Lattès Maps and Solving Polynomial Equations

Quinn Brussel

December 15, 2024

1 Introduction

Throughout history, much effort has been put into finding the roots of polynomial equations. This is an old and difficult problem with broad uses in mathematics. Techniques such as Newton's method allow one to approximate the roots of quadratic polynomials, but prior to 1987 it was unknown whether a similar method existed for polynomials of higher degrees.

Curtis McMullen, in his 1987 paper 'Families of Rational Maps and Iterative Root-Finding Algorithms' [1], proves that there is no generally convergent algorithm for finding the roots of a polynomial of degree 4 or more, and exhibits a generally convergent algorithm for cubics.

This result makes use of the study of elliptic curves and a family of rational maps on $\mathbb{P}^1$ first studied by Samuel Lattès and summarized in John Milnor's 'On Lattès Maps' [3]. We will investigate the basic theory of Lattès maps and use this to understand McMullen's result.

2 Elliptic Curves

Understanding Lattès maps requires an understanding of elliptic curves. In the following material, let $\Lambda \subset \mathbb{C}$ be a discrete additive subgroup. Then $\mathbb{C}/\Lambda$ is a Riemann surface. If Λ is rank 1, then $\mathbb{C}/\Lambda$ isomorphic to an infinite cylinder. If Λ is rank 2, then $\mathbb{C}/\Lambda$ is isomorphic to a complex torus, and called an *elliptic curve*. We are particularly interested in the rank 2 case.

We begin with some basic facts about Elliptic curves. Until specified otherwise, let Λ be a rank two lattice and let $\mathcal{T} = \mathbb{C}/\Lambda$ be the associated torus. There is a simple classification of elliptic curves up to isomorphism.

Proposition 2.1. *Let $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ be elliptic curves. Then $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$ if and only if $\mu \cdot \Lambda_1 = \Lambda_2$ for some $\mu \in \mathbb{C}^*$.*

Proof. Suppose that $\mu \cdot \Lambda_1 = \Lambda_2$ for some $\mu \in \mathbb{C}^*$. Then consider the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \mu z$. Observe that f is an automorphism of $\mathbb{C}$. Furthermore, $\mu x \equiv \mu y \bmod \Lambda_2$ if and only if $x \equiv y \bmod \Lambda_1$, so f descends to an isomorphism from $\mathbb{C}/\Lambda_1$ to $\mathbb{C}/\Lambda_2$.

Conversely, suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is an isomorphism. Then f lifts to an isomorphism $\tilde{f} : \mathbb{C} \to \mathbb{C}$, such that $\tilde{f}(0) = 0$. This is necessarily of the form $\tilde{f}(z) = \mu z$ for $\mu \in \mathbb{C}^*$. Since f must map lattice points to lattice points, it follows that $\mu \cdot \Lambda_1 = \Lambda_2$. ■

In fact, we will later see that the symmetries of Λ play an important role in classifying rational functions. It follows from Proposition 2.1 that we can assume Λ to be a normalized lattice, that is a lattice with shortest vector 1.

Any any holomorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ lifts to a function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ and as such, it must be doubly periodic with respect to Λ. Of course this rules out the possibility of non-constant holomorphic functions on $\mathbb{C}/\Lambda$. This is because any double periodic function on $\mathbb{C}$ must be bounded and therefore constant. More generally, we can say the following about the space of 1-forms on $\mathbb{C}/\Lambda$.

Proposition 2.2. *The space $\Omega(\mathbb{C}/\Lambda)$ of holomorphic 1-forms is 1-dimensional and spanned by the form dz.*

Proof. Since dz on $\mathbb{C}$ is translation invarient, it descends to a nowhere zero 1-form on $\mathbb{C}/\Lambda$. For any other 1-form η, $\eta/dz = f$ is a holomorphic function and therefore constant. Thus $\Omega(\mathbb{C}/\Lambda) = \mathbb{C} \cdot dz$. ∎

This is what we expect given that on any compact Riemann surface X, $\dim \Omega(X) = g$, the genus of X. There is more to say about $\mathcal{M}(\mathbb{C}/\Lambda)$, the space of meromorphic functions on $\mathbb{C}/\Lambda$; or rather doubly periodic (or *elliptic*) meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$. To help us find such functions, we have a useful criterion.

