PARK CITY MATHEMATICS INSTITUTE EXPERIMENTAL MATHEMATICS LAB 2024

JACKSON MORRIS AND J.D. QUIGLEY

Theme

Symmetry in Algebra, Geometry, and Topology

Tentative Schedule

Week 1: Monday: introduction to group and category theory. Tuesday and Wednesday: Brief overviews of possible projects and time to work on exercises. Thursday and Friday: form teams and start working on projects.

Week 2: Working on projects. Teams update each other on progress on Wednesday. Start preparing/practicing presentations on Friday.

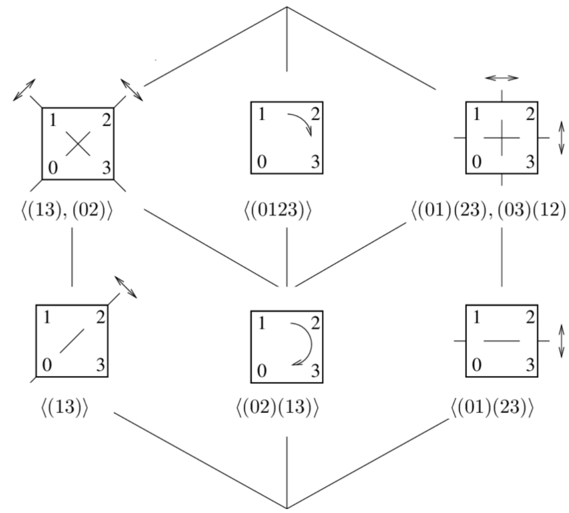
Week 3: Preparing/practicing presentations on Monday. Giving presentations on Tuesday–Friday.

Contents

Date: July 2024.

1. Background: encoding symmetries with groups and categories

References: Dummit and Foote, *Abstract Algebra*
tom Dieck, *Transformation Groups*



1.1. **Groups.**

Definition 1.1.1. A *group* is a set G equipped with a binary operation $\cdot : G \times G \to G$ satisfying

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- There exists $e \in G$ such that $e \cdot g = g$ for all $g \in G$.
- For each $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = G$.

A group G is *abelian* if $a \cdot b = b \cdot a$ for all $a, b \in G$.

Multiplication will often be denoted by concatenation, i.e., $ab := a \cdot b$.

A *group homomorphism* is a function $f : G \to H$ such that $f(gh) = f(g)f(h)$ for all $g, h \in G$.

A group homomorphism $f : G \to H$ is a *group isomorphism* is there exists a group homomorphism $g : H \to G$ such that $gf = \mathrm{id}_G$ and $fg = \mathrm{id}_H$.

Definition 1.1.2. Let G be a group. A subset $H \subseteq G$ is a *subgroup* if the pair $(H, \cdot)$ is a group, where $\cdot$ is the multiplication in G restricted to H. We will often write $H \leq G$ when H is a subgroup of G.

A subgroup $H \leq G$ is *normal* if it is closed under conjugation, i.e., $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$.

If $H \leq G$ is a subgroup and $g \in G$, the *left coset* gH is defined by

$$gH := \{gh : h \in H\}.$$

Exercise 1.1.3. If $H \leq G$ is a normal subgroup, the multiplication on G induces a multiplication on the set G/H of left cosets of H in G. (The resulting group is called the *quotient group* of G by H.)

Construction 1.1.4. Let S be a set of symbols and let F_S be the free group on S, i.e., the set of words formed from symbols in S and their formal inverses, modulo the relation that $s^{-1}s$ and ss^{-1} are the empty word for each $s \in S$; multiplication is given by concatenation. Let R be a set of relations and let N be the smallest normal subgroup of F_S such that $R \subseteq N$. We define

$$\langle S | R \rangle := F_S/N.$$

A *presentation* for a group G is a choice of isomorphism $G \cong \langle S|R\rangle$.

Exercise 1.1.5. Find presentations for the following groups:

(1) Cyclic group of order n, C_n.
(2) Integers under addition, $\mathbb{Z}$.
(3) Symmetric group on n letters, Σ_n.
(4) Alternating group on n letters, A_n.
(5) Dihedral group of order $2n$, D_{2n}.

Exercise 1.1.6. Discuss your favorite group with your group.

Construction 1.1.7. If Q and N are groups, we say that a group G is an *extension* of Q by N if there is a short exact sequence

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1.$$

"Short exact sequence" means that ι is injective, the kernel of π is equal to the image of ι, and π is surjective. Equivalently, the image of each map is equal to the kernel of the map following it.

An extension is called *split* if there is a group homomorphism $s : Q \to G$ such that $\pi \circ s = id_Q$. A group extension is split if and only if there is an isomorphism with a semidirect product $G \cong N \rtimes Q$. Further, semidirect products are in bijection with the set of group homomorphisms $Q \to \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ is the set of all isomorphisms from N to itself.

1.2. **Categories.**

Definition 1.2.1. A *category* $\mathcal{C}$ consists of the following data:

- A collection of objects $\mathrm{ob}(\mathcal{C})$.
- For every pair of objects $X, Y \in \mathrm{ob}(\mathcal{C})$ a collection of morphisms denoted $\mathcal{C}(X,Y)$ or $Hom_{\mathcal{C}}(X,Y)$. For an element $f \in \mathcal{C}(X,Y)$, one usually writes $f : X \to Y$.

The objects and morphisms of a category satisfy the following properties:

- For any three objects $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, there is a composition map
$$\circ : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z),$$
where we use the notation $\circ(g, f) = g \circ f$.
- Composition of morphisms is associative: if $f \in \mathcal{C}(W,X), g \in \mathcal{C}(X,Y)$, and $h \in \mathcal{C}(Y,Z)$, then
$$(h \circ g) \circ f = h \circ (g \circ f).$$
- For every object $X \in \mathrm{ob}(\mathcal{C})$, there is an identity morphism $1_X \in \mathcal{C}(X,X)$ such that, for any morphisms $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(W,X)$ we have
$$1_X \circ g = g \text{ and } f \circ 1_X = f.$$

Example 1.2.2. The category of sets, *Set*, has objects sets and morphisms functions.

Example 1.2.3. The category of groups, *Grp*, has objects groups and morphisms group homomorphisms.

Example 1.2.4. Let k be a field. The category of k-vector spaces, $Vect_k$, has objects k-vector spaces and morphisms k-linear maps.

Example 1.2.5. Let G be a group. We can view G as a category with one object $*$ where a morphism is given by an element $g \in G$. Composition of morphisms then is given by the group operation.

Exercise 1.2.6. Discuss your favorite category with your group.

Definition 1.2.7. A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ associates to each object $X \in \mathrm{ob}(\mathcal{C})$ an object $F(X) \in \mathrm{ob}(\mathcal{D})$ and to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for each $X \in \mathrm{ob}(\mathcal{C})$ and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms f and g in $\mathcal{C}$.

Example 1.2.8. There are *forgetful functors* $Grp \to Set$, $\mathrm{Vect}_k \to Set$, etc. defined by forgetting structure.

