05-16-19

shroeder - Cantor proof

D Successor

a set S,

$\sigma(S)$ is the successor of S,

$\sigma(S) := S \cup \{S\}$

D a set is inductive iff

$\{\} \in S$ & $T \in S \Rightarrow \sigma(T) \in S$

claim if A inductive and B inductive then $A \cap B$ inductive

equinumerous

↓

correspond

↓

if $\exists$ a bij between 2 sets A, B then A, B are equinumerous and we write $A \simeq B$

claim: let $\omega := \bigcap \{B \mid B \text{ is inductive}\}$

natural inductive set

(Cardinal sets)

R surj ∧ inj rely on $\forall / \exists$ not on $\mathbb{N}$ or cardinality

D Any set that is equinumerous to a cardinal set is called finite

Let $\mathcal{F} := \{S \mid S \text{ is finite}\}$

obs $\simeq$ equinumerosity is an equivalence

$\forall S$ $[S]_{\simeq} := \{ T \in \mathcal{E} \mid S \simeq T \}$

$0 := [\{\}]_{\simeq} = \{ \{\} \}$

$1 := [\sigma(\{\})]_{\simeq} = \{ \{a\}, \{b\}, \ldots \}$ infinite cardinality class

$2 := [\sigma\sigma(\{\})]_{\simeq} =$ pairs

claim $\boxed{\sigma([S]_{\simeq}) = [\sigma(S)]_{\simeq}}$

"uni

tay

$\mathbb{N} := \{0, 1, \ldots\}$ A_0 : $\forall n \in \mathbb{N}$ $n + 0 := n$

A_+ : $\forall n, m \in \mathbb{N}\, \sigma(m+n) = m + \sigma(n)$

O_0 : $\forall n \in \mathbb{N}$, $n \leq n$

O_+ : $\forall m, n \in \mathbb{N}$, if $m \leq n$ then $m \leq \sigma(n)$

$\langle \mathbb{N}, \leq \rangle$ (b-set)

$\mathbb{R}$ not all b-sets are posets

$n+1$

$0 = \emptyset$

$(n \times m) = n \times m + m$

$\langle \mathbb{N}, \times, 1 \rangle$

a multiplicative Monoid

$\sim (c, d)$

iff $a + d = b + c$

$\forall$

② $\nexists x \in \mathbb{N}$ $s(x) = 0$

③ if $\sigma(m) = \sigma(n)$ then $m = n$

fact

$\forall m, n, k \in \mathbb{N}$ if $m + k = n + k$ then $m = n$ (cancellation)

? internally related $\sim$ transitivity?

$$\frac{a+d = b+c \quad \text{and} \quad c+f = d+e}{a+d+c+f = b+c+d+e} \quad \langle \text{cancellation twice} \rangle$$

$$\frac{}{a+f = b+e}$$

yes

⋮

Equiv($\sim$)

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addition $\overline{a,b} + \overline{c,d} = \overline{a+c, b+d}$

$$\Rightarrow \left(\frac{a,b \sim a',b' \quad c,d \sim c',d'}{\overline{a',b'} + \overline{c',d'} = \overline{a+c', b'+d'}} \right)$$

$\mathbb{N}^2/\sim$ addition $\quad$ $\mathbb{N}$ addition

prove : HW

$(\leq_\sim)$ $\quad \overline{a,b} \leq \overline{c,d}$ iff both $a \leq c$ and $d \leq b$

let $\phi := n \mapsto \overline{n,0} : \mathbb{N} \to \mathbb{N}^2/\sim$

Show $\quad \phi(m+n) = \phi(m) +_{\mathbb{N}^2/\sim} \phi(n)$

suppose $m,n \in \mathbb{N}$

(1) $\phi(m+n) = \overline{m+n, 0}$

$= \overline{m+n, 0+0}$

$= \overline{m,0} + \overline{n,0}$

$= \phi(m) + \phi(n)$

observe $\overline{a,b} + \overline{(0,0)} = \overline{a,b}$

$\phi(0) = \overline{0,0}$

so $\langle \mathbb{N}^2/\sim, +, \overline{0,0} \rangle$ is an additive monoid

Def opposite of $\overline{a,b}$ to be $\overline{b,a}$

$$-\overline{a,b} := \overline{b,a}$$

can be proven by contradiction via ...