Proposition 2.3. *Let X be a compact Riemann surface. If $f \in \mathcal{M}(X)$, then f has the same number of zeros as poles. If $X = \mathbb{C}/\Lambda$ is an elliptic curve, and $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are the zeros and poles of f respectively, then $\sum a_i = \sum b_i$ in the group-law on $\mathbb{C}/\Lambda$.*

Proof. Let $f \in \mathcal{M}(X)$ and $\omega = df/f = f'(z)dz/f(z)$. Observe that $\mathrm{Res}_p(\omega) = \mathrm{ord}(f, p)$ for all $p \in X$. By the residue theorem, $\sum_{p \in X} \mathrm{Res}_p(\omega) = 0$, so $\sum_{p \in X} \mathrm{ord}(f, p) = 0$. Therefore f has the same number of zeros and poles with multiplicity.

Suppose $X = \mathbb{C}/\Lambda$. Assume $f : X \to \widehat{\mathbb{C}}$ is non-constant. Then since $\Omega(\widehat{\mathbb{C}}) = 0$, the pushforward $f_*(dz) = 0$. Choose a path $C \subset \widehat{\mathbb{C}}$ running from 0 to ∞. Then $\tilde{C} = f^{-1}(C)$ is a collection of arcs connecting (a_i) and (b_i) in pairs. Assume the indices match. Now we have

$$0 = \int_C f_*(dz) = \int_{\tilde{C}} dz = \sum a_i - b_i \bmod \Lambda$$

so $\sum a_i = \sum b_i$ in the group-law on $\mathbb{C}/\Lambda$ as desired. ∎

We now give a concrete example of a meromorphic function on $\mathbb{C}/\Lambda$ which is of great importance (in fact every other meromorphic function in $\mathcal{M}(\mathbb{C}/\Lambda)$ can be written in terms of it). This function is known as the Weierstrass $\wp$-function, and is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}.$$

Note that $\wp$ is even and doubly periodic with respect to Λ, and has double poles on Λ. Therefore $\wp$ defines a meromorphic function on $\mathbb{C}/\Lambda$ with a unique double pole at 0. Since $\wp'$ is an odd function with a pole of order 3, the zeros of $\wp'$ correspond with the order two elements of $\mathbb{C}/\Lambda$ (considered as a group). Thus, we can explictly find the critical points of $\wp$. If $\Lambda = \alpha\mathbb{Z} \oplus \beta\mathbb{Z}$, then the critical points are $0, \alpha/2$, $\beta/2$, and $(\alpha + \beta)/2$. (Notice then that $\wp$ has a critical value at ∞ and three critical values in $\mathbb{C}$.)

Then since $\wp$ is even of degree 2, $\wp : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ is a 2-sheeted branched covering map, branched over four points. This map corresponds with the quotient map

$$\pi : \mathcal{T} \to \mathcal{T}/(t \sim -t) \cong \widehat{\mathbb{C}}.$$

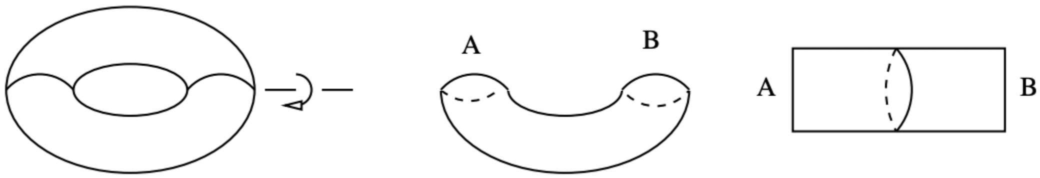

Figure 1. The map $\pi : \mathcal{T} \to \mathcal{T}/(t \sim -t)$
(Graphic courtesy of Curt McMullen [2])

To finish our discussion of the general theory of elliptic curves and functions, we state but do not prove the following:

Remark 2.4. *The function $\wp$ satisfies the differential equation $\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ where e_i are the finite critical values of $\wp$. In fact, if we let*

$$G_n(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^{2n}}$$

$g_2 = 60G_2(\Lambda)$, and $g_3 = 140G_3(\Lambda)$, then an equivalent formulation is

