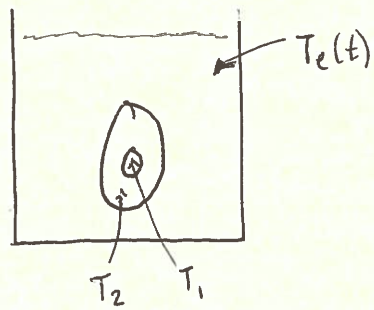
Linear System

$$x' = ax + by + r_1(t)$$
$$y' = cx + dy + r_2(t)$$

$x' = f(x, y, t)$ — x, y dependent, t independent

$x(t_0) = x_0$ [init. conditions]

Mixing Model



define model by Newton's Law

$$\frac{dT_1}{dt} = a(T_2 - T_1)$$ yolk only affected by egg white

$$\frac{dT_2}{dt} = a(T_1 - T_2) + b(T_e - T_2)$$

Re-write system to make more meaningful

$$T_1' = -aT_1 + aT_2$$
$$T_2' = aT_1 - (a+b)T_2 + bT_e(t)$$

Now consider inputs to system

- if we "coddle" the egg then $T_e = 100e^{-kt}$ because we place the egg in the bath when water boils then the egg cools the water exponentially (turn heat off when egg in)
- What if we coddle egg as described above, then place egg in an ice bath, then $T_e = 0$, this is homogeneous.

Now I assign values for the conductivity terms a, b and we solve an example system...

Solving Systems of ODES by Elimination

(1) $T_1' = -2T_1 + 2T_2$

(2) $T_2' = 2T_1 - 5T_2$

$\begin{pmatrix} a = 2 \\ b = 3 \end{pmatrix}$ for ice bath gives this system

$T_1(0) = 40$

$T_2(0) = 45$

1. Eliminate T_2

(3) $$T_2 = \frac{T_1' + 2T_1}{2}$$ [solve (1) for T_2]

$$\left(\frac{T_1' + 2T_1}{2}\right)' = 2T_1 - 5\left(\frac{T_1' + 2T_1}{2}\right)$$ [plug (3) into (2)]

$$\frac{1}{2}T_1'' + \frac{7}{2}T_1' + 3T_1 = 0$$

$$T_1'' + 7T_1' + 6T_1 = 0$$

* Note, we started with 2 1st order equations, now we have a 2nd order eqn. Generally, order is sum of orders of given system. Here 1st + 1st = 2nd. (Law of conservation of mathematical difficulty)

2. Solve for T_1 by usual method

$$r^2 + 7r + 6 = 0$$

$$r = -1, -6 \longrightarrow \underline{T_1 = c_1 e^{-t} + c_2 e^{-6t}} \quad (4)$$

3. Solve for T_2 by substituting (4) into (3)

$$\underline{T_2 = \frac{1}{2}c_1 e^{-t} - 2c_2 e^{-6t}}$$

4. Input initial conditions

$$40 = c_1 + c_2$$
$$45 = \frac{1}{2}c_1 - 2c_2$$
$$\longrightarrow 50 = -5c_2, \quad \underline{c_2 = -10}, \quad \underline{c_1 = 50}$$

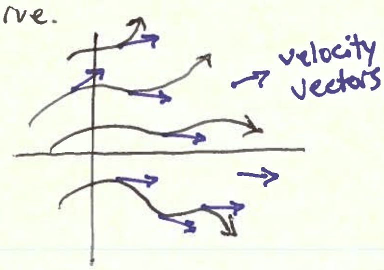
$$\boxed{\begin{aligned} T_1 &= -10e^{-t} + 50e^{-6t} \\ T_2 &= 25e^{-t} - 100e^{-6t} \end{aligned}}$$

If we consider an autonomous system, no time dependence, then a solution to the system is a parameterized curve.

System of 1st order ODEs = Velocity field

Solution = parameterized curve

different curves represent different starting values but all are solutions!



Solve the ~~system~~ same ice bath system by Linear Algebra/Matrix techniques

Now $T_1 \to x$, $T_2 \to y$

$$\boxed{\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned}}, \quad \begin{aligned} x(0) &= 40 \\ y(0) &= 45 \end{aligned} \quad \xrightarrow{\text{Soln by elimination}} \quad \boxed{\begin{aligned} x &= c_1 e^{-t} + c_2 e^{-6t} \\ y &= \tfrac{1}{2} c_1 e^{-t} + 2 c_2 e^{-6t} \end{aligned}}$$

$(c_1 = 50,\ c_2 = -10)$

Re-write system

(1) $$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t}}$$ expected Solution in Matrix form

