We have now concluded our discussion of flow over airfoils. In the remaining lectures, we discuss compressible flow through nozzles and diffusers with application to design of rocket and air-breathing jet engines. Before proceeding, we motivate our discussion with a brief overview of how rocket engines work.

## Basics of Rocket Propulsion:

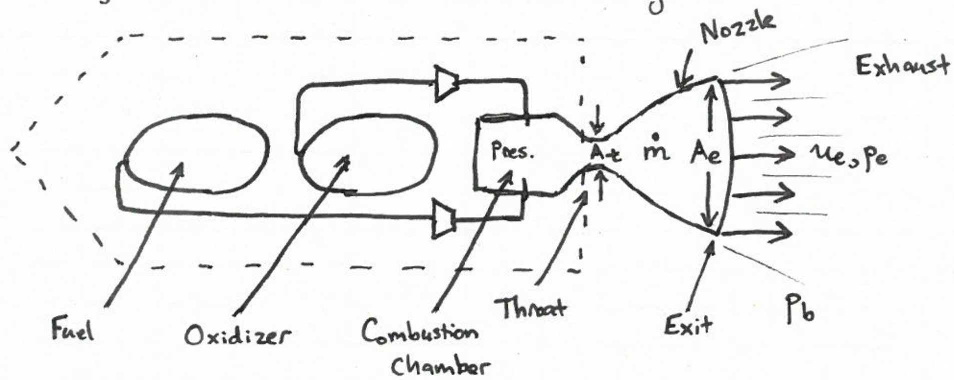
There are two basic classes of rocket engines:

Liquid Rocket Engines: A rocket engine that uses liquid propellants.

Solid Rocket Engines: A rocket engine that uses solid propellants.

The fuel systems differ considerably between these two classes of engines, but the basic mechanisms by which they generate thrust are largely the same. As such, we limit our discussion here to liquid rocket engines.

Below, we show a basic schematic of a liquid rocket engine:




In a liquid rocket, stored fuel and stored oxidizer are pumped into a combustion chamber where they are mixed and burned. The combustion produces great amounts of exhaust gas at high pressure and high temperature. The hot gas is then passed through a nozzle which accelerates the flow.

The thrust developed by a liquid rocket is estimated using the rocket equation which we derived in Homework 2:

$$T = \dot{m} u_e + (p_e - p_b) A_e$$

Above:

$\dot{m}$ = The mass flow rate through the nozzle

$u_e$ = The flow speed at the nozzle exit

$p_e$ = The static pressure at the nozzle exit

$p_b$ = The surrounding ambient or back pressure

$A_e$ = The surface area at the nozzle exit

The rocket equation presented above holds for both liquid and solid rockets, so the ensuing discussion pertains to both cases.

A key goal in rocket design is to maximize thrust. Looking at the rocket equation, this is obtained by:

1) Maximizing $\dot{m}$
2) Maximizing $u_e$
3) Minimizing $|p_e - p_b|$

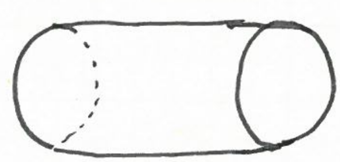
Assuming that the combustion chamber and back pressures are fixed, we can optimize thrust by intelligently choosing the nozzle shape. This requires an in-depth knowledge of how fluid flows through a nozzle, thus we turn exactly to this topic in the remainder of today's lecture.

Note that the details of how to mix and burn the fuel and oxidizer, without blowing out the flame, are very complex. However, these details are beyond the scope of the class.

## Quasi-One-Dimensional Flow:

Nozzle flow is an example of quasi-one-dimensional flow. A quasi-one-dimensional flow is one in which all variables vary primarily along one direction, referred to as $x$. A flow in a duct with slowly varying area $A(x)$ is the case of interest here. A one-dimensional flow, by comparison, is simply a quasi-one-dimensional flow in which the area $A$ is constant. Visually:

