CME241 Assignment9

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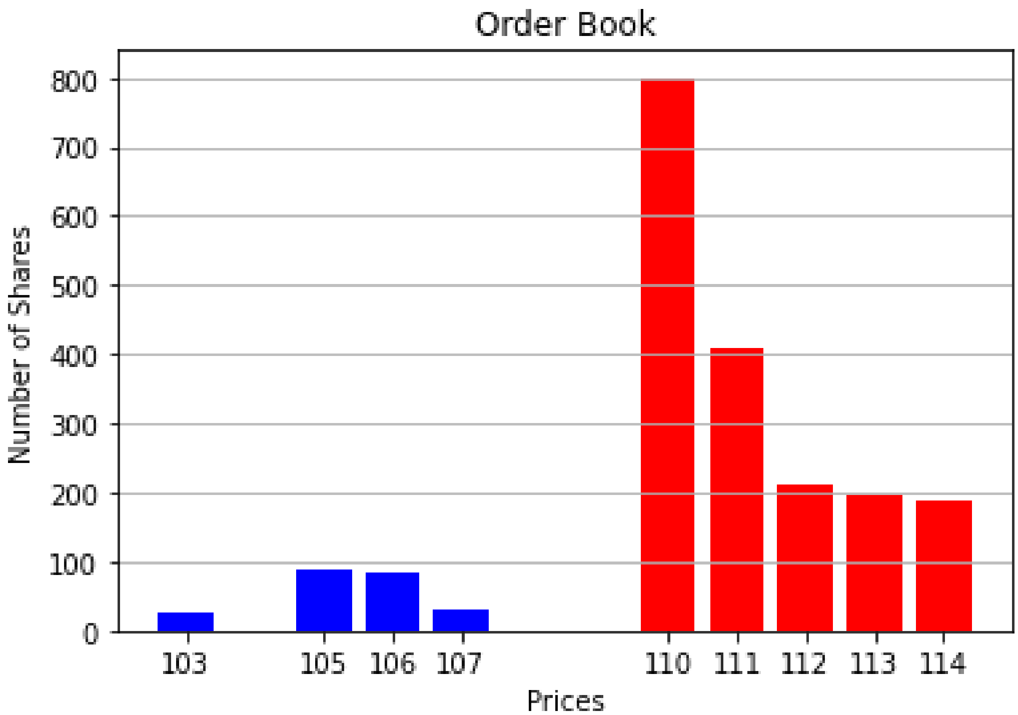
1 Problem 1

Build a simple simulator of Order Book Dynamics and experiment with different models for random arrivals of Market Orders and Limit Orders.