1.3. **Group actions.**

Definition 1.3.1. Let G be a group. A *(left) group action* on an object X in a category $\mathcal{C}$ is a collection of morphisms $\alpha_g : X \to X$ such that

(1) $\alpha_g \alpha_h = \alpha_{gh}$,
(2) $\alpha_e = id_X$.

Example 1.3.2. A *G-set* (X, α) is a set X equipped with a group action $\alpha : G \times X \to X$.

A *map of G-sets* $f : (X, \alpha) \to (Y, \beta)$ is a function $f : X \to Y$ such that $f \circ \alpha = \beta \circ f$.

The category of G-sets, Set^G, has objects G-sets and morphisms maps of G-sets.

1.4. **Transformation groups.**

Definition 1.4.1. Let X be an object in a category $\mathcal{C}$. The *transformation group* of X, denoted $\mathrm{Aut}(X)$, is the group of self-automorphisms of X, i.e., the subgroup of $\mathrm{Hom}_{\mathcal{C}}(X, X)$ consisting of invertible morphisms from X to itself.

Exercise 1.4.2. Describe $\mathrm{Aut}(\{1, \ldots, n\})$.

Exercise 1.4.3. Describe $\mathrm{Aut}(P_n)$ where P_n is a regular n-gon.

Exercise 1.4.4. Draw a few graphs with at least 8 vertices. Compute their transformation groups. (See the section on Graph Theory for the definition of a homomorphism between graphs.)

Exercise 1.4.5. Let k be a field and let Vect_k denote the category of k-vector spaces and k-linear maps. Describe $\mathrm{Aut}(k^n)$.

More generally, if V is a k-vector space, we write

$$GL(V) := \mathrm{Aut}(V)$$

for the *general linear group of V*.

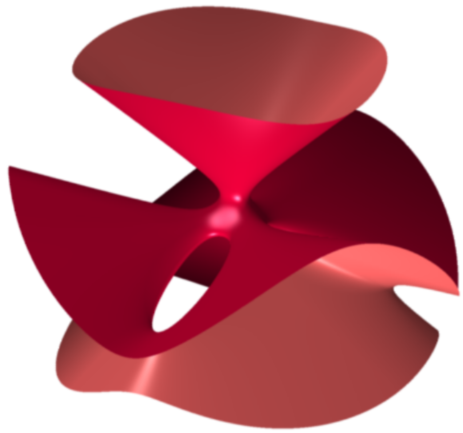
Exercise 1.4.6. Let V be a k-vector space and let G be a group. A *k-representation of G* is a homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the *general linear group of V*.

Explain how a representation $\rho : G \to GL(V)$ gives rise to a G-action on V.

Exercise 1.4.7. Let $\mathcal{C} = \mathrm{Vect}_{\mathbb{R}}^{isom}$ be the subcategory of $\mathrm{Vect}_{\mathbb{R}}$ where morphisms are required to be isometries. Compute $\mathrm{Aut}_{\mathcal{C}}(\mathbb{R}^n)$.

2. Algebraic geometry

References: Hartshorne, *Algebraic Geometry*



2.1. Varieties.

Definition 2.1.1. Let k be a field and let $f \in k[x_1, \dots, x_n]$ be a polynomial in n variables. The *vanishing locus of f*, denoted $V(f)$, is the subset of points $\bar{x} = (x_1, \dots, x_n) \in k^n$ such that $f(\bar{x}) = 0$.

More generally, if $I = (f_1, \dots, f_r)$ is an ideal in $k[x_1, \dots, x_n]$, then $V(I) = \cap_{i=1}^r V(f_i)$. An *affine variety* is any subset of $\mathbb{A}^n = k^n = V(0)$ of the form $V(I)$ for some ideal I.

Exercise 2.1.2. Describe $V(f)$ for the following polynomials, for $k = \mathbb{R}, \mathbb{C}, \mathbb{F}_q$:

(1) $f = 0 \in k[x]$.
(2) $f = x - y \in k[x, y]$.
(3) $f = x^2 - y \in k[x, y]$.
(4) $f = x^2 + y^2 + z^2 - 1 \in k[x, y, z]$.
(5) Come up with one more example.

Definition 2.1.3. Let k be a field. Then *projective n-space* over k, denoted $\mathbb{P}^n$, is defined as the set of all $(n+1)$-tuples $(x_0, \dots, x_n) \in k^{n+1}$ with $x_0, \dots, x_n$ not all 0, modulo the equivalence relation

$$(x_0, \dots, x_n) \sim \lambda(x_0, \dots, x_n)$$

for any $\lambda \in k \setminus \{0\}$. We denote the equivalence class of such a tuple by $[x_0 : \dots : x_n]$, and call the entries $x_0, \dots, x_n$ its homogeneous coordinates.

If $f \in k[x_0, \dots, x_n]$ is a homogeneous polynomial, meaning that each monomial of f is of the same degree, then its vanishing locus $V(f)$ is a subset of $\mathbb{P}^n$. More generally, a *projective variety* is any subset of $\mathbb{P}^n$ of the form $V(I)$ for $I = (f_1, \dots, f_r)$ an ideal of $k[x_0, \dots, x_n]$ where each f_i is homogeneous.

Exercise 2.1.4. Describe $V(I)$ for the following ideals, for $k = \mathbb{R}, \mathbb{C}, \mathbb{F}_q$:

(1) $I = (f), f = x + y + z \in k[x, y, z]$.
(2) $I = (f), f = ax + by \in k[x, y], a, b \in k$ not both 0.
(3) $I = (f), f = xy - x^2 - y^2 \in k[x, y]$.
(4) $I = (f, g), f = x^2 + y^2 + z^2, g = xyz \in k[x, y, z]$.

Definition 2.1.5. Let Y be an affine (or projective) variety in $\mathbb{A}^n$ (or $\mathbb{P}^n$). A function $f : Y \to k$ is called *regular* if for every point $P \in Y$ there is an open neighborhood U with $P \in U \subseteq Y$ and (homogeneous) polynomials $g, h \in k[x_1, \dots, x_n]$ (or $k[x_0, \dots, x_n]$) such that h is nonzero at all points of U and $f = g/h$ at all points of U.

A continuous function of varieties $\varphi : X \to Y$ is *regular* if, for every open set $V \subseteq Y$ and every regular function $f : Y \to k$, the composite function $f \circ \varphi^{-1} : \varphi^{-1}(V) \to k$ is regular.

Definition 2.1.6. Let k be a field. The category of varieties over k, denoted Var_k, has objects algebraic and projective varieties and morphisms regular maps.

Exercise 2.1.7. Determine the largest open set $U \subset X$ on which f is regular:

(1) $X = \mathbb{A}^2$ and $f(x, y) = (x + y)/(x^2 + xy)$.
(2) $X = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ and $f(x, y) = (x + y)/(xy)$
(3) $X = \mathbb{P}^2$ and $f(x, y, z) = (x^2 + y)/(x + y + z)$

Exercise 2.1.8. Compute the transformation groups for each of the varieties in the Exercises 3.1.2 and 3.1.5.