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Proof. See [2]. ∎

If we expand our expression for $\wp'(z)^2$, we have, as a corollary, that $\sum e_i = 0$. A related fact is the following remark:

Remark 2.5. *The map $\pi : \mathbb{C} \to \mathbb{P}^2$ given by*

$$\pi(z) = (\wp(z), \wp'(z))$$

desends to an isomorphism of Riemann surfaces from $\mathbb{C}/\Lambda$ to the smooth cubic curve V of the form $y^2 = 4x^3 + ax + b$.

Proof. See [2] ∎

Additionally, one can prove that $\mathcal{M}(\mathbb{C}/\Lambda)$ is generated by $\wp$ and $\wp'$, so that $\mathcal{M}(\mathbb{C}/\Lambda) \cong \mathbb{C}(x, y)/(y^2 - 4(x - e_1)(x - e_2)(x - e_3))$.

3 Lattès Maps

We now turn to the discussion of rational functions on the Riemann sphere. Specifically, we are interested in those which lift to affine maps on the torus. Such maps, known as *Lattès maps*, have interesting dynamical properties which have been leveraged to prove important results.

In the following definition, recall that the the *exceptional set* of a rational map f is denoted $\mathcal{E}_f$ and is the set of points in $\widehat{\mathbb{C}}$ with finite grand orbit. The cardinality of $\mathcal{E}_f$ can be at most 2, hence the name "exceptional." Let $\Lambda \subset \mathbb{C}$ be a disrecte additive subgroup of rank 1 or 2.

Definition 3.1 (Finite quotient of an affine map)**.** *A rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least 2 is a finite quotient of an affine map if there is a flat surface $\mathbb{C}/\Lambda$, an affine map $L(t) = at + b$ from $\mathbb{C}/\Lambda$ to itself, and a finite-to-one holomorphic map $\Theta : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}} \setminus \mathcal{E}_f$ satisfying the semiconjugacy relation $f \circ \Theta = \Theta \circ L$.*

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "L"] \arrow[d, "\Theta"'] & \mathbb{C}/\Lambda \arrow[d, "\Theta"] \\ \widehat{\mathbb{C}} \setminus \mathcal{E}_f \arrow[r, "f"] & \widehat{\mathbb{C}} \setminus \mathcal{E}_f \end{tikzcd}$$

We sometimes write $f = \Theta \circ L \circ \Theta^{-1}$. We remark that the dynamics of f are closely related to the dynamics of L. Any periodic orbit of L must be a periodic orbit of f; and conversely, any periodic orbit of f outside of the exceptional set is the image of a perioidic orbit of L. (Note that the periods are not necessarily the same.) We can classify finite quotients of affine maps in three ways:

Power Maps. A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least 2 is a *power map* if it is conformally conjugate to a map of the form

$$f_a(z) = z^a$$

where $a \in \mathbb{Z}$. The exceptional set of f_a is given by $\mathcal{E}_{f_a} = \{0, \infty\}$, so $\widehat{\mathbb{C}} \setminus \mathcal{E}_{f_a} = \mathbb{C}^*$. Notice that $f_a|_{\mathbb{C}^*}$ is conjugate to the linear map $t \mapsto at$ on the cylinder $\mathbb{C}/2\pi\mathbb{Z}$. In fact, the conjugacy $t \mapsto e^{it}$ gives an isomorphism from $\mathbb{C}/2\pi\mathbb{Z}$ to $\mathbb{C}^*$. Furthermore, we have the semiconjugacy $f_a(e^{it}) = e^{iat}$.

For such maps, the Julia set is given by $J(f_a) = S^1$, which is a rare example of a smooth Julia set. Furthermore $\mathcal{E}_{f_a} = \{0, \infty\}$ is also the set of critical points of f_a as well as the set of postcritical points.

