Math 113 Course Notes

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1 Tuesday - 1/28/2025

We will introduce the complex numbers and holomorphic functions, which are our main objects of interest.

Complex Numbers. Let $\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}\}$. We identify $\mathbb{C}$ with the plane $\mathbb{R}^2$, letting $x+iy$ correspond with $(x, y) \in \mathbb{R}^2$. We define addition by

$$(x+iy)+(u+iv) = (x+u)+i(y+v)$$

and multiplication by

$$(x+iy)(u+iv) = xu + i(xv - yu) + i^2yv = (xu - yv) = i(xv + yu),$$

so that it obeys the distributive property. This gives $\mathbb{C}$ the structure of a commutative ring with identity. We introduce three important operations on $\mathbb{C}$.

- Complex conjugation: If $z = x+iy$, we define the complex conjugate of z as $\bar{z} := x-iy$.
- Absolute value: We define $|z| := \sqrt{x^2+y^2}$, so that $|z|^2 = z\bar{z}$. Therefore any non-zero z has a multiplicative inverse given by $1/z = \bar{z}/|z|^2$. Hence, $\mathbb{C}$ is a field. Also notice that the absolute value is multiplicative: $|zw|^2 = zw\overline{zw} = z\bar{z}w\bar{w} = |z|^2|w|^2$.
- Argument of a complex number: If z is any non-zero complex number, we can write $z = r(\cos\theta + i\sin\theta)$, with $r > 0 \in \mathbb{R}$ and some angle θ. Now $r = |z|$, and we call θ the *argument* of $\mathbb{C}$.

  Warning: The argument θ is only well defined modulo 2π; that is θ makes sense only as an element of $\mathbb{R}/2\pi\mathbb{Z}$. This is because $\cos\theta = \cos(2\pi n + \theta)$ and $\sin\theta = \sin(2\pi n + \theta)$ for any $n \in \mathbb{Z}$.

Notation. Going forward, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z; that is, $\mathrm{Re}(x+iy) = x$ and $\mathrm{Im}(x+iy) = y$. Notice that $\mathrm{Re}(z) = (z+\bar{z})/2$ and $\mathrm{Im}(z) = (z-\bar{z})/2i$.

Let S^1 denote the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Similarly, let $S^1_r = \{z \in \mathbb{C} \mid |z| = r\}$, $S^1(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = 1\}$, and $S^1_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$.

Let Δ denote the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. We define to $\Delta_r, \Delta(z_0)$, and $\Delta_r(z_0)$ similarly as the case of the circle.

Let $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$ and $\Delta^* := \Delta\setminus\{0\}$. Let $\mathbb{H}$ denote the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. For any set $S \subset \mathbb{C}$, let $\bar{S}$ denote the topological closure of S.

Exponentials and roots.

By convention, we write $re^{i\theta} := r(\cos\theta + i\sin\theta)$. This will make more sense next week. If $z = re^{i\theta}$, then $|z| = r$ and $\bar{z} = re^{-i\theta}$. Moreover, $re^{i\theta} \cdot se^{i\psi} = rse^{i(\theta+\psi)}$. In this way, we see that when multiplying two complex numbers, their absolute values multiply, and their arguments add. Therefore, $1/z = (1/r)e^{-i\theta}$.

As an example, notice that $z^2 = r^2e^{2i\theta}$. Now if $z \neq 0 \in \mathbb{C}$, then z has two square roots. These are the complex numbers with absolute value $\sqrt{|z|}$ and half the argument of z. But since the argument is only defined modulo 2π, we have $\sqrt{z} = \sqrt{r} \cdot e^{i\theta/2}$ and $\sqrt{z} = \sqrt{r} \cdot e^{i(\theta/2+\pi)}$, two distinct complex numbers. This is similar to the case of the real numbers, where each $x \neq 0 \in \mathbb{R}$ has two distinct square roots $\pm\sqrt{x}$.

In general, if $z = re^{i\theta}$ is non-zero, then z has n nth roots

$$\sqrt{r}e^{i\theta/n}, \sqrt{r}e^{i(\theta+2\pi)/n}, \ldots, \sqrt{r}e^{i(\theta+2\pi(n-1))/n}.$$

Therefore, we can't consistently choose an nth root of a non-zero complex number.

As an aside, let $\Gamma := \{(z, w) \in \mathbb{C}^* \times \mathbb{C}^* \mid w^2 = z\}$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consider the projection map $(z, w) \mapsto z$ from Γ to $\mathbb{C}^*$. This projection is a degree two covering map, and makes Γ the unique connected, two-sheeted covering space of $\mathbb{C}^*$.

Holomorphic functions. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{C}$ be a function. We can write $f(z) = u(z) + iv(z)$, where u and v are real valued. We say that f is *holomorphic* or *analytic* on Ω if the derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite for all $z \in \Omega$. In other words, f is holomorphic if for all $z \in \Omega$, there exists $f'(z) \in \mathbb{C}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\left|\frac{f(z+h) - f(z)}{h} - f'(z)\right| < \epsilon.$$

We will give some special properties of holomorphic functions in the next lecture.

2 Thursday - 1/30/2025

The Cauchy-Riemann equations. We continue discussing holomorphic functions $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is an open set. We can write $f(x + iy) = u(x, y) + iv(x, y)$, with $u, v : \mathbb{R}^2 \to \mathbb{C}$. Recall that by definition, f is holomorphic if

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. Since h is a complex number and may approach 0 from any direction in the above expression, the limit has to be the same regardless of the path chosen. If h approaches 0 along the real axis, write $h = \epsilon$. Then our limit becomes

$$\lim_{\epsilon \to 0} \frac{u(x+\epsilon, y) - u(x,y)}{\epsilon} + i\frac{v(x+\epsilon, y) - v(x,y)}{\epsilon} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{1}$$

Now suppose h approaches 0 along the imaginary axis. Write $h = i\epsilon$. Now the limit becomes

$$\lim_{\epsilon \to 0} \frac{u(x, y+\epsilon) - u(x,y)}{i\epsilon} + i\frac{v(x, y+\epsilon) - v(x,y)}{i\epsilon} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{2}$$

Since equations (1) and (2) must have the same value, we conclude that the partial derivatives satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are called the *Cauchy-Riemann equations*, and give us a useful criterion for identifying holomorphic functions.

Properties of holomorphic functions. We will now state some basic facts about holomorphic functions which we will prove over the next few months.

1. There are many simple functions like $f(z) = \mathrm{Re}(z)$ ($\mathrm{Re}(z)$ denotes the real part of z) which are not holomorphic. To see this, observe that f does not satisfy the Cauchy-Riemann equations, or notice that the limit definition of the derivative is not satisfied. (Check this!)

2. If f is holomorphic, then the derivative f' is holomorphic. This implies that holomorphic functions have derivatives of all orders, which we denote $\frac{d^n f}{dz^n}$ or $f^{(n)}$. Moreover, if we identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider f as a function from $\mathbb{R}^2$ to $\mathbb{C}$, then $f \in C^\infty(\mathbb{R}^2)$.

3. Suppose $f(z) = u(z) + iv(z)$ is holomorphic. Then $f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$ as a result of the Cauchy-Riemann equations. Since f' is holomorphic the Cauchy-Riemann equations imply that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. This is to say that $u(z) = u(x,y)$ satisfies Laplace's equation; i.e., u is *harmonic*.

   Given any harmonic function $u : \mathbb{R}^2 \to \mathbb{R}$, the Laplace equation implies that the concavity of u at any point along a horizontal line is the negative of the concavity of u along an intersecting vertical line. This implies that non-constant harmonic functions may not have local extrema.

4. Holomorphic functions are *conformal*, which we define as follows. Suppose we have a holomorphic function $f : \Omega \to \mathbb{C}$ and two arcs $\alpha(t)$ and $\beta(t)$, which intersect at a point $z_0 \in \Omega$. Let θ be the angle between α and β at z_0. If $f'(z_0) \neq 0$, then the angle between $f \circ \alpha$ and $f \circ \beta$ at $f(z_0)$ is still θ.

We conclude that the Jacobian matrix $D_z f : \mathbb{R}^2 \to \mathbb{R}^2$ of f, considering f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, is of the form

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. This is to say that $D_z f$ is a rotation composed with a scaling. Notice that this is the same as the Cauchy-Riemann equations.

5. Non-constant holomorphic maps are open: given a holomorphic map $f : \Omega \to \mathbb{C}$, $f(U)$ is open in $\mathbb{C}$ for each U open in Ω. This gives further evidence that $z \mapsto \mathrm{Re}(z)$ is not holomorphic, since the real axis is not open in $\mathbb{C}$. Another non-example of a holomorphic function is $x + iy \mapsto x^2 + iy$ defined on $\mathbb{C}$. The image of $\mathbb{C}$ is the closed half-plane $\{x + iy \mid x \geqslant 0\}$. This is not open in $\mathbb{C}$, so our mapping is not holomorphic.

6. Holomorphic functions are rigid. Suppose f and g are holomorphic functions on a connected domain Ω, and suppose that $\Gamma \subseteq \Omega$ is open. If $f = g$ on Γ, then $f = g$ on all of Ω.

   By contrast, C^∞ functions are *flabby*: we can deform smooth functions on a small open subset of their domain without violating smoothness. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 0, & x \leqslant 0 \\ e^{-x^2} - 1, & x > 0 \end{cases}$$

   is smooth but not identically equal to zero.

7. We have Liouville's theorem, which states that when f is holomorphic on all of $\mathbb{C}$, then if $|f(z)|$ is bounded, f is constant.

3 Tuesday - 2/4/2025

We will continue illustrating the rigidity of holomorphic functions, give examples of holomorphic functions, and begin the discussion of power series. From here on, we adopt the convention that z and w are complex variables, and x, y, u, v are real variables. Moreover, when we see $a > b$, this means that a and b are real, since there is no total ordering of the complex numbers.

Bump functions. Recall that holomorphic functions are *rigid*: a holomorphic function $f : \Omega \to \mathbb{C}$, with Ω connected, is determined by its values on any open subset of Ω. This is quite different from C^∞ functions, which are *flabby*. We illustrate this point using bump functions.

We start with

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Note that $f \in C^\infty(\mathbb{R})$. Moreover, all of the derivatives of f at 0 are 0. We can also take

$$g(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leqslant 0. \end{cases}$$

Finally, we define

$$h(x) = \begin{cases} e^{\frac{-1}{(x-1)^2} - \frac{1}{(x+1)^2}}, & x \in (-1,1), \\ 0, & x \in (-\infty,-1] \cup [1,\infty). \end{cases}$$

The Taylor series of f at 0 is just the zero function, even though the function is only zero at 0. Hence, f is not *analytic* (referring to functions represented by a power series). Moreover, g is zero on the entire non-positive real axis and f is zero outside of $(-1,1)$. These functions are certainly not rigid, even though they are C^∞.

We say a class of functions $\mathcal{D}$ on Ω is *flabby* if for all open covers $\Omega = \bigcup U_\alpha$, there exists an $f_\alpha \in \mathcal{D}$ whose support is contained in U_α and $\sum_\alpha f_\alpha(x) = 1$ for all $x \in \Omega$ (recall that the support of a function is the locus on which the function is nonzero).

Examples of holomorphic functions. We now return to the discussion of holomorphic functions. We give a few important examples of such functions.

1. For any $c \in \mathbb{C}$, the constant function $f(z) = c$ is holomorphic and $f'(z) \equiv 0$.
2. $f(z) = z$ is holomorphic and $f'(z) \equiv 1$.
3. The sum of two holomorphic functions is holomorphic. Moreover, $(f+g)' = f' + g'$.
4. The product of two holomorphic functions is holomorphic. The product rule for real valued functions extends to holomorphic functions: we have $(fg)' = f'g + g'f$.
5. If f and g are holomorphic and g is non-vanishing, then f/g is holomorphic and obeys the ordinary quotient rule $(f/g)' = (gf' - g'f)/g^2$.
6. The composition $f \circ g$ of holomorphic functions f and g is holomorphic where defined, and follows the chain rule: $(f \circ g)' = f'(g)g'$.
7. Inverses of holomorphic functions are holomorphic. If $f : \Omega \to \Delta$ is holomorphic and invertible, then f^{-1} is holomorphic. Moreover, $(f^{-1})'(z) = 1/f'(f^{-1}(z))$. This follows if we write $f(f^{-1}(z)) = z$ and take the derivative of both sides.

By virtue of what we have already said, all polynomials $p(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ are holomorphic. We will later prove the *fundamental theorem of algebra*, which says that $p(z)$ may be factored into degree one terms; i.e., $p(z) = c_n(a - a_1)(z - a_n)\dots(z - a_n)$, with each $a_i \in \mathbb{C}$. In other words, every polynomial has a zero in $\mathbb{C}$.

Likewise, rational functions, functions of the form $r(z) = p(z)/q(z)$ where p and q are polynomials, are holomorphic where $q(z) \neq 0$.

There is one more basic holomorphic function we need: the exponential $f(z) = \exp(z) = e^z$. In order to do so, we must introduce the notion of a power series. But first, we must review some concepts from real analysis.

Interlude about $\mathbb{R}$. Recall that $\mathbb{R}$ has the *least upper bound property* (l.u.b.): if $\Gamma \subset \mathbb{R}$ is any bounded subset, then there exists a unique $c \in \mathbb{R}$ such that for all $a \in \Gamma$, $c \geqslant a$, and for all $\epsilon > 0$, there exists $a \in \Gamma$ with $a > c - \epsilon$. We call $\mathrm{lub}(\Gamma) = \sup(\Gamma) := c$ the least upper bound of Γ. If we don't assume boundedness, then the least upper bound still exists, but may be equal to ∞.

We can similarly define the greatest lower bound $\mathrm{glb}(\Gamma) = \inf(\Gamma)$ as the number $c \in \mathbb{R}$ such that $c \leqslant x$ for all $x \in \Gamma$, and for all $\epsilon > 0$, there exists $x \in \Gamma$ such that $x < c + \epsilon$.

While the limit of an arbitrary sequence of real numbers may not exist, we can always define lim inf and lim sup. Let $x_1, x_2, x_3 \dots \in \mathbb{R}$ be an arbitrary sequence. We define

$$\begin{aligned} y_1 &:= \mathrm{lub}\{x_1, x_2, \dots\} \\ y_2 &:= \mathrm{lub}\{x_2, x_3, \dots\} \\ y_3 &:= \mathrm{lub}\{x_3, x_4, \dots\} \\ &\vdots \end{aligned}$$

Notice that $y_1, y_2, y_3, \dots$ is a decreasing sequence, and therefore has a limit (allowing $\pm\infty$). Define $\limsup\{x_1, x_2, \dots\} := \lim_{n\to\infty} y_n$. Similarly, let

$$\begin{aligned} u_1 &:= \mathrm{glb}\{x_1, x_2, \dots\} \\ u_2 &:= \mathrm{glb}\{x_2, x_3, \dots\} \\ u_3 &:= \mathrm{glb}\{x_3, x_4, \dots\} \\ &\vdots \end{aligned}$$

which is an increasing sequence. Therefore $\liminf\{x_1, x_2, \dots\} := \lim_{n\to\infty} u_n$ exists. We can check that $\limsup_{n\in\mathbb{N}} x_n = \liminf_{n\in\mathbb{N}} x_n$ if and only if $\lim_{n\to\infty} x_n$ exists.

Convergence of functions. We wish to define holomorphic functions via power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \lim_{m\to\infty} \sum_{n=0}^{m} a_n z^n.$$

We know that polynomials are holomorphic, but the question remains of whether the limit of a sequence of holomorphic functions is still holomorphic. Indeed, this does not hold in the setting of continuous or differentiable functions: a limit of continuous/differentiable functions need not be continuous/differentiable. Thus, we need the strong notion of *uniform convergence*.

Let $f_1, f_2, f_3, \ldots$ be a sequence of functions on a domain Ω. We say that $f_1, f_2, f_3, \ldots$ *converges* to f if for all $x \in \Omega$ and for all $\epsilon > 0$, there exists N such that for all $n \geqslant N$, $|f_n(x) - f(x)| < \epsilon$.

We say that $f_1, f_2, f_3, \ldots$ *converge uniformly* to f if for all $\epsilon > 0$, there exists N such that for all $n \geqslant N$ and for all $x \in \Omega$, $|f_n(x) - f(x)| < \epsilon$.

An example of a sequence of functions which converge non-uniformly is the sequence $f_n : (0,1) \to \mathbb{R}$ given by $f_n(x) = x^n$. Notice that as $n \to \infty$, for each $x \in (0,1)$, $f_n(x) \to 0$. However, the f_n do not uniformly converge to the zero function.

4 Thursday - 2/6/2025

We will introduce power series and define the exponential function.

Power series. We wish to know when an expression of the form

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}$$

defines a holomorphic function in z. We define the *radius of convergence* of such a sum to be

$$R := \frac{1}{\limsup_n \sqrt[n]{|a_n|}}.$$

This could be 0, ∞, or anything in between.

Theorem 4.1 (Chapter 2 Theorem 2 in Ahlfors)**.** Given a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R,

1. For z with $|z| < R$, the sum $\sum a_n z^n$ converges absolutely.
2. For any z with $|z| > R$, the sum diverges.
3. For any $\rho < R$, the sum converges uniformly for $|z| < \rho$. Hence, $f(z) := \sum a_n z^n$ is holomorphic on $\{z : |z| < R\}$.

Remark 4.2. An example of a sum which converges, but does not converge absolutely is $\sum (-1)^n/n = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots$

We have some immediate consequences:

1. The theorem says we can rearrange the terms:

$$\sum a_n z^n + \sum b_n z^n = \sum (a_n + b_n) z^n$$
$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k+\ell=n}(a_k b_\ell)\right) z^n$$

2. We can also divide. Given $\sum a_n z^n, \sum b_n z^n$ with $b_0 \neq 0$, there exists a power series $\sum c_n z^n$ such that $(\sum c_n z^n)(\sum b_n z^n) = \sum a_n z^n$. Indeed, we want

$$(c_0 + c_1 z + c_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) = a_0 + a_1 z + a_2 z^2 + \dots$$
$$c_0 b_0 + (c_0 b_1 + c_1 b_0) z + (c_0 b_2 + c_1 b_1 + c_2 b_0) z^2 = a_0 + a_1 z + a_2 z^2 + \dots$$

so we can solve for the c_n:

$$c_0 = a_0/b_0, \quad c_1 = \frac{a_1 - c_0 b_1}{b_0}, \quad \dots$$

The exponential. Consider the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Notice that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$. Hence, $\sum z^n/n!$ has radius of convergence $R = \infty$. We define $e^z := \sum z^n/n!$, which is holomorphic on all of $\mathbb{C}$.