Problem 2.1.9. Let k be a finite field. Let X be a variety over k of degree d with transformation group G. How many such varieties are there? How does this number change as you alter the degree, transformation group, or base field? Can you develop a computer program (e.g., using Macaulay2 or Sage) to answer these questions?

One nice fact is that if two regular functions $\varphi, \psi : X \to k$ agree on some open set $U \in X$ and X is irreducible, then in fact $\varphi = \psi$. This motivates a looser definition: a *rational* map of varieties $\varphi : X \to Y$ is an equivalence class (U, φ_U) where $U \subseteq X$ is open and $\varphi_U : U \to Y$ is a regular map, under the relation $(U, \varphi_U) \sim (U', \varphi_{U'})$ if $\varphi_U = \varphi_{U'}$ when restricted to $U \cap U'$.

2.2. **Equivariant Enumerative Geometry. References:**

`https://arxiv.org/abs/2210.08622`
`https://scholarcommons.sc.edu/etd/5937/`

The field of *enumerative geometry* counts the number of solutions to geometric problems. For example:

(1) How many lines are on a complex cubic surface? There are 27.
(2) How many circles are tangent to any given 3 circles in the plane? There are 8.
(3) How many lines are on a general quintic threefold? There are 2875.

There is a plethora of historical background on enumerative geometry. More recently, there have been explorations into *equivariant* enumerative geometry. This is the study of enumeration problems with the presence of a group action.

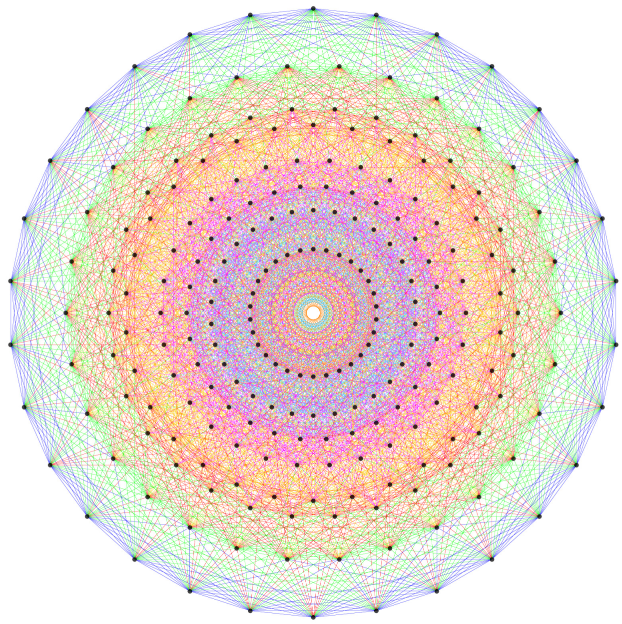
Example 2.2.1. A cubic surface $X = V(f) \subseteq \mathbb{P}^3$ is *symmetric* is $f(x_0 : x_1 : x_2 : x_3 :)$ is a symmetric polynomial. Another interpretation is: there is a natural action of Σ_4 on $\mathbb{P}^3$ by permuting coordinates, and symmetric cubics are fixed under this action. One can further ask how many lines are on a symmetric cubic surface, and how this symmetry helps answer the question when k is not algebraically closed.

Problem 2.2.2. Take an enumerative problem and adapt it to classes of varieties with a symmetry. How do the classical answers change? How does this change as you vary your base field or symmetry?

3. Representation theory

Reference: `https://math.berkeley.edu/~teleman/math/RepThry.pdf`

Reference: `https://archive.org/details/springer_10.1007-978-1-4757-6804-6/mode/2up`



Definition 3.0.1. Let G be a finite group and k be a field. A *representation* of G over k is a group homomorphism $\rho : G \to GL(V)$ for some $V \in Vect_k$. Equivalently, it is a group action of G on some $V \in Vect_k$. We will often just refer to V as a representation of G.

A representation is *faithful* if the map $\rho : G \to GL(V)$ is injective. In terms of group actions, this means that the linear transformations $\rho(g) : V \to V$ are distinct for each $g \in G$: if $\rho(g) : V \to V$ is given by the identity matrix, then g is the identity element $e \in G$.

Example 3.0.2. Let $G = C_n$ be the cyclic group of order n. There is a 2-dimensional real representation of C_n given by rotation matrices. That is, there is a group homomorphism $\rho : C_n \to GL(\mathbb{R}^2)$ where

$$\rho(1) = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$$

for 1 a generator of C_n.

Example 3.0.3. Let $V \in Vect_k$ be an n-dimensional vector space. There is a representation $\rho : \Sigma_n \to GL(V)$ defined by

$$\rho(\sigma) = \begin{pmatrix} \vdots & & \vdots \\ e_{\sigma(1)} & \cdots & e_{\sigma(n)} \\ \vdots & & \vdots \end{pmatrix}$$

where $\sigma \in \Sigma_n$ and $\{e_i\}$ are the standard basis vectors. This recovers the permutation matrices. For example, if $\rho : \Sigma_3 \to GL(k^3)$ is the permutation representation, then

$$\rho(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3.0.4. Give examples of faithful representations over $\mathbb{C}$ for the groups Σ_3, C_6, and D_6. How do the dimensions of these representations compare?

Definition 3.0.5. For a ring R, a (left) *module* M over R is an abelian group with an action of R which is distributive, associative, and unital. For any finite group G and any field k, the *group ring* kG is the vector space with basis $\{e_g\}_{g\in G}$ and a multiplication $e_g e_h = e_{gh}$. In other words, there is a basis element of kG for every element of G, and the multiplication is inherited from G and extended linearly.

Exercise 3.0.6. Show that every G-representation is equivalent to a module over the ring kG and vice versa.

Exercise 3.0.7. Recall that we can view a group G as a category with one object that has morphisms given by elements of G. Show that every G-representation is equivalent to a functor $F : G \to Vect_k$ and vice versa.

Definition 3.0.8. If V is a G-representation, then a subspace W is a *subrepresentation* if it is closed under the action of G. That is, if $w \in W$ then $gw \in W$ for all $g \in G$. We say that a representation is *reducible* if it contains a nontrivial subrepresentation and *irreducible* otherwise.

Exercise 3.0.9. Show that the permutation representation $\rho : \Sigma_n \to GL(V)$ is reducible.

Exercise 3.0.10. Give an example of a representation of Σ_n which is irreducible.

Theorem 3.0.11. ***(Maschke's theorem)*** *Suppose G is a finite group such that $char(k)$ does not divide $|G|$. If V is a G-representation, then*

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$$

where each W_i is irreducible.

Problem 3.0.12. Enumerate the number of irreducible representations of various groups G over a field k of characteristic 0. How does the answer change if k is algebraically closed?

Definition 3.0.13. The category of G-representations, denoted $Rep(G)$, has objects pairs (V, ρ) where $\rho : G \to GL(V)$ is a G-representation and morphisms linear transformations $\varphi : V \to W$ which commute with the G-action, meaning that $\varphi(gv) = g\varphi(v)$ for all $g \in G$ and $v \in V$.

