EXERCISES

J.D. QUIGLEY

Abstract. Exercises for Topology and Algebras Workshop Philippines 2026.

1. Lecture 1

Exercise 1.1. Show that the functor $- \otimes \mathbb{Q} : Ab \to Ab$ is exact.

Exercise 1.2. Show that the functor $(-)^G : \mathbb{Z}[G] - Mod \to Ab$ is left exact.
Show that the functor $(-)_G : \mathbb{Z}[G] - Mod \to Ab$ is right exact.

Exercise 1.3. Show that $(-)^G = Hom_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ and that $(-)_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} (-)$.

Exercise 1.4. Show that a module is projective if and only if it is a summand of a free module.

Exercise 1.5. Construct projective resolutions of the following modules:

(1) $\mathbb{Z}/n$ as a $\mathbb{Z}$-module;
(2) $\mathbb{Z}$ as a $\mathbb{Z}[C_2]$-module, where C_2 acts on $\mathbb{Z}$ trivially;
(3) $\mathbb{Z}$ as a $\mathbb{Z}[x]$-module, where the $\mathbb{Z}[x]$-action on $\mathbb{Z}$ is defined via the projection $\mathbb{Z}[x] \to \mathbb{Z}$ which sends x to 0.

Exercise 1.6. Using the previous exercise, compute the following groups:

(1) $Tor_*^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$
(2) $Tor_*^{\mathbb{Z}[C_2]}(\mathbb{Z}, \mathbb{Z})$
(3) $Tor_*^{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z})$

Exercise 1.7. Compute the following group homology and cohomology groups, both with trivial action on the coefficients:

(1) $H_*(C_n; \mathbb{Z})$ and $H^*(C_n; \mathbb{Z})$.
(2) $H_*(C_2 \times C_2; \mathbb{Z})$ and $H^*(C_2 \times C_2; \mathbb{Z})$.

Exercise 1.8. Let $p, q \geq 0$ be integers. A (p, q)-*shuffle* is a permutation σ of the set $\{1, 2, \dots, p + q\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p),$$
$$\sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(p + q).$$

Let (B_*, d) denote the normalized bar resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module. Define the *shuffle product*

$$* : B_p \otimes B_q \to B_{p+q}$$

by

$$a[g_1|\cdots|g_p] * b[g_{p+1}|\cdots|g_{p+q}] = \sum_{(p,q)-shuffles} (-1)^\sigma ab[g_{\sigma^{-1}(1)}|\cdots|g_{\sigma^{-1}(p+q)}].$$

Prove the following:

(1) If G is an abelian group, then the shuffle product makes (B_*, d) into a differential graded algebra.
(2) For every abelian group G and commutative $\mathbb{Z}[G]$-algebra R, the shuffle product equips $H_*(G; R)$ with the structure of a graded commutative ring.

2. Lecture 2

Exercise 2.1. Sketch the standard n-simplices Δ^n for $n \leq 3$. Explicitly describe the simplices and face maps.

Exercise 2.2. Describe at least three different semisimplicial complexes which are homotopy equivalent to the circle S^1. Compute the simplicial homology of each complex - what do you observe?

Exercise 2.3. Show that simplicial homology is natural with respect to simplicial maps, i.e., if $f_\bullet : X_\bullet \to Y_\bullet$ is a map of semisimplicial complexes, then it induces a map $f_* : H_*(X; A) \to H_*(Y; A)$.

Exercise 2.4. Compute the following homology groups:

(1) $H_*(S^1; A)$
(2) $H_*(T^2; \mathbb{Z})$
(3) $H_*(\mathbb{R}P^2; \mathbb{F}_2)$ and $H_*(\mathbb{R}P^2; \mathbb{Z})$

Exercise 2.5. Compute the cohomology groups of the spaces from the previous exercise, either directly or using the universal coefficient theorem.

Exercise 2.6. Compute $H^*(T^2; \mathbb{Z})$ and $H^*(\mathbb{R}P^2; \mathbb{Z})$ as graded commutative rings, i.e., calculate the cup products of the generators from the previous exercise. Use the latter to show that $\mathbb{R}P^2$ and the wedge product $S^1 \vee S^2$ are not homotopy equivalent.

Exercise 2.7. Show that $H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$, $|x| = 1$, by generalizing the $\mathbb{R}P^2$ example above.

University of Virginia
Email address: `mbp6pj@virginia.edu`