyperreal Johnson–Wilson theory splits into a sum of fixed points of simpler hyperreal Johnson–Wilson theories.

4. The Mahowald invariant

The degree n self-maps of the sphere, the Hopf elements, and their composites are nice geometric representatives for classes in the low-dimensional stable stems, but more elaborate constructions are necessary to understand higher dimensions. The *Mahowald invariant*, $M(-)$, takes a nontrivial class in the stable stems and produces a nontrivial class in a higher stable stem.

The Mahowald invariant interacts with many other objects in algebraic topology. For example, the Mahowald invariant can be used to reproduce the Hopf elements starting with just the degree two self-map of the sphere: $\eta = M(2)$, $\nu = M(\eta)$, and $\sigma \in M(\nu)$. It is also connected to large-scale periodicity in the stable stems: Mahowald and Ravenel [MR87] conjectured that the Mahowald invariant of a v_n-periodic class is v_{n+1}-periodic. Heuristically, this means that if we begin with a family of elements in the stable stems of some short periodicity, their Mahowald invariants will form a new family of elements with some longer periodicity. For example, the degree 2^n-self maps of the sphere form a 0-periodic family (they are all elements in the zero-th stable stem), while their Mahowald invariants assemble into 8-periodic families.

4.1. Homotopy Greek letter elements.

Greek letter elements are infinite, periodic families in the stable homotopy groups of spheres. Adams [Ada66], Smith [Smi70], and Toda [Tod71] constructed early examples, which led to a general procedure which is effective in many cases, but only works if certain finite complexes can be defined. An alternative construction, due to Miller, Ravenel, and Wilson [MRW77], defines *algebraic Greek letter elements* in the E_2-term of a particular spectral sequence converging to the stable stems, and then defines the Greek letter elements to be whatever those classes detect. This definition always makes sense, but some of the resulting Greek letter elements are zero.

In [MR93], Mahowald and Ravenel defined homotopy Greek letter elements. The *p-primary i-th homotopy Greek letter element* is defined by $\alpha_i^h := M(p^i)$, $\beta_i^h := M(M(p^i))$, and so on. These elements are always defined and always nonzero. Furthermore, empirical evidence suggests that homotopy Greek letter elements coincide with the Greek letter elements as defined above whenever the latter are defined and nonzero. The homotopy α-family was computed by

Mahowald–Ravenel [MR93] and Sadofsky [Sad92], and the homotopy β-family was studied extensively in the odd-primary setting [Beh06] and low dimensions at the prime two [Beh07] by Behrens.

A full computation of the 2-primary homotopy β-family could shed light on the existence of exotic spheres because almost every element in the family is expected to be a nontrivial element in coker J with trivial Kervaire invariant. In [Qui21a, Thm. 1.2], I carried out the first step in a program to compute the 2-primary homotopy β-family. More precisely, I computed a *tmf*-based approximation to the β-family using extensive computations related to the *tmf*-homology of infinite real projective space.

The corresponding computation for the α-family was made using *bo* by Mahowald and Ravenel [MR93]. Behrens [Beh07] showed that the *bo*-based approximations can be lifted to a full computation of the α-family using the *bo*-resolution. In future work, I plan to lift the *tmf*-based approximations to the β-family using the *tmf*-resolution (cf. Section 3):

Goal 4.1. Completely compute the 2-primary homotopy β-family using the *tmf*-resolution.

4.2. **Generalized Mahowald invariants.** Periodicity in the motivic and equivariant stable stems is still not well-understood. All classical forms of periodicity exist in the motivic and equivariant stable stems, but new, exotic forms of periodicity also occur. For instance, the first Hopf map $\eta \in \pi_1^{\mathrm{st}}$ is nilpotent, but its motivic counterpart η_F is not. Andrews [And18] produced $\mathbb{C}$-motivic exotic families analogous to the classical periodic families appearing in the image of the J-homomorphism, but instead of the classical 8-fold periodicity, Andrews' families are 20-periodic.

In [Qui19, Qui21c, Qui21b], I defined the *motivic Mahowald invariant* and used it to construct new periodic families in the motivic stable stems.

Theorem 4.2 (Q., [Qui19, Thms. 5.12 and 5.17], [Qui21c, Thms. A and B], [Qui21b, Thms. 4.5 and 4.6])**.** Over any base field of characteristic not two, the motivic Mahowald invariants of the elements $(2+\rho\eta)^i$ where $i \geq 1$ and $i \equiv 2, 3 \mod 4$, contain motivic lifts of Adams' classical periodic families.