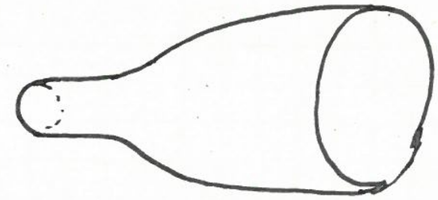
$A = \text{const.}$



$p = p(x)$
$\rho = \rho(x)$
$T = T(x)$
$u = u(x)$

One-Dimensional Flow

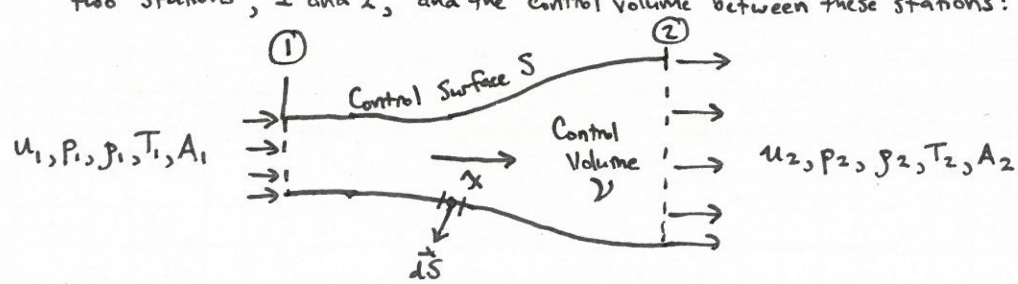
$A = A(x)$



$p = p(x)$
$\rho = \rho(x)$
$T = T(x)$
$u = u(x)$

Quasi-One-Dimensional Flow

Let us now derive governing equations for such a flow. In what follows, consider two stations, 1 and 2, and the control volume between these stations:



We further assume the following:

1. The flow is steady.
2. The flow is isentropic (adiabatic and inviscid).
3. The flow is subject to no body forces.

As per usual, we turn to conservation of mass, momentum, and energy. Let us first consider the integral form of conservation of mass over our control volume. As the flow is steady, we have:

$$\oiint_S \rho \vec{V} \cdot d\vec{S} = 0$$

However:

$$\oiint_S \rho \vec{V} \cdot d\vec{S} = -\rho_1 u_1 A_1 + \rho_2 u_2 A_2$$

so:

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

or equivalently:

$$\rho u A = \text{const.}$$

Differentiating, we obtain:

$$\boxed{d(\rho u A) = 0} \quad \text{Mass Equation}$$

The above is the mass equation for quasi-one-dimensional flow, and it states that the mass flow rate in a duct is independent of $x$ (i.e., $\dot{m} = \text{const.}$).

We next consider conservation of momentum. Since the flow is steady, inviscid, and subject to no body forces, we simply turn to Euler's equation:

$$\frac{1}{2} d(V^2) = -\frac{1}{\rho} dp$$

which holds along a streamline. As the flow simply goes from left to right, the above holds between stations 1 and 2. Then, exploiting the product rule and the fact that $V = u$, we have:

$$\boxed{dp = -\rho u\, du} \quad \text{Momentum Equation}$$

The above is the momentum equation for quasi-one-dimensional flow.

We finally turn to conservation of energy. As the flow is adiabatic in addition to steady and inviscid, we have:

$$h + V^2/2 = \text{const.}$$

along streamlines and hence between stations 1 and 2. Differentiating, we obtain:

$$\boxed{dh + u\,du = 0} \quad \text{Energy Equation}$$

The above is the energy equation for quasi-one-dimensional flow.

Now that we have derived the governing equations for quasi-one-dimensional flow, let us use these to study some physical characteristics of duct flow. By the mass equation:

$$d(\rho u A) = d\rho\, uA + \rho\, du\, A + \rho u\, dA = 0$$

Dividing by $\rho u A$ yields:

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

The above relates changes in density, velocity, and area. We seek to simplify this relationship so that it only relates changes in velocity and area. Towards this goal, we write:

$$\frac{d\rho}{\rho} = \frac{dp}{\rho}\frac{d\rho}{dp}$$

Since the flow is isentropic:

$$\frac{d\rho}{dp} = \left(\frac{\partial p}{\partial \rho}\right)_s^{-1} = \frac{1}{a^2}$$

By the momentum equation:

$$\frac{dp}{\rho} = -u\,du$$

Thus:

$$\frac{d\rho}{\rho} = -\frac{u\,du}{a^2} = -\frac{u^2}{a^2}\frac{du}{u} = -M^2\frac{du}{u}$$

and consequently:

$$-M^2\frac{du}{u} + \frac{du}{u} + \frac{dA}{A} = 0$$

Re-arranging terms yields:

$$\boxed{\frac{dA}{A} = (M^2 - 1)\frac{du}{u}} \quad \text{Area-Velocity Relation}$$

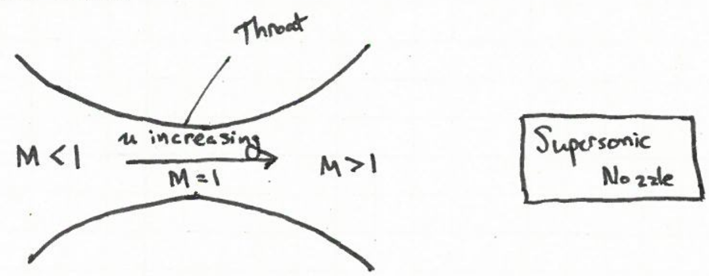
The above is the area-velocity relation, and it tells us three very important things regarding duct flows:

1. For $0 \le M < 1$ (subsonic flow), an increase in velocity is associated with a decrease in area and vice versa.

2. For $M > 1$ (supersonic flow), an increase in velocity is associated with an increase in area. Likewise, a decrease in velocity is associated with a decrease in area.

