Infinite Series - James Hislop
Topology - James Dugundji
Theory & Application of Infinite Series - Knapp

- Every Field 'F' has a subdomain $F_{\mathbb{Z}} \subseteq F$ that is domain isomorphic to $\mathbb{Z}$.
- Every Field "F" has a subfield $F_{\mathbb{Q}} \subseteq F$ that is field isomorphic to $\mathbb{Q}$.

Assume F is ordered and Complete $\therefore$ isomorphic to $\mathbb{R}$

- if $E \subseteq F$ is not bounded above $\sup(E) = +\infty$
- if $E \subseteq F$ is not bounded below $\inf(E) = -\infty$
- $\sup(\{\}) = -\infty$, $\inf(\{\}) = +\infty$

Let $\mathbb{F}$ be a "complete", normed Field

X is a set of points

$d: X \times X \to \mathbb{R}$ is a metric(X)

Metric

if 1) $\forall x,y \in X \quad d(x,y) \geq 0$ ⟵ positive definite

2) $\forall x,y \in X \quad d(x,y) = 0$ iff $x = y$ ⟵ discernability

3) $\forall x,y \in X \quad d(x,y) = d(y,x)$ ⟵ symmetry

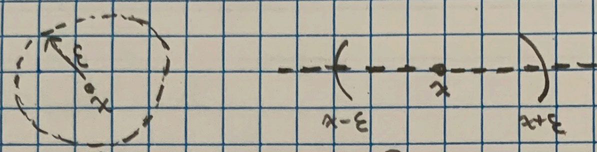
4) $\forall x,y,z \in \mathbb{Z} \quad d(x,y) \leq d(x,z) + d(z,y)$ ⟵ triangle inequality

Let $\langle X, d\rangle$ be a metric space

$\forall x \in X$ and $\varepsilon > 0$

the ε-neighborhood (x), $N_\varepsilon(x)$

$$N_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$$



in $\mathbb{C}$ in $\mathbb{R}$

Suppose $E \subseteq X$ we say $x \in X$ is an accumulation point of E

If $\forall \varepsilon > 0$ $\exists y \in E$ $y \neq x$ where $y \in N_\varepsilon(x)$

$$Acc(E) := \{ x \in X \mid x \text{ is acc. pnt of } E \}$$

if $x \in E$ $x \notin Acc(E)$ then x is an isolated pt of E

$x \in E$ is interior pnt of E

if $\exists \varepsilon > 0$ $N_\varepsilon(x) \subseteq E$

$$Int(E) = \{ x \in E \mid x \text{ is interior to } E \}$$

$E \subseteq X$ is OPEN if $E = Int(E)$
$E \subseteq X$ is CLOSED if $Acc(E) \subseteq E$

Thm: 1) if $E \subseteq X$ is OPEN, E^c is CLOSED
2) if $E \subseteq X$ is CLOSED, E^c is OPEN

Facts: $\forall \varepsilon > 0$ $x \in X$ $N_\varepsilon(x)$ is OPEN

if $x \in Acc(E)$, $\forall \varepsilon > 0$ $N_\varepsilon(x)$ contains infinite members of E

Define Closure of $E \subseteq X$ is $E \cup Acc(E)$
represented as $\overline{E}$.

$$\overline{E} := E \cup Acc(E)$$

SET Theory & Metric Spaces

Irving Kaplansky

Obvs. $\forall E \subseteq X$ 1) $Int(E)$ is OPEN
2) $\overline{E}$ is CLOSED

Boundary Set of $E \subseteq X$ is $\overline{E} \setminus Int(E)$
this is refered to $Bd(E)$

Thms 1) E is closed $\Leftrightarrow$ $E = \overline{E}$
2) if F is closed, $E \subseteq F$, $\overline{E} \subseteq F$

Thms
1) An arbitrary union of open sets is open
2) A finite intersection of finite sets is open
3) A finite union of closed sets is closed
4) A arbitrary intersection of closed sets is closed.