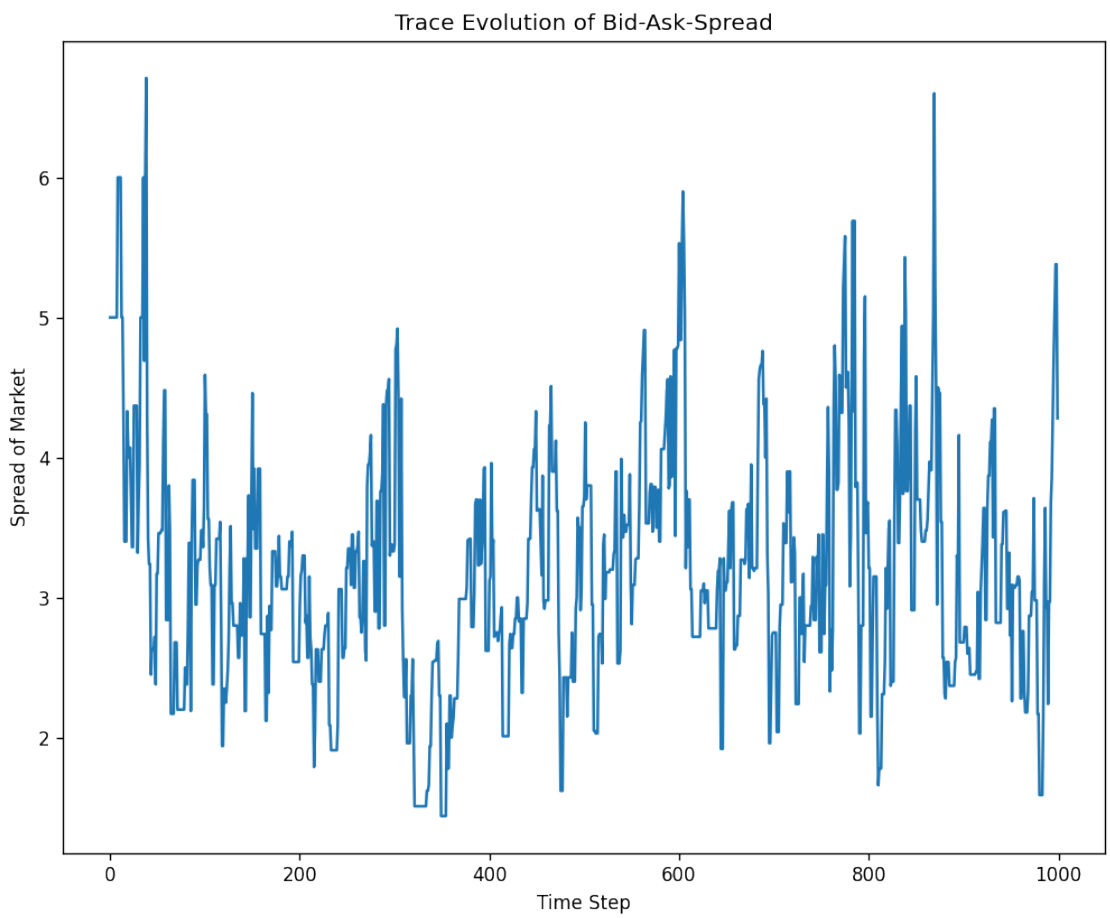
This order book simulator will be a model based on the Markov process class, so we must implement the abstract method transition. In order to do that, we must specify the probability model with which those transitions depend. First, at each time step, we will use a RNG to determine which kind of order we will process. Note, we will not handle multiple orders at a given time step, but this shouldn't matter since we need to execute orders in order, otherwise the structure of the order book is undefined. Next, depending on which kind of order it is (Market or Limit) we will again use a RNG (with different distributions) to determine the next state. If it's a market order, we just need to define a distribution that gives us the number of shares this order wants. We'll use a gamma distribution with user-defined parameters to model this share count. In order to model a Limit Order, we can use this gamma distribution for number of shares, but we must add an additional distribution to model the price. For the price, we'll use another single gamma distribution to model the distance "away" from the mid price that the limit order is set at. Thus, depending if its a sell LO or buy LO, we can +/- the sampled value to get this price.

Using this class, we created 5 instances of the OrderBookSimulator, each with different parameter values that described the model. First, I'll show a trace of the mid-point and bid ask price points for one "experience" of the order book.