Over the complex numbers, the motivic Mahowald invariants of the elements η^i where $i \geq 1$ contain Andrews' exotic periodic elements. Over the real numbers, the same motivic Mahowald invariants contains $\mathbb{R}$-motivic lifts of Andrews' elements.

These computations relied on an interesting technique in motivic and equivariant stable homotopy theory. There are comparison maps from the motivic stable stems to the classical stable stems which are natural with respect to the Mahowald invariant. I was able to understand these maps well enough to use the classical results of Mahowald and Ravenel [MR93] to bound the dimension of the motivic Mahowald invariants. It should be possible to push these comparison techniques further after I compute the classical homotopy β-family:

Goal 4.3. Produce an "exotic homotopy β-family" by computing the iterated motivic Mahowald invariants of powers of η_F.

5. Trace methods

The Lichtenbaum–Quillen Conjecture relates the stable stems, zeta functions, and algebraic K-groups. As discussed above, Adams showed that the stable stems and special values of the Riemann zeta function are related by the J-homomorphism. Quillen [Qui72] showed that the algebraic K-groups of finite fields are related to special values of their Dedekind zeta functions.[9] Quillen [Qui73] also proved that the algebraic K-groups of rings of integers in number fields are

[9]More precisely, $|K_{2i-1}(\mathbb{F}_q)| = |\zeta(\mathbb{F}_q, -i)|^{-1}$.

finitely generated, which combined with work of Borel [Bor72], implies that the rank of these algebraic K-groups equals the order of the zero of $\zeta(F, -i)$.

Hesselholt and Madsen [HM03] proved the Lichtenbaum–Quillen Conjecture for local fields using *trace methods*. Algebraic K-theory is extraordinarily difficult to compute in general, but it can often be analyzed using a series of maps to simpler invariants,

$$K \to TC \to THH \to HH.$$

Dundas, Goodwillie, and McCarthy [DGM12] showed that $K \to TC$ is quite close to an isomorphism after p-completion. Trace methods proceed by computing these invariants from right-to-left, beginning with the completely algebraic HH, using stable homotopy theory to pass to THH, and then using equivariant stable homotopy theory to pass to TC.

5.1. **The Red-Shift Conjecture.** The special values of the Riemann zeta function which appear in the Lichtenbaum–Quillen Conjecture also appear in the image of J-homomorphism discussed in Section 3. This is not a coincidence: Quillen showed that the composite

$$\operatorname{im} J_{4k-1} \hookrightarrow \pi^{\mathrm{st}}_{4k-1} \to K_{4k-1}(\mathbb{Z})$$

is injective. Therefore the periodic families in the stable stems detected by the J-homomorphism are also detected in the algebraic K-theory of the integers.

Using the "brave new algebra" developed by Elmendorff–Kriz–Mandell–May [EKMM07], algebraic K-theory extends to an invariant of multiplicative cohomology theories.[10] The Ausoni–Rognes red-shift philosophy [AR08] generalizes Quillen's connection between periodic families in the stable stems and algebraic K-groups. It suggests that algebraic K-theory increases chromatic complexity. In practice, this means that the algebraic K-groups of a cohomology theory detect more elements in the stable stems than the cohomology theory itself.

In [AKQ19], G. Angelini-Knoll and I studied red-shift phenomena for the algebraic K-theory of $y(n)$, $n \geq 0$, a cohomology theory with chromatic complexity n. For certain n, $y(n)$ is more familiar: $y(0)$ represents stable cohomotopy and $y(\infty)$ represents mod two cohomology. We showed that red-shift occurs in relative topological periodic cyclic homology, TP, which is used to compute topological cyclic homology TC.

Theorem 5.1 (Angelini-Knoll–Q., [AKQ19, Thm. 1.2])**.** The chromatic complexity of the relative topological periodic cyclic homology

$$TP(y(n), H\mathbb{F}_2) := \mathrm{fib}(THH(y(n))^{tS^1} \to THH(H\mathbb{F}_2)^{tS^1})$$

is at least $n+1$.

The algebraic K-groups of stable cohomotopy have important applications in geometric topology [WJR13]. Dundas and Rognes [DR18] proposed that these groups can be computed using the algebraic K-groups of complex cobordism. Angelini-Knoll and I [AKQ21] carried out part of the trace methods program for another family of cohomology theories, $X(n)$, which interpolate between stable cohomotopy and complex cobordism.