A crucial fact is that for all $z, w \in \mathbb{C}$, we have $e^z \cdot e^w = e^{z+w}$. Indeed,

$$\begin{aligned} e^z \cdot e^w = \left(\sum \frac{z^n}{n!}\right)\left(\sum \frac{w^n}{n!}\right) &= \sum_{n=0}^{\infty}\left(\sum_{k+\ell=n} \frac{z^k}{k!}\cdot\frac{w^\ell}{\ell!}\right) \\ &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= e^{z+w}, \end{aligned}$$

by the binomial theorem. In particular, $e^z \cdot e^{-z} = 1$ for all $z \in \mathbb{C}$. Hence e^z is always a unit in $\mathbb{C}$, so e^z is never zero. Also note that

$$e^{\bar{z}} = \sum \frac{\bar{z}^n}{n!} = \overline{\left(\sum \frac{z^n}{n!}\right)} = \overline{e^z}.$$

Thus, if $z = iy$, then

$$|e^z|^2 = |e^{iy}|^2 = e^{iy}\overline{e^{iy}} = e^{iy}e^{\overline{iy}} = 1.$$

Hence e^z maps the imaginary axis to the unit circle S^1. The following figure helps us visualize the behavior of the exponential.

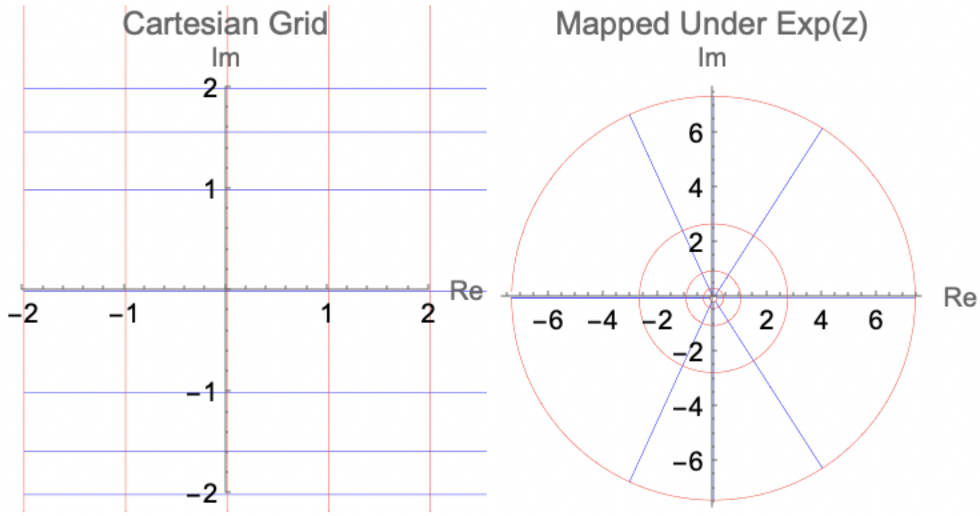


Figure 1. The exponential function.

Trigonometric functions. We define $\cos(z) := \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) := \frac{e^{iz}-e^{-iz}}{2i}$. As a consequence, $e^{iz} = \cos(z) + i\sin(z)$, which is known as Euler's formula. Moreover, $\sin^2(z) + \cos^2(z) \equiv 1$. Using the power series for the exponential, we can write

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$
$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots.$$

This tells us that our notions for sin and cos agree with the usual sin and cos functions on the real axis. Moreover, we have $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. This explains that as z moves along a vertical line $\{x_0 + it \mid t \in \mathbb{R}\}$ at constant speed, e^z moves around the circle $\{|w| = e^{x_0}\}$ at constant speed.

Moreover, notice that $\cos(z + 2\pi) = \cos(z)$, $\sin(z + 2\pi) = \sin(z)$, $\cos(-z) = \cos(z)$, and $\sin(-z) = -\sin(z)$.

The complex logarithm. We wish to define a function $\log(z)$, which will act as an inverse function of e^z for $z \neq 0$. Note that any complex number $z \neq 0$ can be written as

$$z = r(\cos\theta + i\sin\theta), \quad r > 0.$$

Hence, $z = re^{i\theta} = e^{\log r + i\theta}$, where $\log r$ denote the usual logarithm of a positive real number. Therefore any $z \neq 0$ can be expressed as e^w for some $w \in \mathbb{C}$, but w is only determined up to an integral multiple of $2\pi i$; that is, $e^a = e^b$ if and only if $a = b + 2\pi i n$ for some $n \in \mathbb{Z}$.

Thus, $\log z$ is naturally a multi-valued function. We can deal with this in two distinct ways.

1. We can think of log as a function $\log : \mathbb{C}^* \to \mathbb{C}/2\pi i\mathbb{Z}$.

2. Alternatively, we can define $\log z$ as a single-valued function if we restrict the domain. We can define $\log(z)$ consistently as a holomorphic function $\log : \mathbb{C} \setminus \{x \leqslant 0 \in \mathbb{R}\} \to \{x + iy \mid -\pi < \arg(z) < \pi\}$. We set $\log(z) = \log|z| + i\arg(z)$, with $\arg(z) \in (-\pi, \pi)$.

5 Tuesday - 2/11/2025

Logarithm reprise. We continue our discussion of the complex logarithm. We define $\log(z)$ as the inverse function of e^z. That is, if $\log(w) = z$, then $w = e^z$. The problem is that there are infinitely many values of z which satisfy this equation. Indeed, if $e^z = w$, then $e^{z+2\pi in} = w$ for all $n \in \mathbb{Z}$. Hence, $\log(z)$ is only defined modulo $2\pi i\mathbb{Z}$. There are two ways of resolving this.

1. We can make $\log(z)$ single-valued by restricting its domain. Let

$$\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leqslant 0\}.$$

For $z \in \Omega$, we can define $\log(z)$ unambiguously by requiring that $-\pi < \mathrm{Im}(\log(z)) < \pi$. Now $\log(z) = \log|z| + i\arg(z)$, where $\arg : \Omega \to (-\pi, \pi)$. Notice that

$$e^{\log(z)} = e^{\log|z|}e^{i\arg(z)} = z$$

Moreover, because we have chosen the branch of arg in this way, for $z = x + iy$ with $-\pi < y < \pi$, we have

$$\log(e^z) = \log|e^z| + i\arg(e^z) = \log|e^x| + i\arg(e^{iy}) = x + iy = z$$

Warning: If $z = x + iy$ with $y < -\pi$ or $y < \pi$, then $\log(e^z) \neq z$. For example, $\log(e^{2\pi i}) = 0$, instead of $2\pi i$. Luckily, it is always true that $e^{\log(z)} = z$.

2. Alternatively, let

$$\Gamma := \{(z, w) \in \mathbb{C} \times \mathbb{C}^* \mid w = e^z\}$$

Consider the projections $p_1 : (z, w) \to z$ and $p_2 : (z, w) \to w$. The following diagram commutes.

$$\begin{tikzcd} \Gamma \arrow[d, "p_1"] \arrow[dr, "p_2"] & \\ \mathbb{C} \arrow[r, "e^z"] & \mathbb{C}^* \end{tikzcd}$$

The map p_1 is one to one. On the other hand, p_2 is countably infinite to one. We can consistently define $\log : \Gamma \to \mathbb{C}$ by $\log = p_1$.

With the logarithm, we can define other exponentials. We can let

$$a^b = e^{b\log(a)}.$$

This is not well defined in general for the same reason that log is not well defined. But if $a \in \mathbb{R}$, then this is well defined because we set can set $a^b = e^{b\log|a|}$ by convention. Moreover, if b in $\mathbb{Z}$, then it is also well defined, since we can write $a^b = 1 \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{b \text{ times}}$ if $b \geqslant 0$ and $a^b = 1/a^{-b}$ if b is negative.

Inverse trig functions. Let $w = \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$. We multiply both sides by $2e^{iz}$ and obtain

$$(e^{iz})^2 - 2we^{iz} + 1 = 0.$$

We solve for e^{iz} and find that $e^{iz} = w \pm \sqrt{w^2-1}$, so

$$\begin{aligned} z = \arccos(w) &:= -i\log(w \pm \sqrt{w^2-1}) \\ &= \pm i\log(w + \sqrt{w^2-1}). \end{aligned}$$

This is even more ambiguous than log: There is a choice of positive or negative, a choice of log, and a choice of square-root. We would need to be specific about the domain and range to make this function precise.

Integration. We will begin to prove some properties of holomorphic functions. One of the main tools for doing so is integration. For more details, consult Professor Harris's lecture notes for the class, rather than Ahlfors.

We first define the *Riemann integral*. Let $I = (a,b) \subset \mathbb{R}$ and $f \in C[a,b]$ be a continuous real-valued function. For each $n \in \mathbb{N}$, we partition $[a,b]$ into n pieces of size $(b-a)/n$. Choose $x_{n,i}$ in the ith interval. We let

$$I_a^b(f) := \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_{n,i}).$$

Theorem 5.1. The expression $I_a^b(f)$ is well defined regardless of the choices of $x_{n,i}$, and converges to a finite limit.

We can also consider functions $f : [a,b] \to \mathbb{R}$ which are not necessarily continuous, but on which $I_a^b(f)$ is well defined and finite. Such functions are said to be *Riemann integrable*. Similarly, if $\Omega \in \mathbb{R}^2$ is a bounded open set, and $f(x,y)$ is a continuous function on $\overline{\Omega}$, then we can define the Riemann integral $I_\Omega(f)$ similarly, but with squares instead of intervals.

6 Thursday - 2/13/2025

Today we discuss integration and differential forms.

Review. We start with the Riemann integral of f, a continuous real valued function on $[a,b]$. For each $n \in \mathbb{N}$ and for m such that $[m/n,(m+1)/n] \cap [a,b] \neq \emptyset$, choose $x_m \in [m/n,(m+1)/n] \cap [a,b]$. We define the *Riemann integral* to be

$$I_{[a,b]}(f) := \lim_{n\to\infty} \frac{1}{n} \sum_m f(x_m).$$

As stated previously, the limit exists regardless of choices.

Riemann integration in the plane. Let Ω be an open region in $\mathbb{R}^2$ with smooth boundary. Suppose $f(x,y)$ is continuous on $\overline{\Omega}$. For each $n \in \mathbb{N}$, we tile $\mathbb{R}^2$ with squares of length and width $1/n$. For each square $S_{n,m}$ with $S_{n,m} \cap \Omega \neq \emptyset$, we choose $p_m \in S_{n,m} \cap \Omega$. We define the *Riemann integral* in this case to be

$$I_\Omega(f) := \lim_{n\to\infty} \frac{1}{n^2} \sum_m f(p_{n,m}).$$

1-forms. Again, let Ω be an open region in $\mathbb{R}^2$ with smooth boundary. A *1-form* on Ω is an expression of the form

$$\omega = f(x,y)dx + g(x,y)dy.$$

Assume for now that f and g are differentiable. While this is technically incorrect, it is sometimes helpful to think of a 1-form as a vector field on $\mathbb{R}^2$, assigning the vector $(f(x,y), g(x,y))$ to (x,y).

If h is a differentiable function on Ω, the *exterior derivative* of h is the 1-form

$$dh := \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

Path integrals. Suppose $\gamma : [a,b] \to \Omega$ is a differentiable arc. We write $\gamma(t) = (\alpha(t), \beta(t))$. For any 1-form $\omega = f(x,y)dx + g(x,y)dy$, we define

$$\int_\gamma \omega := I_{[a,b]}\left[f(\gamma(t)) \cdot \alpha'(t) + g(\gamma(t)) \cdot \beta'(t)\right] = I_{[a,b]}\left[F(\gamma(t)) \cdot \gamma'(t)\right], \tag{3}$$

where $F(x,y) = (f(x,y), g(x,y))$.

We state three crucial facts about path integrals.

1. $\int_\gamma \omega$ is independent of the choice of parametrization of γ.

2. For any differentiable function $h : \Omega \to \mathbb{R}^2$ and path $\gamma : [a,b] \to \Omega$, we have $\int_\gamma dh = h(\gamma(b)) - h(\gamma(a))$. This is the *fundamental theorem of calculus.*

3. We have Green's theorem (Stokes' theorem for the plane): Let $\gamma : [a, b] \to \Omega$ be a path, and $\omega = f \cdot dx + g \cdot dy$ a 1-form on Ω. Suppose γ is a simple closed curve; that is, $\gamma(a) = \gamma(b)$, but γ is injective on (a, b). Furthermore, assume γ is oriented such that its interior is always "on its left." Say γ bounds a subregion $D \subset \Omega$. Then
$$\int_\gamma \omega = I_D\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right).$$

Integration with complex numbers. We now identify $\mathbb{R}^2$ with $\mathbb{C}$. We introduce complex valued 1-forms $\omega = f(x, y)dx + g(x, y)dy = f(z)dx + g(z)dy$, where f and g are complex valued.

As defined, a 1-form ω associates to each $p \in \Omega$ an element of the two dimensional complex vector space $\mathbb{C}\langle dx, dy\rangle$. We will propose a different basis for $\mathbb{C}\langle dx, dy\rangle$ and express 1-forms as linear combinations of the new basis. Specifically, we take the basis
$$dz := dx + idy, \qquad d\bar{z} := dx - idy.$$
We can invert this change of basis:
$$dx = \frac{dz + d\bar{z}}{2}, \qquad dy = \frac{dz - d\bar{z}}{2i}.$$
Now the exterior derivative of a differentiable function $h(x+iy) = u(x, y)+iv(x, y)$ becomes
$$df = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) d\bar{z}.$$
There are two things to observe. Firstly, if f is holomorphic, then the Cauchy-Riemann equations imply that the coefficient of $d\bar{z}$ vanishes. Secondly, the coefficient of dz is equal to $f'(z)$.

Theorem 6.1 (Cauchy-Riemann equations Version 2)**.** A function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if f is differentiable and $df = f'dz$.

A *holomorphic 1-form* on Ω is a complex valued 1-form of the form $f(z)dz$, where f is holomorphic. When integrating a holomorphic 1-form over a path $\gamma : [a, b] \to \mathbb{C}$, equation (3) becomes
$$\int_\gamma f(z)\, dz = I_{[a,b]}\left[f(\gamma(t)) \cdot \gamma'(t)\right].$$

The key observation is the following.

Theorem 6.2 (Cauchy's theorem)**.** Suppose γ is a simple closed curve which bounds a subregion D, and ω is a holomorphic 1-form on Ω. Then $\int_\gamma \omega = 0$.

Proof. If $\omega = f(x + iy)dx + g(x + iy)dz$, then we have $\int_\gamma \omega = I_D\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)$ by Green's theorem. The Cauchy-Riemann equations imply that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0$ everywhere in Ω. ∎

7 Tuesday - 2/18/2025

We begin with two remarks about integration. Firstly, if $\omega = f(x,y)dx + g(x,y)dy$ and $\sqrt{|f|^2 + |g|^2} < M$ everywhere, then

$$\left|\int_\gamma \omega\right| < M \cdot \text{length}(\gamma).$$

Secondly, if $f(x,t)$ is a function on $[a,b] \times [c,d]$, and f is continuous with respect to x and differentiable with respect to t, then

$$\frac{d}{dt}\int_a^b f(x,t)\, dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\, dx.$$

Tweaked Cauchy's theorem. Cauchy's theorem states that if Ω is a region in $\mathbb{C}$, $f(x)dz$ is a holomorphic 1-form on Ω, and $\gamma : [a,b] \to \Omega$ is a simple closed curve which bounds a region D of $\mathbb{C}$ contained in Ω, then $\int_\gamma f(z)dz = 0$.

We now give an altered version.

Proposition 7.1. Let $a \in D$. Assume that $f(z)$ is holomorphic on $\Omega \setminus \{a\}$ and that $\lim_{z\to a}(z-a)f(z) = 0$. Then

$$\int_\gamma f(z)dz = 0.$$

Hence, f may not be defined at a, but $|f|$ does not grow to quickly as z approaches a.

Proof of Proposition 7.1. We describe a new path γ_ϵ (referring to Figure 2): we travel around γ, until we reach the radial path L, which we take until reaching the circle of radius ϵ C_ϵ around a. We then follow C_ϵ counter clockwise until we return to L, at which point we follow L the opposite direction back to γ. Hence, $\gamma_\epsilon = \gamma * L * C_\epsilon * L^{-1}$, where $*$ represents concatenation.

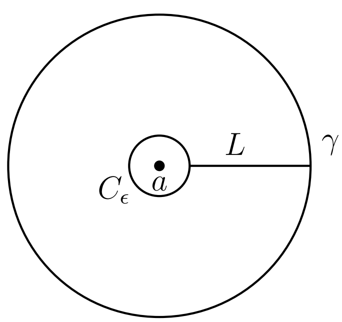


Figure 2. The path γ_ϵ.

Notice that γ_ϵ bounds a region on which f is holomorphic. Therefore $\int_{\gamma_\epsilon} f(z)dz = 0$ by Cauchy's theorem. Now

$$\int_{\gamma_\epsilon} f(z)dz = \int_{\gamma} f(z)dz + \int_{L} f(z)dz + \int_{C_\epsilon} f(z)dz + \int_{L^{-1}} f(z)dz.$$

Since $\int_{L^{-1}} f(z)dz = -\int_L f(z)dz$, we have cancellation. Moreover, observe that

$$\int_{C_\epsilon} f(z)\, dz \leqslant \max_{|z-a|=\epsilon} |f(z)| \cdot 2\pi\epsilon = \max_{|z-a|=\epsilon} |f(z)| 2\pi |z-a| = \max_{|z-a|=\epsilon} 2\pi |f(z)(z-a)|,$$

which tends to zero as $\epsilon \to 0$. Hence, the integral $\int_{C_\epsilon} f(z)dz$ tends to zero. Therefore

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)\, dz = \int_{\gamma} f(z)\, dz.$$

But since $\int_{\gamma_\epsilon} f(z)dz = 0$ for all ϵ, we see that $\int_\gamma f(z)dz = 0$, as desired. ■

Winding numbers. We wish to formalize the notion of "how many times a path winds around a point." Say $\gamma : [s,t] \to \mathbb{C} \setminus \{a\}$ is a path with $\gamma(s) = \gamma(t)$.