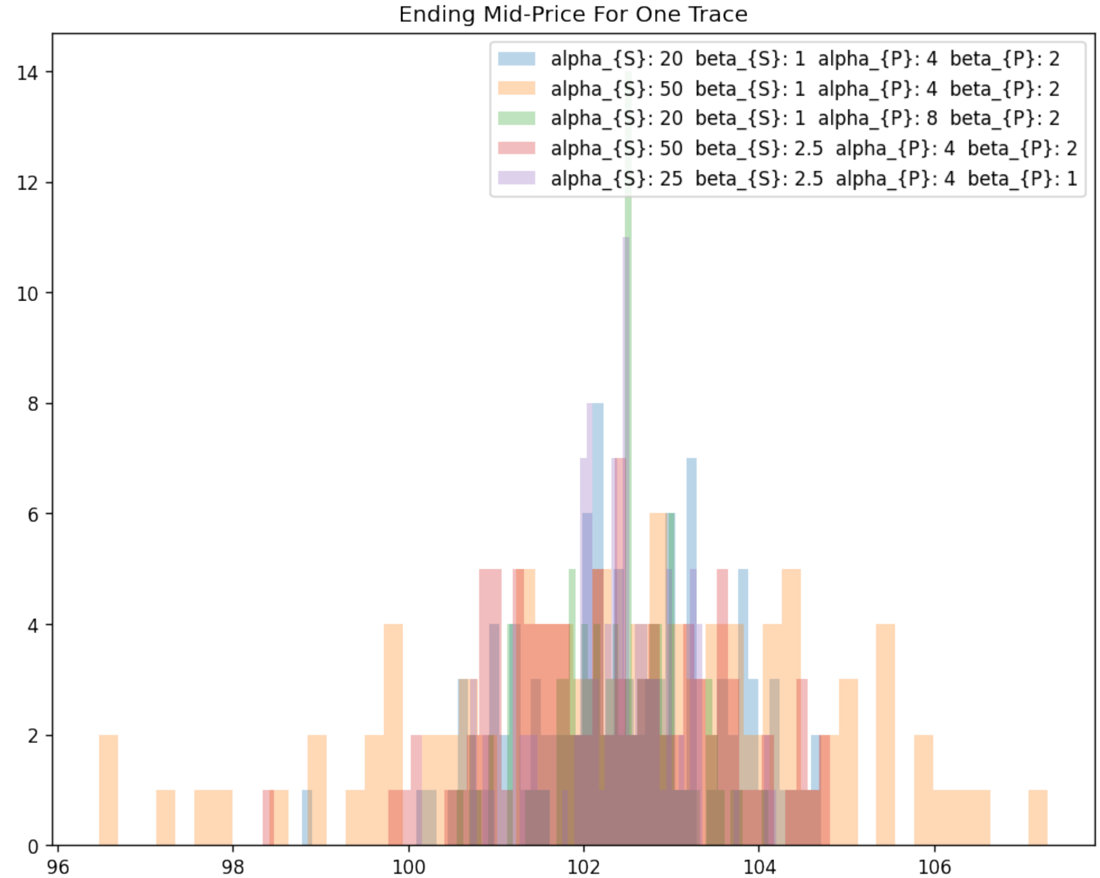
And this is a snapshot of the order book at the finish of that experience. You can see that having prices as float's can lead to some strange behaviour, and would look more "reasonable" if we infused some discretization in terms of the allowable prices for limit orders.