$\varphi(a) \neq \varphi(b)$

$\overline{0,1} \in \mathbb{N}^2/\sim, \quad \overline{0,1} \notin \mathrm{Im}(\varphi)$

$\mathrm{Im}(\varphi) \neq \mathbb{N}^2/\sim = \mathrm{codom}(\varphi)$

~~Epi~~ (φ)

2x2 matrices form a zero divisor

φ is monic, non-epic

Claim every member of $\mathbb{N}^2/\sim$ is either the image of some number under φ or the opposite of such an image

let $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\langle \mathbb{Z}, +, \times, 0, 1 \rangle \quad \forall n \in \mathbb{Z}\ \exists m \in \mathbb{Z}, (n + m = 0)$

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let $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$

zero $= \overline{(0,1)}$

unit $= \overline{1,1}$

a new $\sim_r$ called <u>rational</u>

$\overline{a,b}^r \times \overline{c,d}^r = \overline{ac, bd}^r \iff ad = bc$

$\overline{a,b}^r + \overline{c,d}^r = \overline{ad+bc, bd}^r$

$\psi : a \mapsto \overline{a,1} ; \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}^* / \sim_r$

$-\overline{a,b} = \overline{-a,b}$

you can show that ψ is monic, not epi

mult inv. (reciprocal)

$\frac{\cup}{\overline{a,b}} = \overline{b,a}$

Fraction notation of <u>ratio class</u> $\frac{a}{b} := \overline{a,b}$

$\mathbb{Q} := \left\{ \frac{a}{b} \;\middle|\; a,b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$

$\langle \mathbb{Q}, +, \times, 0, 1, -, \cup \rangle$; <u>field</u>

also a <u>loset</u>

<u>Fact</u> $\mathbb{N}$ is not <u>Bounded Above</u>

assume otherwise, then

$\exists b \;\; \forall n \in \mathbb{N} \; n \leq b$

let $b = \text{lub}(\mathbb{N})$

$n+1 \leq b$

$n \leq b-1 \leq b \Rightarrow \bot$

$b-1 \leq b$

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Archimedean Property

$\forall a,b \in \mathbb{N}$ $a,b>0$ $\exists n \in \mathbb{N}$ s.t. $na>b$

Dfn

Positive Cone :

in a field $\mathbb{F}$ $\exists P \subseteq \mathbb{F}$ s.t. $\forall x,y \in P$

① $xy \neq 0$
② $x+y \in P$, $x \cdot y \in P$
③ $\forall z \in \mathbb{F}$ exactly one of the following is true
 ① $z = 0$
 ② $z \in P$
 ③ $-z \in P$

$\mathbb{F}^+$ is positive cone of $\mathbb{F}$

set $\leq$

sets A, B

$A \leq B \Leftrightarrow$ ~~$\forall a \in A \; \forall b \in B$~~ $a \leq b$

Claim $\mathbb{Q}^+$ is positive cone $(\mathbb{Q})$

$= \{ q \in \mathbb{Q} \mid q > 0 \}$

Claim

$x < y \Leftrightarrow y - x \in \mathbb{F}^+$

BIG IDEA

There are indeed subsets of $\mathbb{Q}$ that ~~have~~ are bounded above but have no lub

is $h \in \mathbb{Q}$? proof by contradiction : if $h \in \mathbb{Q}$, then $\exists a,b \in \mathbb{Z}$ $GCD(a,b) = 1$

A is an initial segment of $\mathbb{Q}$ Dfn whenever $a \in A$ then $\forall q < a$ then $q \in A$ and $A \neq \mathbb{Q}$ and $A \neq \{\}$

Two initial segments A and B are cut-equivalent $A \sim B$ if $A \leq B$ and $B \leq A$