Quadratic Dynamics, the Mandelbrot Set, and Local Connectivity

A dissertation presented
by
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to
The Department of Mathematics

In partial fulfillment of the requirements
for the degree of
Bachelor of Arts with Honors
in the subject of
Mathematics

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Harvard University
Cambridge, Massachusetts
May 2025

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Acknowledgments

Firstly, I want to thank Laura DeMarco, who, aside from being a fantastic advisor, has been a guide throughout most of my mathematical journey at Harvard. As a green-in-the-gills sophomore, I took Laura's topology course as my first ever upper-level math class. Despite my floundering in the first half of the semester, Laura's excellent teaching led me to feel competent in a collegiate math class for the first time. I left this course with more spring in my step and confidence in my decision to study mathematics. Since then, Laura, has helped me enormously in my study of complex analysis and dynamics, guiding me through the material needed to write this thesis. It's been a real treat.

I would like to thank Curtis McMullen, who has been a guiding light for me at Harvard. Curt inspired me to pursue complex analysis in the first place, with his superb teaching and excitement for the subject. Since my first class with him sophomore year, I've taken every course he's taught. Curt has also been a great help to me as I've prepared for this thesis.

Were it not for my parents, both of whom are mathematicians, it is highly unlikely that I would be a mathematician. Having witnessed the excellent lifestyle that comes with being a professor, and out of a desire to understand the family dinner conversations, I was inclined to join the family trade. My parents have encouraged me all the way, while never applying too much pressure. I thank them for their continued love and support.

I would also like to thank Gwendolyn, Finn, Henry, John, Kerem, Vik, Ben, Keegan, the Mather House community, and all my other friends for being awesome. Keep up the good work.

While working on this thesis, conversations with Jeremy Kahn and Alex Kapiamba have been incredibly helpful, and I thank them for their time and insight.

Chapter 1

Introduction

The goal of this work is to understand the dynamics of quadratic polynomials. We will introduce the *space of quadratic polynomials*, along with its special subset, *the Mandelbrot set*. Answers to the deepest remaining questions in quadratic dynamics lie in the properties of the Mandelbrot set. Many of these questions are tied to the conjectured *locally connectivity* of the Mandelbrot set (MLC). We will introduce the requisite background from the theory of complex analysis, geometry, and general complex dynamics, before turning to a study of the Mandelbrot set and MLC.

The Mandelbrot set. Consider the family of quadratic polynomials $f_c(z) = z^2 + c$, indexed by $c \in \mathbb{C}$. We denote by $f_c^n(z)$ the n-fold composition $\underbrace{f(f(\ldots f(z)))}_{n \text{ times}}$. The *Mandelbrot set* is the subset of $\mathbb{C}$ given by

$$M := \{c \in \mathbb{C} \mid \sup_{n \in \mathbb{N}} |f_c^n(0)| < \infty\},$$

that is, the parameters c for which the orbit of zero remains bounded under iteration of f_c. We will see that the complex plane naturally parametrizes the *space of quadratic polynomials* up to conformal equivalence, which we denote M_2. The boundary of the Mandelbrot set is the *bifurcation locus* within M_2, where the dynamical behavior is in flux [MSS82]. To understand the Mandelbrot set is to understand the possible dynamics which can be exhibited by a quadratic polynomial.

Despite its simple definition, the Mandelbrot set is incredibly complex. In the 1980s, computer graphics allowed the first pictures of M, bringing interest to the subject. Douady and Hubbard stated the following conjecture [DH85], which remains open 40 years later.

Conjecture 1 (MLC)**.** The Mandelbrot set is locally connected.

Dynamics and hyperbolicity. If proven, MLC would complete the topological picture of the Mandelbrot set. MLC implies that *hyperbolic dynamics* are dense in the Mandelbrot set [DH85]. A rational function is *hyperbolic* if it is geometrically expanding on its domain of

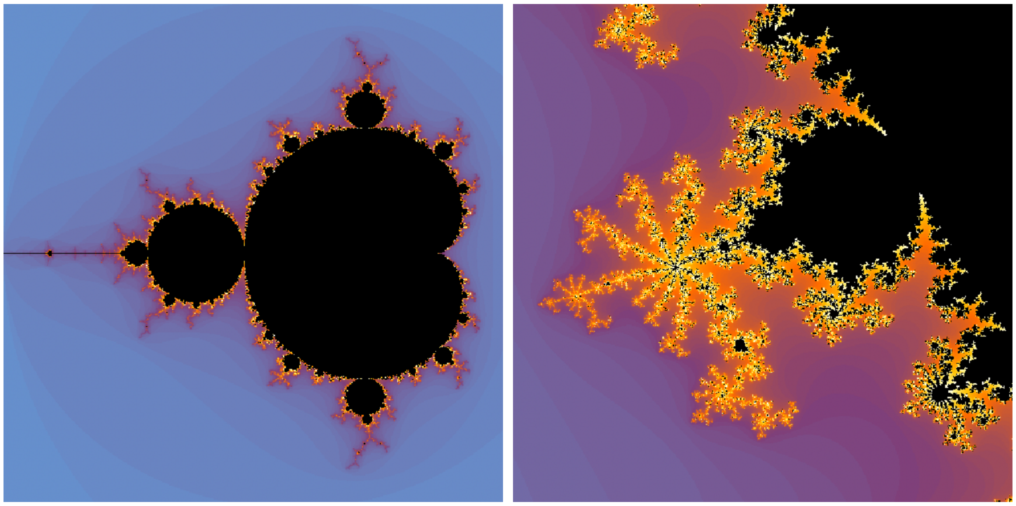

Figure 1.1: The Mandelbrot set

chaos, which we call the *Julia set*. Hyperbolic maps exhibit a variety of nice properties, and are *structurally stable*, meaning that a small perturbation of a hyperbolic map is still hyperbolic (see Proposition 10). MLC implies that every interior component of the Mandelbrot set contains only hyperbolic parameters (Theorem 37).

Geometry. Suppose $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function. As defined in [McM94b], an *invariant line field* for f, defined on a measurable set $E \subset \widehat{\mathbb{C}}$, is a choice of 1-dimensional real subspace L_z in the tangent space $T_z\widehat{\mathbb{C}}$ for each $z \in E$, such that

1. E has positive measure,
2. $f^{-1}(E) = E$,
3. the slope of L_z varies measurably with respect to z, and
4. the derivative f' transforms L_z into $L_{f(z)}$ for almost all $z \in E$.

We can also think of a line field as a measurable section of the real-projectivization of the tangent bundle, defined on E. It is conjectured that no degree two polynomial admits an invariant line field. We will show in Theorem 38 that this is equivalent to density of hyperbolicity, and is therefore implied by MLC.

Rigidity. MLC is related to rigidity theory for quadratic polynomials. It is often helpful to show that an object is classified up to some equivalence by a set of criteria. In this case, we say that the object is *rigid*. For example, the *Mostow rigidity theorem* proves that finite volume hyperbolic manifolds of dimension three and higher are determined up to isometry by their homotopy type. In the case of quadratic dynamics, there are several rigidity theorems which are implied by and equivalent to MLC.

We can associate to any quadratic polynomial f a *combinatorial class*, which records data about *periodic points* under iteration of f (see Section 5.3.2). It is conjectured that quadratic polynomials, which satisfy a few additional assumptions, are *rigid* with respect to their combinatorics: any two such polynomials with the same combinatorics are conformally equivalent. One can show that *combinatorial rigidity* is equivalent to MLC.

A second categorization of quadratic polynomials is their *quasiconformal class*. We say two maps are *quasiconformally equivalent* if they are conjugated by a homeomorphism which distorts the geometry of its domain to a bounded extent (see Chapter 4). It has been conjectured that non-hyperbolic quadratic polynomials are *quasiconformally rigid*, meaning that each $f \in M_2$ has a neighborhood on which quasiconformally equivalence implies equality. This is implied by density of hyperbolicity, and therefore by MLC.

Structure of this exposition. In Chapter 2, we will introduce *Riemann surfaces* and *conformal metrics*. In particular, the *Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the natural setting on which to consider rational functions as a dynamical system. We will also introduce the basic theory of hyperbolic Riemann surfaces, which will serve as a tool in proving several results. Moreover, we will review the *Riemann mapping theorem* and its relationship with local connectivity.

In Chapter 3, we will provide background in *complex dynamics*, the study of iterated holomorphic maps. The focus will be on the case of rational function on the Riemann sphere.