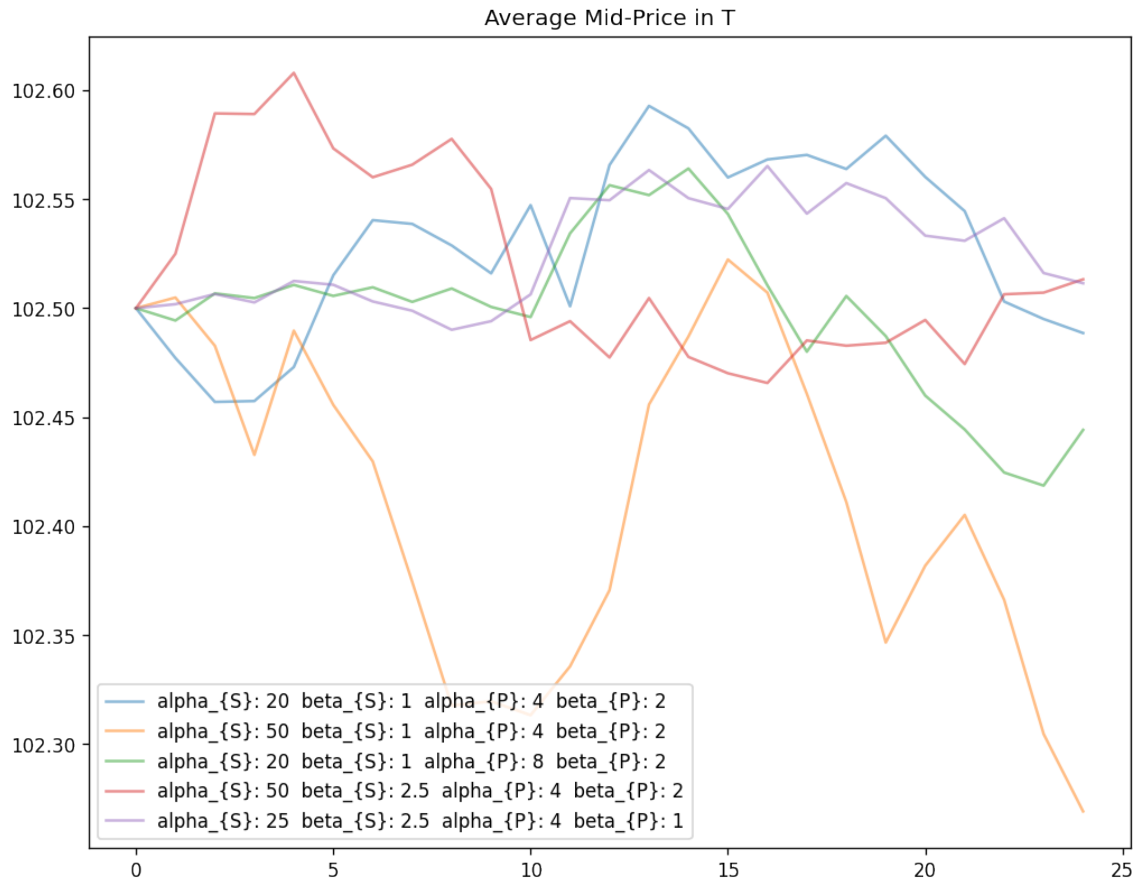
I can also show the evolution of the market spread:



Now I ran 5 different instances of the orderbook with different parameter values for about 50 time steps and 100 traces each. The following is a histogram of the ending mid-prices



Then I also calculated the average mid-price at each time step for these different models. We can see that larger scale parameters for our share amount gamma distributions leads to more volatility in the mid-price as we would expect, since higher levels of market orders should eat more of the book, and "move" the market more. The inverse can be said for the parameters describing the price gamma distribution, since greater values in the magnitude of delta price over the midpoint should create more limit orders "outside" of what will be filled by the market.



We should note that there is an incredible amount of flexibility with this simulator, and there exist so many ways to make it more realistic. First, we should actually make price points discrete, as the share price in the real world is denominated in multiples of \$0.00625 (or something similar). Also, we use the same distribution to model the share count for both Limit and Market orders when they could easily be different. Also, our delta in share price from the midpoint (how we get limit orders) is symmetric between the sell and buy limit orders which doesn't need to be the case. We also don't have any price evolution that our market and limit orders should probably depend on.

2 Problem 2

Derive the expressions for the Optimal Value Function and the Optimal Policy for the Linear-Percentage Temporary Price Impact Model

Our Model is described below:

$$\begin{aligned} P_{t+1} &= Pt \cdot e^{Z_t} \\ X_{t+1} &= \rho \cdot X_t + \nu_t \\ Q_t &= P_t \cdot (1 - \beta \cdot N_t - \theta \cdot X_t) \end{aligned}$$

We know we can write the Value Function for this problem as a function of the policy π as (and note that R_t is the variable describing the remaining shares we need to sell):

$$V_t^{\pi}((P_t, R_t)) = \mathrm{E}_{\pi}[\sum_{i=t}^{T} N_i \cdot Q_i | (P_t, R_t)] \tag{1}$$

Also, the optimal value function is:

$$V_t^*((P_t, R_t)) = \max_{\pi} V_t^*((P_t, R_t)) \tag{2}$$

which must satisfy the bellman equation:

$$V_t^*((P_t, R_t)) = \max_{N_t}\{N_t \cdot Q_t + \mathrm{E}[V_{t+1}^*((P_{t+1}, R_{t+1}))]\} \tag{3}$$