Chebyshev maps We call f a *Chebyshev map* if f is conjugate to $\pm\text{Ч}_n$, where Ч_n is the degree n Chebyshev polynomial defined by

$$\text{Ч}_n(u + u^{-1}) = u^n + u^{-n}.$$

For example,

$$\text{Ч}_2(z) = z^2 - 2 \quad \text{Ч}_3(z) = z^3 - 3z \quad \text{Ч}_4(z) = z^4 - 4z^2 + 2.$$

Notice that $\mathcal{E}_{\text{Ч}_n} = \{\infty\}$. This formula for Ч_n is reminiscent of cosine, and for good reason. Let $\Theta(t) = e^{it} + e^{-it} = 2\cos(t)$. This is a proper map of degree 2 from the cylinder $\mathbb{C}/2\pi\mathbb{Z}$ to $\mathbb{C} = \widehat{\mathbb{C}} \setminus \mathcal{E}_{\text{Ч}_n}$. Furthermore, Ч_n is the quotient of the affine map $t \mapsto nt$ on $\mathbb{C}/2\pi\mathbb{Z}$, with the semiconjugacy given by $\text{Ч}_n(2\cos t) = 2\cos(nt)$.

Further note that $J(\pm\text{Ч}_n) = [-2, 2]$ and the postcritical set of $\pm\text{Ч}$ is $\{\pm 2, \infty\}$.

Lattès maps We now turn to the case of rank two lattices $\Lambda \subset \mathbb{C}$, so that the quotient $\mathcal{T} = \mathbb{C}/\Lambda$ is a complex torus. The rational map $f = \Theta \circ L \circ \Theta^{-1}$ is called a *Lattès map*. Here, L will be an affine self-map of the torus and Θ will be a holomorphic map to $\widehat{\mathbb{C}}$. Since $\mathbb{C}/\Lambda$ is compact, Θ will necessarily be surjective (and therefore Lattès maps have no exceptional points).

Lattès maps are the most interesting example, and exhibit a variety of behaviors. We will categorize those which admit smooth deformations as *flexible*, and those which do not as *rigid* (see Section 4). Furthermore, since repelling periodic cycles of L are dense in $\mathcal{T}$, the Julia set of a Lattès map is the entire Riemann sphere.

The following theorem will allow us to more explictly understand the possible Lattès maps.

Theorem 3.2. *A rational map is Lattès if and only if it is conformally conjugate to a map of the form $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ where:*

- $\mathcal{T} \cong \mathbb{C}/\Lambda$ *is a complex torus,*
- G_n *is the group of n-th roots of unity acting on $\mathcal{T}$ by rotation around a basepoint, with n equal to 2,3,4, or 6,*
- $\mathcal{T}/G_n$ *is the quotient space provided with its natural structure as a Riemann surface of genus zero*
- L *is an affine self-map of $\mathcal{T}$ which commutes with a generator of G_n, and*
- L/G_n *is the induced holomorphic map from $\mathcal{T}/G_n$ to itself.*

Milnor proves this theorem in his paper [3]. We note that the map $\Theta : \mathcal{T} \to \mathcal{T}/G_n \cong \widehat{\mathbb{C}}$ can be formulated in terms of familiar elliptic functions. For $n = 2$, we have already seen that the Weierstrass $\wp$-function is the desired map. We remarked that $\wp$ is a two-sheeted branched cover of the Riemann sphere. The typical preimages of a point are $\wp^{-1}(p) = \{t, -t\}$. The ramification points are where $t = -t$ in the group law on $\mathcal{T}$.

When $n = 3$, the semiconjugacy is $\wp'$. When $n = 6$, we may use $(\wp')^2$ or $\wp^3$. The two are related by the equation $(\wp')^2 = 4\wp(z)^3 + c$ for some $c \in \mathbb{C}$. For $n = 4$, the semiconjugacy is $\wp^2$. We are unsurprised to see the $\wp$-function in all of these formulas because we know that $\mathcal{M}(\mathcal{T})$ is generated by $\wp$ and $\wp'$.