One long-term project in this direction is to understand red-shift for the algebraic K-theory of fixed points of Real oriented cohomology theories. Hahn and Wilson [HW20] have recently proven red-shift results for Brown–Peterson spectra, a family of cohomology theories which generalize connective unitary topological K-theory. *Real Brown–Peterson spectra* are their C_2-equivariant refinements. In my work with Li and Lorman (cf. Section 3), we proved "blue-shift" for the Tate construction of fixed points of real Johnson–Wilson theories by proving a C_2-equivariant result and then passing to C_2-fixed points. Using *real algebraic K-theory* (discussed

[10] The algebraic K-theory of a ring can be recovered as the algebraic K-theory of singular cohomology with coefficients in that ring.

in Section 6), it may be possible to prove red-shift results for algebraic K-theory using similar ideas.

Goal 5.2. Understand red-shift for the algebraic K-theory of fixed points of real Brown–Peterson spectra following the techniques developed with Li and Lorman from Section 3.

6. Real algebraic K-theory

Hesselholt and Madsen [HM15] defined *real algebraic K-theory*, $K\mathbb{R}^{\mathrm{alg}}$, a C_2-equivariant refinement of algebraic K-theory. As with Atiyah's $K\mathbb{R}^{\mathrm{top}}$, we can extract interesting invariants from $K\mathbb{R}^{\mathrm{alg}}$: forgetting structure recovers algebraic K-theory, taking C_2-fixed points recovers *Grothendieck–Witt theory*, and taking geometric fixed points[11] recovers *L-theory* [CDH$^+$20, Sch17]. Grothendieck–Witt groups are used to study quadratic forms and arise in algebraic geometry and number theory, while L-groups arise in geometric topology via the surgery exact sequence.

6.1. Real cyclotomic spectra.

Genuine cyclotomic spectra [BHM93, HM97, BM15] axiomatize the structure present in topological Hochschild homology THH required to construct topological cyclic homology TC. These are incredibly important in trace methods, but they contain a huge amount of data which makes them somewhat difficult to handle.

Recent work of Nikolaus and Scholze [NS18] on cyclotomic spectra has simplified many classical computations and has led to deep connections between number theory and stable homotopy theory [BMS19, AN21]. They defined *Borel cyclotomic spectra* and showed that in many interesting cases, Borel cyclotomic structures are rich enough to define topological cyclic homology. This simplification led to a new formula for TC in terms of THH.

Høgenhaven [Høg16] introduced *genuine real cyclotomic spectra*, generalizing the notion of genuine cyclotomic spectra to the C_2-equivariant setting. She used genuine real cyclotomic spectra to define *real topological cyclic homology* and computed it for certain group rings which arise in geometric topology.

Motivated by the work of Nikolaus and Scholze, J. Shah and I [QS19] developed the theory of *Borel real cyclotomic spectra*. We proved the following:

Theorem 6.1 (Q.–Shah, [QS19, Thm. 0.11])**.** The forgetful functor from genuine to Borel real cyclotomic spectra restricts to an equivalence on the full subcategories of real cyclotomic spectra whose underlying spectrum is bounded below.

Classically, there are two notions of equivariance in stable homotopy theory: Borel equivariance and genuine equivariance. A Borel equivariant cohomology theory is an ordinary cohomology theory with a group action. A genuine equivariant cohomology theory is a more intricate object with additional desirable properties, such as suspension isomorphisms for spheres with group actions.

One of the theoretical difficulties Shah and I encountered was understanding what kind of equivariance our theory should use. Genuine cyclotomic spectra are genuine S^1-equivariant, while Borel cyclotomic spectra are Borel S^1-equivariant. Høgenhaven's genuine real cyclotomic spectra are genuine $O(2)$-equivariant, but it turns out that Borel $O(2)$-equivariance is not rich enough to recover genuine real cyclotomic structures. Shah and I developed a theory of "mixed genuineness" to solve this problem: Borel real cyclotomic spectra are genuine C_2-equivariant objects with a 'twisted' S^1-action (cf. [QS19, Def. 0.7]).

There are many directions for future work with Borel real cyclotomic spectra. For instance, real algebraic K-theory can be defined for extremely general objects called *Poincaré categories*

[11] Another notion of fixed points specific to equivariant stable homotopy theory.