In Chapter 4, we will introduce the theory of quasiconformal mappings and almost complex structures. In particular, we will prove the *measurable Riemann mapping theorem* and the *straightening theorem*. The straightening theorem allows us to discuss *renormalizable* polynomials, which explain self-similarity in the Mandelbrot set.

Chapter 5 will begin the study of the Mandelbrot set and *renormalization theory*. We will first prove basic properties of M, and then introduce the conventions used to precisely discuss its complexities. Next, we introduce the conjectures alluded to above, and prove the stated relationships between them.

Finally, Chapter 6 will give an overview of the modern study of MLC, including partial results and techniques used by researchers.

Chapter 2

Complex Analysis

We begin with some of the necessary background in complex analysis and geometry. For a full treatment of topics discussed here, we recommend [McM23].

Notation. We let $\widehat{\mathbb{C}}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We denote the upper half-plane $\mathbb{H} = \{x + iy \mid y > 0\}$. For disks and circles, we write

1. $\Delta = \{|z| < 1\}$,
2. $\Delta_r = \{|z| < 1\}$,
3. $\Delta(p) = \{|z - p| < 1\}$,
4. $\Delta_r(p) = \{|z - p| < r\}$,

and S^1, S^1_r, $S^1(p)$, and $S^1_r(p)$ for their respective boundaries.

2.1 Riemann Surfaces

A *Riemann surface* is a connected 1-dimensional complex manifold for which the transition maps are holomorphic. More specifically, a Riemann surface X is a connected Hausdorff topological space equipped with a family of *charts* which are homeomorphisms

$$\phi_i : V_i \to U_i \subset \mathbb{C}$$

defined on an open cover $\{V_i\}_i$ of X and with $\phi_j \circ \phi_i^{-1} : \phi_i(V_i \cap V_j) \to \phi_j(V_i \cap V_j)$ holomorphic for each i and j.

It then makes sense to define holomorphic maps between Riemann surfaces. Let X and Y be Riemann surfaces with charts $\{\phi_i\}$ and $\{\psi_i\}$ respectively. Then $f : X \to Y$ is holomorphic if $\psi_j \circ f \circ \phi_i^{-1}$ is holomorphic where defined for each i, j. A holomorphic map with holomorphic inverse is called a *conformal isomorphism*.

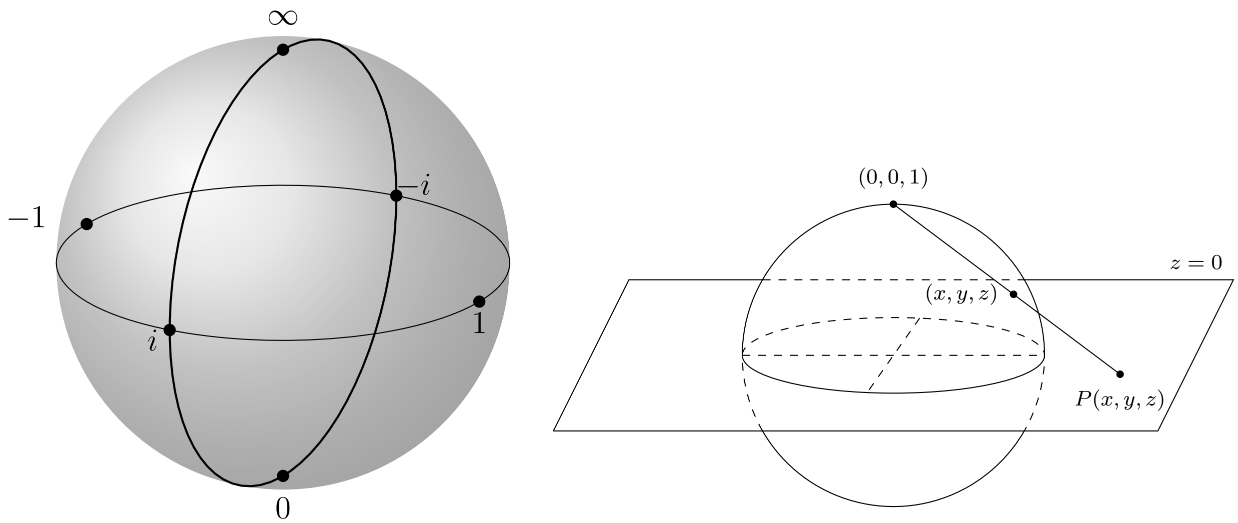

Figure 2.1: The Riemann sphere

Example 1. Any open subset $U \subseteq \mathbb{C}$ is a Riemann surface, with a global chart given by inclusion.

Example 2. The first nontrivial example of a Riemann surface is the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, given the topology of the one-point compactification of $\mathbb{C}$. Stereographic projection shows that $\widehat{\mathbb{C}}$ is homeomorphic to the unit sphere S^2. We give $\widehat{\mathbb{C}}$ the charts $\phi_1 : \mathbb{C} \to \mathbb{C}$ and $\phi_2 : \widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$, defined by $\phi_1(z) = z$ and $\phi_2(z) = 1/z$, letting $\phi_2(\infty) = 0$.

Local coordinates. When dealing with a chart ϕ on a Riemann surface X, we will often write $w = \phi(z)$ and call w a *local coordinate*. Given a function $f : X \to Y$ between Riemann surfaces with $f(z_0) = w_0$, and charts ϕ and ψ at z_0 and w_0 respectively, we write f *in coordinates* as $g(w)$, where $w = \phi(z)$. Specifically, $g(w) = \psi(f(z))$.

Remark 1 (Derivatives)**.** Any holomorphic function $f : X \to Y$ can be assigned a *derivative* $f'(z_0)$ at each point $z_0 \in X$. Let w be a local coordinate at z_0, and write f in coordinates as $g(w)$. Consider the holomorphic 1-form

$$\omega = g'(w)\, dw$$

on X. We let $f'(z) := g'(w)$. This is well defined, as if u is another local coordinate at z_0, then $u = \phi(w)$, for some holomorphic transition map ϕ. In coordinates, f takes the form $h(u) = g(\phi^{-1}(u))$. Observe that

$$h'(u)\, du = g'(\phi^{-1}(u))/\phi'(w) \cdot \phi'(w)\, dw = g'(w)\, dw.$$

Uniformization. We have the following structure theorem for Riemann surfaces, proven in [Hub06].

Theorem 1 (Uniformization theorem)**.** Any simply connected Riemann surface is conformally isomorphic either to $\mathbb{C}$, Δ, or $\widehat{\mathbb{C}}$.

Liouville's theorem implies that there can be no isomorphism $f : \mathbb{C} \to \Delta$, and $\widehat{\mathbb{C}}$ is compact, while $\mathbb{C}$ and Δ are not. Therefore $\mathbb{C}$, Δ, and $\widehat{\mathbb{C}}$ are pairwise not conformally isomorphic. It is common to interchange Δ and $\mathbb{H}$, as they are conformally isomorphic via the function $f : \Delta \to \mathbb{H}$ given by $f(z) = \frac{z+i}{z-i} \cdot \frac{1-i}{1+i}$.

Any Riemann surface X has a universal cover $\tilde{X}$. The covering map $p : \tilde{X} \to X$, restricted to a fine enough open cover of $\tilde{X}$, allows us to transfer the Riemann surface structure of X to $\tilde{X}$, via the charts $\phi \circ p$, where ϕ is a chart on X. The deck transformations must be *conformal automorphisms* of $\tilde{X}$; that is, isomorphisms from X to X.

Moreover, if X is a simply connected Riemann surface and Γ is a subgroup of the *automorphism group* $\mathrm{Aut}(X)$, then we form the quotient space X/Γ, identifying $x, y \in X$ if and only if $y = \gamma(x)$ for some $\gamma \in \Gamma$. If this quotient map is a covering map, then we can similarly transfer the Riemann surface structure of X to X/Γ.

Theorem 2 (Quotients of surfaces)**.** Let X be a Riemann surface and Γ a subgroup of $\mathrm{Aut}(X)$. Then $X \to X/\Gamma$ is a covering map if

1. Γ acts freely; that is each nonidentity element of Γ acts on X without fixed points.
2. Γ acts properly discontinuously; that is each compact $K \subseteq X$ intersects only finitely many translates $\gamma(K)$ under the action of Γ.

In this case, X/Γ inherits the structure of a Riemann surface from X.

Corollary 1 (Uniformization of surfaces)**.** Every Riemann surface is isomorphic to a quotient X/Γ, where $X = \widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}$ and Γ is a subgroup of $\mathrm{Aut}(X)$.

Riemann surfaces covered by $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$ are called *spherical*, *euclidean*, or *hyperbolic* respectively.