So we can just use backward induction on this Finite MDP, beginning with time T-1, knowing that we must sell all remaining shares by this time:

$$V_{T-1}^*((P_{T-1}, R_{T-1})) = N_{T-1} \cdot Q_{T-1} = R_{T-1} \cdot Q_{T-1} \tag{4}$$

Now we can actually plug in our model's form of Q_t and solve for the optimal policy.

$$V_{T-2}^* = \max_{N_{T-2}}\{N_{T-2}P_{T-2}(1-\beta N_{T-2}-\theta X_{T-2}) + \mathrm{E}\left[R_{T-1}P_{T-1}(1-\beta R_{T-1}-\theta X_{T-1})|(P_{T-2}, N_{T-2}, X_{T-2})\right]\} \tag{5}$$

But we know that $R_{T-1} = R_{T-2} - N_{T-2}$

$$\begin{aligned} V_{T-2}^* = \max_{N_{T-2}}\{&N_{T-2}P_{T-2}(1-\beta N_{T-2}-\theta X_{T-2})+ \\ &\mathrm{E}[(R_{T-2}-N_{T-2})P_{T-1}(1-\beta\cdot(R_{T-2}-N_{T-2})-\theta X_{T-1})|(P_{T-2}, R_{T-2}, X_{T-2})\} \end{aligned} \tag{6}$$

and we can make the following observations that $\mathrm{E}\left[P_{T-1}\right] = \mathrm{E}\left[e^{Z_{T-2}}P_{T-2}|P_{T-2}\right] = e^{\mu_z+\frac{1}{2}\sigma_z^2}P_{T-2}$, and that $\mathrm{E}\left[X_{T-1}\right] = \mathrm{E}\left[\rho X_{T-2} + \eta_{T-2}|X_{T-2}\right] = \rho X_{T-2}$. Thus, our equation finally becomes:

$$\begin{aligned} V_{T-2}^* = \{&-\beta P_{T-2}N_{T-2}^2 + N_{T-2}P_{T-2}(1-\theta X_{T-2})+ \\ &(R_{T-2}-N_{T-2})P_{T-2}e^{\mu_z+\frac{1}{2}\sigma_z^2}(1-\beta(R_{T-2}-N_{T-2})-\theta\rho X_{T-2})\} \end{aligned} \tag{7}$$

Now we'll rewrite in terms of N_{T-2}, because in order to find the optimal value we'll need to differentiate this expression with respect to N_{T-2}.

$$\begin{aligned} V_{T-2}^* = \max_{T-2}\{&N_{T-2}^2(-\beta P_{T-2} - \beta e^{\mu_z+\frac{1}{2}\sigma_z^2}P_{T-2})+ \\ &N_{T-2}(P_{T-2}(1-\theta X_{T-2}) + 2\beta e^{\mu_z+\frac{1}{2}\sigma_z^2}P_{T-2}R_{T-2} - P_{T-2}e^{\mu_z+\frac{1}{2}\sigma_z^2}(1-\theta\rho X_{T-2})) \\ &-\beta e^{\mu_z+\frac{1}{2}\sigma_z^2}P_{T-2}R_{T-2}^2 + P_{T-2}e^{\mu_z+\frac{1}{2}\sigma_z^2}R_{T-2}(1-\theta\rho X_{T-2})\} \end{aligned} \tag{8}$$

So we end up seeing that if $-\beta P_{T-2} - \beta P_{T-2}e^{\mu_z+\frac{1}{2}\sigma_z^2} \geq 0$, then the optimal choice would just be to sell all of our shares. But, this would imply that $e^{\mu_z+\frac{1}{2}\sigma_z^2} \leq -1$, which is never true. Thus, we would never sell all of our shares at once.