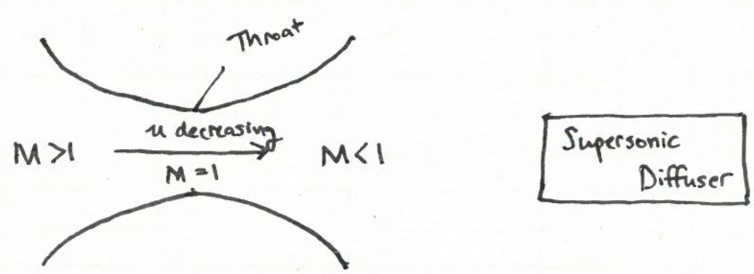
3. For $M = 1$ (sonic flow), $dA = 0$ for any $du$. Mathematically, this corresponds to a local minimum or maximum in the area distribution. Physically, this corresponds to a minimum area.

We are now in a position to examine how a nozzle is able to accelerate a flow. The convergent portion of a nozzle accelerates a gas initially at rest subsonically. If the flow is accelerated quickly enough, it reaches sonic speeds at the minimum area of the nozzle, referred to as the throat. After the flow passes through the throat, it accelerates supersonically through the divergent portion of the nozzle. This is illustrated below:




Note that the above flow scenario only occurs for specific design conditions. Namely, it is possible that the flow does not reach sonic speeds at the throat, or that it passes through shock waves in the divergent portion of the nozzle. These scenarios occur if the back pressure is not [redacted] ideal, as we discuss shortly.

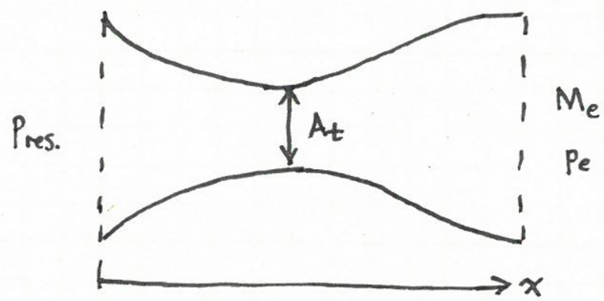
We can decelerate a supersonic in a similar manner. If we wish to take a supersonic flow and slow it down to subsonic speeds, we must decelerate it first in a convergent duct, and then as soon as sonic flow is obtained, we must further decelerate it in a divergent duct. This is illustrated below:



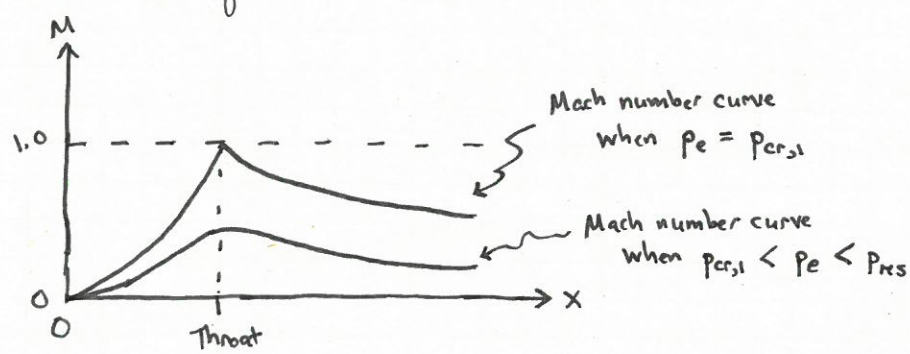

The above configuration is known as a supersonic diffuser, and it acts to isentropically compress a flow. As with a nozzle, the flow through a supersonic diffuser is isentropic only for specific design conditions. Otherwise, shock waves will appear. We return to the topic of diffusers when we discuss supersonic inlets for jet engines.

