CME241 Assignment5

Quinn Hollister

January 2022

1 Problem 1

Assume $U(x) = x - \frac{\alpha x^2}{2}$. Assuming $x \sim \mathcal{N}(\mu, \sigma^2)$, calculate Expected utility, x_{CE}, and π_A

1. Calculate $E[U(x)]$:

$$E[U(x)] = E[x - \frac{\alpha x^2}{2}] = E[x] - \frac{\alpha}{2}E[x^2] \tag{1}$$

But, we know that for a normally distributed random variable x, that $E[x] = \mu$, $E[x^2] = Var(x) + E[x]^2 = \sigma^2 + \mu^2$. So, out equation (1) becomes:

$$E[U(x)] = \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2) \tag{2}$$

2. Calculate x_{CE}:

$$x_{CE} = U^{-1}(E[U(x)]) \tag{3}$$

$$U(x_{CE}) = E[U(x)] \tag{4}$$

$$x_{CE} - \frac{\alpha}{2} \cdot x_{CE}^2 = \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2) \tag{5}$$

And we can use the quadratic equation here to solve for x_{CE}, where $a = \frac{\alpha}{2}$, $b = -1$, $c = \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2) = E[U(x)]$. This leaves us with two solutions of the form:

$$x_{CE} = \frac{1 \pm \sqrt{1 - 2 \cdot \alpha \cdot E[U(x)]}}{\alpha} \tag{6}$$

3. Calculate π_A: This is simply $E[X] - x_{CE}$, so we have:

$$\pi_A = \mu - \frac{1 \mp \sqrt{1 - 2 \cdot \alpha \cdot E[U(x)]}}{\alpha} \tag{7}$$

4. How much would you invest in this risky asset based on this utility function:

Based off of this model, we can model our wealth W (based on a riskless asset with return r) as a normal distribution process with:

$$W \sim \mathcal{N}(1 + r + \pi(\mu - r), (\pi\sigma)^2) \tag{8}$$

In order to find the optimal proportion of our money to invest in the risky asset, we need to maximize $E[U(W)]$, or equivalently x_{CE} with respect to the proportion π. Essentially, find the π^* such that

$$\frac{\partial x_{CE}}{\partial \pi} = 0 \tag{9}$$

$$\frac{\partial x_{CE}}{\partial \pi} = \frac{\partial \mu(\pi)}{\partial \pi} \pm \frac{1}{2} \cdot \frac{1}{\alpha} \cdot (f(\pi))^{-\frac{1}{2}} \cdot \frac{\partial f(\pi)}{\partial \pi} \tag{10}$$

$$f(\pi) = 1 - 2\alpha(\mu(\pi) - \frac{\alpha}{2}(\sigma(\pi)^2 + \mu(\pi)^2)) \tag{11}$$

$$\mu(\pi) = 1 + r + \pi(\mu - r), \sigma(\pi) = \sigma \cdot \pi \tag{12}$$

Thus, in order for our partial to equal zero, either $f(\pi)$ approaches infinity–which is possible only when either σ or μ go to infinity, and thus not worth considering–or the partial of f with respect to pi is zero. This is the condition that we will consider:

$$\frac{\partial f(\pi)}{\partial \pi} = -2\alpha(\mu - r - \alpha\pi\sigma^2 - \alpha(\mu - r)(1 + r + \pi(\mu - r))) = 0 \tag{13}$$

$$(\mu - r - \alpha(\mu - r)(1 + r)) - \pi^*(\alpha\sigma^2 + \alpha(\mu - r)^2) = 0 \tag{14}$$

$$\pi^* = \frac{\mu - r - \alpha(\mu - r)(1 + r)}{\alpha(\sigma^2 + (\mu - r)^2} \tag{15}$$

Thus, when we compare the amount of money we should invest in the risky asset $z = (\$1 \text{ mil})\pi^*$ with our risk aversion α, we get the plot:

2 Problem 2

In this exercise, you will be deriving the kelly criterion, i.e. the optimal amount to bet to maximize the expected utility of wealth after a single bet

1. Write down the two outcomes for wealth W at the end of your bet $f \cdot W_0$:

$$W_1 = \begin{cases} W_0 + f \cdot W_0(1 + \alpha), & \text{Bet wins} \\ W_0 + f \cdot W_0(1 - \beta), & \text{Bet loses} \end{cases} \tag{16}$$

2. Write down two outcomes for Utility of W:

$$\log W_1 = \begin{cases} \log W_0(1 + f \cdot (1 + \alpha), & \text{Bet wins} \\ \log W_0(1 + f \cdot (1 - \beta)), & \text{Bet loses} \end{cases} \tag{17}$$

3. Write down $E[U(W)]$:

$$E[\log W] = p\cdot(\log W_0+\log(1+f(1+\alpha)))+q\cdot(\log W_0+\log(1+f(1-\beta))) \tag{18}$$

But we know $p+q=1$, so

$$E[\log W] = \log(W_0) + p\cdot\log(1+f(1+\alpha)) + q\cdot\log(1+f(1-\beta)) \tag{19}$$

4. Take the derivative with respect to f

$$\frac{\partial E[\log W_1]}{\partial f} = \frac{\partial}{\partial f}(p\cdot\log(1+f(1+\alpha)) + q\cdot\log(1+f(1-\beta))) \tag{20}$$

$$\frac{\partial E[\log W_1]}{\partial f} = p\frac{1+\alpha}{1+f(1+\alpha} + q\frac{1-\beta}{1+f(1-\beta} \tag{21}$$

5. Set this to zero to find f^*:

$$0 = p\frac{1+\alpha}{1+f^*(1+\alpha)} + q\frac{1-\beta}{1+f^*(1-\beta)} \tag{22}$$

$$-p\frac{1+\alpha}{1+f^*(1+\alpha)} = q\frac{1-\beta}{1+f^*(1-\beta)} \tag{23}$$

$$-p(1+\alpha)(1+f^*(1-\beta)) = q(1-\beta)(1+f^*(1+\alpha)) \tag{24}$$

$$f^*\cdot((1+\alpha)(1-\beta)q + (1-\beta)(1+\alpha)p) = -p(1+\alpha) - q(1-\beta) \tag{25}$$

$$f^* = \frac{-p(1+\alpha) - q(1-\beta)}{((1+\alpha)(1-\beta)q + (1-\beta)(1+\alpha)p)} \tag{26}$$

Thus, since $p+q=1$ our kelly criterion is given as:

$$f^* = \frac{-p}{1-\beta} + \frac{-q}{1+\alpha} \tag{27}$$

This makes intuitive sense in the relationship between the parameters and the wager size. We exepct β to be greater than 1 as that would indicate an actual loss, and with that condition, the denominator is always negative. This means that we increase the wager size in a positive relationship with p, and decrease the wager size as β increases in magnitude (when its greater than 1). Also, we decrease the wager size as q increases, and we increase the wager size as we increase α.