DIAGONALIZE MATRIX $S^{-1}AS = \Lambda$

POWERS OF A , EQN $u_{k+1} = Au_k$

Previous lecture on Eigenvalues:

$A - \lambda I$ singular

$Ax = \lambda x$ (x is eigenvector, λ is eigenvalue)

Today we will look at

- $S^{-1}AS = \Lambda$ · $\underline{\Lambda}$ is diagonal <u>eigenvalue matrix</u>
- where $\underline{S}$ is <u>matrix of eigenvectors</u> and is invertible, so we need n independent eigenvectors

Suppose we have n lin. indep. eigenvectors of A, put them in columns of S.

$$AS = A\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ \downarrow & & \downarrow \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{\text{original matrix, } S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & & \vdots \\ & & & \lambda_n \end{bmatrix}}_{\text{Eigenvalue Matrix, } \Lambda} = S\Lambda$$

So far we see $\boxed{AS = S\Lambda}$

With S invertible, $\boxed{\begin{aligned} S^{-1}AS &= \Lambda \\ A &= S\Lambda S^{-1} \end{aligned}}$

Example POWERS OF A

If $Ax = \lambda x$ then $A^2x = \lambda Ax = \lambda^2 x$

Using our new formula,

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

and,

$$A^k = S\Lambda^k S^{-1}$$

A will have n independent eigenvectors (and is diagonalizable) if all the λ's are different (that is, no repeated eigenvalues).

Example : TRIANGULAR MATRIX

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \dots \quad \det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2, \quad \underline{\lambda = 2,2} \text{ Eigenvalue}$$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0, \quad \underline{x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \text{ Eigenvector (only 1, wanted 2)}$$

- Triangular matrices are tough to work with, cannot diagonalize!
- Here we did not have distinct eigenvalues so only 1 eigenvector

Equation : $u_{k+1} = Au_k$

Start with given vector u_0 and multiply by A each iteration.

$$u_1 = Au_0, \quad u_2 = A^2u_0, \quad \dots \quad u_k = A^k u_0$$

- To solve u_0 we write it as a linear combination of eigenvectors

$$u_0 = c_1x_1 + c_2x_2 + \dots c_nx_n \qquad = Sc$$

then multiplying by A (and recall $Ax = \lambda x$)

$$Au_0 = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \dots + c_n\lambda_nx_n$$

doing this k times

$$A^ku_0 = c_1\lambda_1^kx_1 + c_2\lambda_2^kx_2 + \dots + c_n\lambda_n^kx_n = \boxed{\Lambda^k Sc = u^k}$$

Now we have this nice formula, let's do an example.

FIBONACCI :

$F_0 = 0, \quad F_1 = 1 \qquad [0, 1, 1, 2, 3, 5, 8, 13, \ldots]$

$F_{k+2} = F_{k+1} + F_k$ "Standard" form of Fibonacci Eqn

Let's put this in a more familiar form

make into a system

$F_{k+2} = F_{k+1} + F_k$

$F_{k+1} = F_{k+1}$

Let $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

then $u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$

$$u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

So,

$$\boxed{u_{k+1} = A\, u_k \quad \text{where, } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \; u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}$$

FIBONACCI SEQUENCE IN LIN. ALGEBRA FORM

Let's find the EIGENVALUES

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \qquad \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}}$$

From the eqn $u^k = \lambda^k S c$ we see that the EIGENVALUE controls the growth of this function.

Here, $\lambda_1 > 1$ while $\lambda_2 < 1$ so λ_1 term will dominate as k increases.

Let's find the EIGENVECTORS

$$A - \lambda_n I = \begin{bmatrix} 1-\lambda_n & 1 \\ 1 & -\lambda_n \end{bmatrix} x_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}}$$

Let's put this all together

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{u_0}$$

we need to solve

$$c_1 x_1 + c_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{c_1 = \frac{1}{\lambda_1 - \lambda_2}}, \quad \underline{c_2 = \frac{1}{\lambda_2 - \lambda_1}}$$

S is our EIGENVECTOR MATRIX, Λ is our EIGENVALUE IDENTITY MATRIX

$$S = \begin{array}{c} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array} \end{array}, \quad \Lambda = \begin{array}{c} \lambda_1 \\ \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \\ \uparrow \\ \lambda_2 \end{array}$$

C is our vector of coefficients

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{5}}$$

Our final formula /system is :

$$u_k = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^k$$