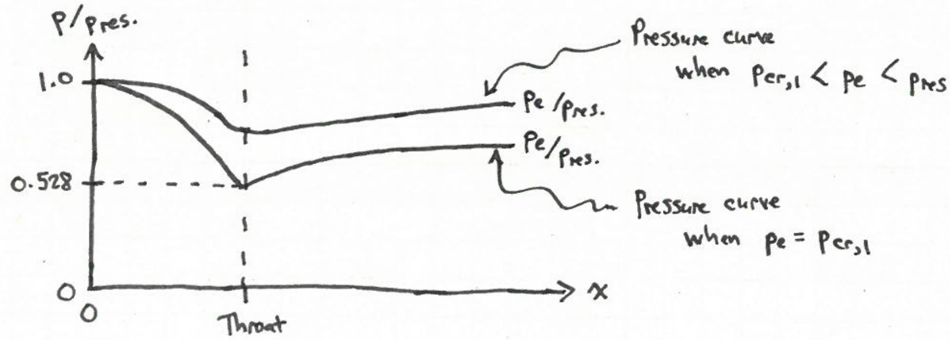
## Nozzle Flow Characteristics:

Ideally, a nozzle accelerates a flow to supersonic conditions as detailed previously. However, if you take a convergent-divergent nozzle and place it on a table in front of you, air is simply not going to start flowing through it on its own accord. We have to impose a force on the gas to produce any acceleration. For the inviscid flows considered here, the only acceleration mechanism is a pressure gradient. Thus, returning to the nozzle on the table, a pressure difference must be created between the inlet and exit. In particular, recognizing the inlet is a reservoir, we must have $p_e < p_{res}$ where $p_e$ is the exit pressure and $p_{res}$ is the reservoir pressure. Visually:




If the exit pressure $p_e$ is just below the reservoir pressure $p_{res}$, then the flow is subsonic throughout the nozzle. Moreover, the flow accelerates through the convergent region, reaches its max speed at the throat, and then decelerates through the divergent region. If we then decrease the exit pressure $p_e$, the flow speed increases throughout the nozzle but remains subsonic. Eventually if the exit pressure is decreased to a certain critical value, the flow at the throat reaches sonic conditions while the flow throughout the remainder of the nozzle continues to be subsonic. This critical exit pressure is called the first critical pressure and is denoted as $p_{cr,1}$. Visually:



When $p_e = p_{cr,1}$, the pressure at the throat, $p_t$, is equal to the sonic pressure. Thus, for this case:

$$\frac{p_t}{p_{res.}} = \frac{p^*}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0.528 \quad \text{for } \gamma = 1.4$$

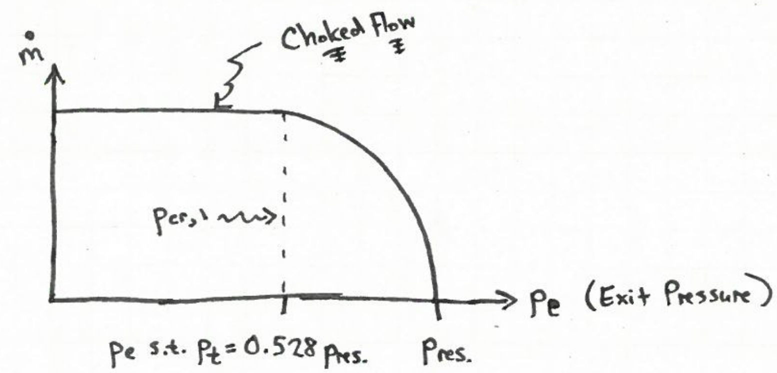
as illustrated in the above figures.

If we continue to decrease the exit pressure below the first critical pressure, then the flow just downstream of the throat reaches supersonic conditions. This represents a significant change in the flow characteristics. As opposed to subsonic flow, supersonic flow accelerates as it passes through the divergent portion of the nozzle. Moreover, as mentioned previously, information cannot pass upstream in an isentropic supersonic flow. There are two consequences of this:

Consequence #1: The flow conditions at and upstream of the throat remain unchanged as $p_e$ is reduced below $p_{cr,1}$.

Consequence #2: The flow just downstream of the throat is unaware of the exit pressure if $p_e < p_{cr,1}$.

