Trace methods for real algebraic K-theory

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I. Real algebraic K-theory

Recall: Given a $\left\{\begin{array}{l}\text{ring (spectrum)} \\ \text{Waldhausen category} \\ \text{CGW category} \\ \text{etc..}\end{array}\right\} \xrightarrow{K(-)}$ algebraic K-theory spectrum.

Natural extension: Given a $\left\{\begin{array}{l}\text{ring (spectrum) with anti-involution} \\ \text{Waldhausen cat with duality}\end{array}\right\} \xrightarrow{KR(-)}$ C_2-spectrum "real algebraic K-thy"

Ex. $A = \mathbb{Z}[G] \circlearrowleft C_2 : g \mapsto g^{-1}$.

$K(\mathbb{Z}[G])$, $K(\mathbb{S}[G])$ appear in surgery theory.

Turns out: Fixed points of KR contain important info:

- $KR(A)^e \simeq K(A^e)$
- $KR(A)^{C_2} \simeq GW(A)$
- $\Phi^{C_2} KR(A) \simeq L^g(A)$

cf. Hesselholt–Madsen; Schlichting; Calmès et. al.

Question: Trace methods for KR?

II. Real trace methods

$\xrightarrow{\text{Dotto–Ogle}}$

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$$\begin{tikzcd}
KR \arrow[r, "DGM", "\text{Harpaz-Nikolaus-Shah}"'] \arrow[rrr, bend left, "\text{Dotto-Ogle}"] \arrow[dr, "\text{Goodwillie}"'] & TCR \arrow[rr] \arrow[d] & & THH\ THR \arrow[d, "\text{Bökstedt}", "\text{Angelini-Knoll-Gerhardt-Hill}"'] \\
& HCR \arrow[rr] & & HHR
\end{tikzcd}$$

Remaining theoretical questions: (1) KR(CGW categories with ___?___) ?

(2) Categorical Dundas-Goodwillie-McCarthy? cf. Dotto for exact categories

(3) What is KR(C_2-spectrum) "really"?

III. Calculations

Recall: Some formulas for (p-typical) TC(A): Let $\mu_{p^n} \leq S^1$.

(1) [Bökstedt-Hsiang-Madsen] $TR(A) := \lim\limits_{\overleftarrow{n}} THH(A)^{\mu_{p^n}}$

$$TC(A) \simeq \mathrm{fib}\left(TR(A) \xrightarrow{1-F} TR(A)\right).$$

(2) [Nikolaus-Scholze] If A is bounded below, then

$$TC(A) \simeq \mathrm{fib}\left(\underbrace{THH(A)^{hS^1}}_{\text{homotopy fixed pts}} \xrightarrow{can - \phi_p^{hS^1}} \underbrace{THH(A)^{tS^1}}_{\text{Tate construction}}\right)$$

Spectral sequences: eg HFPSS

$$E_2^{s,t} = H^s(S^1; \pi_t THH(A)) \Rightarrow \pi_{t-s} THH(A)^{hS^1}.$$

Ex. $TC(\mathbb{F}_p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1} H\mathbb{Z}_p$.

Sketch: (i) [Bökstedt] $THH(\mathbb{F}_p) \simeq \bigvee_{i \geq 0} \Sigma^{2i} H\mathbb{F}_p$.

(ii) $H^s(S^1; \mathbb{F}_p) \cong H^s(\mathbb{CP}^\infty; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & s \geq 0 \text{ even}, \\ 0 & \text{else}. \end{cases}$

(iii) HFPSS & Tate SS collapse:

$$\pi_* THH(\mathbb{F}_p)^{hS^1} \underset{\uparrow}{\cong} \left\{ \begin{matrix} \mathbb{Z}_p & * = \text{even}, \\ 0 & * = \text{odd} \end{matrix} \right\} \cong \pi_* THH(\mathbb{F}_p)^{tS^1}.$$

(iv) Use LES to get answer. //

With reality: Two formulas for (p-typical) $TCR(A)$: $O(2) = S^1 \rtimes C_2$.

(1) [Høgenhaven] $TRR(A) \simeq \varprojlim_n THR(A)^{\mu_{p^n}} \in Sp^{C_2}$.

$$TCR(A) \simeq \mathrm{fib}\left(TRR(A) \xrightarrow{1-F} TRR(A) \right) \in Sp^{C_2}.$$

(2) [Q-Shah] If A^e is bounded below, then

$$TCR(A) \simeq \mathrm{fib}\left(THR(A)^{h_{C_2}S^1} \xrightarrow{can - \varphi_p^{h_{C_2}S^1}} THR(A)^{t_{C_2}S^1} \right).$$

"parametrized" HFP and Tate construction

Thm (Q-Shah; Dotto-Moi-Patchkoria) For all primes p,

$$TCR(\underline{\mathbb{F}_p}) \simeq H\underline{\mathbb{Z}_p} \vee \Sigma^{-1} H\underline{\mathbb{Z}_p}.$$

<u>Many</u> computations are open:

(1) Algebraic side:

(i) (Mehrle–Q) Hochschild–Kostant–Rosenberg Thm

(ii) (Mehrle–Q–Stahlhauer) $HH_*^G(\underline{A}[x]/\underline{A})$.

(iii) Amitsur–Dress–Tate cohomology (input to HFPSS)

(iv) Input to Bökstedt SS for $HH_*^G(A)$

(cf. Adamyk–Gerhardt–Hess–Klang–Kong)

(2) Topological computations

(i) $TCR(A)$ with A = perfect $\mathbb{F}_2$-algebra.

cf. (Hesselholt–Madsen) $TCR(A)^e$

(Dotto–Moi–Patchkoria) $\Phi^{C_2} TCR(A)$.

(ii) $THR(R)$ known for some Thom spectra, eg $H\mathbb{F}_p$, $H\mathbb{Z}_p$, MR...

What about $TCR(R)$? cf. Horev–Klang–Zou

(iii) $KR(k\mathbb{R})$, $KR(ko_{C_2})$, $KR(BP\mathbb{R}\langle n\rangle)$

Ausoni–Rognes noneqvt   Hahn–Wilson noneqvt

(iv) $KR(k[x]/x^n)$, etc.   cf Hesselholt–Madsen, Angeltveit–Gerhardt–Hesselholt, Speirs, ... noneqvt.

(v) Etc.