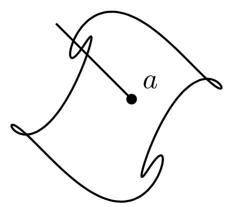


Figure 3.

Intuitively, for any ray coming out from a, we want to add 1 every time γ crosses the ray from the counterclockwise direction, and subtract 1 every time γ crosses the other way. The example in Figure 3 has winding number 1.

To make this rigorous, let $\eta : [s,t] \to \mathbb{C} \setminus \{a\}$ be a closed curve with $\eta(s) = \gamma(s)$ such that $[\eta]$ generates $\pi_1(\mathbb{C} \setminus \{a\}, \eta(s))$. Moreover, choose η to be oriented counterclockwise. Consider the isomorphism $\phi : \pi_1(\mathbb{C} \setminus \{a\}, \eta(s)) \to \mathbb{Z}$ defined by $\phi([\eta]) = 1$. Then the *winding number* of γ around a is $\phi([\gamma])$, which we denote $n(\gamma, a)$.

A special case is when γ is a simple closed curve, bounding a region $D \subset \mathbb{C}$. If γ goes counterclockwise then the winding number is given by

$$n(\gamma, a) = \begin{cases} 0, & a \notin D \\ 1, & a \in D. \end{cases}$$

Recall from the homework that

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i.$$

We claim that if γ is any closed loop in $\mathbb{C} \setminus \{0\}$, then

$$\int_\gamma \frac{dz}{z} = 2\pi i \cdot n(\gamma, 0).$$

More generally, for any point $a \in \mathbb{C}$, we have

$$\int_\gamma \frac{dz}{z-a} = 2\pi i \cdot n(\gamma, a).$$

This is actually how Ahlfors defines the winding number.

Aside on Cauchy's theorem. Professor Harris did not mention this in class, it is worth noting a consequence of Cauchy's theorem. Recall that two paths $\eta, \gamma : [a, b] \to \Omega$ which agree on their start and end points are *path homotopic* if η can be continuously deformed to γ in Ω while fixing the base-point. More precisely, if there exists a continuous function $H : [a, b] \times [0, 1] \to \Omega$ such that $H(t, 0) = \eta(t)$ for all t, $H(t, 1) = \gamma(t)$ for all t, $H(a, s) = \eta(a)$ for all s, and $H(b, s) = \eta(b)$ for all s.

Proposition 7.2 (Path independence)**.** Let $\Omega \subseteq \mathbb{C}$ be open, and let ω be a holomorphic 1-form on Ω. Suppose that $\eta, \gamma : [a, b] \to \Omega$ are two paths in Ω which are path-homotopic. Then

$$\int_\eta \omega = \int_\gamma \omega.$$

Proof. Consider the region $D = H([a, b], [0, 1]) \subseteq \Omega$, which is bounded by the concatenation $\sigma = \eta * \gamma^{-1}$, which is a closed loop beginning and ending at $\eta(0) = \gamma(0)$. Since H maps to Ω, D is contained in Ω. By Cauchy's theorem, we have $\int_\sigma \omega = 0$. But since $\int_\sigma = \int_\eta \omega - \int_\gamma \omega$, the result follows. ■

In general, we say that two closed loops $\eta, \gamma : [a, b] \to \Omega$ are *homotopic* if there is a continuous function $H : [a, b] \times [0, 1] \to \Omega$ such that $H(t, 0) = \eta(t)$ and $H(t, 1) = \gamma(t)$ for all t. Intuitively, η can be continuously deformed to γ in Ω.

Proposition 7.3. Let $\Omega \subseteq \mathbb{C}$ be open, and let ω be a holomorphic 1-form on Ω. Suppose that $\eta, \gamma : [a, b] \to \Omega$ are closed loops which are homotopic. Then

$$\int_\eta \omega = \int_\gamma \omega.$$

Proof. Let $\sigma : [0, 1] \to \Omega$ be given by $\sigma(s) = H(0, s)$. Then σ is a path from $\eta(0) = \eta(1)$ to $\gamma(0) = \gamma(1)$. Let $\sigma^{-1} : [0, 1] \to \mathbb{C}$ be given by $\sigma^{-1}(s) = H(0, 1 - s)$, which is just the reversal of σ. Let p be the concatenation $p = \eta * \sigma * \gamma^{-1} * \sigma^{-1}$. Now p is a curve which bounds the region $H([a, b], [0, 1]) \subset \Omega$, so Cauchy's theorem implies that $\int_p \omega = 0$. But

$$0 = \int_p \omega = \int_\eta \omega + \int_\sigma \omega - \int_\gamma \omega - \int_\sigma \omega = \int_\eta \omega - \int_\gamma \omega.$$

Therefore $\int_\eta \omega = \int_\gamma \omega$, as desired. ■

As a consequence, let γ be any simple closed curve, and take a in the region bound by γ. Then $\int_\gamma 1/(z-a)\, dz = 2\pi i$. This is because γ is homotopic in $\mathbb{C}\setminus\{a\}$ to a circle $\eta : [0,1] \to \mathbb{C}$ given by $\eta(t) = a + e^{2\pi i t}$. Now

$$\int_\gamma \frac{1}{z-a}\, dz = \int_\eta \frac{1}{z-a}\, dz = \int_0^1 \frac{2\pi i e^{2\pi i t}}{a + e^{2\pi i t} - a}\, dt = \int_0^1 2\pi i\, dt = 2\pi i.$$

8 Thursday - 2/20/2025

We will prove Cauchy's integral formula and some of its corollaries. Recall that given a path $\gamma : [0,1] \to \Omega \subseteq \mathbb{C}$, and a family of functions $f(\zeta, z)$ on Ω which are holomorphic with respect to ζ and differentiable with respect to z, we have

$$\frac{d}{dz}\int_\gamma f(\zeta, z)\, d\zeta = \int_\gamma \frac{\partial f}{\partial z}(z,\zeta)\, d\zeta.$$

Cauchy integral formula. Suppose $f(z)$ is a holomorphic on a region $\Omega \subseteq \mathbb{C}$. Let γ be a simple closed curve in Ω, bounding a region $D \subset \Omega$. Take $a \in D$. We introduce the function

$$F(z) := \frac{f(z) - f(a)}{z - a},$$

which is defined and holomorphic in $\Omega \setminus \{a\}$. The 1-form $F(z)dz$ satisfies the hypotheses of tweaked Cauchy, so

$$0 = \int_\gamma F(z)\, dz = \int_\gamma \frac{f(z) - f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)}{z-a}\, dz - \int_\gamma \frac{f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)}{z-a} - 2\pi i \cdot f(a).$$

This implies that $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\, dz$. This is known as the Cauchy integral formula:

Theorem 8.1 (Cauchy integral formula)**.** Suppose $f(z)$ is a holomorphic on a region $\Omega \subseteq \mathbb{C}$. Let γ be a simple closed curve in Ω, bounding a region $D \subset \Omega$. For any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

This is remarkable, as it implies that the value of a holomorphic function at a point z in its domain are determined by the values of the function away from this point. This is not true in general for C^∞ functions, and reinforces the rigidity of holomorphic functions.

The first consequence of the Cauchy integral formula is called the Cauchy integral formula for derivatives.

Corollary 8.2 (Cauchy integral formula for derivatives)**.** Suppose $f(z)$ is a holomorphic on a region $\Omega \subseteq \mathbb{C}$. Let γ be a simple closed curve in Ω, bounding a region $D \subset \Omega$. For any $z \in D$, the k-th derivative satisfies

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta.$$

Consequently, holomorphic functions have derivatives of all orders.

Proof. We use induction. Cauchy's integral formula implies the result when $k = 0$. Assume the formula holds for k. Then

$$\begin{aligned} f^{(k+1)}(z) &= \frac{d}{dz}\frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta \\ &= \frac{k!}{2\pi i} \int_\gamma \frac{d}{dz} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta \\ &= \frac{(k+1)k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+2}}\, d\zeta \\ &= \frac{(k+1)!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+2}}\, d\zeta, \end{aligned}$$

as desired. ■

Another important consequence is Liouville's theorem. We call a holomorphic function defined on all of $\mathbb{C}$ an *entire function.*

Theorem 8.3 (Liouville's theorem)**.** If $f : \mathbb{C} \to \mathbb{C}$ is an entire function and $|f(z)|$ is bounded, then f is constant.

This is unlike C^∞ functions. There exist smooth functions $f : \mathbb{R} \to \mathbb{R}$ which are bounded but not constant. For example, $f(x) = \sin x$ is bounded above by 1. Another example is $f(x) = \frac{1}{1+x^2}$. If we replace x with a complex variable z, then $f(z) = \sin z$ is not bounded, since $\sin(z)$ goes to ∞ as the imaginary part of z goes to ∞. The function $f(z) = 1/(1+z^2)$ is not an entire function, since it is not defined at $\pm i$. Moreover, f is not bounded in a neighborhood of $\pm i$.

Proof of Liouville's theorem. Assume that f is an entire function with $|f(z)| < M$ for all $z \in \mathbb{C}$. For any $z \in \mathbb{C}$, let $S_r^1(z)$ be the circle of radius r centered at z, oriented counterclockwise.

$$f'(z) = \frac{1}{2\pi i} \int_{S_r^1(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

The length of $S_r^1(z)$ is $2\pi r$, and $|f(\zeta)/(\zeta - z)^2| < M/r^2$ for all $\zeta \in S_r^1(z)$. Therefore

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{S_r^1(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| < \frac{M}{r}.$$

Since we could choose any $r \gg 0$, we see that $|f'(z)| \leqslant \epsilon$ for all $\epsilon > 0$. Therefore $f'(z) = 0$. Since the derivative of f is 0 everywhere in $\mathbb{C}$, we conclude that f is constant. ■

The fundamental theorem of algebra. A corollary of Liouville's theorem is the fundamental theorem of algebra.

Theorem 8.4 (Fundamental theorem of algebra)**.** If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with coefficients in $\mathbb{C}$ and degree $n > 0$, then p has a root in $\mathbb{C}$; that is, there exists $z \in \mathbb{C}$ such that $p(z) = 0$.

Proof. Suppose that p has no roots. Then the function $f(z) = 1/p(z)$ is defined and holomorphic on all of $\mathbb{C}$. Since $p(z) \to \infty$ as $z \to \infty$, we know $f(z) \to 0$ as $z \to \infty$. Therefore f is bounded, and must be constant. But this means that p is constant, a contradiction since we required p to have degree at least 1. ■

This theorem implies that $p(z)$ may be written as $p(z) = \prod_{i=1}^{n}(z - r_i)$, for some $r_i \in \mathbb{C}$. If $n = 1$, then certainly this is true. If the claim holds for n, then any $n+1$ degree monic polynomial q has a zero r_{n+1}. Therefore $p(z) = q(z)/(z - r_{n+1})$ is a polynomial of degree n, and can be written $p(z) = \prod_{i=1}^{n}(z - r_i)$. Thus, $q(z) = \prod_{i=1}^{n+1}(z - r_i)$.

9 Tuesday - 2/25/2025

We will discuss *singularities* of holomorphic functions. Let Ω be an open domain in $\mathbb{C}$, and suppose that f is holomorphic on $\Omega \setminus \{a\}$. We wish to know how f behaves as $z \to a$. There are three possible types of behavior: a could be a *removable singularity*, a *pole*, or an *essential singularity*.

Removable singularities. A singularity of f at a is *removable* if $\lim_{z\to a}(z - a)f(z) = 0$. In this case, we can apply tweaked Cauchy: for all $z \neq a \in \Delta_r(a)$,

$$f(z) = \frac{1}{2\pi i}\int_{S^1_r(a)} \frac{f(\zeta)}{\zeta - z}\, dz.$$

The right hand side is defined and holomorphic for *all* $z \in \Delta_r(a)$, including the point a. Therefore f extends to a holomorphic function on all of Ω. In essence, a removable singularity is not really a singularity at all. It is just a value of f which we forgot to define.

Example 9.1. Consider the function $f(z) = \sin(z)/z$ which appears to have a singularity at zero. Notice that $\lim_{z\to 0} zf(z) = \sin(0) = 0$. Therefore 0 is a removable singularity of f. Indeed, notice that $\lim_{z\to 0} \sin(z)/z = 1$. Another example is the function $(e^z - 1)/z$.

Now let f be any holomorphic function on Ω. We apply this theory to the function

$$g(z) = \frac{f(z) - f(a)}{z - a}.$$

for all $z \neq a \in \Omega$. Notice that g satisfies $\lim_{z \to a}(z-a)g(z) = 0$, so g is defined and holomorphic at a. What this says is that if f is defined and holomorphic in Ω and $f(a) = 0$, then $f(z) = (z-a)g(z)$ for some holomorphic $g : \Omega \to \mathbb{C}$.

Remark 9.2. A basic fact about polynomials is that if $p(x)$ is a polynomial and $p(a) = 0$, then there exists a polynomial $q(x)$ such that $p(x) = (x-a)q(x)$. To prove this, we can use polynomial long division. What we have just shown is an analog of this for holomorphic functions.

Remark 9.3. By contrast, if f is a continuous function and $f(a) = 0$, it is not necessarily true that $f(x) = (x-a)g(x)$ where g is continuous. A standard example is $f(x) = \sqrt[3]{x}$, defined on $\mathbb{R}$. We have $f(x) = 0$, but $f(x)/x = x^{-2/3}$, which blows up at $x = 0$.

If $g(a) = 0$, then by the same token, we can write $f(z) = (z-a)^2 h(z)$ for some holomorphic h. We can continue this process until we reach a quotient function which does not vanish. A crucial fact which we will not prove today, is that if f is not identically the zero function, this process terminates.

Proposition 9.4. If $f : \Omega \to \mathbb{C}$ is holomorphic and $f \not\equiv 0$, then there is $m \in \mathbb{N}_0$ such that $f(z) = (z-a)^m g(z)$, where $g : \Omega \to \mathbb{C}$ is holomorphic and $g(a) \neq 0$. We say that f has a zero of order m at the point a.

Proposition 9.5. If $f : \Omega \to \mathbb{C}$ is holomorphic, then its zeros are isolated: If $f(a) = 0$, then there is an open neighborhood N of a such that $f(z) \neq 0$ for all $z \neq a \in N$.

Proof. Write $f(z) = (z-a)^m g(z)$ for some holomorphic $g : \Omega \to \mathbb{C}$ with $g(a) \neq 0$ and $m \in \mathbb{N}$. Say $|g(a)| = \epsilon > 0$. Since $|g|$ is continuous, the preimage N of the interval $(0, \infty)$ under $|g|$ is open in $\mathbb{C}$. Moreover, N is non-empty since $a \in N$. By construction, g does not vanish on N. ∎

As a consequence, a holomorphic function which is zero along any non-discrete set is identically zero. Examples of such sets include open sets, arcs, or sequences of points converging to a limit in the domain.

Moreover, this proves that holomorphic functions are *rigid*: If f and g agree on any non-discrete set S, then $h(z) = f(z) - g(z)$ is holomorphic and takes the value zero on S. Therefore $h \equiv 0$, so $f(z) = g(z)$.

Another consequence is that the zero set of a holomorphic function cannot be uncountable. This is because the topology on $\mathbb{C}$ has a countable basis $\{\Delta_r(p) \mid r \in \mathbb{Q}, p \in \mathbb{Q}(i)\}$. For any $p \in \mathbb{Q}(i)$. There is r_p small enough such that $\Delta_{r_p}(p)$ contains at most one zero of f. The set $\{\Delta_{r_p}(p) \mid p \in \mathbb{Q}(i)\}$ is countable and covers $\mathbb{C}$. Moreover, the number of zeros of f is bounded above by the cardinality of this set.

Poles. We say that $f : \Omega \setminus \{a\} \to \mathbb{C}$ has a *pole* at $z = a$ if $\lim_{z \to a} f(z) = \infty$. In other words, for all $N \in \mathbb{N}$, there exists $\epsilon > 0$ such that $|f(z)| > N$ when $|z - a| < \epsilon$.

If f has a pole at z, then $\lim_{z\to a} 1/f(z) = 0$, so $g(z) := 1/f(z)$ has a removable singularity at $z = a$. Therefore can write $g(z) = (z-a)^m h(z)$ near z, where h is holomorphic and $h(a) \neq 0$. Now let $k(z) = 1/h(z)$. Then k is holomorphic and non-zero at $z = a$. Finally, we see that $f(z) = (z-a)^{-m}k(z)$ near a. In this case, we say that f has a pole of *order* m at a.

From this, we see that a function with a pole of order m locally looks like the reciprocal of a holomorphic function with a zero of order m.

A function on Ω is *meromorphic* if for each $z \in \Omega$, either f is holomorphic at z, or f has a pole at z. By analogous reasoning to Proposition 9.5, the set of poles of a meromorphic function must be discrete.

Remark 9.6. We will later discuss the *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which we give the topology of the 1-point compactification of $\mathbb{C}$. Without discussing the details, a meromorphic function $f : \Omega \to \mathbb{C}$ is well defined as a holomorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$.

Remark 9.7. The set of holomorphic functions on Ω forms a ring $\mathcal{O}(\Omega)$ with the operations $(f+g)(z) = f(z) + g(z)$ and $(fg)(z) = f(z)g(z)$. In fact this ring is an integral domain. The ring of meromorphic functions on Ω is denoted $\mathcal{M}(\Omega)$, and is a field under the same operations.

10 Thursday - 2/27/2025

Accumulation points. Let $\Gamma \subset \mathbb{C}$ be any subset, and take $z \in \mathbb{C}$. We say a is an *accumulation point* of Γ if there exists $z_1, z_2, z_3, \dots \in \Gamma \setminus \{a\}$ such that $\lim z_i = a$. Equivalently, every neighborhood of a in Γ contains some z_i. From our discussion last lecture, we have the following.

Theorem 10.1. If $f : \Omega \to \mathbb{C}$ is holomorphic and the zero set of f has an accumulation point a in Ω, then $f \equiv 0$ in Ω.