Instead, we need to differentiate and set to zero to find the optimal action

$$\begin{aligned} \frac{\partial f(N_{T-2})}{\partial N_{T-2}} = 2N_{T-2}(&-\beta P_{T-2} - \beta e^{\mu_z+\frac{1}{2}\sigma_z^2}P_{T-2}+ \\ &P_{T-2}[(1-\theta X_{T-2}) + 2\beta e^{\mu_z+\frac{1}{2}\sigma_z^2}R_{T-2} - e^{\mu_z+\frac{1}{2}\sigma_z^2}(1-\theta\rho X_{T-2})] = 0 \end{aligned} \tag{9}$$

Thus, we get the following action for T-2, and it holds in general (doing for V_{T-3} and so on results in the same expression:

$$N^*_{T-2} = c_t^{(1)} + c_t^{(2)} R_{T-2} + c_t^{(3)} X_{T-2} \tag{10}$$

$$c_t^{(1)} = \frac{1 - e^{\mu_z + \frac{1}{2}\sigma_z^2}}{2\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}} \tag{11}$$

$$c_t^{(2)} = \frac{e^{\mu_z + \frac{1}{2}\sigma_z^2}}{(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2})} \tag{12}$$

$$c_t^{(3)} = \frac{-\theta(1 - \rho e^{\mu_z + \frac{1}{2}\sigma_z^2})}{2\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2})} \tag{13}$$

So now when we plug this back into our optimal value function and group terms, we get the following optimal value function, which also holds for general t:

$$V^*_{T-2} = P_{T-2}[c_t^{(4)} + c_t^{(5)} R_{T-2} + c_t^{(6)} X_{T-2} + c_t^{(7)} R^2_{T-2} + c_t^{(8)} X^2_{T-2} + c_t^{(9)} R_{T-2} X_{T-2}] \tag{14}$$

$$c_t^{(4)} = -\beta c_{T-2}^{(1)^2} (1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}) + c_{T-2}^{(1)} (1 - e^{\mu_z + \frac{1}{2}\sigma_z^2}) \tag{15}$$

$$c_t^{(5)} = -2\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}) c_{T-2}^{(1)} c_{T-2}^{(2)} + 2\beta e^{\mu_z + \frac{1}{2}\sigma_z^2} c_{T-2}^{(1)^2} + c_t^{(2)} (1 - e^{\mu_z + \frac{1}{2}\sigma_z^2}) + e^{\mu_z + \frac{1}{2}\sigma_z^2} \tag{16}$$

$$c_t^{(6)} = -2\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}) c_{T-2}^{(1)} c_{T-2}^{(3)} - \theta c_t^{(1)} + c_{T-2}^{(1)^2} e^{\mu_z + \frac{1}{2}\sigma_z^2} \theta\rho + c_{T-2}^{(3)} - c_{T-2}^{(3)} e^{\mu_z + \frac{1}{2}\sigma_z^2} \tag{17}$$

$$c_t^{(7)} = -\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}) c_{T-2}^{(2)^2} - e^{\mu_z + \frac{1}{2}\sigma_z^2} \beta \tag{18}$$

$$c_t^{(8)} = -\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}) c_{T-2}^{(3)^2} - \theta c_{T-2}^{(3)} + e^{\mu_z + \frac{1}{2}\sigma_z^2} \theta\rho c_{T-2}^{(3)} \tag{19}$$

$$c_t^{(9)} = -2\beta(1 + e^{\mu_z + \frac{1}{2}\sigma_z^2}) c_{T-2}^{(2)} c_{T-2}^{(3)} - \theta c_t^{(2)} + c_{T-2}^{(2)} e^{\mu_z + \frac{1}{2}\sigma_z^2} \theta\rho + 2\beta e^{\mu_z + \frac{1}{2}\sigma_z^2} c_t^{(3)} - e^{\mu_z + \frac{1}{2}\sigma_z^2} \theta\rho \tag{20}$$

These values are validated by the paper referenced by Bertsimas and Lo, so we know this is correct.