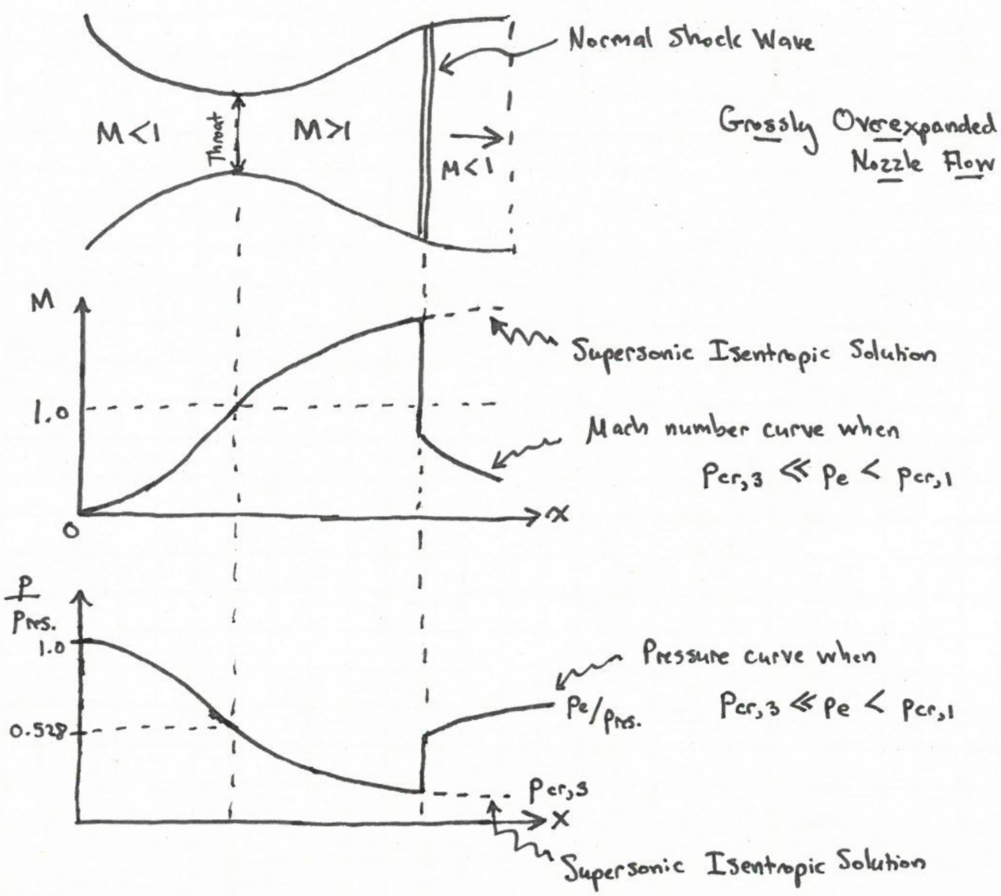
Due to consequence #1, the flow at the "throat" becomes "frozen" as $p_e$ is reduced below $p_{cr,1}$. Thus, the mass flow rate through the throat, and hence throughout the nozzle, remains constant no matter how low $p_e$ is reduced as depicted below:




This situation is referred to as choked flow.

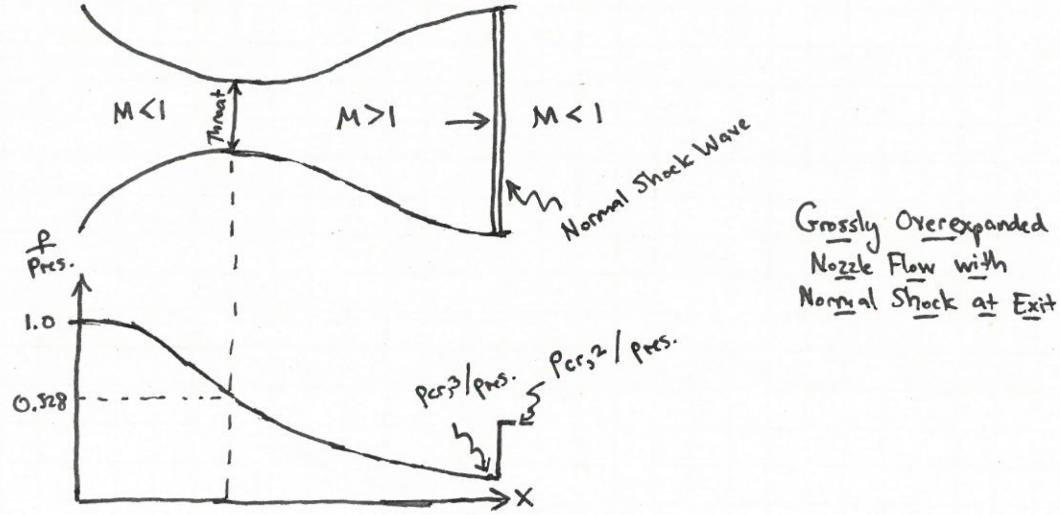
Due to consequence #2, if the exit pressure is too high to allow an isentropic supersonic flow throughout the entire divergent section, then a normal shock wave must form downstream of the throat to compress the flow and increase the flow pressure. In fact, there is only one exit pressure that yields a supersonic isentropic solution. This pressure is known as the third critical pressure and is denoted as $p_{cr,3}$.