Essential singularities. We again assume that f is a holomorphic function on $\Omega \setminus \{a\}$ for some $a \in \Omega$. If $\lim_{z\to a}(z-a)f(z) = 0$, then a is a removable singularity, and f extends over a as a holomorphic function. If $\lim_{z\to a} f(z) = \infty$, then we say a is a pole of f, and can write $f(z) = (z-a)^{-m}g(z)$ for some $m \in \mathbb{N}$ and where g is holomorphic and non-zero in a neighborhood of a.

We now consider the case when a is neither removable nor a pole of f. If there exists $m \in \mathbb{Z}$ such that $\lim_{z\to a}(z-a)^m f(z)$ exists in $\mathbb{C}$, then $g(z) := (z-a)^{m-1}f(z)$ is holomorphic. Now $f(z) = (z-a)^{-m}g(z)$ is meromorphic. If f is not meromorphic at a, then $\lim_{z\to a}(z-a)^m f(z)$ does not exist for any m. In this case, we say that f has an *essential singularity* at a.

Theorem 10.2. If f is holomorphic on $\Omega \setminus \{a\}$, and f has an essential singularity at a, then $f(N)$ is dense in $\mathbb{C}$ for any neighborhood N of a in Ω.

In other words, for all $w \in \mathbb{C}$, $\epsilon, \delta > 0$, there exists $z \in \Omega$ such that $|z - a| < \epsilon$ and $|f(z) - w| < \delta$. We could also write this as $\overline{f(N)} = \mathbb{C}$ for every neighborhood N of a in Ω.

Consider any sequence $z_1, z_2, z_3, \cdots \in \Omega \setminus \{a\}$ such that $\lim z_i = a$. If f is meromorphic, then $\lim f(z_i)$ is fixed for any sequence: it is either a point of $\mathbb{C}$, or it is infinity. If f has an essential singularity at a, then for any $w \in \mathbb{C}$, we can find a sequence such that $\lim f(z_i) = w$. Thus, holomorphic functions are all are nothing at their singularities: either the singularity is relatively trivial, or f exhibits the wildest possible behavior at a.

Example 10.3. Consider the function $f(z) = e^{1/z}$, which is defined on $\mathbb{C}^*$. The sequence $z_n = 1/n$ converges to 0, and $f(z_n)$ converges to ∞. However, for the sequence $w_n = -1/n$, we have $f(w_n) \to 0$. Hence f has an essential singularity at 0. Observe that $z \mapsto 1/z$ takes $\mathbb{C}^*$ bijectively to itself, and maps Δ_r to $\mathbb{C} \setminus \overline{\Delta}_{1/r}$. For each $n \in \mathbb{Z}$, e^z maps $S_n = \{n \leqslant \operatorname{Im}(z) < n + 2\pi i\}$ surjectively to $\mathbb{C}^*$. For any r, we can choose n large enough so that S_n does not intersection $\overline{\Delta}_{1/r}$. Hence, f maps Δ_r onto $\mathbb{C}^*$ for each $r > 0$. Therefore, $f(\Delta_r)$ is not only dense in $\mathbb{C}$ for every r, but omits only a single point of $\mathbb{C}$.

Proof of theorem 10.2. Suppose that f has an essential singularity at a and assume there is $w \in \mathbb{C}$ and $\epsilon, \delta > 0$ such that $|f(z) - w| > \delta$ for all z with $|z - a| < \epsilon$. In this case, consider the function

$$g(z) := \frac{1}{f(z) - w}$$

Now $|g(z)| < 1/\delta$ for all $z \in \Delta_\epsilon(a)$, which means that g is holomorphic at a:

$$\lim_{z \to a} |(z - a)g(z)| \leqslant \lim_{z - a} |z - a|/\delta = 0,$$

so a is a removable singularity of g. Now $f(z) = 1/g(z) + w$, so f is meromorphic. This is a contradiction. ∎

Preview of Taylor's theorem. Recall that if f is holomorphic in Ω and $f(a) = 0$ for some $a \in \Omega$, then $f(z) = (z - a)g(z)$ for some holomorphic g. Start with any f holomorphic on Ω, and $a \in \Omega$ ($f(z)$ not necessarily zero). Then $f(z) - f(a)$ is zero at a, so

$$f(z) - f(a) = (z - a)f_1(z),$$

for some f_1 holomorphic on Ω. Observe that $f_1(a) = f'(a)$, as $f_1(a) = \lim_{z \to a}(f(z) - f(a))/(z - a)$. We repeat the process, and write

$$f_1(z) - f_1(a) = (z - a)f_2(z).$$

Then $f_2(a) = f''(a)$. Now

$$f(z) = f(a) + f'(a)(z - a) + f_2(z)(z - a)^2.$$

If we continue this process, we obtain

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n f_n(z).$$

In fact, we can use the Cauchy integral formula and explicitly write

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)(\zeta-z)}\,dz,$$

where C is a circle with a and z in its interior. The goal of Taylor's theorem will be to discover what happens when we let $n \to \infty$.

The argument principle. Suppose that Ω is an open subset of $\mathbb{C}$ and C is a circle bounding a region D inside Ω. Suppose $a \in D$. If f is holomorphic in Ω, then for any $z \in D$, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,dz.$$

Hence, any question about the value of f in D should be answerable in terms of the values of f around the boundary. Specifically, we pose the following question: how many zeros does f have in D? Assume that $f(z) \neq 0$ for all $z \in C$. If follows that there will be finitely many $z_i \in D$ such that $f(z_i) = 0$. This is because $\overline{D}$ is compact, and an infinite sequence of zeros would have an accumulation point in $\overline{D}$.

For any $a \in D$, we can write $f(z) = (z-a)^m g(z)$ where $m \in \mathbb{N}_0$ and $g(a) \neq 0$. The value m is the *multiplicity* of the zero of f at a (we also call this the order). For each zero z_i of f in D, there is a multiplicity m_i. We will derive the *argument principle* to count $\sum m_i$, the number of zeros in D *with multiplicity.*

11 Tuesday - 3/4/2025

The argument principle. Suppose that f is holomorphic in $\Omega \subset \mathbb{C}$. Let γ be a simple closed curve in Ω, bounding a region D. We wish to find a method of counting the zeros of f in Ω. We make a few remarks:

1. We will assume that $f \neq 0$ on γ, so that we don't have to worry about f having zeros on the boundary of D.
2. There are necessarily finitely many zeros, as the region $\overline{D}$ is compact because its boundary γ is compact.
3. We will count zeros *with multiplicity.* If $f(a) = 0$, then we can write $f(z) = (z-a)^m g(z)$, where g is holomorphic and non-zero at a. Then number m is the *multiplicity of the zero at a.*

Suppose the zeros of f in D are $z_1, \dots, z_k$, with multiplicities $m_1, \dots, m_k$ respectively. We can write

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \dots (z - z_k)^{m_k} g(z), \tag{4}$$

where g is holomorphic and non-zero in D.

Definition 11.1 (Logarithmic derivative)**.** The *logarithmic* derivative of f is

$$\frac{f'(z)}{f(z)} = \frac{d}{dz} \log f(z).$$

Recall that in $\mathbb{C}^*$, the log function is only well defined modulo $2\pi i$. However, taking the derivative kills constants, so the derivative of log is well-defined on $\mathbb{C}^*$.

Observe that if $h(z) = f(z)g(z)$, then

$$\frac{h'(z)}{h(z)} = \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}.$$

Hence, the logarithmic derivative converts products to sums. Moreover, if $f(z) = (z - a)^m$, then

$$\frac{f'(z)}{f(z)} = \frac{m}{z - a}.$$

Therefore the logarithmic derivative may have poles at the zeros of f, but these poles are always simple, regardless of the order of the pole of f. We now apply the logarithmic derivative to both sides of (4). We have

$$\frac{f'(z)}{f(z)} = \frac{m_1}{z - z_1} + \dots + \frac{m_k}{z - z_k} + \frac{g'(z)}{g(z)}.$$

Integrating both sides around the loop γ, we have

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_\gamma \frac{m_1}{z - z_1}\, dz + \dots + \int_\gamma \frac{m_k}{z - z_k}\, dz + \int_\gamma \frac{g'(z)}{g(z)}\, dz.$$

By Cauchy's integral formula, we have $\int_\gamma m_i/(z - z_i) dz = 2\pi i m_i$. Since g is non-vanishing in D, $g'(z)/g(z)$ is holomorphic, so $\int_\gamma g'(z)/g(z) dz = 0$ by Cauchy's theorem. Therefore we conclude that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i \sum_{i=1}^{k} m_i.$$

This is known as the argument principle.

Theorem 11.2 (Argument principle)**.** Let $\Omega \subset \mathbb{C}$ be open, and γ a simple closed curve in Ω, bounding a region $D \subset \Omega$. Suppose that $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function with no zeros in γ. If f has zeros $z_1, \ldots, z_k$ in D with multiplicities $m_1, \ldots, m_k$ respectively, then

$$\sum_{i=1}^{k} m_i = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Another way of thinking about the argument principle is to consider the image curve $\Gamma = f \circ \gamma$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w} = n(\Gamma, 0),$$

which is the winding number of Γ around zero. This makes sense, because the number of zeros of f, counted with multiplicity, is equal to the number of times f 'folds' the region D over itself around zero.

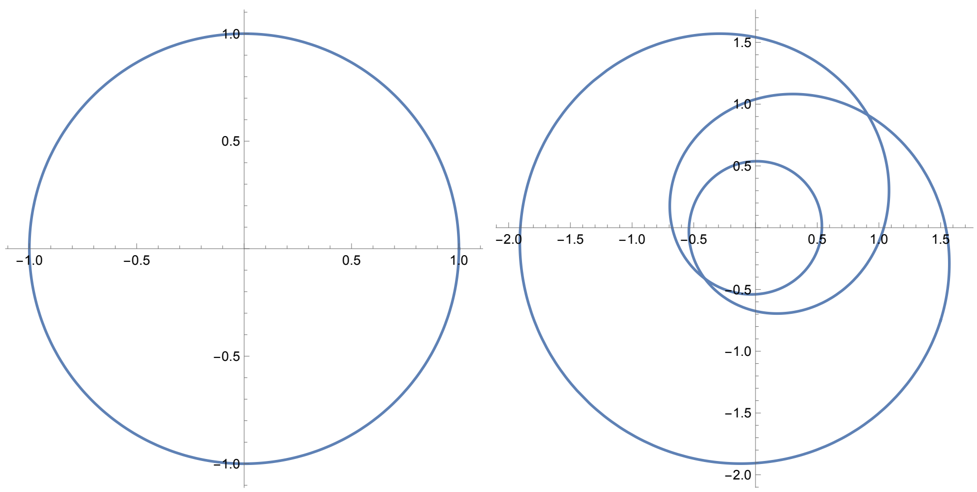


Figure 1. The image of the unit circle under a holomorphic map with three zeros in Δ.

Now, we wish to know the number of times f assumes a given value a. This is given by

$$\frac{2}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a}\, dz.$$

Now let $\epsilon = \min_{z \in \gamma} |f(z)|$. We will restrict ourselves to values of a such that $|a| < \epsilon$. As long as $a \in \Delta_\epsilon$, function

$$a \mapsto \int_\gamma \frac{f'(z)}{f(z) - a}\, dz$$

is a continuous function of a, since the integrand can never have a pole. But this function must be an integer, since it counts the number of times f assumes the value a. A continuous, integer-valued function must be constant. Therefore, in any small disk D, the function f assumes any value in D the same number of times.

Remark 11.3. This property is not true for smooth functions on $\mathbb{R}$. Consider function $f(x) = x^2$, and let I be any neighborhood of zero. The function f assumes each non-negative value in I two times with multiplicity. However, $f(x)$ is never a negative number.

The argument principle has a notable corollary which gives a geometric description of holomorphic functions.

Theorem 11.4 (Open mapping theorem)**.** Any non-constant holomorphic function is an open map; that is, $f(U)$ is open in $\mathbb{C}$ for each open set U.

Proof. Let U be an open set. For any $a \in f(U)$, the argument principle implies that that there is a small open neighborhood N of a such that f assumes all values in N the same number of times that it assumes a. In particular, $N \subset f(U)$. ■

12 Thursday - 3/6/2025

Let $D \subset \mathbb{C}$ be open and f holomorphic in $\overline{D}$, meaning f is defined and holomorphic on some open set $\Omega \supset \overline{D}$. Assume that D is bounded by a simple closed curve C. If ϵ is the minimum value of $f(z)$ on C and $|a| < \epsilon$, then the number of zeros of $f(z) - a$, counted with multiplicity is given by

$$\#(f, a) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - a}\, dz,$$

which is continuous in a. Hence, $\#(f, a)$ is *locally constant* in a neighborhood of a.

Visualizing holomorphic functions. The map $f(z) = z^d$ is a branched cover which wraps $\mathbb{C}$ around it self d times, fixing zero. The graph of f $\Gamma = \{(z, w) \in \mathbb{C}^* \times \mathbb{C}^* \mid w = z^d\}$ is the unique, connected degree d covering space of $\mathbb{C}^*$. As we shall soon see, this is the local behavior of any holomorphic function with a critical point.

Critical points. When we have $f(z_0) = w_0$, then $f(z) - w_0 = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$ and $k \geqslant 1$. Therefore,

$$f'(z) = k(z - z_0)^{k-1} g(z) + (z - z_0)^k g'(z),$$

so $f'(z_0) = 0$ if and only if $k > 1$. Hence critical points, where $f'(z) = 0$ are precisely the points where $f(z) = w$ with multiplicity greater than one.

Theorem 12.1 (Inverse function theorem)**.** If $f'(z_0) \neq 0$ and $f(z_0) = w_0$, then there are neighborhoods U of z_0 and V of w_0 such that $f : U \to V$ is *biholomorphic*; that is, invertible with holomorphic inverse. Moreover,

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}.$$

Proof. By the argument principle, there is a neighborhood U of z_0 which does not contain any other preimage point of w_0, and where $\#(f, w)$ is constant. Since $f'(z_0) \neq 0$ and z_0 is the only preimage point of w_0 in U, $\#(f, w_0) = 1$. Therefore f is one to one on U. Let $V = f(U)$. Then $f : U \to V$ is a bijection.

Since $f(f^{-1}(w)) = w$, we have $f'(f^{-1}(w))(f^{-1})'(w) = 1$. Therefore,

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}.$$

■

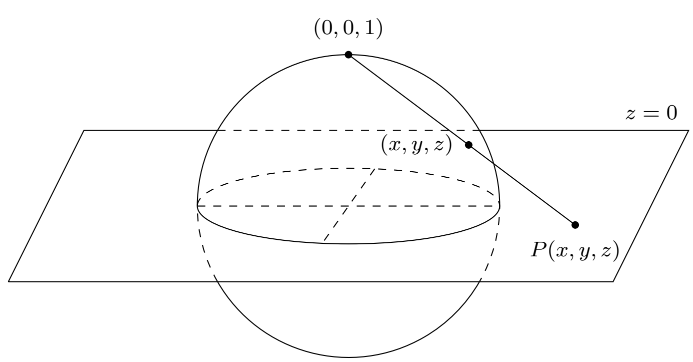
Corollary 12.2. If f is a holomorphic function with no critical points, then f is a local homeomorphism.

If $f(z_0) = w_0$ with multiplicity greater than one, then the argument principle implies that on any neighborhood N of w_0, we have $\#(f, w_0) > 1$. Thus, a holomorphic function with critical points cannot be injective.

The Riemann sphere. Consider the set $\mathbb{C} \cup \{\infty\}$. We will describe a topology (a set of open sets) and call the resulting *topological space* $\widehat{\mathbb{C}}$ the *Riemann sphere*. Our open sets will be

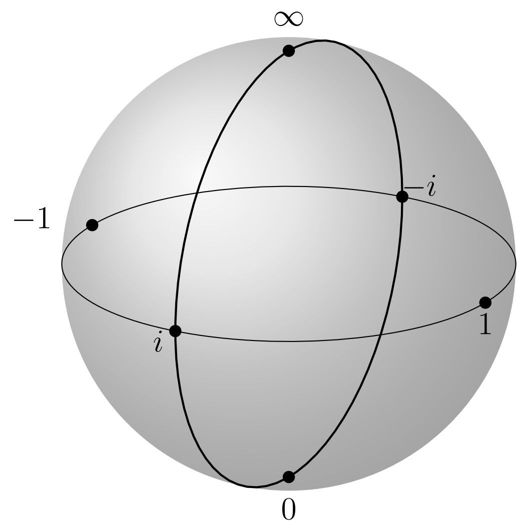
- Open subsets $U \subset \mathbb{C}$
- $\{\infty\} \cup V$, where $V \subset \mathbb{C}$ and $\mathbb{C} \setminus V$ is compact (closed and bounded).

The Riemann sphere is the *one point compactification* of $\mathbb{C}$, and is identified with the unit sphere S^2 in $\mathbb{R}^3$. The map from $P : S^2 \to \widehat{\mathbb{C}}$ is given by stereographic projection.



Stereograph projection.

Under stereographic projection, the north pole of S^2 maps to ∞, and the south pole maps to zero. The lower hemisphere maps to the unit disk Δ, and the upper hemisphere maps to $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. The rear hemisphere $\{(x, y, z) \in S^2 \mid y > 0\}$ maps to the upper half plane $\mathbb{H}$.



Visualization of $\widehat{\mathbb{C}}$.

For an open set $\Omega \subset \mathbb{C}$, a map $f : \Omega \to \widehat{\mathbb{C}}$ is *holomorphic* if f is meromorphic. If $U \subset \widehat{\mathbb{C}}$ is any open set and $g : U \to \mathbb{C}$ is a function, there are two possibilities.

1. If U does not contain ∞, then we say g is *holomorphic* if it is holomorphic in the usual sense.

2. If $\infty \in U$, then g is holomorphic if $g : U \setminus \{\infty\} \to \mathbb{C}$ is holomorphic and there is a neighborhood V of zero on which $g(1/z)$ is holomorphic.