Proper maps. For any Riemann surface X, let $\mathcal{O}(X)$ denote the ring of holomorphic functions $f : X \to \mathbb{C}$.

A holomorphic map $f : X \to Y$ between locally compact metric spaces is *proper* if $f^{-1}(K)$ is compact for each compact $K \subseteq Y$. Equivalently, f is proper if $f(z)$ escapes all compact compact sets as z escapes all compact sets. This often written as $\lim_{z\to\infty} f(z) = \infty$. Proper maps have several good qualities:

1. Proper functions map closed sets to closed sets.
2. A proper local homeomorphism is a covering map.

Moreover, proper analytic maps are the tamest analytic functions, and satisfy properties such as:

1. f is either constant or surjective.

2. If f' never vanishes, then f is a covering map.

3. If f' never vanishes and Y is simply connected, then f is an isomorphism.

Euclidean surfaces The classification of singularities tells us that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is either a polynomial or has an essential singularity at ∞. The limit $\lim_{z\to\infty} f(z)$ does not exist if f has an essential singularity at ∞, but $\lim_{z\to\infty} f(z) = \infty$ if f is a polynomial. Therefore the set of proper maps $f : \mathbb{C} \to \mathbb{C}$ is given by the set of polynomials $\mathbb{C}[z]$. We then obtain the following.

Theorem 3 (Automorphisms of $\mathbb{C}$)**.** The Automorphisms of $\mathbb{C}$ are given by the affine transformations $z \mapsto az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Proof. Clearly, an affine transformation is an automorphism. Conversely, suppose $f \in \mathrm{Aut}(\mathbb{C})$. Then f must be proper, so f is a polynomial. An automorphism must be one-to-one, so $\deg(f) = 1$. Hence $f(z) = az + b$, where $a \neq 0$. $\square$

The only affine transformations without fixed points are those of the form $f(z) = z + b$ with $b \in \mathbb{C}^*$.

Corollary 2 (Euclidean surfaces)**.** Every Riemann surface covered by $\mathbb{C}$ is isomorphic to $\mathbb{C}/\Gamma$ where Γ is a discrete additive subgroup of $(\mathbb{C}, +)$, generated by translations.

If Γ is rank 1, then $\mathbb{C}/\Gamma$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}$. If Γ is rank 2, then $\mathbb{C}/\Gamma$ is homeomorphic to a torus. A subgroup of higher rank will not be discrete, so these are the only possible surfaces.

Spherical Riemann surfaces. Suppose $f : \widehat{\mathbb{C}} \to \mathbb{C}$ is holomorphic and non-constant. Since $\widehat{\mathbb{C}}$ is compact, $f(\widehat{\mathbb{C}})$ must be a compact subset in $\mathbb{C}$. Moreover, $f(\widehat{\mathbb{C}})$ is open by the open mapping theorem. There are no compact and open subsets of $\mathbb{C}$. Thus, we have the following.

Proposition 1. Any holomorphic function $f : \widehat{\mathbb{C}} \to \mathbb{C}$ is constant. Hence, $\mathcal{O}(\widehat{\mathbb{C}}) \simeq \mathbb{C}$.

More interesting is the ring of *meromorphic functions* $\mathcal{M}(X)$, consisting of holomorphic functions $f : X \to \widehat{\mathbb{C}}$. A *rational function* is a function $f(z) = p(z)/q(z)$, where p and q are polynomials with no common roots. Rational functions are well defined on the Riemann sphere, taking the value ∞ at the zeros of q, and extending by continuity to ∞. We denote by $\mathbb{C}(z)$ the set of rational functions. It is easy to see that $\mathbb{C}(z) \subseteq \mathcal{M}(\widehat{\mathbb{C}})$, as any $f \in \mathbb{C}(z)$ is holomorphic as a map from $\mathbb{C} \setminus f^{-1}(\infty) \to \mathbb{C}$, and must extend to a holomorphic map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by the Riemann removable singularities theorem.

Proposition 2 (Meromorphic functions on the Riemann sphere)**.** Any holomorphic function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is given by a rational function, Hence, $\mathcal{M}(\widehat{\mathbb{C}}) \simeq \mathbb{C}(z)$.

Proof. Suppose $f \in \mathcal{M}(\widehat{\mathbb{C}})$. If f is constant, then $f \in \mathbb{C}(z)$. Otherwise, f has finitely many poles $p_1, \dots, p_n$ in $\mathbb{C}$. Let $g_i(z) = a_m/(z - p_i)^m + \cdots + a_1/(z - p_i)$ be the Laurent tail of f at any p_i. Then $h = f - \sum g_i$ is an entire function. Since f has a well defined value at ∞ and $\lim_{z\to\infty} h(z) = f(\infty)$, we deduce that h must be a polynomial. We conclude that $f = h + \sum g_i$ is a rational function. $\square$

A *Möbius transformation* is a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

In other words, a Möbius transformation is a degree 1 rational function.

Theorem 4 (Automorphisms of the Riemann sphere)**.** The Automorphism group of $\widehat{\mathbb{C}}$ is given by the group of Möbius transformations. Hence, $\mathrm{Aut}(\widehat{\mathbb{C}}) \simeq \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$.

Proof. Evidently, Möbius transformations are automorphisms of $\widehat{\mathbb{C}}$. Conversely, suppose f is an automorphism of $\widehat{\mathbb{C}}$. Then f is a degree 1 rational function, so it must be a Möbius transformation.

It follows that $\mathrm{Aut}(\widehat{\mathbb{C}}) \simeq \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$, as the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d}$$

has kernel $\{\pm I\}$. $\square$

There is a simple characterization of Möbius transformations.

Proposition 3 (Simple 3-transitivity)**.** Given any triples of points $z_1, z_2, z_3, w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ with $z_i \neq z_j$ and $w_i \neq w_j$ for all i, j, there is a unique Möbius transformation taking (z_1, z_2, z_3) to (w_1, w_2, w_3).

Proof. It suffices to show that there is a unique Möbius transformation mapping (z_1, z_2, z_3) to $(0, 1, \infty)$. If $z_1, z_3, z_3 \in \mathbb{C}$, then consider the function

$$f(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

If z_1, z_2, or z_3 are ∞, then respectively give f the form

$$f(z) = \frac{z_2 - z_3}{z - z_3}, \quad f(z) = \frac{z - z_1}{z - z_3}, \quad f(z) = \frac{z - z_1}{z_2 - z_1}.$$

Notice that $f(z_1) = 0$, $f(z_2) = 1$, and $f(z_3) = \infty$. Moreover, if g is another Möbius transformation with this property, then f/g is a rational function with no zeros and no poles, which must then be a constant. Since $f(z_2)/g(z_2) = 1$, we deduce that $f/g = 1$, so that $f = g$. $\square$

As a corollary, we deduce that the dimension of $\operatorname{Aut}(\widehat{\mathbb{C}}) \simeq \mathrm{PSL}_2(\mathbb{C})$ as a complex Lie group is three. We leave the following fact as an exercise (also see [Mil06a]).

Proposition 4 (Action of $\mathrm{PSL}_2(\mathbb{C})$ on lines and circles)**.** Let C denote the set of lines and circles in $\mathbb{C}$. Then the group of Möbius transformations acts transitively on C.

This is to say that for each $L \in C$ and Möbius transformation f, $f(L)$ is a line or circle. Moreover, given any $L' \in C$, there is a Möbius transformation taking L to L'. For the proof, it suffices to show that the generators of $\mathrm{PSL}_2(\mathbb{C})$ preserve C.

Proposition 5 (Spherical Riemann surfaces)**.** Every Riemann surface covered by $\widehat{\mathbb{C}}$ is isomorphic to $\widehat{\mathbb{C}}$.

Proof. Let

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

Then f has fixed points at the roots of $cz^2 + (d-a)z - b$. This has roots as long as $c \neq 0$ or $d \neq a$. If $c = 0$ and $a = d$, then $ad - bc = a^2$, so $a \neq 0$. Now $f(z) = z + b/a$, which has a fixed point at ∞. Therefore f must have a fixed point. Any deck transformation for a covering map $p : \widehat{\mathbb{C}} \to S$ must be fixed point free, so there can be no holomorphic covering map. $\square$

Hyperbolic Riemann surfaces. It follows that any Riemann surface not homeomorphic to a torus, a sphere, a cylinder, or the plane is hyperbolic.

Proposition 6 (The triply punctured sphere)**.** $\widehat{\mathbb{C}} \setminus \{a, b, c\}$ is hyperbolic for any three distinct points $a, b, c \in \widehat{\mathbb{C}}$.