Ex. $\left(\frac{1}{n}, \frac{1}{n}\right)$ $n \in \mathbb{Z}^+$

Def: We say collection of $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ where $G_\alpha \subseteq X$ and open is an open cover of $E \subseteq X$ if $E \subseteq \bigcup_{\alpha \in I} G_\alpha$

Ex. $(n-1, n+1)$ $n \in \mathbb{Z}$

Def: We say an open cover $\mathcal{G}$ of E admits a subcover $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{G}$ and $E \subseteq \bigcup \mathcal{H}$

Def: We say $K \subseteq X$ is compact if every open cover of K admits a finite subcover of K.

Fact All compact sets are closed.

Thm Suppose $K \subseteq X$ is compact:

1) $E \subseteq K$ and E is closed, E is compact
2) $F \subseteq X$ and F is closed, $F \cap K$ is compact

$E \subsetneq X$ is <u>disconnected</u>

if $\exists$ E_1, E_2 where

1) $E_1 \subsetneq E$ $E_2 \subsetneq E$
2) $E_1 \cup E_2 = E$
3) $E_1 \cap \overline{E_2} = \{\} = \overline{E_1} \cap E_2$

A set is connected if not <u>disconnected</u>.

Ex. $(0,1) \cup (1,2)$ ← disconnected

Ex. $(0,1) \cup [1,2)$ ← connected

(get this bit later)

Thm <u>Balzano - Weierstraß Thm</u>

Every bounded, infinite set $E \subseteq \mathbb{R}$ has at least one $Acc(E)$.

Pf: Assume $Acc(E) = \{\}$

$\forall x \in E$ x is an isolated point

this means $\forall x \in E$ $\exists \varepsilon > 0$ $N_{\varepsilon_x}(x) \cap E = \{x\}$

and $\{N_\varepsilon(x)\}$ is an infinite open cover of E

Since $\{\} \subseteq E$, E is closed and cover does not admit a finite subcover

E is not compact

$\perp$

But E is compact.

Series & Sequences

Observation: if $J \subseteq \mathbb{N}$, J is "countable"

$\forall k \in \mathbb{N}$ define $k..\infty := \{n \in \mathbb{N} \mid n \geq k\}$

Def: For a metric space / field X

$$\alpha : n \mapsto \alpha_n : (k..\infty) \to X \qquad k \in \mathbb{N}$$

is an X-valued sequence

$\underline{\alpha : \mathbb{N} \to X}$ or $\alpha : \mathbb{Z}^+ \to X$ are sequences

most are this

$$(\alpha_n)_{n \in \mathbb{N}} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$$

$$Img(\alpha) = \{\alpha_n\}_{n \in \alpha} \subseteq X$$

Ex. $\alpha := x \mapsto \begin{cases} -1, & \text{odd} \\ 1, & \text{even} \end{cases} = (1, -1, 1, -1, 1 \ldots)$

$Img(\alpha) = \{-1, 1\}$

Def if $\varkappa$ is an strictly increasing sqnc of $\mathbb{N}$, $(Im(\varkappa) \subseteq \mathbb{N})$
then $(\alpha \circ \varkappa) := n \mapsto \alpha_{\varkappa_N}$

an (α_{K_N}) is a subsequence of (α_n).

Thm If β_n is a subsqnc α_n we write

$$\beta_n \sqsubseteq \alpha_n$$

Thm If $(\alpha_n) \longrightarrow \alpha_\infty$ then $\forall \beta \sqsubseteq \alpha$ $(\beta_n) \rightarrow \alpha_\infty$

Cor If $\exists \beta \sqsubseteq \alpha$ and β diverges, then α diverges.