Exercise 3.0.14. Let $\varphi : V \to W$ be a morphism of G-representations. Show that the linear subspaces $\ker\varphi \subseteq V$ and $\mathrm{im}\varphi \subseteq W$ are subrepresentations.

Theorem 3.0.15. ***(Schur's Lemma)*** *Let V and W be irrreducible G-representations. If $\varphi : V \to W$ is a morphism of G-representations, then either φ is an isomorphism or φ is the 0 map.*

Definition 3.0.16. Let $\rho : G \to GL(V)$ be a G-representation. The *character* of ρ is the function $\chi : G \to k$ given by

$$\chi(g) = \mathrm{tr}(\rho(g)),$$

where $\mathrm{tr}(\rho(g))$ is the trace of the matrix assigned to g. We call the values $\chi(g)$ the *characters* of the representation.

Exercise 3.0.17. For each representation listed above, compute its characters.

Exercise 3.0.18. Show that if two elements g, h are conjugate in G (meaning $g = khk^{-1}$ for some $k \in G$), then $\chi(g) = \chi(h)$.

Problem 3.0.19. Let G be a finite group such as Σ_n, C_n, D_n (or finite products of such groups). Compute the characters for all the irreducible complex representations of G. Are there ways to make this process easier than by brute force?

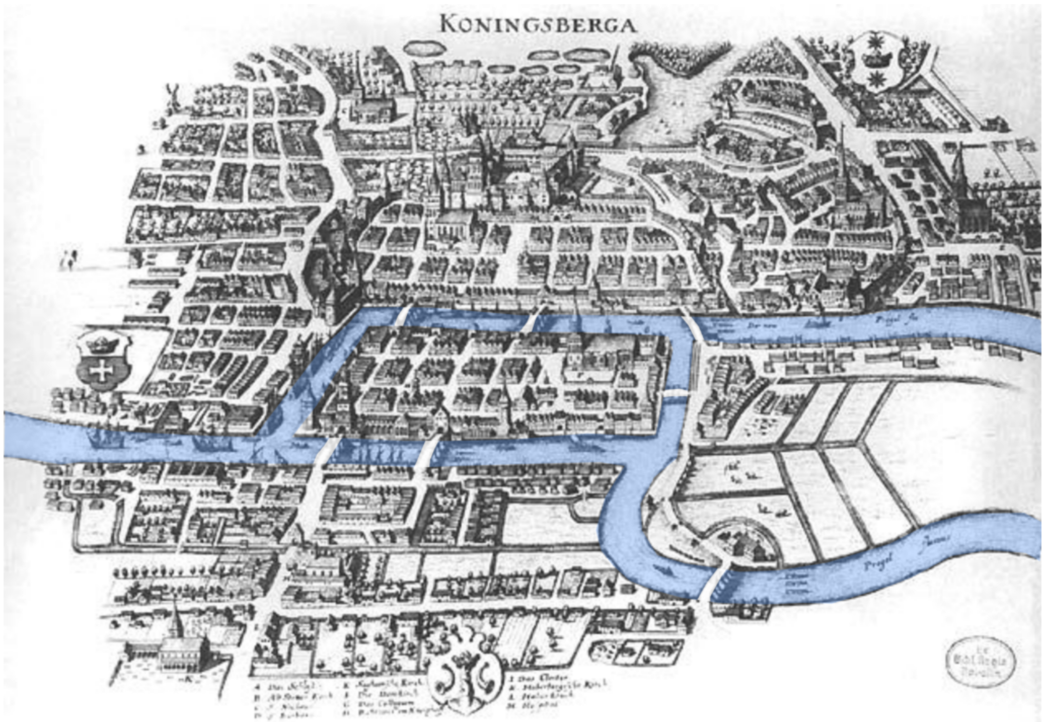
Problem 3.0.20. Suppose k is a field with characteristic $p > 0$ and that G is a finite group. Classify the irreducible representations of G over k. How does this answer change if $|G|$ is divisible by $char(k)$, meaning that you cannot use Maschke's theorem?

Problem 3.0.21. What if we work instead with different categories than vector spaces? Let G be a finite group and X an object of a category $\mathcal{C}$ such as Top, Man, $VB(X)$, or $Graph$. Can you classify or characterize group homomorphisms $\rho : G \to \mathrm{Aut}(X)$?

4. Graph theory

Reference: `https://www.maths.ed.ac.uk/~v1ranick/papers/wilsongraph.pdf`

Example 4.0.1. Here is a map of the city of Konigsberg in the 1700s:



Can you find a path which crosses all seven bridges in Konigsberg precisely once?

Definition 4.0.2. An *undirected graph* is a pair $\Gamma = (V, E)$ where $V = \{v_1, v_2, \ldots\}$ is a set of *vertices* and $E = \{\{v_i, v_j\}\}$ is a set of unordered pairs of vertices called *(undirected) edges*.

A *directed graph* is the same data, except we take $E = \{(v_i, v_j)\}$ to be a set of ordered pairs of vertices called *(directed) edges*.

A *path* is a sequence of adjacent edges. A *cycle* is a path whose first and last vertex is the same.

Example 4.0.3. A *tree* is a graph without any cycles.

Definition 4.0.4. A *graph homomorphism* $f : \Gamma \to \Gamma'$ is a function from $V(\Gamma)$ to $V(\Gamma')$ which preserves edges.

Exercise 4.0.5. Compute the number of graph isomorphism classes of binary rooted trees with n leaves.

Exercise 4.0.6. Construct directed and/or undirected graphs with the following transformation groups, or show that no such graph exists:

(1) The cyclic group C_n.
(2) The dihedral group D_n.
(3) The symmetric group Σ_n.
(4) The integers $\mathbb{Z}$.

Problem 4.0.7. What proportion of graphs with n vertices have a nontrivial transformation group? What happens if you restrict to certain kinds of graphs (e.g., trees) or a particular class of transformation groups? What if you let n tend to infinity? Can you develop a computer program to answer these questions?

Problem 4.0.8. *Simplicial complexes* generalize graphs to higher dimensions; instead of just vertices and edges, simplicial complexes also contain (two-dimensional) faces and hyperfaces (faces of dimension greater than two). Study the previous problem with "graph" replaced by "simplicial complex of dimension at most d" for some $d > 1$.

Definition 4.0.9. An *Eulerian path* is a path which visits each edge exactly once (revisiting vertices is allowed).

An *Eulerian circuit* is an Eulerian path which starts and ends at the same vertex.

Exercise 4.0.10. Give necessary and sufficient conditions for when graphs admit Eulerian paths and/or circuits. Using this, determine the proportion of graphs with n vertices which admit an Eulerian path/circuit.

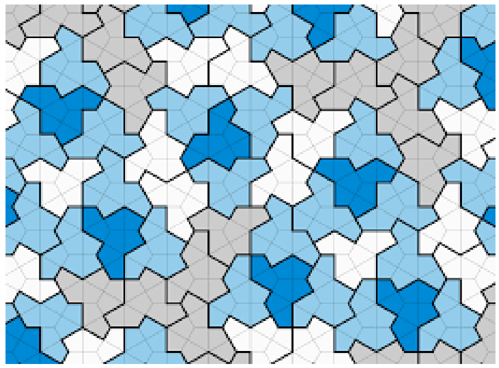
Problem 4.0.11. Formulate what it means for an Eulerian path/circuit to be symmetric. What proportion of Eulerian paths in a given graph are symmetric? How does the proportion of symmetric Eulerian paths in the collection of all graphs with n vertices vary as n varies? What about Eulerian circuits? (Note that this problem is probably much simpler in directed graphs.)