[CDH+20], which encompass rings with anti-involution, multiplicative C_2-equivariant cohomology theories, and various categories of topological spaces. Shah and I are currently working on the following:

Goal 6.2. Define real topological Hochschild and cyclic homology for Poincaré categories. Construct trace maps between them and show that real topological cyclic homology is a close approximation to real algebraic K-theory.

6.2. **Real topological cyclic homology.** The connections between cyclotomic spectra and number theory mentioned above were motivated by deep computations of topological cyclic homology. For example, the Witt vectors and de Rham–Witt complex naturally arose in the computations of Hesselholt and Madsen [HM97, HM03] on the topological cyclic homology of local fields. One of their first computations was the topological cyclic homology of finite fields,

$$TC_*(\mathbb{F}_p) \cong \mathbb{Z}_p \oplus \Sigma^{-1}\mathbb{Z}_p,$$

which they made using genuine cyclotomic methods. The work of Nikolaus–Scholze greatly simplified this computation.

The real topological cyclic homology of finite fields $\mathbb{F}_p$ was surprisingly difficult to compute. Dotto, Moi, Patchkoria, and Reeh [DMPR21] computed their real topological Hochschild homology, $THR(H\mathbb{F}_p)$, and Shah and I computed their real topological cyclic homology for p odd in [QS19]. The case $p = 2$ was handled in a recent preprint of Dotto–Moi–Patchkoria [DMP21] using genuine real cyclotomic methods. Shah and I obtained the same result using Borel real cyclotomic methods:

Theorem 6.3 (Dotto–Moi–Patchkoria; Q.–Shah, [QS19, Thm. 7.61] for p odd, forthcoming work for $p = 2$)**.** There is an equivalence of genuine C_2-spectra,

$$TCR(H\mathbb{F}_p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p,$$

where HM represents C_2-equivariant Bredon cohomology with coefficients in the constant Mackey functor at M.

Goal 6.4. Compute the real topological cyclic homology of more complicated rings. For instance, relate TCR of local fields to an equivariant refinement of the de Rham–Witt complex following [HM03].

6.3. **The Borel Conjecture.** A topological manifold M is called *aspherical* if $\pi_k(M) = 0$ for $k > 1$. An aspherical manifold M is uniquely determined, up to homotopy equivalence, by its fundamental group $\pi_1(M)$. The *Borel Conjecture* states that every homotopy equivalence between closed, aspherical topological manifolds is homotopic to a homeomorphism. This implies that apherical topological manifolds are uniquely determined, up to homeomorphism (not just homotopy equivalence), by their fundamental groups.

For topological manifolds of dimension $d \geq 5$, the Borel Conjecture holds if and only if certain assembly maps for symmetric L-theory are isomorphisms. Since these L-groups appear as the geometric fixed points of real algebraic K-theory, Shah and I hope to use real cyclotomic spectra to study these assembly maps. This approach has a very successful precedent in (ordinary) algebraic K-theory: Bökstedt, Hsiang, and Madsen [BHM93] used cyclotomic spectra to prove rational injectivity of the K-theory assembly map in great generality.

Goal 6.5. Prove rational injectivity for the assembly map in real algebraic K-theory for a large class of groups using the trace methods developed with Shah.

7. Equivariant algebra

Goodwillie [Goo86] showed that *cyclic homology* is a close approximation to algebraic K-theory rationally. Cyclic homology is completely algebraic, and moreover, it is intimately connected with algebraic geometry: the cyclic homology of a smooth $\mathbb{Q}$-algebra may be identified with algebraic de Rham cohomology via the *Hochschild–Kostant–Rosenberg Theorem* [HKR09].

Using my work with Shah, it should be possible to prove an analogous result for the real cyclic homology of rings with anti-involution. Such rings are examples of more general objects, *Mackey functors* and *incomplete Tambara functors* [BH18], which arise in representation theory, homological algebra, and equivariant stable homotopy theory. The computability of real cyclic homology requires the development of commutative and homological algebra for Mackey and incomplete Tambara functors.

7.1. **Hochschild–Kostant–Rosenberg Theorem.** The *Hochschild–Kostant–Rosenberg Theorem* identifies the Hochschild homology of a smooth algebra with its module of differential forms. When the ground ring contains the rationals, the HKR Theorem extends to identify cyclic homology with algebraic de Rham cohomology.[12] This facilitates the passage of information between homological algebra and algebraic geometry.