Proposition 12.3 (Holomorphic functions on the Riemann sphere)**.** A map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic if and only if f is a rational function, and is biholomorphic if and only if

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Proof. Suppose $f \in \mathcal{M}(\widehat{\mathbb{C}})$. If f is constant, then $f \in \mathbb{C}(z)$. Otherwise, $f|_{\mathbb{C}}$ has finitely many poles $p_1, \dots, p_n$ in $\mathbb{C}$ with multiplicity. Consider $h(z) = (z - p_1) \dots (z - p_n) f(z)$, which has no poles in $\mathbb{C}$, and is thus an entire function. Evidently, $h(z) \to \infty$ as $z \to \infty$, so h is a polynomial. It follows that $f(z)$ is rational. ■

The automorphisms of $\widehat{\mathbb{C}}$ are called *Möbius transformations*, or *linear fractional transformations*. They form a group under composition, which is isomorphic to $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$. This is realized by the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d},$$

which has kernel $\{\pm I\}$

13 Tuesday - 3/11/2025

The maximum principle. Recall that a non-constant holomorphic function is an open map, i.e. $f(U)$ is open if U is open. Therefore, any point $f(z_0) \in f(U)$ is the center of a ball of some radius r which is contained in $f(U)$. But then, there exists a point $f(w_0)$ with $|f(w_0)| > |f(z_0)|$.

If f is defined and holomorphic on the closure of Ω and $\overline{\Omega}$ is compact, then $|f(z)|$ must attain a maximum in $\overline{\Omega}$. Thus, we have the following.

Theorem 13.1 (Maximum principle)**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Then $|f(z)|$ does not achieve a maximum in Ω. If f is defined continuously on $\overline{\Omega}$ and $\overline{\Omega}$ is compact, then $|f(z)|$ achieves a maximum on $\partial\Omega$.

Example 13.2. Suppose $f : \overline{\Delta_2} \to \mathbb{C}$ is holomorphic and non-constant. Then for all $z \in \Delta_2$, $|f(z)| < \max_{S^1_2} |f(z)|$. Now say $f(1) = f(-1) = 0$. Then

$$g(z) = \frac{f(z)}{z^2 - 1}$$

is defined and holomorphic in $\overline{\Delta_2}$. Now

$$|g(0)| < \max_{z \in S^1_2} \left| \frac{f(z)}{z^2 - 1} \right| \leqslant \max_{z \in S^1_2} \frac{|f(z)|}{|z|^2 - 1} = \max_{z \in S^1_2} \frac{|f(z)|}{3}.$$

Now $|f(0)| = |g(0)|$, so $|f(0)| < \max_{S^1_2} |f(z)|/3$. Thus, we can control the absolute value of f at zero just by knowing its zeros in Δ_2.

Schwarz lemma. We now state a theorem which at first sounds rather specific, but in fact will be of great use, as it will allow us to classify holomorphic functions between disks.

Suppose that $f : \Delta \to \Delta$ is holomorphic; that is, $|f(z)| < 1$ for all $z \in \Delta$. Assume that $f(0) = 0$.

Theorem 13.3 (Schwarz lemma)**.** Under these condition, either f is a rotation $f(z) = e^{i\theta} z$, or $|f(z)| < |z|$ for all $z \in \Delta$ and $|f'(0)| < 1$.

Proof. Consider the function $g(z) = f(z)/z$, which has a removable singularity at $z = 0$, since $f(0) = 0$. Indeed,

$$g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f'(z)}{1} = f'(0).$$

Then for all $z \in \Delta_r$, with $r < 1$,

$$|g(z)| \leqslant \max_{z \in S^1_r} |g(z)|. = \max_{z \in S^1_r} \frac{|f(z)|}{r} < 1,$$

by the maximum principle. Therefore, for all $z \in \Delta$

$$|g(z)| \leqslant \lim_{r \to 1} \max_{z \in S^1_r} \frac{|f(z)|}{r} \leqslant 1.$$

But since $|g(z)| = |f(z)|/|z|$, we have $|f(z)| \leqslant |z|$. If $|g(z)| = 1$ for some $z \in \Delta$, then g is constant with absolute value 1. Therefore $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. ■

We can reformulate the Schwarz lemma in the following way. Suppose that $f : \Delta \to \mathbb{C}$ is holomorphic and $|f'(0)| > 1$. Then there exists $z \in \Delta$ with $|f(z)| > 1$. This is not true in general for functions of a real variable. For example, the function $f(x) = \sin(x)$ has derivative 1 at $x = 0$, but takes values in $[-1, 1]$.

Rouché's theorem. This is a theorem about counting the number of zeros of a holomorphic function.

Theorem 13.4 (Rouché's theorem)**.** Suppose that f and g are holomorphic on a compact region $\overline{\Omega}$ with boundary curve γ. If $|f(z) - g(z)| < |f(z)|$ for all $z \in \gamma$, then f and g have the same number of zeros in Ω.

Notice that the hypothesis implies that neither g nor f are zero on the arc γ, and therefore f and g have finitely many zeros in Ω. To prove Rouché's theorem, we must upgrade the argument principle. Denote by $\#(f, \infty)$ the number of poles of f with multiplicity; that is, the number of zeros of $1/f$.

Theorem 13.5 (Argument principle version two)**.** Suppose that $f : \overline{\Omega} \to \mathbb{C}$ is meromorphic with no zeros or poles on the boundary curve γ. Then

$$\#(f, 0) - \#(f, \infty) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz.$$

Proof. Say f has zeros $a_1, \ldots, a_k$ with multiplicities $m_1, \ldots, m_k$ and f has poles $b_1, \ldots, b_\ell$ of multiplicities $n_1, \ldots, n_\ell$. We can write

$$f(z) = \frac{(z - a_1)^{m_1} \ldots (z - a_k)^{m_k}}{(z - b_1)^{n_1} \ldots (z - b_\ell)^{n_\ell}} g(z),$$

where f is defined, holomorphic, and has no zeros in Δ. We apply the logarithmic derivative to both sides. Now

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{k} \frac{m_i}{z - a_i} - \sum_{j=1}^{\ell} \frac{n_j}{z - b_j} + \frac{g'(z)}{g(z)}.$$

We integrate both sides along γ. Since g has no zeros or poles, $\int_\gamma g'/g = 0$. Therefore

$$\int_\gamma \frac{f'(z)}{f(z)} = 2\pi i \left(\sum_{i=1}^{k} m_i - \sum_{j=1}^{\ell} n_j \right).$$

Therefore, the number of zeros of f minus the number of poles of f is given to us by

$$\#(f, 0) - \#(f, \infty) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

■

Notice that this can be rephrased as $\#(f, 0) - \#(f, \infty) = \frac{1}{2\pi i} \int_\Gamma dw/w$, where Γ is the image of γ under f. Therefore $\#(f, 0) - \#(f, \infty) = n(\Gamma, 0)$, the winding number of Γ around zero.

Proof of Rouché's theorem. We will count the number of zeros minus the number of poles of $h(z) := g(z)/f(z)$.

Since $|f(z) - g(z)| < |f(z)|$ for $z \in \gamma$, we have that $|h(z) - 1| < 1$. Therefore $h(\gamma)$ has winding number zero around $z = 0$, so the number of zeros of h equals the number of poles. Since the zeros correspond with the zeros of f, and the poles correspond with the zeros of g, the result follows. ■

We will now give an application of Rouché's theorem.

Example 13.6. Suppose that $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions which converge uniformly to a holomorphic $f : \Omega \to \mathbb{C}$. Say that $f(a) = 0$ for some $a \in \Omega$, and let N be a neighborhood of a such that f has no zeros on ∂N. Then for sufficiently large n, f_n has a zero in N.

Let $\epsilon = \max_{z \in \partial N} |f(z)|$. By uniform convergence, $|f(z) - f_n(z)| < \epsilon$ for n sufficiently large. In this case, f and f_n satisfy the hypothesis of Rouché's theorem, so f and f_n have the same number of zeros in N.

This fact does not hold in general for functions of a real variable. Consider the functions $g_n(x) = x^2 + 1/n$, which converge uniformly to $g(x) = x^2$. While g has a zero at $x = 0$, the g_n never have a zero.

14 Thursday - 3/13/2025

Happy housing day!

Example 14.1. Consider $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$. The fundamental theorem of algebra tells us that g has seven zeros in $\mathbb{C}$, but not their location.

To apply Rouché's theorem, let $f(z) = 6z^3$, the cubic term of g. We choose this term because the coefficient 6 is greater than the sum of all other coefficients of g. Thus, on S^1, $|g(z) - f(z)| = |z^7 - 2z^5 - z + 1|$. This value is at most 5 by the triangle inequality, so $|g(z) - f(z)| < 6 = |f(z)|$. Now, the number of zeros of g in Δ is equal to the number of zeros of f in Δ, which is 3.

Residue calculus. We will now develop tools which will allow us to evaluate integrals, including real integrals such as

$$\int_0^\infty \frac{\sin x}{x}\, dx, \quad \text{and} \quad \int_0^\infty \frac{x^{\sqrt{2}}}{1+x^2},$$

which are quite difficult to evaluate with the usual techniques of calculus.

Residues. Suppose that f is holomorphic in the punctured disk $\Delta_r^*(a)$ of radius r centered at a. For any $0 < \epsilon < r$, we define the *residue*

$$\mathrm{Res}_a(f) := \frac{1}{2\pi i} \int_{S^1_\epsilon(a)} f(z)\, dz$$

of f at a. By Cauchy's theorem, this is the same regardless of ϵ.

Now suppose Ω is open and bounded by a simple closed curve γ, and that

$$f : \overline{\Omega} \setminus \{a_1, \dots, a_k\} \to \mathbb{C}$$

is holomorphic. We tweak the path γ as in Figure 1, so that it deviates and makes loops C_i around each of the k_i, and then returns. We orient the C_i counterclockwise, but travel around them clockwise. Let the resulting path be given by γ_t.

Then γ_t bounds a region on which f is holomorphic, so we have

$$0 = \int_{\gamma_t} f(z)\, dz = \int_\gamma f(z)\, dz - \int_{C_i} f(z)\, dz = \int_\gamma f(z)\, dz - 2\pi i \sum_{j=1}^k \mathrm{Res}_{a_j}(f).$$

The result is the following.

Theorem 14.2 (Residue theorem)**.** Suppose that $f : \overline{\Omega} \setminus \{a_1, \dots, a_k\} \to \mathbb{C}$ is holomorphic. Then

$$\int_{\partial\Omega} f(z)\, dz = 2\pi i \sum_{j=1}^k \mathrm{Res}_{a_j}(f).$$

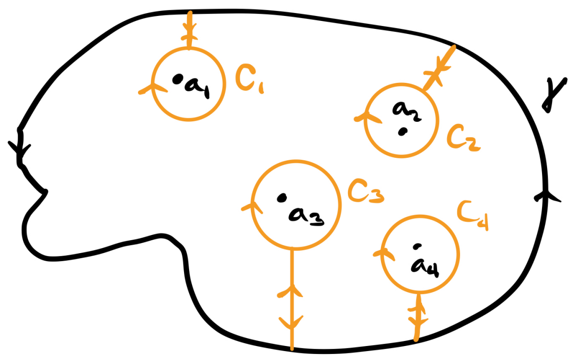


Figure 1: The path γ_t

15 Tuesday - 3/25/2025

Calculating residues. Suppose f is holomorphic on a punctured disk $\Delta_r(a) = \{|z-a| < r\}$. Recall that the *residue* of f at a is by definition

$$\mathrm{Res}_a(f) := \frac{1}{2\pi i}\int_{C_\epsilon} f(z)\,dz,$$

where $C_\epsilon = S^1_\epsilon(a) = \{|z-a| = \epsilon\}$. To calculate the residues, we have the following.

Proposition 15.1. Suppose that f is holomorphic in a punctured neighborhood of a.

- If f has a removable singularity at a, then $\mathrm{Res}_a(f) = 0$.
- If f has a pole of order h at a, then

$$\mathrm{Res}_a(f) = \lim_{z\to a}\frac{((z-a)^h f(z))^{(h-1)}}{(h-1)!}.$$

Proof. If f has a removable singularity at a, then $\mathrm{Res}_a(f) = \frac{1}{2\pi i}\int_{C_\epsilon} f(z)\,dz = 0$ by Cauchy's theorem. Suppose that f has a pole of order h at a. Then

$$f(z) = \frac{c_h}{(z-a)^h} + \frac{c_{h-1}}{(z-a)^{h-1}} + \cdots + \frac{c_1}{(z-a)} + u(z),$$

where u is holomorphic at a. Then

$$\int_{C_\epsilon} f(z)\,dz = \int_{C_\epsilon}\frac{c_h}{(z-a)^h}\,dz + \cdots + \int_{C_\epsilon}\frac{c_1}{z-a}\,dz + \int_{C_\epsilon} u(z)\,dz = \int_{C_\epsilon}\frac{c_1}{z-a}\,dz = 2\pi i c_1.$$

Therefore $\mathrm{Res}_a(f) = c_1$. To recover c_1, let $g(z) = (z-a)^h f(z)$, which is holomorphic at a and

$$g(z) = c_h + c_{h-1}(z-a) + \cdots + c_1(z-a)^{h-1} + u(z)(z-a)^h.$$

Now $g^{(h-1)}(a) = (h-1)! \cdot c_1 = (h-1)! \cdot \mathrm{Res}_a(f)$. Thus, we conclude that

$$\mathrm{Res}_a(f) = \frac{g^{(h-1)}(a)}{(h-1)!} = \lim_{z\to a} \frac{((z-a)^h f(z))^{(h-1)}}{(h-1)!},$$

as desired. ■

Notice that if f has a simple pole at a, then $\mathrm{Res}_a(f) = \lim_{z\to a}(z-a)f(z)$.

Example 15.2. Let $f(z) = e^z/(z-a)^2$, which has a pole of order two at a. Then

$$\mathrm{Res}_a(f) = \lim_{z\to a} \exp'(a) = e^a.$$

Warning. If f has an essential singularity at a, then this singularity might not be isolated. For example, let $f(z) = \frac{1}{e^{1/z}-1}$, which has poles at the points $z = 1/(2\pi n i)$ and an essential singularity at the origin. This singularity at 0 is not isolated. At such points, the residue is not defined.

However, if f has an isolated essential singularity, then we will see next week that f has a *Laurent series expansion*; that is, in a neighborhood of a,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

As in the case of a pole, we have $\mathrm{Res}_a(f) = c_{-1}$. Therefore we can upgrade the residue theorem to include this case.

Theorem 15.3 (Expanded residue theorem)**.** If C is a simple closed curve in $\mathbb{C}$, bounding a region D, and f is holomorphic in $\overline{D} \setminus \{a_1, \dots, a_k\}$, then

$$\int_C f(z)\, dz = 2\pi i \sum_{n=1}^{k} \mathrm{Res}_a(f).$$

Residue calculus. We can use the residue theorem solve problems from classical calculus.

Example 15.4 (Reciprocals of trig functions)**.** Choose $a > 1$, and consider the integral

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}.$$

Make the change of variables $z = e^{i\theta}$. Now $\theta = -i \log z$, and $d\theta = -i\frac{dz}{z}$. Moreover,

$$a + \cos\theta = \frac{z + 2a + z^{-1}}{2},$$

since $\cos\theta = (z + z^{-1})/2$. Now

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = -2i \int_{S^1} \frac{dz}{z^2 + 2az + 1}.$$

Let $f(z)$ be the new integrand. Then f has singularities at $z = -a \pm \sqrt{a^2 - 1}$, which are real and distinct, since $a > 1$. The unit circle only encloses the singularity at $z_0 = -a + \sqrt{a^2 - 1}$. The residue theorem implies that

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = -2i \int_{S^1} \frac{dz}{z^2 + 2az + 1} = 4\pi \cdot \operatorname{Res}_{z_0}(f).$$

To find the residue at z_0, we take

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{1}{z + a + \sqrt{a^2 - 1}} = \frac{1}{2\sqrt{a^2 - 1}}.$$

We conclude that

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

16 Thursday - 3/27/2025

We will give more examples of *Residue calculus*, the application of the Residue theorem in solving real integrals.

Example 16.1 (Rational functions)**.** Let $R(z) = P(z)/Q(z)$ be a rational function, where P and Q are polynomials which do not share roots. Consider the integral

$$\int_{-\infty}^{\infty} R(x)\, dx.$$

Assume that $\deg(Q) \geqslant \deg(P) + 2$ and R has no poles in $\mathbb{R}$, so that the integral converges. By convergence,

$$\int_{-\infty}^{\infty} R(x)\, dx = \lim_{\rho \to \infty} \int_{-\rho}^{\rho} R(x)\, dx.$$

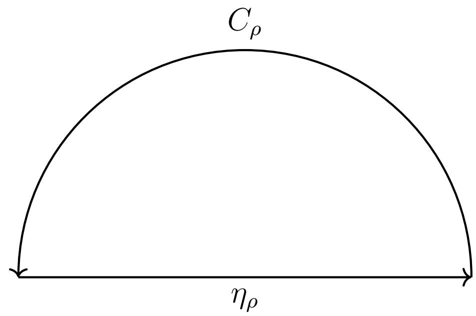


Figure 2: The path $\eta_\rho * C_\rho$

Let $C_\rho : [0, \pi] \to \mathbb{C}$ parametrize the upper half circle $C_\rho(t) = \rho e^{it}$. Let $\eta_\rho : [-1, 1] \to \mathbb{C}$ be given by $\eta_\rho(t) = \rho t$. Then the concatenation $\eta_\rho * C_\rho$ is a simple closed curve, bounding the upper half disk $D_\rho = \{|z| < \rho,\ \mathrm{Im}(z) > 0\}$ of radius ρ. Moreover,

$$\int_{\eta_\rho * C_\rho} R(z)\ dz = \int_{-\rho}^{\rho} R(z)\ dz + \int_{C_\rho} R(z)\ dz = 2\pi i \sum_{a \in D_\rho} \mathrm{Res}_a(R).$$

We take the limit as $\rho \to \infty$. By the hypothesis that $\deg(Q) \geqslant \deg(P) + 2$, there is constant K such that, for large values of ρ, $|R(z)| \leqslant K/\rho^2$ on C_ρ. However, the length of C_ρ is always $\pi\rho$. Now

$$\lim_{\rho\to\infty} \int_{C_\rho} R(z)\ dz \leqslant \lim_{\rho\to\infty} \pi\rho \cdot K/\rho^2 = \lim_{\rho\to\infty} \pi K/\rho = 0.$$

We conclude that

$$\lim_{\rho\to\infty} R(z)\ dz = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} R(z)\ dz + 0 = \int_{-\infty}^{\infty} R(z)\ dz = 2\pi i \cdot \sum_{a \in \mathbb{H}} \mathrm{Res}_a(R),$$

where $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$.