Problem 4.0.12. There are various algorithms for constructing Eulerian paths in a given graph, e.g., Fleury's algorithm, Hierholzer's algorithm. Can you improve these algorithms if your graph possesses some kind of symmetry? How does the computational complexity of your modified algorithm compare to the complexity of the original algorithm? Can you write code implementing your algorithm in real time?

Problem 4.0.13. Study the previous problems for *weighted graphs*, i.e., graphs with weights attached to their edges.

5. Tilings

References: `https://math.mit.edu/~rstan/papers/tilings.pdf`
`https://math.uchicago.edu/~may/REU2020/REUPapers/Sasse.pdf`



Definition 5.0.1. A *tiling* is a covering of a surface using countably many closed subsets called *prototiles* with no overlaps (except along edges) and no gaps.

An *edge* is the intersection between two tiles and a *vertex* is a point where three or more tiles meet.

A tiling is *isogonal* if the arrangement of polygons about each vertex is the same.

A tiling is *monohedral* if it has only one prototile. More generally, a tiling is n-*hedral* if it has n prototiles.

A tiling is *periodic* if you can outline a region of the tiling and tile the plane by translating copies of that region (without rotating or reflecting). A tiling which is not periodic is *aperiodic*.

Exercise 5.0.2. Determine which *regular* polygons can be used as prototiles for monohedral tilings of the plane. Which polygons (not necessarily regular) can be used as prototiles for monohedral tilings of the plane?

Problem 5.0.3. Given a finite group, can you find a prototile with that group as its symmetry group which gives a monohedral tiling of the plane? What if you work with tesselations (i.e., tilings in higher dimensions)? Can you enumerate (or approximate) all tilings by tiles with symmetry if you restrict to a surface with finite area?

Definition 5.0.4. Let E_2 denote the group of isometries of the Euclidean plane. Note that any isometry can be encoded by a translation vector $\vec{v}$ and a 2-by-2 matrix M, giving an isometry $\vec{x} \mapsto M\vec{x} + \vec{v}$.

Let $G \leq E_2$ be a subgroup.

- Its *translation group* is the subgroup
$$T := \{(\vec{v}, I) \in G\}.$$
- Its *point group* is the subgroup
$$H := \{M : (\vec{v}, M) \in G \text{ for some } \vec{v} \in \mathbb{R}^2\}.$$

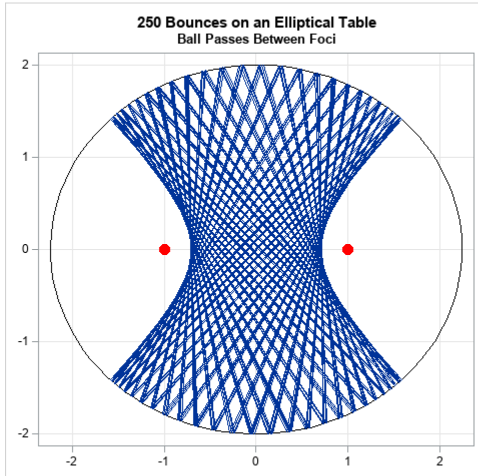
A *wallpaper group* is a subgroup of E_2 with a finite point group and a translation group generated by two linearly independent translations.

Exercise 5.0.5. Classify all wallpaper groups. (See Sasse, "Classification of the 17 wallpaper groups.")

Problem 5.0.6. Generalize wallpaper groups to higher-dimensional Euclidean spaces.

6. Dynamical billiards

Reference: `https://math.uchicago.edu/~may/REU2014/REUPapers/Park.pdf`



Definition 6.0.1. A *billiard table* $Q \in \mathbb{R}^2$ is an open bounded connected domain such that its boundary ∂Q is a finite union of smooth compact curves.

We wish to understand a moving particle in a billiard table Q. Suppose a moving particle has position $q \in Q$ and velocity $\vec{v} \in \mathbb{R}^2$. To model the motion of the particle, we make the following assumptions:

(1) The curves which comprise the boundary of the billiard table are disjoint but may have common endpoints.
(2) The particle travels in a straight line parallel to $\vec{v}$ until it hits ∂Q.
(3) The *billiard trajectory* is the segment $\overline{p_1 p_2}$, where $p_1, p_2 \in \partial Q$ where the particle consecutively hits the boundary.
(4) Define $n(p)$ as the inward pointing normal vector at point $p \in \partial Q$.
(5) Let $p_1, p_2, p_3 \in \partial Q$ be three consecutive points of contact with ∂Q. At point p_2, the *angle of incidence* is the angle between $n(p_2)$ and and the billiard trajectory $\overline{p_1 p_2}$. The *angle of reflection* is the angle between $n(p_2)$ and the billiard trajectory $\overline{p_2 p_3}$.
(6) Whenever the particle hits the boundary, the angle of incidence is equal to the angle of reflection.

Exercise 6.0.2. Describe the possible trajectories in a regular n-gon-shaped billiard table. Can you find periodic trajectories which hit the boundary arbitrary numbers of times?

Problem 6.0.3. Can you find a relation between the symmetries of a billiard table and the possible trajectories of a ball on that table? What if the table is not convex?

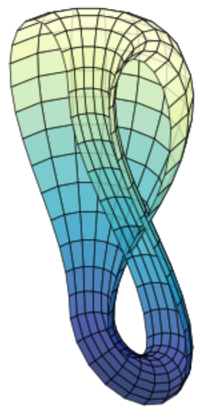
Problem 6.0.4. Use DynamicalBilliards.jl (`https://juliadynamics.github.io/DynamicalBilliards.jl/dev/`) or similar software to study the problem above.

Problem 6.0.5. *Outer billiards* is a variant of billiards in which the particle travels *outside* of the table. Study outer billiards when the table is a regular polygon, cf. Bedaride–Cassaigne, "Outer billiards outside regular polygons." What can you say if the table is not a regular polygon, but still possesses some nontrivial symmetry?

Problem 6.0.6. Study billiards on a table with nonzero curvature and some nice symmetry, e.g., an upper hemisphere.

7. Basics of Topology

References: Gamelin–Greene, *Introduction to Topology*
Hatcher, *Algebraic Topology*



Definition 7.0.1. A *topological space* is a set X equipped with a collection of subsets $\mathcal{U} = \{U_i \subseteq X : i \in I\}$ such that

(1) $\emptyset, X \in \mathcal{U}$,
(2) $\mathcal{U}$ is closed under arbitrary unions.
(3) $\mathcal{U}$ is closed under finite intersections.

The members of $\mathcal{U}$ are called *open sets*.

A *map of topological spaces*, or *continuous map* $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a function $f : X \to Y$ such that $f^{-1}(V) \subseteq X$ is open for every open $V \subseteq Y$.