Proof. The discussion above implies that any surface with non-abelian fundamental group is a hyperbolic Riemann surface. $\square$

A *Blaschke product* is a rational function of the form.

$$f(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \bar{a}_k z},$$

for $a \in \Delta$. Observe that if $|z| = 1$, then

$$\left| \frac{z - a_k}{1 - \bar{a}_k z} \right| = 1,$$

Moreover, each factor maps zero to $-a_k \in \Delta$. Therefore Blaschke products are proper self-maps of Δ. Alternatively, if $g : \Delta \to \Delta$ is proper, then let $a_1, a_2, \ldots, a_d$ be the zeros of g with multiplicity. If

$$\prod_{k=1}^{d} \frac{z - a_k}{1 - \bar{a}_k z},$$

then f/g has no zeros in Δ and $|f(z)/g(z)| \to 1$ as $|z| \to 1$. By the maximum principle, $f(z)/g(z) = e^{i\theta}$ is constant, so we conclude that g is a Blaschke product. Thus, we have the following.

Theorem 5 (Automorphisms of the disk)**.** The proper maps $f : \Delta \to \Delta$ are given by Blaschke products. Hence $\mathrm{Aut}(\Delta)$ comprises degree one Blaschke products.

Therefore $\mathrm{Aut}(\Delta)$ acts transitively on Δ. To see this, note that for any $a \in \Delta$, the automorphism

$$z \mapsto \frac{z - a}{1 - \bar{a} z}$$

takes a to zero. Moreover, this function is unique up to composition with a rotation.

One can show that the automorphisms of $\mathbb{H} \simeq \Delta$ are Möbius transformations with real coefficients. Thus, $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{PSL}_2(\mathbb{R})$.

2.2 Conformal Metrics

We now give natural metrics which turn Riemann surfaces into Riemannian manifolds and are compatible with the holomorphic framework. More details can be found in [Mil06a] and [McM23].

A Riemannian metric on a real smooth 2-manifold is specified by an expression of the form

$$ds^2 = g_{11} dx^2 + 2 g_{12} dx dy + g_{22} dy^2$$

where $[g_{jk}]$ is a positive definite matrix which depends smoothly on $z = x + iy$. A Riemannian metric is said to be *conformal* if $g_{11} = g_{22}$ and $g_{12} = 0$. Hence, $[g_{jk}]$ evaluated at any point z is a positive multiple of the identity matrix, so that the unit circle in the tangent space under the norm given by ds^2 is an ordinary circle. Moreover, angles as measured by ds^2 correspond with the ordinary measure of angles.

In other words, a conformal metric is one which can be written as $ds^2 = \rho(x+iy)^2(dx^2+dy^2)$, or more briefly as $ds = \rho(z)|dz|$. Given a holomorphic function $f : X \to Y$ between Riemann surfaces, where Y has conformal metric $ds = \rho(z)|dz|$, we can define the *pullback* f^*ds of ds under f as

$$f^*ds = \rho(f(z))|f'(z)||dz|,$$

which specifies a conformal metric on X. A conformal metric $ds = \rho(z)|dz|$ on X is said to be *invariant* under a conformal automorphism $f : X \to X$ if $f^*ds = ds$. Or equivalently, $\rho(w)|dw| = \rho(z)|dz|$, where $w = f(z)$. In this case, f is said to be an *isometry* with respect to ds.

Theorem 6 (Hyperbolic metric)**.** There exists one and, up to multiplication by a positive constant, only one Riemannian metric on $\mathbb{H}$ which is invariant under under every conformal automorphism of $\mathbb{H}$.

An immediate corollary is that an equivalent statement holds for Δ or any other surfaces isomorphic to $\mathbb{H}$.

Proof of Theorem 6. Suppose that a conformal metric $ds = \rho(z)|dz|$ on $\mathbb{H}$ is invariant under every linear automorphism $f(z) = az + b$ where $a > 0$. Notice that $f(i) = ai + b$. Moreover, we can rewrite $f^*ds = ds$ as $\rho(f(z)) = \rho(z)/|f'(z)|$. Thus, $\rho(ai + b) = \rho(i)/|a|$. After multiplying ds by a positive constant, we may assume that $\rho(i) = 1$. Hence, we obtain that $\rho(x + iy) = 1/y$ for all $x + iy \in \mathbb{H}$, so that $ds = |dz|/y$.

Moreover, this metric is invariant under every conformal automorphism of $\mathbb{H}$. If $g \in \mathrm{Aut}(\mathbb{H})$, take any $w_1 \in \mathbb{H}$ and let $w_2 = f(w_1)$. Then g can be expressed as the composition of a linear automorphism $g_1(w) = aw + b$ which maps w_1 to w_2 and an automorphism g_2 of $\mathbb{H}$ which fixes w_2. We know g_1 is an isometry by construction. It follows from the Schwarz lemma that $|g_2'(w_2)| = 1$, so g_2 is an isometry at w_2. Now, our metric is invariant at an arbitrary point of an arbitrarily chosen automorphism.

To complete the proof, we must show that a metric which is invariant under all automorphisms of $\mathbb{D}$ or $\mathbb{H}$ is necessarily conformal. Suppose that $ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$ is a Riemannian metric which is invariant under all automorphisms of $\mathbb{H}$. For any $z_0 \in \mathbb{H}$, take an automorphism f which fixes z_0 and with $f'(z_0) = i$. Then $f^*ds^2 = g_22dx^2 - 2g_{12}dxdy + g_{11}dy^2$. This shows that $g_{11} = g_{22}$ and $g_{12} = 0$, as desired. ☐

The metric $ds = |dz|/y$ is called the *Poincaré metric* on $\mathbb{H}$. The corresponding metric on Δ is given by

$$ds = \frac{2|dz|}{1 - |z|^2}, \quad z \in \Delta.$$

Corollary 3 (The Poincaré metric on hyperbolic surfaces)**.** For any hyperbolic surface X, there is a unique Riemannian metric ρ_X on X such that the projection $\mathbb{H} \to X$ is a local isometry with respect to the Poincaré metric on $\mathbb{H}$.

Proof. We may assume that $X = \mathbb{H}/\Gamma$, where Γ is a subgroup of $\mathrm{Aut}(\mathbb{H})$. Let $p : \mathbb{H} \to \mathbb{H}/\Gamma$ be the holomorphic covering map. Choose an open cover $\{U_\alpha\}$ of X by neighborhoods evenly covered by p. Let $\{V_\alpha\}$ be neighborhoods in $\mathbb{H}$ such that $p^{-1} : U_\alpha \to V_\alpha$ is a conformal isomorphism.

We construct a conformal metric ρ_X on X as follows. Let $\rho_X = (p^{-1})^* \rho_{\mathbb{H}}$, where $\rho_{\mathbb{H}}$ denotes the Poincaré metric on $\mathbb{H}$. To see that this is well defined, suppose U_α and U_β are overlapping neighborhoods. Let $g_1 = p^{-1} : U_\alpha \to V_\alpha$ and $g_2 = p^{-1} : U_\beta \to V_\beta$. Then on $U_\alpha \cap U_\beta$, $g_2 = \gamma \cap g_1$, for some $\gamma \in \Gamma$. Since γ is an isometry of $\mathbb{H}$, we have

$$g_2^* \rho_{\mathbb{H}} = (\gamma \circ g_1)^* \rho_{\mathbb{H}} = g_1^*(\gamma^* \rho_{\mathbb{H}}) = g_1^* \rho_{\mathbb{H}}.$$

Thus, ρ_X is well defined, and p is a local isometry by construction. Moreover, if ρ is any other Riemannian metric on X with $p^* \rho = \rho_{\mathbb{H}}$, then locally $(p^{-1})^* \rho_{\mathbb{H}} = \rho$, so ρ_X is unique. ☐

Also notice that since $\rho_{\mathbb{H}}$ is a conformal metric and p is conformal, ρ_X is a conformal metric. We call the metric ρ_X the *Poincaré metric* on X.

Since there is an isometry carrying any point of $\mathbb{H}$ to any other point, it follows that the Poincaré metric has constant Gaussian curvature. In fact, we may compute that the Poincaré metric has constant curvature $K \equiv -1$ [Mil06a]. It then follows that every hyperbolic surface with its Poincaré metric has constant curvature -1.