For $p_e$ just less than $p_{cr,1}$, there is a standing normal shock wave near the throat. As we continue to decrease $p_e$, this normal shock wave moves downstream as illustrated below:



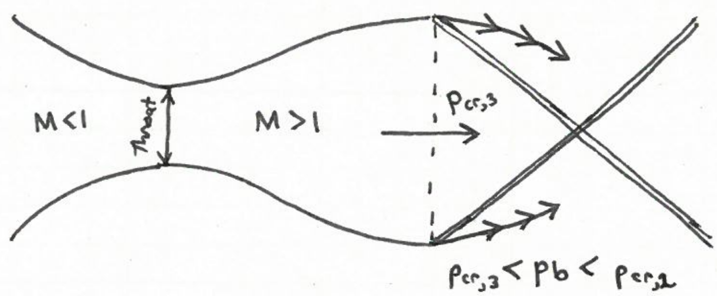

This flow scenario is referred to as grossly overexpanded nozzle flow as the nozzle expands the flow far too much and normal shocks inside the nozzle must compress the flow so that it meets the exit pressure boundary condition.

Eventually, as the exit pressure reaches a certain value, the normal shock wave stands precisely at the exit. This pressure is referred to as the second critical Mach number and is denoted as $p_{cr,2}$. In this scenario, there is a pressure discontinuity at the exit. To the left, the pressure is $p_{cr,3}$, and to the right, the pressure is $p_{cr,2}$:



To this stage, we have dealt with $p_e$, the pressure at the exit. However, now we have a pressure discontinuity at the exit. Upstream of the discontinuity, the flow is isentropic. Downstream, the flow is atmospheric. Hence, we will now refer to the back pressure $p_b$, which is simply the surrounding ambient pressure. For $p_e > p_{cr,2}$, $p_b = p_e$, so our previous discussion also applies replacing with $p_b$ the exit pressure.

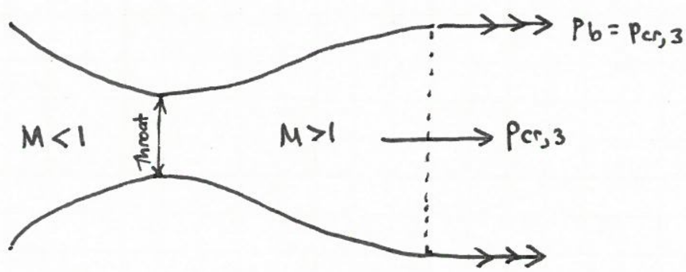
For the remainder of our discussion, assume we have control of $p_b$ and that we are going to continue to decrease $p_b$. When $p_{cr,3} < p_b < p_{cr,2}$, the back pressure is still above the isentropic pressure at the nozzle exit. Hence, the jet of gas from the nozzle must somehow be compressed such that it pressure matches $p_b$. This compression takes place across oblique shocks attached to the exit as shown below:




Overexpanded Nozzle Flow

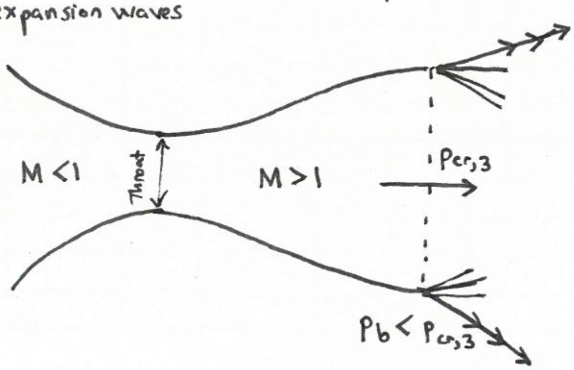
This flow scenario is referred to as overexpanded nozzle flow as the nozzle flow leaving the exit is expanded too much and must be compressed via oblique shocks. The patterns formed by these oblique shocks are known as shock diamonds.

When $p_b$ is reduced to a value such that $p_b = p_{cr,3}$, there is no mismatch between the exit pressure and the back pressure. The nozzle jet then exhausts smoothly into the surroundings without passing through any waves. This is known as perfectly expanded nozzle flow and is shown below:




Perfectly Expanded Nozzle Flow

Finally, if $p_b$ is reduced below $p_{cr,3}$, the jet of gas from the nozzle must expand further to match the lower back pressure. This expansion takes place across centered expansion waves




Underexpanded Nozzle Flow

This flow scenario is referred to as underexpanded nozzle flow as the nozzle flow is not expanded enough at the exit and must be expanded via Prandtl-Meyer expansion waves.