Example 16.2 (Mixed rational and exponential functions)**.** Consider the integral

$$\int_{-\infty}^{\infty} e^{iz} R(z)\ dz,$$

where $R(Z) = P(z)/Q(z)$ is a rational function. Assume that P and Q have no common zeros, and that $\deg(Q) \geqslant \deg(P) + 2$. Since $|e^{iz}| < 1$ for all $z \in \mathbb{H}$, the same strategy as last time applies. Therefore, we conclude that

$$\int_{-\infty}^{\infty} e^{iz} R(z)\ dz = 2\pi i \cdot \sum_{a \in \mathbb{H}} \mathrm{Res}_a(e^{iz} R(z)).$$

Remark 16.3. If $R(z)$ is a rational function, say $R(z) = P(z)/Q(z)$ with $\deg(Q) \geqslant \deg(P)+2$ and no poles in $\mathbb{R}$, then $\sum_{a\in\mathbb{C}} \operatorname{Res}_a(R(z)) = 0$.

To see this, note that $\int_{-\infty}^{\infty} R(z)\, dz = 2\pi i \cdot \sum_{a\in\mathbb{H}} \operatorname{Res}_a(R)$. If we tweaked our approach to integrate around the semi-circle in the *lower half plane*, then we would obtain $\int_{-\infty}^{\infty} R(z)\, dz = -2\pi i \sum_{a\in-\mathbb{H}} \operatorname{Res}_a(R)$. Hence,

$$\sum_{a\in\mathbb{C}} \operatorname{Res}_a(R) = \sum_{a\in\mathbb{H}} \operatorname{Res}_a(R) + \sum_{a\in-\mathbb{H}} \operatorname{Res}_a(R) = 0.$$

We now look at a more specific example of mixed rational and exponential functions.

Example 16.4. By our previous arguments,

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}\, dz = 2\pi i \cdot \sum_{a\in\mathbb{H}} \operatorname{Res}_a(e^{iz}/(1+z^2)).$$

The integrand has simple poles $\pm i$, of which just i is in $\mathbb{H}$. Now

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}\, dz = 2\pi i \cdot \lim_{z\to i} (z-i)\frac{e^{iz}}{1+z^2} = 2\pi i \cdot \lim_{z\to i} \frac{e^{iz}}{z+i} = 2\pi i \cdot e^{-1}/(2i) = \pi/e.$$

Moreover, observe that

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2}\, dz = \int_{-\infty}^{\infty} \frac{\cos z}{1+z^2}\, dz + i\int_{-\infty}^{\infty} \frac{\sin z}{1+z^2}\, dz.$$

However, since the integrand $\sin(z)/(1+z^2)$ is an odd function, the integral $\int_{-\infty}^{\infty} \sin(z)/(1+z^2)\, dz = 0$. Therefore,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}.$$

17 Tuesday - 4/1/2025

Last time, we exhibited a method to calculate integrals of the form

$$\int_{-\infty}^{\infty} e^{iz} P(z)/Q(z)\, dz,$$

where $P, Q \in \mathbb{C}[z]$ with $\deg(Q) \geqslant \deg(P)+2$, and Q has no zeros in $\mathbb{R}$. In fact, when this factor of e^{iz} is present, the integral converges even if $\deg(Q) = \deg(P)+1$, while not absolutely. Instead of integrating along the semicircle C_ρ, we integrate along the boundary

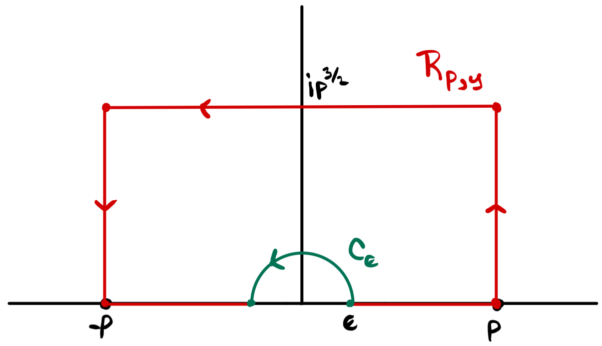


Figure 3: The path $\Gamma_{\rho,\epsilon}$

of the rectangle $[-x_1, x_2] \times [0, y]$. As y gets large, the exponential decays rapidly, and we proceed similarly to the previous example. For example, consider

$$\int_{-\infty}^{\infty} \frac{z}{1+z^2} e^{iz}\, dz$$

If $f(z) = ze^{iz}/(1+z^2)$, then this integral is equal to

$$2\pi i \cdot \sum_{a \in \mathbb{H}} \mathrm{Res}_a(f) = 2\pi i \mathrm{Res}_i(f) = i\pi/e.$$

Example 17.1. Consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

To evaluate this integral, we can use a similar method as before, and instead try to evaluate

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz.$$

Unlike the previous examples, $z \mapsto 1/z$ has a pole in $\mathbb{R}$. Moreover, the degree of the denominator is only one greater than the degree of the numerator. We propose to evaluate this integral by integrating e^{iz}/z along a cleverly chosen path which avoids the origin.

Consider the following path $\Gamma_{\rho,\epsilon} = R_{\rho,\epsilon} * -C_\epsilon$, as seen above. The integrand $f(z) = e^{iz}/z$ is holomorphic in the region bound by $\Gamma_{\rho,\epsilon}$, so by Cauchy's theorem,

$$0 = \int_{\Gamma_{\rho,\epsilon}} \frac{e^{iz}}{z}\, dz = \int_{R_{\rho,\epsilon}} \frac{e^{iz}}{z}\, dz + \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz.$$

Let

$$\begin{aligned} \gamma_1(t) &= \rho + it, \quad t \in [0, \rho^{3/2}] \\ \gamma_2(t) &= i\sqrt{\rho} - t, \quad t \in [-\rho, \rho] \\ \gamma_3(t) &= -\rho + i(1-t) \quad t \in [0, \rho^{3/2}] \end{aligned}$$

be three of the segments of $R_{\rho,\epsilon}$. We can check that as $\rho \to \infty$, $\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz \to 0$ and $\int_{\gamma_2} f(z)dz \to 0$. Therefore,

$$0 = \lim_{\epsilon \to 0} \lim_{\rho \to \infty} \int_{R_{\rho,\epsilon}} \frac{e^{iz}}{z}\, dz - \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz - \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz = \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz.$$

Suppose f is meromorphic with a simple pole at a. If γ_ϵ is the circle of radius ϵ around a, then

$$\int_{\gamma_\epsilon} f(z)\, dz = 2\pi i \cdot \mathrm{Res}_a(f)$$

Now suppose $\gamma_{\epsilon,\alpha}$ is the path $t \mapsto a + \epsilon e^{it}$, where $0 \leqslant t \leqslant 2\pi\alpha$. We claim that

$$\lim_{\epsilon \to 0} \int_{\gamma_{\epsilon,\alpha}} f(z)\, dz = 2\pi i\alpha \cdot \mathrm{Res}_a(f).$$

Since f has a simple pole at a, write

$$f(z) = \frac{c}{z-a} + j(z),$$

where $j(z)$ is holomorphic. Now

$$\int_{\gamma_{\epsilon,\alpha}} f(z)\, dz = \int_{\gamma_{\epsilon,\alpha}} \frac{c}{z-a}\, dz + \int_{\gamma_{\epsilon,\alpha}} j(z)\, dz$$

Since j is holomorphic, and thus bounded in a neighborhood of a, the integral of j along $\gamma_{\epsilon,\alpha}$ goes to zero as $\epsilon \to 0$. Therefore,

$$\lim_{\epsilon \to 0} \int_{\gamma_{\epsilon,\alpha}} f(z)\, dz = \int_{\gamma_{\epsilon,\alpha}} \frac{c}{z-a} = 2\pi i\alpha \cdot c = 2\pi i\alpha \cdot \mathrm{Res}_a(f),$$

as desired. In our case, we have $C_\epsilon = \gamma_{\epsilon,1/2}$, so

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \pi i \cdot \mathrm{Res}_0(f) = \pi i.$$

Thus, we conclude that

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz = i\pi.$$

It then follows that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} = \pi.$$

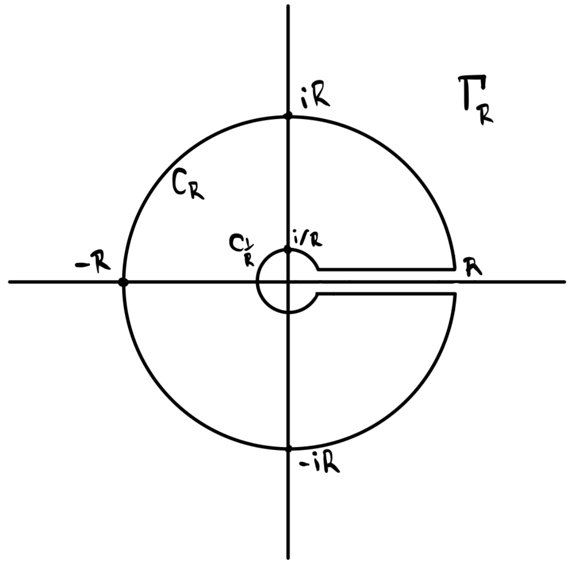


Figure 4: The path Γ_R.

Example 17.2 (Non-integral powers)**.** Consider the integral

$$\int_0^\infty x^\alpha R(x)\,dx,$$

where $R(z) = P(z)/Q(z)$ is a rational function with no poles in $\mathbb{R}$. Assume $0 < \alpha < 1$ and $\deg(Q) \geqslant \deg(P) + 2$. If $\alpha = m/n$ is a rational number, then we can use the substitution $w^n = z$, which will convert our integral to a rational function, which we know how to compute. If α is irrational, then we have another method.

The function $z \mapsto z^\alpha$ can locally be defined holomorphically away from the origin. However, there is no global definition. We will exploit this fact to evaluate the integral. Indeed, we write

$$z^\alpha = e^{\alpha \log z}.$$

The problem is that log is not well defined unless we choose a branch for the argument function. In our case, assume that $\arg z$ takes values in $[0, 2\pi)$. Therefore log maps to $\{0 \leqslant \mathrm{Im}(z) < 2\pi\}$. Consider the path $t \mapsto e^{it}$, with $0 \leqslant t < 2\pi$. At $t = 0$, $\log(e^{it}) = \log(1) = 0$. As $t \to 2\pi$, then point $e^{it} \to 1$, but the argument tends to 2π, so $\log(e^{it}) \to 2\pi i$.

Consider the following keyhole path Γ_R, with outer radius R and inner radius $1/R$, as seen above. Specifically, let

$$\begin{aligned} C_R(t) &= Re^{it}, \quad t \in [\epsilon, 2\pi - \epsilon] \\ \gamma_1(t) &= e^{2\pi i - i\epsilon}(t/R + (1-t)R), \quad t \in [0,1] \\ C_{1/R} &= e^{it}/R, \quad t \in [\epsilon, 2\pi - \epsilon] \\ \gamma_2(t) &= e^{i\epsilon}(tR + (1-t)/R), \quad t \in [0,1] \end{aligned}$$

so that $\Gamma_R = C_R * \gamma_1 * -C_{1/R} * \gamma_2$. The integrand $f(z) = z^\alpha R(z)$ is meromorphic in the region bound by Γ_R, which eventually contains all poles of f for large enough radius. By the residue theorem,

$$\begin{aligned} 2\pi i \cdot \sum_{a\in\mathbb{C}} \operatorname{Res}_a(f) &= \int_{\Gamma_R} f(z)\,dz \\ &= \lim_{R\to\infty}\left(\int_{C_R} f(z)\,dz + \int_{\gamma_1} f(z)\,dz - \int_{C_{1/R}} f(z)\,dz + \int_{\gamma_2} f(z)\,dz\right). \end{aligned}$$

Observe that

$$\left|\int_{C_R} \frac{z^\alpha P(z)}{Q(z)}\,dz\right| \leqslant \int_{C_R} \left|\frac{e^{\alpha\log(z)}P(z)}{Q(z)}\right|\,dz \lesssim \int_{C_R} \frac{R^\alpha}{R^2}\,dz \leqslant 2\pi R^{\alpha-1} \to 0,$$

as $R\to\infty$. Moreover, using the fact that $R(z)$ is bounded in a neighborhood of zero,

$$\left|\int_{C_{1/R}} \frac{z^\alpha P(z)}{Q(z)}\,dz\right| \lesssim \int_{C_{1/R}} \frac{1}{R^\alpha}\,dz \leqslant \frac{2\pi}{R^{1+\alpha}} \to 0$$

as $R\to\infty$. Now

$$2\pi i \cdot \sum_{a\in\mathbb{C}} \operatorname{Res}_a(f) = \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = \lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz + \lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz.$$

Notice that $\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = \int_0^\infty x^\alpha R(z)\,dz = I$, our desired integral. Moreover,

$$\lim_{R\to\infty}\int_{\gamma_1} \frac{e^{\alpha(\log|z|+i\arg(z))}P(z)}{Q(z)}\,dz = -\lim_{R\to\infty}\int_{\gamma_2} \frac{e^{\alpha(\log|z|+i\arg(z)+2\pi i)}P(z)}{Q(z)}\,dz = -e^{2\pi\alpha i}I.$$

Now

$$2\pi i \cdot \sum_{a\in\mathbb{C}} \operatorname{Res}_a(f) = I(1-e^{2\pi\alpha i}),$$

so we conclude that

$$\int_0^\infty x^\alpha R(x)\,dx = \frac{2\pi i}{1-e^{2\pi\alpha i}} \sum_{z\in\mathbb{C}} \operatorname{Res}_a(f),$$

which can be easily calculated.

18 Thursday - 4/3/2025

Power series. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic and $z_0 \in \Omega$. We can write $f(z) - f(z_0) = (z - z_0)f_1(z)$, where $f_1(z)$ is holomorphic. Then $f_1(z) - f_1(z_0) = (z - z_0)f_2(z)$. We obtain

$$\begin{aligned} f(z) &= f(z_0) + (z - z_0)f_1(z) \\ f_1(z) &= f_1(z_0) + (z - z_0)f_2(z) \\ &\vdots \\ f_n(z) &= f_n(z_0) + (z - z_0)f_{n+1}(z), \end{aligned}$$

where f_j is holomorphic in Ω for each j. Combining these expressions, we arrive at

$$f(z) = f(z_0) + (z - z_0)f_1(z_0) + (z - z_0)^2 f_2(z_0) + \cdots + (z - z_0)^n f_n(z_0) + (z - z_0)^{n+1} f_{n+1}(z).$$

Take the kth derivative of both sides, and set $z = z_0$. We conclude from this that $f_k(z_0) = f^{(k)}(z_0)/k!$. Now our expression becomes

$$\begin{aligned} f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \\ + (z - z_0)^{n+1} f_{n+1}(z). \end{aligned}$$

The first $n + 1$ terms, $f(z_0) + \cdots + f^{(n)}(z_0)(z - z_0)^n/n!$ is a polynomial in z, which we call the nth Taylor polynomial of f at z_0. The question is how the error term $(z - z_0)^{n+1} f_{n+1}(z)$ behaves. If we can show that $f_{n+1}(z) \cdot (z - z_0)^{n+1} \to 0$ as $n \to \infty$, then it will follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

in some neighborhood of z_0.

Up until now, everything we have done can be done for $C^\infty(\mathbb{R})$ functions, but in that case, the error term need not go to zero as $n \to \infty$. For example the function

$$g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

has $f^{(n)}(0) = 0$ for all n, and hence does not equal its power series. But when f is holomorphic, we will see that this sort of phenomenon does not occur.

Theorem 18.1 (Taylor's theorem)**.** Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. Then for each $z_0 \in \Omega$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

on the open disk of largest radius centered at z_0 which is contained in Ω.

Proof. Choose $\rho > 0$ such that the disk $\overline{\Delta}_\rho = \{|z - z_0| < \rho\} \subset \Omega$. Let C be the boundary curve of Δ_ρ. Since f_{n+1} is holomorphic,

$$\begin{aligned} f_{n+1}(z) &= \frac{1}{2\pi i} \int_C \frac{f_{n+1}(\zeta)}{\zeta - z} \, d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)} \, d\zeta \end{aligned}$$

Now

$$\begin{aligned} |f_{n+1}(z)(z - z_0)^{n+1}| &\leqslant \left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)|z - z_0|^{n+1}}{(\zeta - z_0)^{n+1}(\zeta - z)} \, d\zeta \right| \\ &\leqslant \rho \cdot \max_{|\zeta - z_0| = \rho} \left| \frac{f(\zeta)(z - z_0)^{n+1}}{(z - z_0)^{n+1}(\zeta - z)} \right| \\ &\leqslant \frac{M|z - z_0|^{n+1}}{\rho^n(\rho - |z - z_0|)}, \end{aligned}$$

were $M = \max_{|\zeta - z_0| = \rho} |f(z)|$. For any $0 < r < \rho$, and w with $|z - z_0| \leqslant r$, we have

$$\frac{M|z - z_0|^{n+1}}{\rho^n(\rho - |z - z_0|)} \leqslant \frac{Mr^{n+1}}{\rho^n r} = \frac{Mr^n}{\rho^n} \to 0$$

as $n \to \infty$. Therefore $|f_{n+1}(z)(z - z_0)^{n+1}| \to 0$ uniformly as $n \to \infty$ on the closed disk of radius r centered at z_0. On the other hand, ρ can be choose arbitrarily close to the distance between z_0 and $\partial\Omega$, and so then can r. ■

Remark 18.2. Weierstrass's theorem states that if (f_n) is a sequence of holomorphic functions converging uniformly to a limit f, then f is analytic and $f_n' \to f'$ uniformly. From this, it follows that any uniformly converging power series defines a holomorphic function, a converse to Taylor's theorem. We have not proven Weierstrass's theorem.

19 Tuesday - 4/8/2025

We have proven that if $f : \Omega \to \mathbb{C}$ is holomorphic, then at any point $z_0 \in \Omega$, we can express f as a power series $f(z) = \sum a_n(z - z_0)^n$, which converges uniformly on any closed disk centered at z_0 which is contained in Ω. Therefore, the radius of convergence R, given by $1/R = \limsup_n |a_n|^{1/n}$, is at least the distance from z_0 to $\partial\Omega$.