A conformal metric $\rho = \rho(z)|dz|$ on a Riemann surface X allows us to measure the length of paths and the area of regions. For any C^1 path $\gamma : [a, b] \to X$ or region U, we have

$$\begin{aligned} \mathrm{length}(\gamma) &= \int_\gamma \rho = \int_a^b \rho(\gamma(t))|\gamma'(t)|\, dt \\ \mathrm{area}(U) &= \int_\gamma \rho^2 = \int_U \rho(x + iy)^2\, dxdy. \end{aligned}$$

We define the distance between two points x and y as

$$d_\rho(x, y) := \inf_\gamma \mathrm{length}(\gamma)$$

where γ ranges over all paths connecting x and y. A *geodesic* is a curve which contains the shortest path between any two points it contains.

We say a Riemannian manifold M is *complete* with respect to its metric ds if the following conditions hold:

1. Every Cauchy sequence with respect to the metric d_M converges.

2. Any two points of M are joined by at least one minimal geodesic.

Theorem 7 (Completeness of the Poincaré metric)**.** Every hyperbolic surface S is complete with respect to its Poincaré metric.

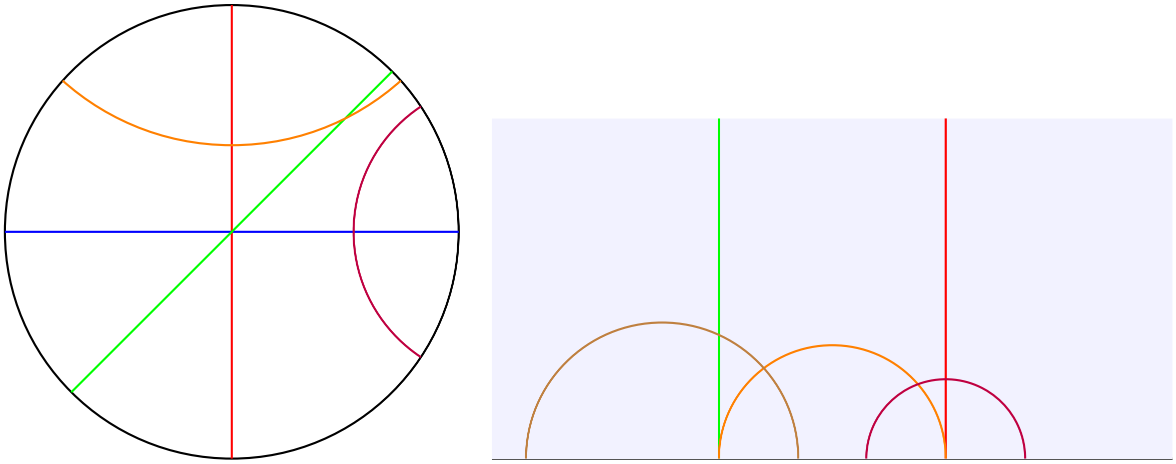

Figure 2.2: Hyperbolic geodesics in Δ and $\mathbb{H}$

Proof. We first consider the case $S = \Delta$. Given two points in Δ, We may choose a conformal automorphism of Δ which takes the first point to the origin and the second point to a point r on the positive real axis. Since each conformal automorphism of Δ is an isometry, we may reduce to this case.

Let γ be a piecewise smooth path between 0 and r within Δ. Then

$$\int_\gamma ds = \int_\gamma \frac{2|dz|}{1-|z|^2} \geqslant \int_\gamma \frac{2|dx|}{1-x^2} \geqslant \int_0^r \frac{2dx}{1-x^2} = \log\frac{1+r}{1-r}.$$

Equality holds if and only if γ is the straight line segment $[0, r]$. Hence, the Poincaré distance from 0 to z for all $z \in \Delta$ is given by $d_\Delta(0, z) = \log\frac{1+|z|}{1-|z|}$, and the unique geodesic minimal geodesic from 0 to z is the straight line segment connecting them. It then follows that each geodesic has infinite length, so all Cauchy sequences converge. Since Δ is complete with respect to ds, we find that every hyperbolic surface is complete with respect to its Poincaré metric. $\square$

Corollary 4 (Hyperbolic geodesics.)**.** The geodesics of Δ with its Poincaré metric are diameters of Δ and circular arcs which meet S^1 orthogonally. Therefore, geodesics of $\mathbb{H}$ are vertical lines and circles centered on $\mathbb{R}$.

Proof. From the proof of Theorem 7, it is clear that diameters of Δ are geodesics. If ℓ is any geodesic in Δ, then choose any point $x_0 \in \ell$ and an automorphism γ of Δ taking x_0 to 0. Again from our previous discussion, $\gamma(\ell)$ is a diameter of Δ. By Proposition 4, $\ell = \gamma^{-1}(\gamma(\ell))$ is a line or a circle. Since γ^{-1} fixes S^1 and is conformal, ℓ must meet S^1 orthogonally.

The isometry $g : \Delta \to \mathbb{H}$ is a Möbius transformation, and thus takes geodesics in Δ to lines and circles which meet $\mathbb{R}$ orthogonally. Thus, the geodesics of $\mathbb{H}$ are vertical lines and circles centered on $\mathbb{R}$. $\square$

Corollary 5 (Metrics of constant curvature)**.** Every Riemann surface admits a complete conformal metric of constant curvature which is either positive, negative, or zero according to whether the surface is spherical, hyperbolic, or Euclidean.

Proof. We have already shown this in the hyperbolic case. Let $X = \mathbb{C}$. Then we have the usual conformal metric $|dz|$, with constant curvature 0. We often refer to $|dz|$ as the *flat metric* on $\mathbb{C}$. Evidently, $\mathbb{C}$ is complete with respect to $|dz|$. Recall that any deck transformation of $\mathbb{C}$ is of the form $f(z) = z+b$, where $b \in \mathbb{C}$. Then $f^*|dz| = |d(z+b)| = |dz|$. Hence each deck transformation is a flat isometry. Therefore, the flat metric descends to any Euclidean surface $\mathbb{C}/\Gamma$.

There is only one spherical Riemann surface up to isomorphism. Using stereograph projection, we can identity $\widehat{\mathbb{C}}$ with the unit sphere in $\mathbb{R}^3$. The metric inherited from $\mathbb{R}^3$ then pulls back to the *spherical metric*

$$ds = \frac{2|dz|}{1+|z|^2}, \quad z \in \widehat{\mathbb{C}}$$

with constant curvature $+1$. □

By construction, the flat metric on $\mathbb{C}$ is unique only up to multiplication by a positive scalar. The spherical metric is quite non-unique. The conformal isometry group of the sphere with respect to this metric is isomorphic to SO(3), which is much smaller than $\mathrm{PSL}_2(\mathbb{C})$. For each $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$ which is not an isometry of $\widehat{\mathbb{C}}$, we obtain a new complete conformal metric of constant curvature $+1$ by pulling back the spherical metric under f.

2.3 Normal Families

We continue our discussion of Riemann surfaces and begin to transition to the study of complex dynamics by introducing *normal families*. In this chapter, they will allow us to prove the *Riemann mapping theorem*, which will be of great importance in chapters to come. Later on, normal families provide us the basic framework for descriptions of the possible behavior of holomorphic maps under iteration.

Theorem 8 (Riemann mapping theorem)**.** Suppose $U \subsetneq \mathbb{C}$ is a simply connected open subset, and take any $z_0 \in U$. Then there is an isomorphism $\phi : U \to \Delta$ with $\phi(z_0) = 0$ and $\phi'(z_0) > 0 \in \mathbb{R}$. The map ϕ satisfying these properties is unique.

Corollary 6. If U is a simply connected open subset of $\widehat{\mathbb{C}}$ with $|\widehat{\mathbb{C}} \setminus U| > 1$, then the Riemann mapping theorem holds for U.

The Riemann mapping theorem is a special case of the uniformization theorem, and is more easily proven than the general case. We call the map ϕ the *Riemann map* form (U, p) to $(\Delta, 0)$.

Normal families. To prove the Riemann mapping theorem, we will introduce the notion of a *normal family*, which will be useful throughout this exposition. We consult [Mil06a].

Definition 1 (Topology of local uniform convergence.)**.** Let X be a locally compact Hausdorff topological space and let Y be a metric space. Let $\mathcal{F}$ be a family of continuous mappings $f : X \to Y$. For any compact $K \subseteq X$, $\epsilon > 0$ and $f \in \mathcal{F}$, consider the subset

$$N_{K,\epsilon}(f) = \{g \in \mathcal{F} \mid \max_{x \in K} \operatorname{dist}(f(x), g(x)) < \epsilon\}$$

The topology generated by the sets $N_{K,\epsilon}(f)$ ranging over all compact subsets $K \subseteq X$, $\epsilon > 0$, and $f \in \mathcal{F}$ is called the *topology of local uniform convergence.*

A sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ converges to $f \in \mathcal{F}$ in the topology of local uniform convergence if and only if for each compact $K \subseteq X$, the f_n converge uniformly to f on K. In this case, we say that the f_n *converge locally uniformly* to f.