Two spaces X and Y are *homeomorphic* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf = id_X$ and $fg = id_Y$.

Example 7.0.2. Any metric space gives rise to a topological space whose topology is generated by the open balls.

Example 7.0.3. The n-disk is homeomorphic to the n-cube.

Exercise 7.0.4.

(1) If X is a set, define the finest and coarsest possible topologies on X, i.e., the discrete and indiscrete topologies.
(2) If Y is a topological space and $X \subseteq Y$ is a subset, define the topology on X inherited from Y, i.e., the subspace topology.
(3) If X and Y are topological spaces, define a topology on their product $X \times Y$.
(4) If $X \subseteq Y$ is a subspace, define a topology on the quotient space Y/X.

Exercise 7.0.5. Compute the transformation group of the the set $\{1, 2, \dots, n\}$ of n points equipped with the discrete topology.

Exercise 7.0.6. Construct infinite subgroups of the transformation groups of the following spaces:

(1) The n-sphere S^n.
(2) The torus T^2.
(3) An oriented genus g surface Σ_g.
(4) Euclidean space $\mathbb{R}^n$.

Definition 7.0.7. Two maps $f, g : X \to Y$ between topological spaces are *homotopic*, denoted $f \simeq g$, if there exists a continuous map

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Two spaces X and Y are *homotopy equivalent*, denoted $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq id_X$ and $fg \simeq id_Y$.

Exercise 7.0.8. Show that $\mathbb{R}^n \simeq *$, i.e., $\mathbb{R}^n$ is *contractible*.

Exercise 7.0.9. Compute $\mathrm{Aut}(S^1)$/homotopy, i.e., the fundamental group of the circle.

Definition 7.0.10. A *G-space* is a topological space X equipped with a continuous map $G \times X \to X$.

If X is a G-space and $H \leq G$ is a subgroup, we write

$$X^H := \{x \in X : hx = x \text{ for all } h \in H\}$$

for the *H-fixed points* of X.

Example 7.0.11. Given a real G-representation V, its one-point compactification S^V is a sphere of dimension $\dim(V)$ with a G-action. The H-fixed points of S^V are given by $(S^V)^H = S^{V^H}$.

Definition 7.0.12. The nth homotopy group of a topological space X, denoted $\pi_n(X)$, is the group of based homotopy classes of based maps $S^n \to X$. The group structure is induced by the pinch map $S^n \vee S^n \to S^n$.

Example 7.0.13. The 1st homotopy group of a space $\pi_1(X)$ is the *fundamental group.*

Example 7.0.14. There is an isomorphism $\pi_4(S^3) \cong \mathbb{Z}/2$, where the nontrivial element in $\pi_4(S^3)$ is represented by the suspension of the quotient map

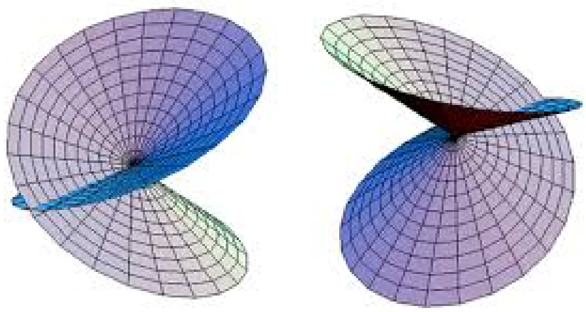
$$S^3 \cong S(\mathbb{C}^2) \to \mathbb{CP}^1 \cong S^2.$$

Definition 7.0.15. Let S^V be the one-point compactification of a G-representation V and let X be a G-space. The *V-th homotopy group of* X is the group (if V contains a rank one trivial summand) of equivariant homotopy classes of maps from S^V to X.

Problem 7.0.16. You will learn about stabilization and stable homotopy groups in the summer school. Define equivariant stable homotopy groups. Find examples of nontrivial elements in the equivariant stable homotopy groups of spheres. Compute the 0-th equivariant stable homotopy group of the sphere S^0 with trivial G-action.

8. Classifying spaces

References: `https://math.mit.edu/~mbehrens/18.906spring10/prin.pdf` and Chapter VII of `https://www.math.uchicago.edu/~may/BOOKS/alaska.pdf`



Definition 8.0.1. Let G be a finite group. A collection of subgroups $\mathcal{F}$ of G is called a *family* if it is closed under conjugation and taking subgroups.

If $\mathcal{F}$ is a family of subgroups of G, the *universal G-space for $\mathcal{F}$*, denoted $E\mathcal{F}$, is the unique G-equivariant homotopy type characterized by the property that

$$(E\mathcal{F})^H \simeq \begin{cases} & \text{if } H \notin \mathcal{F}, \\ * & \text{if } H \in \mathcal{F}. \end{cases}$$

If $\mathcal{F} = \{e\}$ contains only the trivial subgroup of G, then we write $EG := E\mathcal{F}$ for the *universal space of G.*

If $G = N \rtimes Q$ is a semidirect product, let

$$\mathcal{F}(G, N) := \{H \leq G : H \cap N = e\}.$$

The *Q-equivariant classifying space for N* (associated to the extension $N \to G \to Q$) is defined by

$$B_Q N := E\mathcal{F}(G, N)/N.$$

Example 8.0.2. A model for EC_2 is given by taking the unit sphere in $\mathbb{R}^\infty$ with C_2-action given by multiplication by -1.

A model for $E_{C_2}\mu_2$ (associated to the trivial extension $D_4 = \mu_2 \times C_2$) is given by taking the unit sphere in $\mathbb{C}^\infty$, where C_2 acts by complex conjugation and μ_2 acts by multiplication by -1.

Definition 8.0.3. Let B be a topological space and let P be a right G-space equipped with an equivariant map $\pi : P \to B$ (where G acts trivially on B. The pair (P, π) is a *principal G-bundle over B* if B has an open covering $\mathcal{U}$ such that there exist G-equivariant homeomorphisms

$$\phi_U : \pi^{-1}(U) \to U \times G$$

such that the diagram

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\phi_U"] \arrow[d] & U \times G \arrow[dl] \\ U & \end{tikzcd}$$

commutes for each $U \in \mathcal{U}$. In other words, P is a locally free G-space with orbit space homeomorphic to B.

A *morphism* of principal bundles over B is an equivariant map $\sigma : P \to Q$. A principal G-bundle is trivial if it is isomorphic to the bundle $B \times G \to B$.

Exercise 8.0.4. Show that every morphism of principal G-bundles is an isomorphism.

Exercise 8.0.5. Give an example of a nontrivial principal C_2-bundle over the circle S^1.

Exercise 8.0.6. Show that pulling back the universal bundle over BG induces a natural bijection between the set of isomorphism classes of principal G-bundles over a topological space X and homotopy classes of maps from X to BG.

Problem 8.0.7. For various groups G and topological spaces X, classify the isomorphism classes of principal G-bundles over X.