Use trial solution as we did before. Say

(2) $x = a_1 e^{\lambda_1 t}$

(3) $y = a_2 e^{\lambda_2 t}$

* The Soln to left is wrong. Note in soln above that x, y have the same constants so $a_1 = a_2$, $\lambda_1 = \lambda_2$ to get $\longrightarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \quad (4)$$

Substitute trial solution into system

(5) $$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$$

We did the sub. so the exponential terms would drop out. We reduced a calculus problem down to an algebra problem! (same method as Laplace Trans form) Let's write out our equations:

(6) $\lambda a_1 = -2a_1 + 2a_2$

(7) $\lambda a_2 = 2a_1 + 5a_2$

Problem, we have 2 equations and 3 unknowns, is this indeterminant? Also, they are non-linear $(\lambda \cdot a_1)$! Normal 2 eqns, 3 unknowns gives an infinite number of solutions. Let's call a_1, a_2 variables and say λ is just a parameter, that is it's just an unknown constant. Now, these equations are linear AND homogeneous! Let's rewrite:

(8) $(-2-\lambda) a_1 + 2 a_2 = 0$

(9) $2 a_1 + (-5-\lambda) a_2 = 0$

Now we have a pair of simultaneous DE's of 2 variables, we can solve this. This is a square linear system.

We can find non-trivial solutions iff the determinant is zero.

(10) $$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$(\lambda+2)(\lambda+5) - 4 = 0$

$\lambda^2 + 7\lambda + 6 = 0$ this is our characteristic equation (same as using other method)

roots: $\lambda = -1, -6$

Find a_1, a_2 for $\lambda = -1$ (plug into (8))

$$-a_1 + 2a_2 = 0$$
$$2a_1 - 4a_2 = 0$$

* Notice one eqn of system is linear comb. of the other. This <u>must</u> be the case! Else the only solution is trivial, $a_1 = a_2 = 0$. This is why we took determinant equal to zero → so we could get a redundant soln.

We can now solve for a_1, a_2.
Let's do this by simply fixing a_2, then determine a_1. Must be:

$$c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Any multiple of c will be a solution. The soln to the system is

$$c\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

Now solve for a_1, a_2 for $\lambda = -6$ (plug into (8,9))

$$4a_1 + 2a_2 = 0$$
$$2a_1 + 1a_2 = 0 \quad \longrightarrow \quad d\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now I can write the general soln to my system by superposition

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + d\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}}$$

this is the solution we expected!

Procedure to solve System of ODE's

1. Define model using matrix notation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{"A"}} \begin{pmatrix} x \\ y \end{pmatrix}$$

2. Come up w/ trial solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$$

3. Substitute trial soln into system

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

4. The homogenous system will look like

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

5. Check to see if we have a non-trivial soln

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \overset{?}{=} 0$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) \overset{?}{=} 0 \; *$$

[characteristic Equation of matrix A]

* Note we can jump directly here from step 1!
We can think of this using matrix operations to be:

$$\lambda^2 - Tr(A) \cdot \lambda + Det(A) \overset{?}{=} 0$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from system, λ is called eigenvalue of matrix A (characteristic/proper value)

6. For each λ_i, find associated vector $\begin{pmatrix} a_{1_i} \\ a_{2_i} \end{pmatrix}$ of coefficients by solving homogeneous system from (4) with selected λ_i.
* Note we call the vector $\vec{\alpha} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ the eigenvector belonging to λ_i

7. General solution is superposition of arbitrary constant times the eigenvector times the exponential of eigenvalue from trial solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix} e^{\lambda_2 t}$$ ← eigenvalue

We can write this in a still more condensed way.

8. Write vectors in condensed form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \vec{x}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \overleftrightarrow{A}$$

$$\boxed{\vec{x}\,' = \overleftrightarrow{A}\,\vec{x}}$$

* this is our system

9. Our trial solution is now

$$\vec{x} = \vec{\alpha}\, e^{\lambda t}$$

10. Plug trial solution into system

$$\lambda \vec{\alpha} = A \vec{\alpha}$$

11. Solve system from (10)

$$(A - \lambda I)\,\vec{\alpha} = 0$$

* must include Identity matrix to use matrix op. λ is a scalar

12. The characteristic equation says the determinant of (11) is zero so that the system is solvable.

$$|A - \lambda I| = 0$$

13. The roots of the characteristic eqn are called eigenvalues (λ_i) which are used to calculate the eigenvectors ($\vec{\alpha}_i$). The general solution is the superposition of $c_i \vec{\alpha}_i \lambda_i$

$$\vec{x} = \sum_i c_i \vec{\alpha}_i \lambda_i$$