The above flow scenarios are summarized in the table below.

| Flow Scenario | Back Pressure Criteria | Defining Characteristics |
|---|---|---|
| Subsonic Nozzle Flow | $p_b > p_{cr,1}$ | Flow is subsonic throughout nozzle. |
| First Critical Nozzle Flow | $p_b = p_{cr,1}$ | Flow is sonic at throat and subsonic elsewhere. |
| Grossly Overexpanded Nozzle Flow | $p_{cr,2} < p_b < p_{cr,1}$ | Flow is choked and there is a normal shock within nozzle. |
| Second Critical Nozzle Flow | $p_b = p_{cr,2}$ | Flow is choked and there is a normal shock at nozzle exit. |
| Overexpanded Nozzle Flow | $p_{cr,3} < p_b < p_{cr,2}$ | Flow is choked and there are oblique shocks attached to exit. |
| Third Critical / Perfectly Expanded Nozzle Flow | $p_b = p_{cr,3}$ | Flow is choked and jet exhausts smoothly. |
| Underexpanded Nozzle Flow | $p_b < p_{cr,3}$ | Flow is choked and there are expansion fans attached to exit. |

Note that the discussion above is predicated on having a duct of a given shape. Namely, we assumed $A = A(x)$ is given. Later, we will learn how to design the throat and exit areas as to maximize thrust for a rocket engine. However, this theory will not tell us how to design the contour of the nozzle. In reality, if the walls of the nozzle are not curved just right, then oblique shocks can occur within the nozzle. To properly design the nozzle contour, we need to turn to advanced techniques such as the method of characteristics.

Obviously, knowledge of the first, second, and third critical pressures for a given nozzle is of great interest. Momentarily, we will show how to compute these quantities. Before doing so, we first introduce one more concept: sonic throat area.

Sonic Throat Area:

Recall that we earlier defined sonic temperature and speed of sound as the temperature $T^*$ and speed of sound $a^*$ obtained by changing the flow adiabatically to sonic conditions. We also defined sonic density and pressure as the density $\rho^*$ and pressure $p^*$ obtained by changing the flow isentropically to sonic conditions. Similarly, We define the sonic throat area as the area $A^*$ that the nozzle area must be reduced to to bring a steady, isentropic quasi-one-dimensional flow subject to no body forces to sonic conditions.

The sonic throat area $A^*$ is constant through an isentropic flow but it does change across shocks.

By conservation of mass:

$$\rho^* u^* A^* = \rho u A$$

Since $u^* = a^*$, we have:

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_0} \frac{\rho_0}{\rho} \frac{a^*}{u}$$

Now note:

$$\frac{\rho^*}{\rho_0} = \left( \frac{2}{\gamma+1} \right)^{1/(\gamma-1)}$$

$$\frac{\rho_0}{\rho} = \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{1/(\gamma-1)}$$

$$\left( \frac{u}{a^*} \right)^2 = M^{*2} = \frac{((\gamma+1)/2) M^2}{1 + ((\gamma-1)/2) M^2}$$

Thus:

$$\left( \frac{A}{A^*} \right)^2 = \left( \frac{\rho^*}{\rho_0} \right)^2 \left( \frac{\rho_0}{\rho} \right)^2 \left( \frac{a^*}{u} \right)^2$$

$$= \left( \frac{2}{\gamma+1} \right)^{2/(\gamma-1)} \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{2/(\gamma-1)} \left( \frac{((\gamma+1)/2) M^2}{1 + ((\gamma-1)/2) M^2} \right)$$

Algebraic simplifications then yield:

$$\boxed{\left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left( \frac{2}{\gamma+1} \left( 1 + \frac{\gamma-1}{2} M^2 \right) \right)^{(\gamma+1)/(\gamma-1)}}$$

The above is known as the area-Mach number relation. Turned inside out, it says that $M = f(A/A^*)$. That is, the Mach number is simply a function of the ratio of the local duct area to the sonic throat area. There are actually two solutions for $M$ for a given $A/A^* > 1$ : a subsonic solution $M < 1$ and a supersonic solution $M > 1$. These are tabulated in Appendix A in Anderson along with the usual isentropic relations.

## Calculation of the First, Second, and Third Critical Pressures:

We now discuss how to compute the first, second, and third critical pressures for a given nozzle geometry (i.e., $A_t$ and $A_e$) and reservoir pressure $p_{res}$.