Manipulation of power series. There are a couple standard manipulations which can be performed on uniformly convergent power series. Without loss of generality, assume our power series are centered at $z_0 = 0$.

- $(\sum a_n z^n) + (\sum b_n z^n) = \sum (a_n + b_n) z^n$ if both series are absolutely convergent.

- $\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum_n \left(\sum_{j+k=n} a_j b_k\right) z^n$.
- If $f(z) = \sum a_n z^n$ with $a_0 \neq 0$, then there exists $g(z) = \sum b_n z^n$ with $f(z)g(z) = 1$. We can find the coefficients b_n by solving successively. Indeed,

$$f(z)g(z) = a_0 b_0 + (a_1 b_0 + a_0 b_1)z + (a_2 b_0 + a_1 b_1 + a_0 b_2)z^2 + \dots,$$

so $b_0 = 1/a_0$. Then $b_1 = -a_1 b_0 / a_0$, $b_2 = -(a_2 b_0 - a_1 b_1)/a_0$, etc.

The radius of convergence of the sum or product of $\sum a_n z^n$ and $\sum b_n z^n$ is equal to the minimum of the radii of convergence of the two series. The radius of convergence of the reciprocal $1/(\sum a_n z^n)$ is equal to the distance between zero and the nearest zero of $\sum a_n z^n$.

- If $f(z) = a_n z^n$ and $g(z) = b_n z^n$, then we form the power series $g(z)f(z)$, using a combination of the last two techniques.
- Suppose $f(z) = \sum a_n z^n$, with $f(0) = a_0 \neq 0$), and $g(z) = \sum b_n z^n$. Then we can find the power series of the composition $g(f(z))$. Now

$$\begin{aligned} g(f(z)) &= \sum_{n=0}^{\infty} b_n \left(\sum_{m=1}^{\infty} a_m z^m\right)^n \\ &= b_0 + b_1(a_1 z + a_2 z^2 + \dots) + b_2(a_1 z + a_2 z^2 + \dots)^2 + \dots \\ &= b_0 + a_1 b_1 z + (b_1 a_2 + b_2 a_1^2) z^2 + \dots. \end{aligned}$$

If $a_0 \neq 0$, then it is somewhat more difficult to find the coefficients of the composition.
- Say $f(z) = \sum a_n z^n$ with $a_0 = 0$ and $a_1 \neq 0$. Thus, $f'(0) \neq 0$. Then by the inverse function theorem, f has an inverse g defined in some neighborhood of 0. Write $g(z) = \sum b_n z^n$. Then we can solve for the coefficients b_n in the usual way, given that $g(f(z)) = z$.
- If $f(z) = \sum a_n z^n$, then

$$f'(z) = \sum_{n=1}^{\infty} n \cdot a_n z^{n-1}.$$

It follows that *any uniformly convergent power series defines a holomorphic function.*

Multi-valued functions. We cannot write a power series for $\log z$ in any neighborhood of the origin. Instead, consider the function $\log(1 + z)$. Choose the branch $(-\pi, \pi]$ for the argument function. Then we can define $\log(1+z)$ unambiguously in $\mathbb{C}\setminus\mathbb{R}_{\leqslant -1}$. Now $\log(1+z)$ has a power series expansion centered at $z=0$ given by

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

with radius of convergence is 1, as anticipated. To obtain this series, note that

$$\frac{d}{dz}\log(1+z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$$

by the geometric series formula. Integrating term by term, we obtain the desired series for $\log(1+z)$. Alternatively, invert the power series for $e^z - 1$.

We conclude by summarizing our findings on power series.

1. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, and $z_0 \in \Omega$. Let $\rho = d(z_0, \partial\Omega)$. Then on the open disk $\Delta_\rho = \{|z - z_0| < \rho\}$, f can be expressed as

   $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

   This series converges uniformly to f on all closed disks $\overline{\Delta}_r = \{|z - z_0| \leqslant r\}$ for all $r < \rho$.

2. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ defines a uniformly convergent power series in a neighborhood of z_0 with radius of convergence R, then f is holomorphic in $\Delta_R = \{|z - z_0| < R\}$. If $g : \Omega \to \mathbb{C}$ is holomorphic on some open set $\Omega \supsetneq \Delta_R$ and $g = f$ on Δ_R, then g has a singularity on the circle $\{|z - z_0| = R\}$.

20 Thursday - 4/10/2025

We will introduce Laurent series and partial fraction expansions.

Laurent series. A Laurent series is an expression

$$\sum_{n\in\mathbb{Z}} a_n z^n.$$

We can write this expression as the sum of the positive terms and negative terms

$$\sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}.$$

The first sum converges in some disk of radius $R_2 \geqslant 0$. The second sum converges for all z with $|1/z| < 1/R_1 \geqslant 0$, so $|z| > 1/R \geqslant \infty$. Thus, the Laurent series converges in an annulus $\{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$. An arbitrary Laurent series may not converge at all, which is the case when $R_2 = 0$ and $R_1 = \infty$.

Theorem 20.1 (Laurent series)**.** If $f(z)$ is holomorphic on an annulus $R_1 < |z| < R_2$, then it is equal to a Laurent series.

Proof. We claim that if f is holomorphic on an annulus $R_1 < |z| < R_2$, then we can write $f_1(z) + f_2(z)$, where f_1 is holomorphic in the disk $|z| < R_2$, and f_2 is holomorphic in $\widehat{\mathbb{C}} \setminus \overline{\Delta}_{R_1}$, where $\Delta_{R_1} = \{|z| < R_1\}$. Note that the point at infinity is included. Hence, $f_2(1/z)$ is holomorphic in Δ_{1/R_1}.

If the claim holds, then write $f_1(z) = \sum a_n z^n$ and $f_2(1/z) = \sum b_n z^n$. Now

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_n}{z^n},$$

as desired.

Take $\rho > R_1$ and $r < R_2$. Then the annulus $\mathbb{A}_{\rho,r}$ is contained in our original annulus. Let

$$\begin{aligned} C_r(t) &= re^{2\pi i t}, \quad t \in [0,1] \\ C_\rho(t) &= \rho e^{2\pi i t}, \quad t \in [0,1] \\ \gamma(t) &= (1-t)r + r\rho, \quad t \in [0,1]. \end{aligned}$$

Consider the path $\Gamma_{\rho,r} = C_r * \gamma * (-C_\rho) * (-\gamma)$, a closed loop. Then by the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\Gamma_{\rho,r}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} - \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\zeta)}{\zeta - z}\, dz. \end{aligned}$$

The term on the left, f_1, is holomorphic in the disk of radius r. The term on the right, f_2, is holomorphic for all $|z| > \rho$ as well as at ∞. ■

Warning. A given function can have different Laurent series, each converging on a different annulus. For instance, the function $f(z) = \frac{1}{1-z}$. Inside the unit disk, $f(z) = 1 + z + z^2 + z^3 + \ldots$, by the geometric series formula. We can also write f as a function which converges in the complement of the closed unit disk. Indeed,

$$f(z) = \frac{1/z}{1/z - 1} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots.$$

Similarly, the function

$$f(z) = \frac{1}{(z-1)(z-2)},$$

which has poles at $z = 1$ and $z = 2$. There is a Laurent series which converges in the unit disk Δ, a Laurent series which converges in the annulus $\mathbb{A}_{1,2}$, and one which converges for all $|z| > 2$.

21 Tuesday - 4/15/2025

Partial fraction expansion. If $R(z)$ is a rational function, then

$$R(z) = \lambda \frac{\prod(z - a_i)}{\prod(z - b_i)},$$

for some complex numbers a_i, b_i. We can also write R using partial fractions. If the poles $b_1, \ldots, b_n$ are simple; that is the b_i are distinct, then we can write

$$R(z) = \frac{c_1}{1 - b_1} + \frac{c_2}{1 - b_2} + \cdots + \frac{c_n}{z - b_n} + P(z),$$

where P is a polynomial. More generally, if R has a pole of order m_i at the point b_i, then

$$R(z) = \frac{c_1(z)}{(z - b_1)^{m_1}} + \cdots + \frac{c_n(z)}{(z - b_n)^{m_n}} + P(z),$$

where $c_i(z)$ is a polynomial of degree $\leqslant m_i - 1$. We wish to know if a similar expression can be given for meromorphic functions.

Remark 21.1. Recall that if $f(z)$ is a meromorphic function and b is a pole of order m of f, then $g(z) = (z - b)^m f(z)$ is holomorphic at b. Now g has a Taylor series $g(z) = \sum a_n z^n$, so

$$f(z) = \frac{a_0}{(z - b)^m} + \frac{a_1}{(z - b)^{m-1}} + \cdots + \frac{a_{m-1}}{z - b} + \text{something holomorphic.}$$

We call the sum $\sum a_i/(z - b)^{m-i}$ the *polar part* of f.

Given an entire meromorphic function f with poles at $b_1, b_2, b_3, \ldots$ having polar part $P_\nu((z - b_\nu)^{-1})$ at b_ν, can we write $f(z) = \sum P_\nu((z - b_\nu)^{-1}) + g(z)$ with g an entire holomorphic function? Specifically,

1. Given an entire meromorphic function f, can we write it in this way, and how do we find this expression?

2. Given $b_1, b_2, \cdots \in \mathbb{C}$ with $\lim b_\nu = \infty$ and polynomials $P_1, P_2, \ldots$, is there an entire meromorphic function with polar part $P_\nu((z - b_\nu)^{-1})$ at b_ν and no other poles?

Example 21.2. Let $f(z) = \pi^2 \csc^2 \pi z$, which is meromorphic in $\mathbb{C}$ and poles exactly at the integers. To find the polar parts, start with the function

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{6} + \frac{\pi^5 z^5}{120} - \ldots$$

Then

$$\sin^2 \pi z = \pi^2 z^2 \left(1 - \frac{\pi^2 z^2}{3} + \dots\right).$$

Now the reciprocal is given by

$$\frac{1}{\sin^2 \pi z} = \frac{1}{\pi^2 z^2} \left(1 + \frac{\pi^2 z^2}{3} - \dots\right).$$

Finally,

$$\pi^2 \csc^2 \pi z = \frac{1}{z^2} \left(1 + \frac{\pi^2 z^2}{3} + \dots\right).$$

Thus, the polar part of $\pi^2 \csc \pi z$ at $z = 0$ is just $1/z^2$. By the periodicity of csc, the polar part of $\pi^2 \csc^2 \pi z$ at $z = n$ is $1/(z-n)^2$.

Consider the expression

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

This expression converges away from its poles (compare $\sum 1/n^2$). Let $g(z) = \pi^2 \csc^2 \pi z - f(z)$, which must be entire and holomorphic. Now $\pi^2 \csc^2 \pi z = \sum \frac{1}{(z-n)^2} + g(z)$, as desired.

We know the function g is periodic with period 1. This is because both $\pi^2 \csc^2 \pi z$ and f are periodic with period 1. Therefore g is determined by its values in the fundamental domain $0 \leqslant \mathrm{Re}(z) \leqslant 1$.

Observe that as $z \to \infty$ in the strip $0 \leqslant \mathrm{Re}(z) \leqslant 1$, we have $\sin z \to \infty$. Therefore $\pi^2 \csc^2 \pi z \to 0$ as $z \to \infty$ in the strip. Likewise,

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \to 0$$

as $z \to \infty$ in the strip. Therefore $g(z) \to 0$ as $z \to \infty$ in the strip. Thus g is bounded, and constant by Liouville's theorem. Since $\lim g(z) = 0$, we have $g \equiv 0$. Therefore,

$$\pi^2 \csc^2 \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

This gives an example of entire meromorphic function which can be written as an infinite partial fraction expansion. In an attempt to answer the question about the existence of an entire meromorphic function with given polar parts, we can naively consider the sum

$$\sum_{\nu=1}^{\infty} P_\nu \left(\frac{1}{z - b_\nu}\right)$$

and hope it converges. We will first consider the simplest case.

Example 21.3. Does there exists a meromorphic function f with simple poles at $\mathbb{Z}$, all with residue 1? Unfortunately, the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$$

does not converge. However, we can fix this by adding $1/n$ to the nth term when $n \neq 0$. Thus, consider

$$\frac{1}{z} + \sum_{n\neq 0\in\mathbb{Z}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\neq 0\in\mathbb{Z}} \frac{z}{n(z-n)}.$$

The right hand side allows us to compare this expression to $\sum 1/n^2$, which converges.

Theorem 21.4. Let $b_1, b_2, b_3, \ldots \neq 0 \in \mathbb{C}$ with $\lim b_\nu \to \infty$ and polynomials $P_1, P_2, P_3, \ldots$ be given. Write the Taylor series

$$P_\nu\left(\frac{1}{z-b_\nu}\right) = \sum_{k=0}^{\infty} a_{\nu k} z^k$$

centered at zero. Then there exists a sequence $m_1, m_2, m_3, \ldots$ such that if

$$p_\nu(z) = \sum_{k=0}^{m_\nu} a_{\nu k} z^k,$$

then

$$\sum_{\nu=1}^{\infty} P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)$$

converges.

The proof is left to Ahlfors. We return to the expression

$$f(z) = \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Since this converges uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, we can take the derivative term by term, and we have

$$f'(z) = -\frac{1}{z^2} - \sum_{n\neq 0\in\mathbb{Z}} \frac{1}{(z-n)^2} = -\pi^2 \csc^2 \pi z.$$

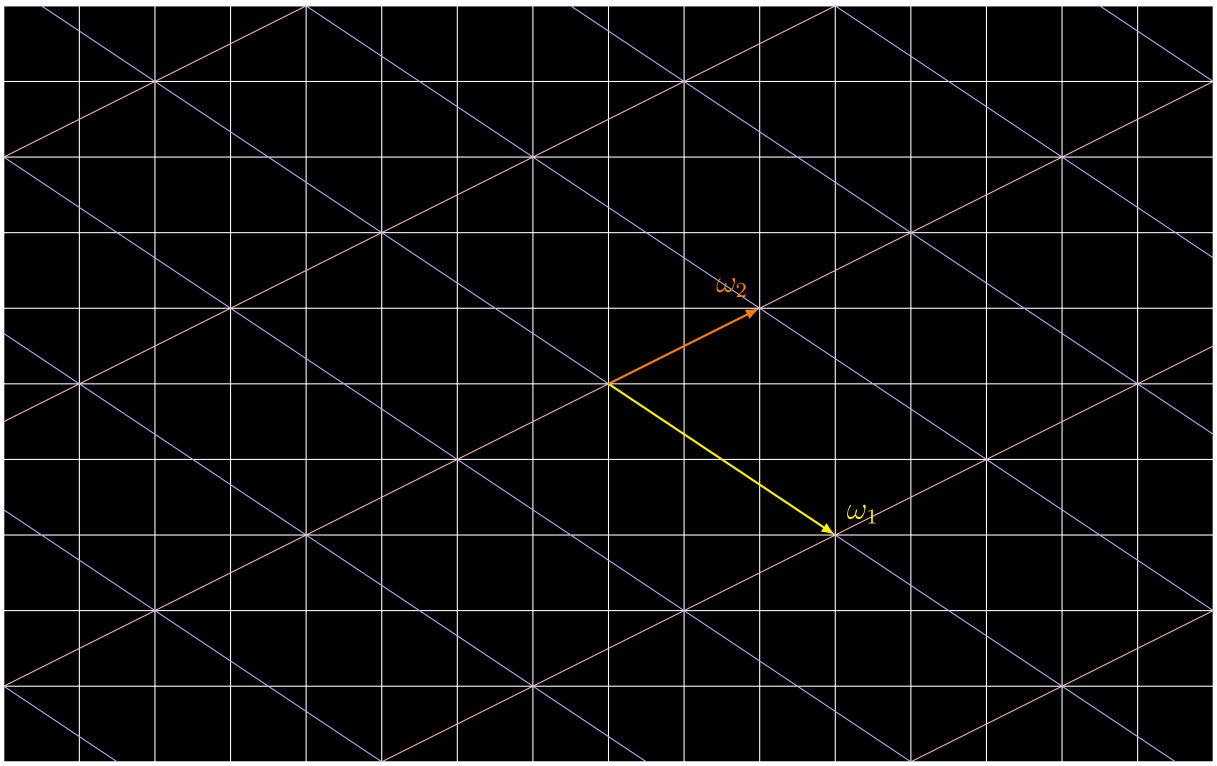
Now f itself is an anti-derivative of $-\pi^2 \csc^2 \pi z$. Therefore $f(z) = \pi \cot \pi z + C$, for some constant C. Since f and $\pi \cot \pi z$ are odd functions, the same must be true of C; i.e., $C = -C$. Therefore $C = 0$, so $f(z) = \pi \cot \pi z$.

22 Thursday - 4/17/2025

Doubly periodic functions. Suppose that $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent as vectors in $\mathbb{C} \simeq \mathbb{R}^2$. In other words, we cannot find $a, b \in \mathbb{R}^*$ such that $a\omega_1 = b\omega_2$. Consider the lattice

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$$

We say any entire meromorphic function f is *doubly periodic* with respect to Λ if $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$ and $z \in \mathbb{C}$.



A doubly periodic function is entirely determined by its values in the parallelogram $F = \{t\omega_1 + s\omega_2 \mid t, s \in [0,1]\}$, which is called the *fundamental domain*.

Since F is compact, $\max_{z\in\mathbb{C}} |f(z)| = \max_{z\in F} |f(z)|$ is achieved somewhere in F. If f is holomorphic, this forces f to be bounded, and hence constant. Thus, a doubly periodic, non-constant, entire meromorphic function must have poles.

In order to find an example of a doubly periodic entire meromorphic function, we naively consider the sum

$$\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^n}$$

for some $n \in \mathbb{N}$, which must be doubly periodic with respect to Λ if it converges. Unfortunately, this only converges when $n \geqslant 3$. However, for $n = 2$, we can add a term in the way prescribed by Theorem 21.4. We now take

$$\frac{1}{z^2} + \sum_{\lambda \neq 0 \in \Lambda} \left(\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2} \right).$$

This gives us a non-constant entire meromorphic function, which is clearly doubly periodic. This function is called the *Weierstrass $\wp$-function* for the lattice Λ, which has great importance in the study of elliptic curves and elliptic functions. We denote this $\wp_\Lambda(z)$, or just $\wp(z)$ when the lattice is unambiguous.