Example 3. Let U and V be open subsets of $\mathbb{C}$. We denote by $\operatorname{Hol}(U, V)$ the set of holomorphic mappings $f : U \to V$, with the topology of local uniform convergence.

By Corollary 5, any Riemann surface may be given a complete conformal metric. Therefore if X and Y are Riemann surfaces, we can similarly define the topological space $\operatorname{Hol}(X, Y)$.

Definition 2 (Normal families version 1)**.** Let X be a locally compact Hausdorff space, and Y a compact metric space. Suppose $\mathcal{F}$ is a family of continuous functions $f : X \to Y$ and $\mathcal{F}' \subseteq \mathcal{F}$. We say that $\mathcal{F}'$ is a *normal family* in $\mathcal{F}$ if $\overline{\mathcal{F}'}$ is compact in the topology of local uniform convergence on $\mathcal{F}$.

Practically, it suffices to show that any sequence (f_n) in $\mathcal{F}'$ has a subsequence which converges locally uniformly to an element of $\mathcal{F}$.

Example 4. Let $f_n : \Delta \to \overline{\Delta}$ be given by $f_n(z) = z^{2n}$. Then $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a normal family in $\operatorname{Hol}(\Delta, \overline{\Delta})$. Indeed, any sequence in $\mathcal{F}$ has a subsequence which is contained in the sequence (f_n), so it suffices to show that this sequence has a limit in $\operatorname{Hol}(\Delta, \overline{\Delta})$. Any compact $K \subset \Delta$ is contained in $\overline{\Delta}_r$, for some $r < 1$. Hence, $f^{2n}(K) \subseteq \overline{\Delta}_{r^{2n}}$. For any $\epsilon > 0$, we can find n large enough that $r^{2n} < \epsilon$, so (f_n) converges uniformly to the zero function on K. We conclude that (f_n) coverges to the zero function locally uniformly, so the result follows.

Example 5. Let $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be given by $f_n(z) = z^{2n}$. Then $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is *not* a normal family in $\mathrm{Hol}(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$. For any $z \in \Delta$, $f_n(z) \to 0$, but for any $z \in \widehat{\mathbb{C}} \setminus \overline{\Delta}$, $f_n(z) \to \infty$. There is no continuous function on $\widehat{\mathbb{C}}$ which is infinite on $\widehat{\mathbb{C}} \setminus \overline{\Delta}$ and zero on Δ.

It is also possible to define normal families when the target is not compact. In a topological space Y, a sequence of points (p_n) *diverges* from Y if for each compact $K \subseteq Y$, $p_n \notin K$ for sufficiently large n. For example, the sequence $(i/n)_{n\in\mathbb{N}}$ diverges from $\mathbb{H}$, even though it converges to zero in $\mathbb{C}$.

We say a sequence of maps $f_n : X \to Y$ *diverges locally uniformly* from Y if for each compact $K \subseteq X$ and $K' \subseteq Y$, we have $f_n(K) \cap K' = \emptyset$ for all n sufficiently large. If Y is compact, this cannot happen.

Definition 3 (Normal family version 2)**.** Let X be a locally compact Hausdorff space and Y a metric space. Let $\mathcal{F}$ be a family of continuous mappings $f : X \to Y$. A family $\mathcal{F}' \subseteq \mathcal{F}$ is normal in $\mathcal{F}$ if every sequence in $\mathcal{F}'$ either contains a subsequence which converges locally uniformly, or a subsequence which diverges locally uniformly from Y.

If Y is compact, the this is the same as our previous definition, since a sequence of maps can never diverge locally uniformly from Y.

Example 6. Consider the family $\mathcal{F} = \{z \mapsto z + n \mid n \in \mathbb{N}\}$ inside $\mathrm{Hol}(\mathbb{C}, \mathbb{C})$. We claim $\mathcal{F}$ is a normal family.

Let $f_n(z) = z + n$. Any sequence of maps in $\mathcal{F}$ contains a subsequence $(f_{n(j)})_j$, where $n(j)$ is increasing. Thus, it suffices to show that the sequence (f_n) converges or diverges locally uniformly from $\mathbb{C}$.

Indeed, for any pair of compact subsets $K, K' \subset \mathbb{C}$, we have $K \subseteq S_r = \{x + iy \mid -r \leqslant x, y \leqslant r\}$ and $K' \subseteq S_{r'}$ (similarly defined) for some $r, r' > 0$. We can find n large enough such that $-r + n > r'$. Hence, for all n sufficiently large, $f_n(K) \cap K' = \emptyset$. Therefore (f_n) diverges locally uniformly from $\mathbb{C}$.

Although the normality of a family of $\mathcal{F}$ on a domain X appears to be a global property of X, it turns out that in some cases, being a normal family is a local property.

Proposition 7. Suppose $\mathcal{F}$ is a family of continuous functions between a locally compact, connected Hausdorff space X and a metric space Y. If each $x \in X$ has a neighborhood U such that $\{f|_U \mid f \in \mathcal{F}\}$ is a normal family on U, then $\mathcal{F}$ is a normal family on Y.

Proof. Suppose that each $x \in X$ has a neighborhood U such that $\{f|_U \mid f \in \mathcal{F}\}$ is a normal family in $\mathrm{Hol}(U, Y)$.

Take any sequence (f_n) in $\mathcal{F}$ and any compact $K \subset K$. For each $z \in S$, let U_z be a neighborhood of z on which $\mathcal{F}$ is a normal family. Since X is locally compact, we can find $V_z \supset U_z$ such that $z \in V_z$ and $\overline{V}_z \subset U_z$.

By assumption, $(f_n|_{U_z})$ has a subsequence which converges locally uniformly or a subsequence which diverges locally uniformly. Let U be the union of all U_z on which (f_{n_j}) has a subsequence which converges locally uniformly, and let V be the union of all the remaining U_z. Then $U \sqcup V$ is a separation of X. Since X is connected, either U is empty or V is empty.

Since K is compact, and $\{V_z\}$ is an open cover, we can find a finite subcover. For simplicity, write $K \subset \overline{V}_1 \cup \cdots \cup \overline{V}_k \subset U_1 \cup \cdots \cup U_k$.

Suppose U is non-empty, so $(f_n|_{U_j})$ has a subsequence which converges locally uniformly. Let Q_1 be the set of indices for such a sequence on U_1. Inductively, let Q_{j+1} be the set of indices of a locally uniformly convergent or divergent subsequence of $\{f_n|_{U_{j+1}} \mid n \in Q_j\}$. Hence, $Q_{j+1} \subseteq Q_j$ for each $1 \leqslant j \leqslant k$.

If V is non-empty, then follow a similar procedure, but for indices of a subsequence which diverges locally uniformly.

Write $Q_k = \{n_1, n_2, \dots\}$ where $n_{j+1} > n_j$. For each U_i, the sequence $(f_{n_j}|_{U_i})$ either converges locally uniformly in $\mathrm{Hol}(U_i, Y)$ or diverges locally uniformly from Y.

If U is non-empty, then let f_i be the limit of (f_{n_j}) on U_i. If $U_i \cap U_j \neq \emptyset$, then $f_i = f_j$ on $U_i \cap U_j$: if $f_i(x) \neq f_j(x)$ for some $x \in U_i \cap U_j$, then f_{n_j} does not converge uniformly on the compact subset $\{x\}$ of U_i. Therefore we let $f : X \to Y$ be given by

$$f(z) = \Big\{ f_i(z), \quad z \in U_i,$$

which is well defined.

Notice that $\overline{V}_i \cap K$ is compact and contained in U_i. Given any $\epsilon > 0$, there is N_i such that $d(f_{n_j}(x), f(x)) < \epsilon$ for all $n_j > N_i$ and $x \in \overline{V}_i$. Let $N = \max\{N_i\}$, which exists since there are finitely many $\overline{V}_i$. Then since the $\overline{V}_i$ cover K, we have $d(f_{n_j}(x), f(x)) < \epsilon$ for all $n_j > N$ and $x \in K$. Therefore (f_{n_j}) converges locally uniformly to f, so we conclude that $\mathcal{F}$ is a normal family.