To start, note that the first and third critical pressures correspond to the back pressure when it is equal to the exit pressure and the flow is isentropic throughout the nozzle and sonic at the throat. In this setting, $A_t = A^*$, and by the area-Mach number relation, the exit Mach number satisfies:

$$\boxed{\left(\frac{A_e}{A_t}\right)^2 = \frac{1}{M_e^2}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right)^{(\gamma+1)/(\gamma-1)}}$$

The above admits two solutions which we denote as $M_{e,\text{subsonic}} < 1$ and $M_{e,\text{supersonic}} > 1$. These can be found using a numerical root-finding technique such as Newton's method or by appealing to Appendix A in Anderson.

$M_{e,\text{subsonic}}$ is the exit pressure for first critical nozzle flow, so we have by the isentropic relations:

$$\frac{p_{cr,1}}{p_{res}} = \left(\frac{p_0}{p_{e,\text{subsonic}}}\right)^{-1} = \left(1+\frac{\gamma-1}{2}M_{e,\text{subsonic}}^2\right)^{-\frac{\gamma}{\gamma-1}}$$

$M_{e,\text{supersonic}}$ is the exit pressure for third critical nozzle flow, so we have by the isentropic relations:

$$\frac{p_{cr,3}}{p_{res}} = \left(\frac{p_0}{p_{e,\text{supersonic}}}\right)^{-1} = \left(1+\frac{\gamma-1}{2}M_{e,\text{supersonic}}^2\right)^{-\frac{\gamma}{\gamma-1}}$$

Finally, the second critical pressure is the pressure behind a normal shock with upstream Mach number $M_{e,\text{supersonic}}$ and pressure $p_{cr,3}$. By the normal shock relations:

$$\frac{p_{cr,2}}{p_{cr,3}} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{e,\text{supersonic}}^2 - 1\right)$$

In summary:

$$\boxed{\begin{aligned}
p_{cr,1} &= \left(1+((\gamma-1)/2)\,M_{e,\text{subsonic}}^2\right)^{-\gamma/(\gamma-1)}\,p_{res}\\
p_{cr,3} &= \left(1+((\gamma-1)/2)\,M_{e,\text{supersonic}}^2\right)^{-\gamma/(\gamma-1)}\,p_{res}\\
p_{cr,2} &= \left(1+((2\gamma)/(\gamma+1))\left(M_{e,\text{supersonic}}^2-1\right)\right)p_{cr,3}
\end{aligned}}$$

We now consider a quick example.

<u>Problem Statement:</u> Consider a nozzle with area ratio $A_e/A_t = 3$ and reservoir pressure $p_{res} = 30$ atm. Compute the first, second, and third reservoir pressures.

<u>Solution:</u> <u>Tables Approach:</u> By Appendix A:

$$M_{e,\text{subsonic}} = 0.2000 \qquad \frac{p_{cr,1}}{p_{res}} = 0.9728$$

$$M_{e,\text{supersonic}} = 2.650 \qquad \frac{p_{cr,3}}{p_{res}} = 0.0464$$

for $A_e/A_t = 3$. By Appendix B:

$$\frac{p_{cr,2}}{p_{cr,3}} = 8.026$$

Thus:

$$p_{cr,1} = \left(\frac{p_{cr,1}}{p_{res}}\right) p_{res} = 29.18 \text{ atm}$$

$$p_{cr,3} = \left(\frac{p_{cr,3}}{p_{res}}\right) p_{res} = 1.39 \text{ atm}$$

$$p_{cr,2} = \left(\frac{p_{cr,2}}{p_{cr,3}}\right) p_{cr,3} = 11.17 \text{ atm}$$

<u>Exact Approach:</u> Newton's method yields:

$$M_{e,\text{subsonic}} = 0.19745\ldots$$
$$M_{e,\text{supersonic}} = 2.63741\ldots$$

By direct calculation:

$$\frac{p_{cr,1}}{p_{res}} = \left(1 + \frac{\gamma-1}{2} M_{e,\text{subsonic}}^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.97318147\ldots$$

$$\frac{p_{cr,3}}{p_{res}} = \left(1 + \frac{\gamma-1}{2} M_{e,\text{supersonic}}^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.04729911\ldots$$

$$\frac{p_{cr,2}}{p_{cr,3}} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{e,\text{supersonic}}^2 - 1\right) = 7.94858675\ldots$$

Thus:

$$p_{cr,1} = \left(\frac{p_{cr,1}}{p_{res}}\right) p_{res} = 29.1954\ldots \text{ atm}$$

$$p_{cr,3} = \left(\frac{p_{cr,3}}{p_{res}}\right) p_{res} = 1.4190\ldots \text{ atm}$$

$$p_{cr,2} = \left(\frac{p_{cr,2}}{p_{cr,3}}\right) p_{res} = 11.2788\ldots \text{ atm}$$