CME241 Assignment8

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1 Problem 1

Model an MDP so you can run the bank in the most optimal manner, i.e., maximizing the Expected Utility of assets less liabilities at the end of a T -day horizon, conditional on any current situation of assets and liabilities. Specify the states, actions, transitions, rewards with precise mathematical notation (make sure you do the financial accounting from one day to the next precisely).

The action space is simpler than specifying the state space in this model, so we'll first delineate the options available to the bank. We can either borrow y from another bank or invest π_t into a risky asset each day. Thus our action's are defined by the tuple:

$$\mathcal{A} = \{y_t, \pi_t\} \tag{1}$$

Our state space is defined by our current cash amount, and the previous day's borrowing terms as well as the previous day's investment into the risky asset. We'll also need to know the previous day's unfulfilled withdrawal's since this information is needed in order to calculate future cash position's. So our state space is:

$$\mathcal{S} = \{(c_t, y_{t-1}, \pi_{t-1}, \delta w_{t-1}\} \tag{2}$$

Our transition's are dictated by the following equation:

$$\begin{aligned}
c_t =& c_{t-1} + y_t - \pi_t - y_{t-1} \cdot (1 - R) + X(\pi_{t-1}) - \mathcal{I}_\sqcup + (d_t - \delta w_{t-1} - w_t) \\
\mathcal{I}_\sqcup =& \begin{cases} -K \cdot \cot \frac{\pi \cdot c}{2 \cdot C}, & c_t \leq C \\ 0, & c_t \geq C \end{cases} \\
\delta w_t =& \begin{cases} 0, & c_t \leq 0 \\ -c_t, & c_t \leq 0 \implies c_t = 0 \end{cases}
\end{aligned}$$

The rewards for each day are just the inter-day returns on the cash position.

$$\mathcal{R}_\sqcup = c_t - c_{t-1} \tag{3}$$

In these equations, we have four undefined parameters, namely C, the required cash position by the regulator, and also X, the pdf describing the distribution of the risky asset. Also not described are the probability distributions describing the behaviour of customers, i.e. the frequency and the magnitude of customer's withdrawals and deposits. This will dictate the probability transitions of the MDP. Also, I've assumed that this risky asset has a memory of step 1. If it instead accumulates, we would need to change the state space to include the sum over all time steps of our π_t, and our transitions between days would require a change in dependency onto that entire history of π_t. In order to maximize this Expected Utility of assets less liabilities, we will essentially just be maximizing c_T, and we can then expand in terms of the actions that we took at each time step T. Because are state space consists of continuous variables $(c_t, \pi_t, \delta w_t)$, and the fact that we don't have explicit transition probabilities between these continuous states, we will need to use Approximate Dynamic Programming in order to find the optimal policy for this MDP. We'll do Finite-Horizon Approximate Value Iteration in order to find these optimal value functions and policies, so we will need to append our state space to include the time step.

2 Problem 2

You are a milk vendor and your task is to bring to your store a supply (denoted S $\in$ R) of milk volume in the morning that will give you the best profits. You know that the demand for milk through the course of the day is a probability distribution function f (for mathematical convenience, assume people can buy milk in volumes that are real numbers, hence milk demand x $\in$ R is a continuous variable with a probability density function). For every extra gallon of milk you carry at the end of the day (supply S exceeds random demand x), you incur a cost of h (effectively the wasteful purchases amounting to the difference between your purchase price and end-of-day discount disposal price since you are not allowed to sell the same milk the next day). For every gallon of milk that a customer demands that you don't carry (random demand x exceeds supply S), you incur a cost of p (effectively the missed sales revenue amounting to the difference between your sales price and purchase price). So your task is to identify the optimal supply S that minimizes your Expected Cost g(S), given by the following

$$g(S) = p \cdot g_1(S) + h \cdot g_2(s) \tag{4}$$

Finding the optimal policy is just a matter of finding the derivative of the expected cost with respect to S, and setting it to zero. The function $g(S)$ is differentiable so this method is valid.

$$\begin{aligned}
\frac{\partial g(s)}{\partial s} &= p \cdot \frac{\partial g_1(s)}{\partial s} + h \cdot \frac{\partial g_2(s)}{\partial s} \\
\frac{\partial g_1(s)}{\partial s} = \frac{\partial}{\partial s}[\int_s^\infty (x-s) \cdot f(x)dx] &= -\int_s^\infty f(x)dx \\
\frac{\partial g_2(s)}{\partial s} = \frac{\partial}{\partial s}[\int_{-\infty}^s (s-x) \cdot f(x)dx] &= \int_{-\infty}^s f(x)dx
\end{aligned}$$

Now seting this to zero:

$$\frac{\partial g(s)}{\partial s} = -p \cdot \int_s^\infty f(x)dx + h \cdot \int_{-\infty}^s f(x)dx = 0 \tag{5}$$

$$\begin{aligned}
p \cdot \int_{s^*}^\infty f(x)dx &= h \cdot \int_{-\infty}^{s^*} f(x)dx \\
\int_s^\infty f(x)dx &= \int_{-\infty}^\infty f(x)dx - \int_{-\infty}^{s^*} f(x)dx \\
p \cdot \int_{-\infty}^\infty f(x)dx &= (h+p) \cdot \int_{-\infty}^{s^*} f(x)dx
\end{aligned}$$

But, because $f(x)$ is a probability distribution, we know that $\int_{-\infty}^{\infty} f(x)dx = 1$, and $\int_{-\infty}^{s^*} f(x)dx = F_X(s^*)$ with X as a random variable. Thus, our equation become

$$\begin{aligned}
p &= (h+p) \cdot F_X(s^*) \\
F_X(s^*) &= \frac{p}{h+p}
\end{aligned}$$

This problem can easily be seen as call/put portfolio problem. It's essentially trying to determine the optimal strike price such that the expected loss on a call and a put at the same strike price is minimized.