Example 3.3. *Using Theorem 3.2, we can give a concrete example of a Lattès map. Let $\mathcal{T} = \mathbb{C}/\Lambda$ be a torus with $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$. Consider the affine self-map $L : \mathcal{T} \to \mathcal{T}$ given by $L(t) = 2t$. We want to find the Lattès map $f = L/G_2 : \mathcal{T}/G_2 \to \mathcal{T}/G_2$. Here $G_2 = \{\pm 1\}$ and $T/G_2 \cong \widehat{\mathbb{C}}$. The quotient map $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$ is given by the Weierstrass $\wp$-function.*

Thus, we want to find f such that $f(\wp(z)) = \wp(2z)$. It is well known that

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z). \tag{1}$$

Furthermore, we know from Remark 2.4 that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

which tells us that

$$2\wp'(z) \cdot \wp''(z) = 12\wp(z) \cdot \wp'(z) - g_2\wp'(z)$$

so that $\wp''(z) = 6\wp(z) - g_2/2$. Now if we expand (1), we have

$$\wp(2z) = \frac{1}{4} \cdot \frac{36\wp(z)^2 - 6g_2\wp(z) + \frac{1}{4}g_2^2}{4\wp(z)^3 - g_2\wp(z) - g_3} - 2\wp(z).$$

Therefore

$$f(z) = \frac{1}{4} \cdot \frac{36z^2 - 6g_2 z + \frac{1}{4}g_2^2}{4z^3 - g_2 z - g_3} - 2z$$

is our desired Lattès map.

The following fact about Lattès maps will be important for the rest of our discussion. For a Lattès map f, let C_f be the set of critical points and $V_f = f(C_f)$ be the set of critical values. Similarly, let V_Θ be the set of critical values of the projection map $\Theta : \mathcal{T} \to \mathbb{C}$.

Lemma 3.4. *Every Lattès map f is postcritically finite. In fact, the postcritical set*

$$P_f = V_f \cup f(V_f) \cup f^2(V_f) \cup \dots$$

is equal to V_Θ.

Proof. Let $\deg_f(z)$ be the local degree of f at a point z. Then $1 \leqslant \deg_f(z) \leqslant deg(f)$. Furthermore, $\deg_f(z) > 1$ if and only if z is a critical point of f. Let $t_j \in \mathcal{T}$ and $z_j \in \widehat{\mathbb{C}}$, related in the following way:

$$\begin{tikzcd} t_1 \arrow[r, mapsto, "L"] \arrow[d, mapsto, "\Theta"] & t_0 \arrow[d, mapsto, "\Theta"] \\ z_1 \arrow[r, mapsto, "f"] & z_0 \end{tikzcd}$$

Note that L is the affine self map of $\mathcal{T}$ which is semiconjugate to f via Θ. Because $\deg_L(t) = 1$ everywhere, $\deg_\Theta(t_0) = \deg_\Theta(t_1) \cdot \deg_f(z_1)$. Since L and Θ are surjective, z_0 is a crtical value of Θ if and only if it is a critical value of f, has preimagine $z_1 \in f^{-1}(z_0)$ which is a critical value of Θ, or both. Thus,

$$V_\Theta = V_f \cup f(V_\Theta)$$

and by induction, $f^n(V_f) \subset V_\Theta$. Therefore $P_f \subset V_\Theta$.

Conversely, if t_0 is a critical point of Θ and $\Theta(t_0)$ is outside of P_f, then the same is true of the infinitely many iterated preimages of $\cdots \mapsto t_2 \mapsto t_1 \mapsto t_0$ under L. But Θ has finitely many critical points so this is not possible. Hence $V_\Theta \subset P_f$ and we're done. ∎

4 Finding the roots of Polynomials

We now turn to a few major results related to the study of Lattès maps. For more details on the following material, see McMullen's 1987 paper "Families of Rational Maps and Iterative Root-Finding Algorithms" [1]. It is well known that Newton's method gives a good method for finding the roots of a quadratic polynomial. Specifically, if $p(z)$ is a quadratic polynomial, then by iterating the rational function

$$T_p(z) = z - \frac{p(z)}{p'(z)}$$

from some starting point z_0, we converge to a root of $p(z)$ for almost all (p, z_0). The question is when does there exist a generally convergent purely iterative algorithm for finding the roots of a polynomial of degree d? To answer this question, we begin with some definitions.

Let $T : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function and let $p(z)$ be a monic polynomial.