D. Mehrle and I are working to prove an HKR Theorem in equivariant algebra. Angelini-Knoll, Gerhardt, and Hill have defined C_2-equivariant refinements of Hochschild and cyclic homology, and Hill [Hil17] and Leeman [Lee19] have defined (for any finite group) equivariant refinements of differential forms. Mehrle and I have shown that these agree for certain simple incomplete Tambara functors and are working to prove a general result:

Goal 7.1. Prove that equivariant Hochschild homology and equivariant differential forms coincide for smooth algebras over incomplete Tambara functors.

In our work on freeness of polynomial algebras described below, M. Hill, Mehrle, and I showed that the homological algebra of Mackey functors is greatly simplified in the 'equivariantly rational' setting [HMQ21, Thm. E]. This suggests that the following is also possible:

Goal 7.2. Prove that equivariant cyclic homology and equivariant de Rham cohomology coincide for smooth algebras over equivariantly rational incomplete Tambara functors.

7.2. **Freeness and polynomial resolutions.** Let R be a commutative ring. The free commutative R-algebra on one generator, $R[x]$, is also free as an R-module: $R[x] \cong R\{1, x, x^2, \ldots\}$. This has an important consequence in commutative and homological algebra: resolutions by free commutative algebras are also resolutions by free modules.

In equivariant algebra, *Mackey functors* play the role of abelian groups, and *incomplete Tambara functors* play the role of commutative rings. Hill, Mehrle, and I [HMQ21] proved the following surprising result:

Theorem 7.3 (Hill–Mehrle–Q., [HMQ21, Thm. D])**.** Free incomplete Tambara functors for finite groups are almost never flat as Mackey functors.

In particular, this implies that "polynomial resolutions" in equivariant algebra are almost never sufficient for computing derived functors.

Goal 7.4. Understand when polynomial resolutions are homologically meaningful in equivariant algebra, i.e. when Mackey functors can be resolved by free incomplete Tambara functors which are projective as Mackey functors.

[12] This provides a nice conceptual interpretation of the trace map from K-theory to cyclic homology: it is an algebraic version of the Chern character from unitary K-theory to rational cohomology.

7.3. **Koszul complex.** In some proofs of the HKR Theorem, the *Koszul complex* appears as a free resolution over a polynomial ring. This is useful for computing Hochschild homology. For instance, there is an isomorphism

$$\mathrm{HH}_*(\mathbb{F}_p[x]) \cong \mathbb{F}_p[x] \otimes_{\mathbb{F}_p} \mathrm{Tor}_*^{\mathbb{F}_p[x]}(\mathbb{F}_p, \mathbb{F}_p),$$

and the right-hand side can be computed using the Koszul resolution. The Koszul complex is also an important tool for studying depth in commutative algebra and the cohomology of Lie algebras.

In equivariant algebra, Tor over a polynomial ring can be quite complicated. Classically, Tor over a polynomial ring on one generator is concentrated in degrees 0 and 1, but in the C_p-equivariant setting, analogous polynomial rings can have nonvanishing Tor groups in infinitely many dimensions. Mehrle, M. Stahlhauer, and I are developing a Koszul complex in equivariant algebra.

Goal 7.5. Use the Koszul complex for free incomplete Tambara functors to compute equivariant versions of Tor and Hochschild homology.

7.4. **Local-to-global principles.** Another important tool in some classical proofs of the HKR Theorem is the *local-to-global principle*, which is used to pass from smooth algebras to local algebras. In its simplest form, the local-to-global principle says that an R-module M is zero if and only if every localization $M_{\mathfrak{p}}$ of M with respect to a prime ideal $\mathfrak{p}$ of R is zero. Many results about modules are easier to check after localization, so the local-to-global principle is a standard tool in commutative algebra.

Lewis [Lew81], Nakaoka [Nak12], and Blumberg–Hill [BH18] have defined localizations for modules over incomplete Tambara functors. A prime ideal in a Tambara functor is a collection of prime ideals for each subgroup, but the localization at a prime ideal is *not* always obtained by levelwise localization. J. Carlisle, Mehrle, and I are working understand these localizations:

Goal 7.6. Express localizations for incomplete Tambara functors in terms of nonequivariant localizations. Use this to prove a local-to-global principle in equivariant algebra.

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