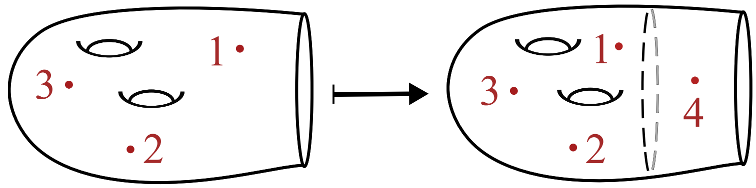
As examples, you might consider principal C_2-bundles over low-dimensional spheres S^n.

Problem 8.0.8. Give a formula for the fixed points $(B_Q N)^H$ for all $H \leq Q$.

Note that the fixed points $(B_Q N)^H$ admit a residual $W_Q(H)$-action. Can you give a formula which incorporates this residual action?

9. Configuration spaces

Reference: `https://scholar.rose-hulman.edu/cgi/viewcontent.cgi?article=1430&context=rhumj`



Definition 9.0.1. Let X be a topological space. The *nth ordered configuration space of* X is

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for all } i \neq j\}.$$

The *nth unordered configuration space of* X is

$$C_n(X) := \mathrm{Conf}_n(X)/\Sigma_n.$$

Exercise 9.0.2. Sketch some elements in $\mathrm{Conf}_n(X)$ and $C_n(X)$ for various spaces X and values of n.

Problem 9.0.3. If $X = \{1, 2, \ldots, n\}^k$, what proportion of configurations are symmetric?

Problem 9.0.4. Study the fundamental groups of graphs in the following spaces:

- Euclidean space $\mathbb{R}^n$
- Graphs Γ

Problem 9.0.5. Suppose X is a G-space. Give formulas for $\mathrm{Conf}_n(X)^G$ and $C_n(X)^G$ in terms of the fixed points of X.

Problem 9.0.6. (For students familiar with operads) Relate ordered configuration spaces to E_n operads. Learn about E_V-operads from the paper of Guillou–May on equivariant loop spaces and study their equivariant homotopy type following Hill's paper on equivariant little disks.

Problem 9.0.7. (For students familiar with homology) The homology groups of configuration spaces are known to satisfy *homological stability*, which relates the homology of $C_{n-1}(M)$ to the homology of $C_n(M)$ for certain manifolds M, and *representation stability*, which relates the homology of $\mathrm{Conf}_n(X)$ to the homology of $\mathrm{Conf}_{n-1}(X)$. Representation stability, in particular, involves symmetric group actions in a fundamental way. Learn about representation stability for ordered configuration spaces (see, e.g., Church's paper).

10. Homological algebra

References: `https://people.math.rochester.edu/faculty/doug/otherpapers/weibel-hom.pdf`

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\, f \arrow[r] & \mathrm{Ker}\,\alpha \arrow[r] \arrow[d] & \mathrm{Ker}\,\beta \arrow[r] \arrow[d] & \mathrm{Ker}\,\gamma \arrow[d] \arrow[dddll, "s"] & & \\
 & & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 & \\
 & 0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & & \\
 & & \mathrm{coker}\,\alpha \arrow[r] & \mathrm{coker}\,\beta \arrow[r] & \mathrm{coker}\,\gamma \arrow[r] & \mathrm{coker}\, g' \arrow[r] & 0
\end{tikzcd}
$$

Definition 10.0.1. Let R be a ring. An *associative algebra* over R is a ring A which is also a module over R. More concretely, A is a ring such that, for $r, s \in R$ and $a, b \in A$:

- $0(a) = 0$
- $1(a) = a$
- $r(a + b) = ra + rb$
- $rs(a) = r(sa)$
- $r(ab) = (ra)b = a(rb)$

Example 10.0.2. The polynomial ring $k[x_1, \ldots, x_n]$ is an algebra over k, where the addition and multiplication is usual polynomial addition and multiplication.

Example 10.0.3. If X is a topological space, then the set $C(X)$ of all continuous functions $X \to \mathbb{R}$ is an algebra over $\mathbb{R}$. If $f, g \in C(X)$, then

$$(f + g)(x) = f(x) + g(x),$$
$$(f * g)(x) = f(x)g(x).$$

Exercise 10.0.4. We can view the multiplication on A as a function $A \otimes_R A \xrightarrow{m} A$. In commutative diagrams, depict what it means for the multiplication on A to be associative, commutative, and unital. Try to write all of the equations in the definition of an algebra in terms of commutative diagrams.

Definition 10.0.5. For any module M over a ring R, there is a functor

$$Hom_R(M, -) : Mod_R \to Ab$$

such that:

- If $N \in Mod_R$, then $Hom_R(M, -)(N) = Hom_R(M, N)$ is the abelian group of morphisms $M \to N$;
- If $f : N \to Q$ is a morphism in Mod_R, then
$$Hom_R(M, -)(f) : Hom_R(M, N) \to Hom_R(M, Q)$$
is given by post-composition with f: if $q \in Hom_R(M, N)$, then
$$Hom_R(M, -)(f)(q) = f \circ q.$$

Definition 10.0.6. A *projective* module over a ring R is a module P such that any of the following equivalent conditions hold:

- For any map of modules $f : M \to N$ and any surjection $g : P \twoheadrightarrow N$ there is a map $h : P \to M$ such that $g = f \circ h$;
- Every short exact sequence $0 \to A \to B \to P \to 0$ ending in P splits;
- The functor $Hom(P, -) : Mod_R \to Ab$ is exact.

Exercise 10.0.7. Show that these conditions are equivalent.

Exercise 10.0.8. Let $R = \mathbb{Z}$. Determine if the following modules are projective:

- $\mathbb{Z}$
- $\mathbb{Z}/2\mathbb{Z}$
- $\mathbb{Z} \oplus \mathbb{Z}$

Exercise 10.0.9. Show that every module over a field is projective.

Definition 10.0.10. Let M be a module over a ring R. Then a *projective resolution* of M is an exact sequence of R-modules:

$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

where each P_i is a projective R-module. We will often use the shorthand $P_\bullet \to M$. The length of a minimal projective resolution for M is the *projective dimension* of M, denoted $pd(M)$.

Problem 10.0.11. Let A be a finite-dimensional algebra over a field k. If M is an A-module, what are the possible values of $pd(A)$? What properties of A and M change these values?

Definition 10.0.12. Let R be a ring and $M, N \in Mod_R$. Define the *Ext groups* $\mathrm{Ext}^n_R(M, N)$ for $n \geq 0$ as follows:

- Take a projective resolution of M:
$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$
- Apply the contravariant functor $Hom_R(-, N)$ to the resolution $P_\bullet$:
$$0 \to Hom_R(P_0, N) \xrightarrow{d_0} Hom_R(P_1, N) \xrightarrow{d_1} Hom_R(P_2, N) \xrightarrow{d_2} \ldots$$
- Define $\mathrm{Ext}^n_R(M, N)$ as the n-th quotient:
$$\mathrm{Ext}^n_R(M, N) = \ker(d_n)/\mathrm{im}(d_{n-1})$$

Problem 10.0.13. Let $A = kG$ for a finite group G and $M = N = k$, where the module structure on k is defined by $g_i \cdot k = k$. In other words, k is a G-representation with trivial G-action. Try to compute $\mathrm{Ext}^n_{kG}(k, k)$. This is called the *cohomology of* kG.