Now suppose V is non-empty. Let K' be a compact subset of Y. Then for each i, there is M_i such that $f_{n_j}(\overline{V}_i) \cap K' = \emptyset$ for $n_j > M_i$. Let $M = \max\{M_i\}$. Then $f_{n_j}(\overline{V}_i) \cap K' = \emptyset$ for each i when $n_j > M$. Since $K \subset \bigcup \overline{V}_i$, it follows that $f_{n_j}(K) \cap K' = \emptyset$ when $n_j > M$. Therefore (f_{n_j}) diverges locally uniformly from Y, so $\mathcal{F}$ is a normal family in this case as well. ☐

Another useful criterion for determining when a family is normal is the *Arzela-Ascoli theorem.*

Theorem 9 (Arzela-Ascoli)**.** Let $\mathcal{F}$ be a family of continuous functions on $\Omega \subseteq \mathbb{C}$ with values in a metric space S. Then $\overline{\mathcal{F}}$ is compact in $C^0(\Omega, S)$ with the topology of local uniform convergence if and only if

(i) $\mathcal{F}$ is equicontinuous on every compact set $K \subset \Omega$

(ii) for any $z \in \Omega$ the values $f(z)$, $f \in \mathcal{F}$, lie in a compact subset of S.

Proof. Suppose that $\mathcal{F}$ is normal on $\Omega \subset S$. We will first prove (i). Suppose that $\mathcal{F}$ is not equicontinuous on some compact $K \subset S$. Then there exists $\epsilon > 0$, points $z_n, z'_n \in K$, and $f_n \in \mathcal{F}$ such that $|z_n - z'_n| \to 0$, but $d(f_n(z_n), f_n(z'_n)) \geqslant \epsilon$ for all n.

Since K is compact and $|z_n - z'_n| \to 0$, there exist subsequences of z_n and z'_n which converge to some common limit $z''_n \in K$. Moreover, since $\mathcal{F}$ is normal, there exists a subsequence of f_n which converges uniformly to a limit f on K. We may use the same subscript n_k for each of these subsequences.

Now there exists some k such that the distances from $f_{n_k}(z_{n_k})$ to $f(z_{n_k})$, $f(z_{n_k})$ to $f(z'_{n_k})$, and from $f(z'_{n_k})$ to $f_{n_k}(z'_{n_k})$ are less than $\epsilon/3$. Now by the triangle inequality,

$$d(f_{n_k}(z_{n_k}), f_{n_k}(z'_{n_k})) < \epsilon$$

contradicting our assumption that $d(f_n(z_n), f_n(z'_n)) \geqslant \epsilon$ for all n. Thus we have proven (i).

Still assuming that $\mathcal{F}$ is normal on Ω, we wish to prove (ii). It will suffice to show that $f(\Omega)$ has compact closure. Let $\{w_n\}$ be any sequence of points in $\overline{f(\Omega)}$. For each n, we can find $f_n \in \mathcal{F}$ such that $d(f_n(z), w_n) < 1/n$ for some $z \in \Omega$. By the normality of $\mathcal{F}$, there is a convergent subsequence $\{f_{n_k}(z)\}$, and $\{w_{n_k}\}$ converges to the same limit. Thus $\overline{f(\Omega)}$ is sequentially compact and therefore compact.

Now assume conditions (i) and (ii). We want to prove that $\mathcal{F}$ is normal. First notice that there exists a dense sequence ζ_k in Ω. (This can be found by taking all points with rational coordinates.) Given a sequence $\{f_n\}$ in $\mathcal{F}$, we will construct a subsequence which converges at all ζ_k.

For any point in particular, we know by (ii) that we can find a subsequence converging at this point. Therefore we have an array

$$\begin{gathered}
n_{11} < n_{12} < \dots \\
n_{21} < n_{22} < \dots \\
\vdots \\
n_{k1} < n_{k2} < \dots \\
\vdots
\end{gathered}$$

with each row contained in the preceding row and $\lim_{j\to\infty} f_{n_{kj}}(\zeta_k)$ existing for all k. Now the diagonal subsequence $\{n_{jj}\}$ is strictly increasing, and is such that $f_{n_{jj}}$ converges at all ζ_k. For convenience, replace n_{jj} with n_j.

Take any compact $K \subset \Omega$. We wish to show that f_j converges uniformly on K. Because $\mathcal{F}$ is equicontinuous on K, given $\epsilon > 0$, we may choose δ such that if $|z - z'| < \delta$, then $d(f(z), f(z')) < \epsilon/3$ for all $f \in \mathcal{F}$. Since K is compact, we may cover K with finitely many $\delta/2$-neighborhoods. Select a ζ_k from each of these neighborhoods. Now there exists i_0 such that for all $i, j > i_0$, we have $d(f_{n_i}(\zeta_k), f_{n_j}(\zeta_k)) < \epsilon/3$ for all these ζ_k.

Furthermore, for all $z \in K$, there is some ζ_k within distance δ of z. Therefore the distance from $f_{n_i}(z)$ to $f_{n_i}(\zeta_k)$ and from $f_{n_j}(z)$ to $f_{n_j}(\zeta_k)$ are less than $\epsilon/3$. Combining the three inequalities and using the triangle inequality, we find that $d(f_{n_i}(z), f_{n_j}(z)) < \epsilon$.

Thus we we have shown that for all $\epsilon > 0$, $d(f_{n_i}(z), f_{n_j}(z)) < \epsilon$ for all $z \in K$ and i, j sufficiently large. Now the sequence $f_{n_j}(z)$ is Cauchy, so since $f_{n_i}(z)$ take values in a compact (and therefore complete) metric space, f_{n_i} converges uniformly on K, and we're done. □

A consequence of note is Montel's theorem, which will enable us to prove the Riemann mapping theorem.

Theorem 10 (Montel's theorem)**.** Suppose that $\mathcal{F}$ is a family of holomorphic functions on a domain $U \subset \mathbb{C}$ which is uniformly bounded; i.e., there exists $M > 0$ such that $|f(z)| \leqslant M$ for all $f \in \mathcal{F}$ and $z \in U$. Then $\mathcal{F}$ is a normal family.

Proof. Let D be a disk of radius r such that $\overline{D}$ is contained in U. Let γ be the boundary of D, parametrized counterclockwise. Then for any $z, w \in D$ within distance $r/2$ of the center of D, the Cauchy integral formula tells us that for all $f \in \mathcal{F}$.

$$\begin{aligned}
|f(z) - f(w)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} d\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{f(\zeta)(z-w)}{(\zeta - w)(\zeta - z)} d\zeta\right| \\
&\leqslant \frac{1}{2\pi} \cdot \frac{2\pi r M|z-w|}{(r/2)^2} \\
&= \frac{M|z-w|}{r}.
\end{aligned}$$

Now take any compact $K \subset U$. Choose $r < d(K, \partial U)$. Fix $\epsilon > 0$ and choose

$$\delta < \min\left\{\frac{r}{4}, \frac{\epsilon r}{4M}\right\}.$$

Now for all $z, w \in K$ with $|z - w| < \delta$ and $f \in \mathcal{F}$, the above bound gives us

$$|f(z) - f(w)| \leqslant \frac{M|z-w|}{r} < \frac{M\delta}{r} < \epsilon.$$

Therefore $\mathcal{F}$ is equicontinuous on all compact subsets of U, and by the Arzela-Ascoli theorem, $\mathcal{F}$ is normal. □

We can generalize Montel's theorem even further.

Theorem 11 (Montel version 2)**.** Let S be a Riemann surface and let $\mathcal{F}$ be a collection of holomorphic maps $f : S \to \widehat{\mathbb{C}}$ which omit three different values. That is, assume that there are distinct points $a, b, c \in \widehat{\mathbb{C}}$ such that $f(S) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family; that is, the closure $\overline{\mathcal{F}} \subset \mathrm{Hol}(S, \widehat{\mathbb{C}})$ is a compact set.

Proof. Notice that the proof of Montel's theorem version 1 can easily by adapted to the setting where $U = \widehat{\mathbb{C}}$. If $\mathcal{F}$ is a collection of holomorphic mappings $f : S \to \widehat{\mathbb{C}}$ which omit three different values, then $f(S)$ is contained in the hyperbolic Riemann surface $T = \widehat{\mathbb{C}} \setminus \{a, b, c\}$.

Let X be the universal cover of S. We can lift each $f \in \mathcal{F}$ to a map $\tilde{f} : X \to \Delta$. By uniformization we can take $X = \widehat{\mathbb{C}}$, $\mathbb{C}$ or Δ, so $\{\tilde{f}\}$ is a normal family by Montel version 1.