Definition 4.1. *We say $T(z)$ is convergent for p if $T^n(z)$ converges to a root of p for all z in a dense open subset of $\mathbb{P}^1$.*

Let Poly_d denote the space of complex polynomials of degree d and Rat_k denote the space of rational functions of degree k. We can parametrize a degree d monic polynomial by its coefficients, so $\mathrm{Poly}_d \cong \mathbb{C}^d$. We can do the same for a degree k rational function $f(z) = p(z)/q(z)$, which has $2k+2$ coefficients. However, to avoid duplication, we may assume $p(z)$ is monic. Thus, $\mathrm{Rat}_k \cong \mathbb{P}^{2k+1}$.

Definition 4.2. *A purely iterative algorithm is a rational function $T : Poly_d \to Rat_k$.*

We allow T to be undefined on an algebraic subvariety of its domain. Write $T_p(z)$ for the mapping under T of $p \in \mathrm{Poly}_d$.

Definition 4.3. *We say T_p is generally convergent if T_p is convergent for all p in a dense open set of Rat_d.*

We can now state the main result. Denote by $C(T)$ the *centralizer* of T, which is the group of Möbius transformations commuting with T.

Theorem 4.4. (McMullen - Classification of Generally Convergent Algorithms)

1. *There are no generally convergent purely iterative algorithms for polynomials of degree $d \geqslant 4$.*
2. *For $d = 3$, every generally convergent algorithm is obtained by specifying a rational map T such that*
   (i) T is convergent for $x^3 - 1$
   (ii) $C(T)$ contains the group of Möbius transformations which permute the cube roots of unity. This is given by $\langle z \mapsto 1/z, z \mapsto \zeta_3 \cdot z\rangle$ where ζ_3 is the primative cube root of unity. The algorithm is given (for p with distinct roots) by $T_p = M_p T M_p^{-1}$, where M_p is a mobius transformation carrying the cube roots of unity to the roots of p.

   Any T with properties (i) and (ii) determines a generally convergent algorithm for cubics.

Before justifying the theorem, let us explictly give a generally convergent algorithm for degree 3 polynomials. We say the algorithm T_p is *superconvergent* if the critical points of T_p are fixed and coincide with the roots of p. Note that superconvergent implies generally convergent. This is because if T_p is superconvergent, then the Julia set is nowhere dense and each stable region attracts a critical point.

In the following, let the degree of an algorithm be the degree of T_p as a rational function.

Proposition 4.5. *There is a unique degree 4 superconvergent algorithm for cubics. If the monic cubic polynomial p is given by*

$$p(z) = z^3 + az + b$$

then the algorithm is given by

$$T_p(z) = z - \frac{(z^3 + az + b)(3az^2 + 9bz - a^2)}{3az^4 + 18bz^3 - 6a^2z^2 - 6abz - 9b^2 - a^3}.$$

To see convergence, notice that this is the same as Newton's method applied to

$$q(z) = \frac{p(z)}{3az^2 + 9bz - a^2}$$

which clearly has the same zeros as p. Furthermore, the points of inflection of q coincide with the roots of p. Therefore Newton's method converges if we take an inflection point of

q as our guess. Then a theorem of Smale gives gives us convergence for a dense open set of initial guesses [4].

It is worth noting that the above algorithm not only converges for z in a dense open set, but for almost every $z \in \mathbb{P}^1$ in a measure theoretic sense.

In order to outline the steps towards Theorem 4.4, we define an *algebraic family* of rational maps to be a rational mapping of an irreducible (quasi-) projective variety V into Rat_k; i.e. a family of rational maps parametrized by V. Notice that any algebraic family is a holomorphic family. A family is *stable* if there is a uniform bound on the period of the attracting cycles of the rational maps. A family is *trivial* if all members are conjugate by Möbius transformations. The following lemma will be helpful:

Lemma 4.6. *A stable algebraic family of rational maps is either trivial or all its members are postcritically finite.*

This lemma is proved by McMullen [1].