$Cnv(\alpha)$ means α convergent

$\cancel{Cnv}(\alpha)$ means α divergent

Arithmetic of Sequences (in $\mathbb{R}$)

Suppose $(\alpha_n) \longrightarrow \alpha_\infty$, $(\beta_n) \longrightarrow \beta_\infty$, $c \in \mathbb{R}$

1) $(\alpha + \beta)_n = (\alpha_n + \beta_n) \longrightarrow (\alpha_\infty + \beta_\infty)$

2) $(c\alpha)_n = (c\alpha_n) \longrightarrow c \cdot \alpha_\infty$

3) $(\alpha \cdot \beta)_n = (\alpha_n \cdot \beta_n) \longrightarrow \alpha_\infty \cdot \beta_\infty$

4) $\left(\frac{\alpha}{\beta}\right)_n = \left(\frac{\alpha_n}{\beta_n}\right) \rightarrow \frac{\alpha_\infty}{\beta_\infty}$ st. $\beta_n \neq 0$, $\beta_\infty \neq 0$

5) $\alpha_n = c$ $\forall n$ $(\alpha_n) \rightarrow c$

6) β_n is bounded $(\alpha_n) \rightarrow 0$, $(\alpha_n \cdot \beta_n) \rightarrow 0$

7) $\alpha_n \leq \beta_n$ $\forall n$, $\alpha_\infty \leq \beta_\infty$

8) Suppose $\alpha_n \leq \gamma_n \leq \beta_n$ $\forall n$, $(\alpha_n) \rightarrow \alpha_\infty$, $(\beta_n) \rightarrow \alpha_\infty$,
the $(\gamma_n) \rightarrow \alpha_\infty$. Squeeze Thm.!

Thm: Let $E = \mathrm{Im}(\alpha)$

If $\exists$ $\mathrm{Acc}(x)$, then there exists $\beta_n \sqsubseteq \alpha_n$ where $(\beta_n) \to x$

Thm Bounded monotonic sqnc theorem BMST

Every bounded monotonic sqnc converges. !!!

Define $\alpha := n \mapsto \begin{cases} 2 & n = 0 \\ \dfrac{\alpha_{n-1} + \frac{2}{\alpha_{n-1}}}{2} & \text{if } n \geqslant 1 \end{cases}$

$\alpha_0 = 2$

$\alpha_1 = \dfrac{2 + \frac{2}{2}}{2} = \dfrac{3}{2}$

$\alpha_2 = \dfrac{\frac{3}{2} \; \frac{2}{\frac{3}{2}}}{2} = \dfrac{\frac{3}{2} + \frac{4}{3}}{2} = \frac{3}{4} + \frac{4}{6} = \frac{17}{12}$

$\forall n, \alpha_n \in \mathbb{Q}$

α_n is decreasing

Claim: α_n is bounded below by 0

$\alpha_0 \geqslant 0$

$0 \leq \alpha_n \leq 2$

$\alpha_\infty = \dfrac{\alpha_\infty + \frac{2}{\alpha_\infty}}{2}$

$2\alpha_\infty = \alpha_\infty + \frac{2}{\alpha_\infty}$

$(\alpha_\infty)^2 = 2$

$\alpha_\infty = \sqrt{2}$

A sequence (α_n) is <u>Cauchy</u>

if $\forall \varepsilon > 0 \ \exists N > 0 \quad \forall m,n > N$

$$d(\alpha_m, \alpha_n) < \varepsilon$$

<u>Thm</u> Every convergent sequence is <u>Cauchy</u>

Def: A space X is <u>Cauchy Complete</u>
is called <u>sequential</u> <u>Complete</u>

if Every Cauchy Sequence converges.

α_n will converge

or diverge to $-\infty$ or $+\infty$

or diverge with bounded oscillation $(\sin(n))$

or diverge with unbounded oscillation $(-n)^n \ n \geq 1$

Notable Results:

"P-sequence" $p > 0, \left(\frac{1}{n^p}\right) \longrightarrow 0$

"root-sequences" $p > 0, (\sqrt[n]{p}) \rightarrow 1 \ \& \ (\sqrt[n]{n}) \longrightarrow 1$

"geometric sequence" $|r| < 1, (r^n) \rightarrow 0$