Aside: Complex tori. For those with some background in topology, recall that the topological quotient $\mathbb{C}/\Lambda$, where two points $z, w \in \mathbb{C}$ are identified if and only if $z = w + \lambda$ for some $\lambda \in \Lambda$, is homeomorphic to a torus $T \simeq S^1 \times S^1$. Since $\wp_\Lambda$ is doubly periodic, it induces a continuous function $f : T \to \widehat{\mathbb{C}}$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\wp"] \arrow[d] & \widehat{\mathbb{C}} \\ \mathbb{C}/\Lambda \arrow[ur, "f"] & \end{tikzcd}$$

One can show that the map f is degree two, with four critical points. Thus, $\wp$ induces a branched double cover from the torus to the 2-sphere, branched over four points.

Infinite products. Another way of representing entire functions is as infinite products. We begin with a useful lemma. The proof requires some knowledge of the theory of covering spaces.

Lemma 22.1. If $f(z)$ is an entire holomorphic function and nowhere 0, then there exists an entire holomorphic function $g(z)$ such that $f(z) = e^{g(z)}$.

Proof. Recall that $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map. This is clear because any holomorphic map is a local homeomorphism away from its critical points. Since exp has no critical points, and is proper when $\mathbb{C}^*$ is considered as its co-domain, the claim follows.

Suppose that f is an entire holomorphic function which is nowhere zero. Then we can consider f as a map $f : \mathbb{C} \to \mathbb{C}^*$. Since $\mathbb{C}$ is simply connected, the lifting lemma (see Munkres §79) implies that f lifts to a holomorphic map $g : \mathbb{C} \to \mathbb{C}$ such that $f = \exp \circ g$. ■

Any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ can be written as a product

$$p(z) = a_n \prod_{n=1}^{k} (z - b_\nu)^{n_\nu}$$

for some k. Since entire holomorphic functions are infinite polynomials, we might hope that a a similar expression may be written down. In other words, if f is entire with zeros at $b_1, b_2, \dots$ of multiplicities $\nu_1, \nu_2, \dots$, can we write

$$f(z) = \prod_{\nu} g(z)(z - b_\nu)^{n_\nu},$$

where g is entire and non-vanishing? Moreover, can we use such an expression to generate new entire functions? Ideally, given any sequence $b_1, b_2, \dots \in \mathbb{C}$ with $\lim b_j = \infty$ and $n_1, n_2, \dots \in \mathbb{N}$, we will be able to find an entire holomorphic function with these zeros and multiplicities.

Convergence of products. Given complex numbers $p_1, p_2, \dots \in \mathbb{C}$, we wish to formulate a definition for $\prod p_i$. A naive first try is to set

$$P_n := \prod_{i=1}^{n} p_i,$$

and ask that $\lim_{n\to\infty} P_n$ exists. The problem is that if a single p_k is zero, all partial products are eventually zero, and hence the limit of the P_n is zero. We wish to create a definition which allows some factors to vanish without annihilating the whole product. We now state the correct definition.

Definition 22.2 (Infinite products)**.** We say that $\prod p_i$ converges if

1. At most finitely many $p_i = 0$,
2. The partial products of the non-zero factors

$$P_n := \prod_{\substack{i \leqslant n \\ p_i \neq 0}} p_i$$

   converge to a non-zero limit.

If these conditions hold, we set $\prod p_i = \lim P_n$.

If the P_n converge to a non-zero limit, then $p_i \to 1$. Therefore, it makes sense to write $p_i = 1 + a_i$. Then $\prod p_i$ converges if $a_i \to 0$ sufficiently fast. Indeed, $\prod(1 + a_i)$ converges if and only if $\sum \log(1 + a_i)$ converges. The requirement that $a_i \to 0$ allows us to set $\arg : \mathbb{C}^* \to (-\pi, \pi]$, and define log unambiguously. From our power series expansion for $\log(1 + z)$, then see that $\sum \log(1 + a_i)$ converges if and only if

$$\sum_{i=1}^{\infty} \left(a_i - \frac{a_i^2}{2} + \frac{a_i^2}{3} - \dots \right)$$

converges, which is the case if and only if $\sum a_i$ converges.

23 Tuesday - 4/22/2025

We continue the discussion of infinite products and entire functions.

Theorem 23.1. If $b_1, b_2, b_3, \dots \in \mathbb{C}$ have no accumulation points and $n_1, n_2, n_3, \dots \in \mathbb{N}$, then there exists an entire holomorphic function f with zeros of order n_ν at b_ν for all ν, and no other zeros.

Before proving the theorem, we consider an example of the type of construction we may encounter.

Example 23.2. Take $f(z) = \sin \pi z$, which is entire and holomorphic, with simple zeros exactly at the integers. We might first try to take the product

$$\prod_{n \in \mathbb{Z}} (z - n).$$

Unfortunately, this product is far from convergent. Indeed, the factors do not converge to 1. Instead, consider

$$z \cdot \prod_{n \neq 0 \in \mathbb{Z}} \left(1 - \frac{z}{n}\right).$$

This product still does not converge, as

$$\sum_{n \neq 0 \in \mathbb{Z}} \log\left(1 - \frac{z}{n}\right) = \sum_{n \neq 0 \in \mathbb{Z}} \left(-\frac{z}{n} + \frac{z^2}{2n^2} - \dots\right)$$

which cannot converge because of the $-z/n$ in each summand. However,

$$\sum_{n \neq 0 \in \mathbb{Z}} \left(\log\left(1 - \frac{z}{n}\right) + \frac{z}{n}\right)$$

does converge. Therefore, consider the product

$$z \cdot \prod_{n \neq 0 \in \mathbb{Z}} e^{z/n} \left(1 - \frac{z}{n}\right).$$

It follows that $\sin \pi z = z \cdot e^{g(z)} \cdot \prod (1 - z/n)\, e^{z/n}$, for some entire function g. To uncover the function g, we take the logarithmic derivative of both sides of the expression, and we have

$$\frac{\pi \cos \pi z}{\sin \pi z} = \pi \cot \pi z = \frac{1}{z} + g'(z) + \sum \left(\frac{1}{z - n} + \frac{1}{n}\right).$$

But we've seen from the partial fractions expansion that $\pi \cot \pi z$ is precisely the right hand side excluding the g' term. Therefore $g' \equiv 0$, so g is a constant function. Now

$$\sin \pi z = cz \cdot \prod_{n \neq 0 \in \mathbb{Z}} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

To find the constant c, divide both sides by z, and take the limit as $z \to 0$. Now

$$c = \lim_{z \to 0} \frac{\sin \pi z}{z} = (\sin \pi z)'|_{z=0} = \pi.$$

Finally, we have

$$\sin \pi z = \pi z \cdot \prod_{n \neq 0 \in \mathbb{Z}} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

where in the final expression, we combine the factors corresponding to $+n$ and $-n$.

We will now prove Theorem 23.1

Proof of Theorem 23.1. We start with the first attempt

$$f(z) = z^c \cdot \prod_{\nu} \left(1 - \frac{z}{b_\nu}\right)^{n_\nu}.$$

In some cases this may converge, for example if the b_ν are the squares of the integers. Indeed, this expression converges if and only if

$$\sum_{\nu} n_\nu \cdot \log(1 - z/b_\nu) = \sum_{\nu} n_\nu \left(-\frac{z}{b_\nu} - \frac{1}{2}\left(\frac{z}{b_\nu}\right)^2 - \frac{1}{3}\left(\frac{z}{b_\nu}\right)^3 - \dots\right)$$

converges. We can find a sequence $m_1, m_2, m_3 \dots$ such that

$$\prod_{\nu} \left(1 - \frac{z}{b_\nu}\right) e^{\frac{z}{b_\nu} + \frac{1}{2}\left(\frac{z}{b_\nu}\right)^2 + \dots + \frac{1}{m_\nu}\left(\frac{z}{b_\nu}\right)^{m_\nu}}$$

converges. This is the same tactic used to find an infinite product for $\sin \pi z$. ■

An important consequence of this theorem is that any meromorphic function on $\mathbb{C}$ is the ratio of two entire holomorphic functions. Indeed, given a meromorphic function f with poles b_ν of orders n_ν, let g be the entire function with zeros at b_ν of order n_ν. Then $f \cdot g$ is an entire function h, so write $f = h/g$.

It then follows that the field $\mathcal{M}(\mathbb{C})$ of meromorphic functions on $\mathbb{C}$ is the field of fractions of the ring $\mathcal{O}(\mathbb{C})$ of entire functions.

24 Thursday - 4/24/2025

We have seen many ways of written a holomorphic function as a sum or product, such as

1. Power series;
2. Laurent series;
3. Partial fraction expansion;
4. and Infinite product expansion.

Other examples of expansions include

1. Dirichlet series and
2. Fourier expansions.

For example, we have written

$$\begin{aligned}
\pi^2 \csc^2 \pi z &= \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \\
\sin \pi z &= \pi z - \frac{\pi^3}{z} z^3 + \dots \\
\sin^2 \pi z &= \pi^2 z^2 - \frac{\pi^4}{3} z^4 + \dots \\
\pi^2 \csc^2 \pi z &= \frac{1}{z^2} + \frac{\pi^2}{3} + \dots .
\end{aligned}$$

As an application, this allows us to express

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The partition function. Let $p(n)$ denote the number of ways to express n as a sum of positive integers, where we ignore order. Then

$$\begin{aligned}
p(1) = 1 : &\quad 1 \\
p(2) = 2 : &\quad 1+1,\ 2 \\
p(3) = 3 : &\quad 2+1,\ 1+1+1,\ 3 \\
p(4) = 5 : &\quad 3+1, 2+2,\ 2+1+1,\ 1+1+1+1,\ 4 \\
p(5) = 7 : &\quad 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1,\ 5
\end{aligned}$$

Another way to express a partition is to specify a sequence $a_1, a_2, a_3, \dots \in \mathbb{N}$ such that $\sum j \cdot a_j = n$. This makes it clear that $p(0) = 0$.

The function $p(n)$ has interesting arithmetic properties. For example, Ramanujan proved that if $n \equiv 4 \mod 5$, then $p(n) = 0 \mod 5$. The proof is beyond the scope of this course.

We wish to look at the asymptotic behavior of $p(n)$. For instance, does $p(n)$ grow as a polynomial (i.e. $p(n) \sim c \cdot n^k$ for some c, k)? For two real functions on $\mathbb{N}$ or $\mathbb{R}_+$, we say $f \sim g$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = n.$$

We may also ask if $p(n)$ grows exponentially, (i.e. $p(n) \sim \lambda e^{\mu n}$ for some λ, μ). To answer these types of questions, we will encode the information of the sequence $p(n)$ by making them the coefficients in the expansion of a holomorphic/meromorphic function. Let

$$P(z) = \sum_{n=0}^{\infty} p(n) z^n.$$

We can also write

$$P(z) := \prod_{n=1}^{\infty} \frac{1}{1 - z^n}.$$

Indeed,

$$\begin{aligned}
\frac{1}{1-z} &= 1 + z + z^2 + \dots \\
\frac{1}{1-z^n} &= 1 + z^n + z^{2n} + \dots \\
\prod_{n=1}^{\infty} \left(\frac{1}{1-z^n} \right) &= (1 + z + z^2 + \dots)(1 + z^2 + z^4 + \dots)(1 + z^3 + z^6 + \dots) \dots \\
&= 1 + z + 2z^2 + 3z^3 + 5z^4 + \dots \\
&= P(z).
\end{aligned}$$

Clearly, this infinite product does not converge beyond the unit circle. Indeed, it would have a pole at any root of unity. Therefore, S^1 is a natural boundary for the domain of P. Already, this gives us some information about the rate of growth of $p(n)$. Indeed, once can show that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2n/3}}.$$

Therefore $p(n)$ grows faster than any polynomial, but slower than the exponential.

Dirichlet series. Consider the sum

$$\sum_{n=0}^{\infty} \frac{a_n}{n^z},$$

where $n^z = e^{z \ln n}$. Notice that $|n^{-z}| = n^{-\mathrm{Re}(z)}$, so for any a_n, $\sum a_n z^{-n}$ converges in some half plane $\mathrm{Re}(z) > R_0$. For example, consider $a_n = 1$ for all n. The resulting function is the Riemann zeta-function

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z},$$

which converges when $\mathrm{Re}(z) > 1$.

Prime numbers. Euclid proved that there are infinitely many primes $p \in \mathbb{N}$. The proof is that if there were only finitely many prime $p_1, \ldots, p_n$, then set $k = \prod p_i + 1$, which is not divisible by any prime, and must be prime itself.

Since there are infinitely many primes, we may wish to know the proportion of natural numbers which are prime. Some information is given by where or not

$$\sum_{p \text{ prime}} \frac{1}{p}$$

converges. If this sum diverges, then there are more primes than squares. To answer this question, we observe that the zeta-function can be expressed as an infinite product. Indeed,

$$\zeta(z) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}} = \prod_{p \text{ prime}} \left(1 + p^{-z} + p^{-2z} + \ldots\right).$$

This expansion comes from the unique factorization of the natural numbers. From this, we have

$$\begin{aligned}
\log(\zeta(z)) &= \log\left(\prod_p \frac{1}{1 - p^{-z}}\right) \\
&= \sum_p \left(\frac{1}{p^z} + \frac{1}{2p^{2z}} + \frac{1}{3p^{3z}} + \ldots\right) \\
&= \sum_p \frac{1}{p^z} + \frac{1}{2}\sum_p \frac{1}{p^{2z}} + \frac{1}{3}\sum_p \frac{1}{p^{3z}} + \ldots.
\end{aligned}$$

Notice that as $z \to 1$, the right hand side becomes

$$\sum_p \frac{1}{p} + \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{3}\sum_p \frac{1}{p^3} + \cdots = \sum_p \frac{1}{p} + K$$

for some constant K. But as $z \to 1$, we know that $\zeta(z)$ becomes the harmonic series, and thus diverges to infinity. Therefore, so does its logarithm. Now $\sum_p 1/p$ must diverge.

The Riemann zeta-function can be extended to a meromorphic function on all of $\mathbb{C}$ with a simple pole at $z = 1$. The region $0 < \mathrm{Re}(z) < 1$ is called the critical strip.

Conjecture 24.1 (Riemann hypothesis)**.** All zeros of the Riemann zeta-function in the critical strip have real part equal to $1/2$.

25 Tuesday - 4/29/2025

On the last day of classes, we will introduce Nevanlinna theory, also known as value distribution theory. A polynomial function $p(z)$ assumes every value in $\mathbb{C}$ the same number of times with multiplicity. Likewise, a rational function.

But suppose that $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic on all of $\mathbb{C}$. Can we say that f assumes all values the same number of times? To quantify, we introduce the function

$$\begin{aligned} n_f(a, r) :=& \#\{z \in \mathbb{C} \mid |z| < r,\ f(z) = a\} \\ =& \sum_{|z|<r} \mathrm{ord}_z(f, a). \end{aligned}$$

Note that values are counted with multiplicity. We can say that f assumes the values a and b the same number of times if $n_f(a, r) \sim n_f(b, r)$. In other words,

$$\lim_{r\to\infty} \frac{n_f(a, r)}{n_f(b, r)} = 1.$$

We will look instead at

$$N_f(a, r) := \int_0^r n_f(a, t)\frac{dt}{t},$$

which encapsulates a similar notion, but fits better with the theorems we will state.

It is possible that f spends a lot of time very close to a certain value without touching it. To quantify this, we introduce the *proximity function* $m_f(a, r)$. Define $\log^+(x) = \max(0, \log x)$. Then let

$$\begin{aligned} m_f(a, r) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left(\frac{1}{f(re^{i\theta}) - a}\right) d\theta \\ m_f(\infty, r) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta. \end{aligned}$$

Let $T_f(a, r) := N_f(a, r) + m_f(a, r)$, which we call the Nevanlinna characteristic.

Theorem 25.1. For all $a, b \in \widehat{\mathbb{C}}$, $T_f(a,r) \sim T_f(b,r)$.

For example, consider the function $f(z) = e^z$. If $r \gg 0$, then the the segment of the circle of radius r with $-\pi/2 + \epsilon < \arg(z) < \pi/2 - \epsilon$ is mapped very close to ∞. Likewise, the segment of the circle of radius r with $\pi/2 + \epsilon < \arg(z) < 3\pi/2 - \epsilon$ is mapped very close to zero. In other words, f spends most of its time in either a small neighborhood of infinity or a small neighborhood of zero.

This is to say that for exp, $m_f(0,r)$ and $m_f(\infty,r)$ are large, but $N_f(0,r) = N_f(\infty,r) = 0$. The function only comes close to any other point only briefly, but obtains each of these values infinitely many times. For all $a \in \mathbb{C}^*$, $N_f(a,r)$ is large and $m_f(a,r) = 0$.

For a given meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ and $a \in \widehat{\mathbb{C}}$, we define the defect $\delta_f(a) := \liminf_{r\to\infty} m_f(a,r)/T_f(a,r) = 1 - \limsup_{r\to\infty} N_f(a,r)/T_f(a,r)$. This describes how much of a contribution to the characteristic function $T_f(a,r)$ is given by the proximity. If $\delta_f(a) \neq 0$, then a is said to be defective.

Theorem 25.2. We have

$$\sum_{a\in\widehat{\mathbb{C}}} \delta_f(a) \leqslant 2.$$

This implies in particular that there only only countably many defective values. For the complement of these values, f does assume each value *the same number of times*. This bound is sharp, since $\sum_{a\in\widehat{\mathbb{C}}} \delta_f(a) = 2$ when $f(z) = e^z$. Indeed, $\delta_{\exp}(0) = \delta_{\exp}(\infty) = 1$, and $\delta_{\exp}(a) = 0$ for all $a \in \mathbb{C}^*$.

A consequence of this is the *Little Picard theorem*:

Theorem 25.3 (Little Picard)**.** If $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is meromorphic and there exist distinct points $a, b, c \in \widehat{\mathbb{C}}$ such that $f(\mathbb{C}) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$, then f is constant.

If f is an entire function and there exist $a \neq b \in \mathbb{C}$ such that $f(\mathbb{C}) \subset \widehat{\mathbb{C}} \setminus \{a, b\}$, then theorem implies that f is constant.