We call a Lattès map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ (with $\mathcal{T} = \mathbb{C}/\Lambda$) *flexible* if we can vary Λ and L continuously so as to obtain other Lattès maps which are not conformally conjugate to it (if not, the map is *rigid*). A more practical formulation is the following lemma:

Lemma 4.7. *A Lattès map $L/G_n : \mathcal{T}/G_n \to \mathcal{T}/G_n$ is flexible if and only if $n = 2$ and the affine map $L(t) = at + b$ has integer derivative $L' = a \in \mathbb{Z}$.*

This is proven in the Milnor paper [3]. In McMullen's paper we assume $b = 0$. Thus, the flexible Lattès maps are quotients of repeated addition on a torus. We now state the critical theorem in the proof of Theorem 4.4

Theorem 4.8 (McMullen - Rigidity of algebraic families)**.** *A stable algebraic family is either trivial or its members are flexible Lattès.*

For a proof of Theorem 4.8, see McMullen's paper [1]. We proved that Lattès maps are postcritically finite, and this fact combined with Lemma 4.6 and the work of William Thurston are used in the proof.

If we assume Theorem 4.8, we can now prove the nonexistence of generally convergent purely iterative algorithms for polynomials of degree at least 4.

Proof of Theorem 4.4 (1). We will first show that a generally convergent algorithm must, by rigidity, be a trivial family. Then we will observe that the sinks of T_p coincide with the roots of p. Finally, we will use the fact that the roots of any two generic p and q in Poly_d are related by a Möbius transformation, which is impossible for degree $d \geqslant 4$ because the cross ratio of four points must be preserved.

Suppose T_p is a generally convergent algorithm. Then T_p must be stable. In fact, if T_p has an attracting cycle of order $\geqslant 2$, then this cycle persists on an open neighborhood of p in

Poly_d, and thus the algorithm is not generally convergent. We remark that the Julia set of T_p is not $\widehat{\mathbb{C}}$, so T_p is not Lattès (recall that Lattès maps have Julia set $\widehat{\mathbb{C}}$). Now by Theorem 4.8, T_p is trivial as desired.

Recall that $x \in \mathbb{P}^1$ is a sink of T if there exists an open set U such that $T^n(u) \to x$ for all $u \in U$. Since T_p is generally convergent, the sinks are contained in the roots of p. Since T_p depends only upon symmetric functions of the roots, if one root is a sink, so are the rest.

To make this precises, replace T_p by T_r where $r \in \mathbb{C}^d$ consists of the roots of p. Then T_r is a well defined rational map for all r outside of a proper algebraic subvariety W, and $T_r = T_{\sigma r}$ for all permutations σ of the roots. For simplicity, adjoin to W the set of r with non-distinct coordinates.

Now associate to each T_r the set I_r of indicies i such that T_r has a sink at the root r_i. Notice that I_r is locally constant because the sinks of T_r are a continuously varying subset of the coordinates of r, and these coordinates are distinct outside of W. But $\mathbb{C}^d \setminus W$ is connected, so T_r is globally constant. Thus, there exists i such that for all r, T_r has a sink at r_i. On the other hand, $T_r = T_{\sigma r}$, for all permutations, so T_r has a sink at all coordinates of r.

Since T_r is a trivial family, the sink locations, and hence the roots, of any two members are related by a Möbius transformation. For $d \geqslant 4$, this is clearly impossible. ■

Acknowledgements and Further Reading

Thus, we conclude our discussions of elliptic curves, Lattès maps, and finding the roots of polynomials. For further reading, I recommend the original Curt McMullen paper [1]. For more introductory material on the theory of finite quotients of rational maps, read Milnor's "On Lattès Maps" [3].

I would like to thank Rafael for teaching an excellent and inspiring class this semester. I will be focusing on complex dynamics in my senior thesis, and this most likely wouldn't be the case if not for this class being so good. I would also like to thank Curt McMullen and Laura DeMarco for their help with this material.

References

[1] Curt T. McMullen. Families of rational maps and iterative root-finding algorithms. *Annals of Mathematics*, 125(3), 1987.

[2] Curt T. McMullen. Advanced complex analysis, 2023.

[3] John W. Milnor. On lattès maps, 2004.

[4] M. Shub and S. Smale. Computational complexity: On the geometry of polynomials and a theory of cost: Ii. *SIAM Journal on Computing*, 15(1):145–161, 1986.