Definition 10.0.14. A *graded ring* is a ring R with a decomposition

$$R = \bigoplus_{i=0}^{\infty} R_i = R_0 \oplus R_1 \oplus \ldots$$

where each R_i is an abelian group such that $R_m R_n \subseteq R_{m+n}$.

A *graded algebra* is an algebra which is a graded ring, and a *graded module* is a module with a direct sum decomposition $M = \oplus M_i$ such that $R_i M_n \subseteq M_{i+n}$. A *graded commutative ring* satisfies $xy = (-1)^{|y||x|} yx$, where $|x| = n$ if $x \in R_n$.

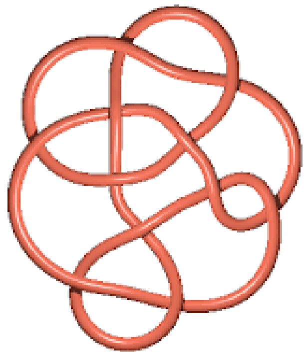
Example 10.0.15. The polynomial ring $k[x_1, \ldots, x_n]$ is a graded algebra: the pieces in the direct sum decomposition are precisely the degree n monomials.

Problem 10.0.16. Let A be a graded commutative algebra over a field k and M an A-module. How can we compare the various homological properties of M and its dual $M^{\vee} = Hom_A(M, k)$? (**Note:** There are a lot of pictures to draw here. Ask Jackson about them if you are interested in this problem.)

References: `https://people.math.rochester.edu/faculty/doug/otherpapers/Beaudry-Campbell.pdf` (This has examples of pictures referenced above.)

11. Knot theory

References: `https://math.uchicago.edu/~may/REU2014/REUPapers/Linov.pdf`
`https://graphics.stanford.edu/courses/cs468-02-fall/projects/desanti.pdf`



Definition 11.0.1. A *knot* is an embedding $S^1 \to \mathbb{R}^3$. We say two knots K_1 and K_2 are *equivalent* if there exists an ambient isotopy between them, i.e., a continuous deformation of $\mathbb{R}^3$ which carries K_1 to K_2.

The *crossing number* of a knot is the smallest number of crossings of any diagram of that knot.

Example 11.0.2. The *(standard) unknot* is the standard embedding $S^1 \to \mathbb{R}^3$ as the unit circle in $\mathbb{R}^2 \subseteq \mathbb{R}^3$.

A knot is called *nontrivial* if it is not equivalent to the unknot.

Remark 11.0.3. A *knot diagram* is a projection of a knot $K \subseteq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ onto the plane $\mathbb{R}^2$ which is 1-to-1 everywhere except at *crossings*, i.e., points where it is 2-to-1. Crossings are drawn so that the part of the knot with larger z-coordinate "passes above" the part with smaller z-coordinate. (See Wikipedia for examples.)

Two knots can be shown to be equivalent if their associated knot diagrams are related by a sequence of *Reidemeister moves* (again, see Wikipedia).

Exercise 11.0.4. Sketch trivial knots with 1, 2, 3, and 4 crossings.

Exercise 11.0.5. Sketch nontrivial knots with 3 and 4 crossings.

Definition 11.0.6. Given two knots K_1 and K_2, their *knot sum* is the connected sum $K_1 \# K_2$ obtained by removing interval from K_1 and K_2, and then connecting the endpoints. (See Wikipedia for a picture.)

A knot is called *prime* if it cannot be written as the knot sum of two nontrivial knots.

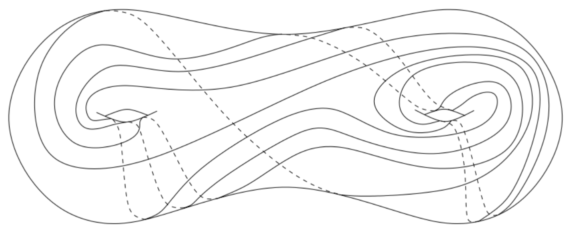
Exercise 11.0.7. Classify prime knots with n crossings for all $n \leq 7$.

Problem 11.0.8. Define what is means for a knot to be *symmetric*. Count the number of knots with particular symmetry groups and crossings. Are there any patterns in the Jones polynomials for symmetric knots?

Problem 11.0.9. There are various algorithms for determining when a given knot is equivalent to the unknot, e.g., Haken's algorithm. Can you modify this algorithm in the presence of symmetry to improve its runtime?

12. Mapping class groups

Reference: Farb–Margalit, *A primer on mapping class groups*, `https://www.labri.fr/perso/vdelecro/books/FarbMargalit-APrimerOnMappingClassGroup.pdf`



Definition 12.0.1. Let S be a surface. The *mapping class group* of S, $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation-preserving self-diffeomorphisms of S which restrict to the identity near the boundary of S (if $\partial S \neq \emptyset$).

Exercise 12.0.2.

(1) Find an order g element in $\mathrm{Mod}(S_g)$, where S_g is a surface of genus g.
(2) Find an order d element in $\mathrm{Mod}(S_g)$ for any d which divides $4g+2$.
(3) Prove the *Alexander lemma*: the group $\mathrm{Mod}(D^2)$ is trivial.
(4) Prove that $\mathrm{Mod}(S^2)$ is trivial.

The mapping class group can be computed in many nontrivial examples.

Proposition 12.0.3. *There is an isomorphism*

$$\mathrm{Mod}(T^2) \cong SL(2,\mathbb{Z}).$$

One can prove the nontriviality of elements in $\mathrm{Mod}(S)$ using the *Alexander method*. A collection $\{\gamma_i\}$ of curves and arcs in S *fills* S if the surface obtained from S by cutting along the γ_i is a disjoint union of disks.

Proposition 12.0.4. *Let S be a compact surface and let $\phi \in Homeo^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of simple closed curves and simple proper arcs in S with the following properties:*

(1) *The γ_i are pairwise in minimal position.*
(2) *The γ_i are pairwise nonisotopic.*
(3) *For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_j \cap \gamma_k$, and $\gamma_i \cap \gamma_k$ is nonempty.*

Then:

(1) *If there is a permutation σ of $\{1, \ldots, n\}$ such that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\bigcup \phi(\gamma_i)$ is isotopic to $\bigcup \gamma_i$ relative to ∂S.*
    If we regard $\bigcup \gamma_i$ as a (possibly disconnected) graph Γ in S, with vertices at the intersection points and at the endpoints of arcs, then the composition of ϕ with this isotopy gives an automorphism ϕ_* of Γ.
(2) *Suppose now that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power that is isotopic to the identity.*

Exercise 12.0.5. Use the Alexander method to prove the triviality and nontriviality of at least two self-homeomorphisms of S_2.

Problem 12.0.6. Study the mapping class groups of some low-genus surfaces.

Problem 12.0.7. The *extended mapping class group* is a variant of the mapping class group in which orientation-reversing automorphisms are included. Fix an orientation-reversing involution of a closed surface of genus g. Describe the elements in the extended mapping class group which commute with that involution.