Let $p_1 : X \to S$ and $p_2 : \Delta \to T$ be the covering maps. Let $(p_2 \circ \tilde{f}_n)$ be any sequence. If $(\tilde{f}_n)$ has a locally uniformly converging subsequence, then so does $(p_2 \circ \tilde{f}_n)$. If $(\tilde{f}_n)$ has a subsequence which locally uniformly diverges from Δ, then since p_2 is proper, $(p_2 \circ \tilde{f}_n)$ has a subsequence which converges locally uniformly to a, b, or c. Therefore $\{p_2 \circ \tilde{f}\}$ is a normal family of maps from X to $\widehat{\mathbb{C}}$.

Recall that to show $\mathcal{F}$ is a normal family, it suffices to show that each $z \in S$ has a neighborhood on which the restriction of $\mathcal{F}$ is normal. This is Proposition 1.19. Any $z_0 \in S$ has a neighborhood U which is evenly covered by p_1. Let p_1^{-1} be the local inverse of p_1 on U. Notice that $f|_U = p_2 \circ \tilde{f} \circ p_1^{-1}$ for each $f \in \mathcal{F}$. Take any sequence (f_n) in f. We know that $(p_2 \circ \tilde{f}_n)$ has a subsequence which converges locally uniformly to a map $f : X \to \widehat{\mathbb{C}}$. Therefore $(f_n) = (p_2 \circ \tilde{f}_n \circ p_1^{-1})$ converges locally uniformly to $f \circ p_1^{-1}$. Thus $\mathcal{F}$ is a normal family in $\mathrm{Hol}(U, \widehat{\mathbb{C}})$, so we conclude that $\mathcal{F}$ is a normal family in $\mathrm{Hol}(S, \widehat{\mathbb{C}})$. $\square$

2.4 Proof of the Riemann Mapping Theorem

Armed with the theory of normal families, we can now prove the Riemann mapping theorem. First, recall that a mapping $f : U \to \mathbb{C}$ is *univalent* if it is holomorphic and injective. Then its inverse is holomorphic, and f is a conformal isomorphism between U and $f(U)$.

Lemma 1. Let $f_n : U \to \mathbb{C}$ be a sequence of univalent maps, converging locally uniformly to $f : U \to \mathbb{C}$. If f is non-constant, then f is univalent.

Proof. A uniform limit of holomorphic functions is always holomorphic, so it suffices to show that f is injective.

Take points $a, b \in U$ with $f(a) = f(b)$, but $a \neq b$. Suppose that f is non-constant. Let $g(z) = f(z) - f(b)$, so that g has a zero at $z = a$. Then $f_n(z) - f_n(b)$ converges locally uniformly to g. Let $g_n(z) = f_n(z) - f_n(b)$. Let V be any neighborhood small enough that $z = a$ is the only zero of g in $\overline{V}$ and ∂V is compact. Then there is $\epsilon > 0$ such that $|g(z)| > \epsilon$ for all $z \in \partial V$. Since g_n converges uniformly to g on ∂V, we have $|g_n(z) - g(z)| < \epsilon$ on ∂V for all n sufficiently large. Now $|g| > |g_n - g|$ on ∂V, so Rouche's theorem says that g and g_n have the same number of zeros in V. Now for each n sufficiently large, there is $a_n \in V$ with $f_n(a_n) = f_n(b)$. This violates the univalence of the f_n. $\square$

Lemma 2. The space $\mathrm{Hol}(U, \overline{\Delta})$ is compact in the topology of local uniform convergence.

Proof. It suffices to show that $\mathrm{Hol}(U, \overline{\Delta})$ is a normal family. Since $|f(z)| \leqslant 1$ for all $z \in U$ and $f \in \mathrm{Hol}(U, \overline{\Delta})$, Montel's theorem immediately implies the result. $\square$

Lemma 3. If $U \subset \mathbb{C}$ is simply connected at $0 \notin U$, then there exists a univalent map $f : U \to \mathbb{C}$ such that $f(z)^2 = z$.

Proof. Take any $z_0 \in U$. Since U is simply connected and $0 \notin U$, the function $g(z) = \int_{z_0}^{z} \frac{dw}{w}$ is well defined and holomorphic on U. Moreover, $g'(z) = 1/z$. We claim that $e^{g(z)} = z/z_0$ for all $z \in U$. Indeed observe that

$$\frac{d}{dz}\left(\frac{e^{g(z)}}{z/z_0}\right) = \frac{d}{dz}\left(\frac{z_0 e^{g(z)}}{z}\right) = \frac{z_0 e^{g(z)} - z_0 e^{g(z)}}{z^2} = 0.$$

Hence, $e^{g(z)} = cz/z_0$ for some constant c. Observe that $g(z_0) = 0$, so letting $z = z_0$, we find that $c = 1$ as desired.

Let $\sqrt{z_0}$ be one of the two square roots of z_0. Let $f(z) = \sqrt{z_0} \cdot e^{g(z)/2}$. Notice that f is holomorphic on U. Moreover,

$$f(z)^2 = z_0 \cdot e^{g(z)} = z_0 \cdot z/z_0 = z,$$

Since $f(z)^2$ is injective, it follows that f is injective. Hence, f is univalent, and is the desired function. $\square$

We are ready to prove the Riemann mapping theorem.

Proof of the Riemann mapping theorem. We begin with the proof of existence. Let U be a simply connected open subset of $\mathbb{C}$ with $U \neq \mathbb{C}$. Take any $p \in U$. We may assume that $0 \notin U$, as we can always precompose with a translation. Let $\mathcal{F}$ denote the space of univalent maps $f : (U, p) \to (\Delta, 0)$. We also impose the condition $f'(p) > 0$.

We will first verify that $\mathcal{F}$ is not empty. This is clear if U is bounded. If U is not bounded, then since $0 \notin U$, there exist a univalent function $f : U \to \mathbb{C}$ with $f(z)^2 = z$ by Lemma 1.22. Now $V = f(U)$ is disjoint from $-V$ (if $v, -v \in f(U)$, then say $v = f(u)$ and $-v = f(w)$. Now $f(u)^2 = f(w)^2$, implying that $u = w$. Now $v = -v$, so $v = 0$. But since $f(u)^2 = u = 0$, we have a contradiction since $0 \notin U$). Therefore V is contained in a half-plane. By composing with a rotation $R(z) = e^{i\theta} z$, we can ensure that $V \subseteq \mathbb{H}$. Now the Möbius transformation $T(z) = (1 + iz)/(i + z)$ takes V into Δ. Now $T \circ R \circ f \in \mathcal{F}$.

Since U is open, it contains a ball $\Delta_r(p)$ with $r > 0$. For any $f \in \mathcal{F}$, $g(z) = f(rz + p)$ is a holomorphic map from Δ to Δ which fixes 0. The Schwarz lemma thus implies that $|g'(0)| \leqslant 1$. But $|g'(0)| = r|f'(p)|$, so we see that $|f'(p)| \leqslant 1/r$ for all $f \in \mathcal{F}$. Therefore $M = \sup_{f \in \mathcal{F}} |f'(p)|$ is finite. Since $\mathcal{F} \subset \mathrm{Hol}(U, \overline{\Delta})$, the equality $|f'(p)| = M$ is achieved for some $f \in \overline{\mathcal{F}} \subset \mathrm{Hol}(U, \overline{\Delta})$ by the compactness of $\mathrm{Hol}(U, \overline{\Delta})$. Moreover, there is a sequence (f_n) of functions in $\mathcal{F}$ which converge locally uniformly to f. Since the f_n are all univalent, f is univalent by Lemma 1.20.

To complete the proof, we will show that $f(U) = \Delta$. If not, then there is $a \in \Delta \backslash f(U)$. Choose an automorphism $A : \Delta \to \Delta$ such that $A(a) = 0$ (for example $A(z) = (z - a)/(1 - \bar{a}z)$). Let $s : \Delta \to \Delta$ be given by $s(z) = z^2$, and let $s^{-1} : A(f(U)) \to \Delta$ be a univalent map with $s(s^{-1}(z)) = z$ (such a map is guaranteed by Lemma 1.22 because $A(f(U))$ is simply connected and $0 \notin A(f(u))$). Let B be an automorphism of Δ such that

$$g = B \circ s^{-1} \circ A \circ f : U \to \Delta$$

satisfies $g(p) = 0$ and $g'(p) > 0$. For example, let $b = s^{-1}(A(f(p)))$, and let

$$\tilde{B}(z) = \frac{z - b}{1 - \bar{b}z}.$$

Choose θ such that $e^{i\theta}\tilde{B}'(p) > 0$, and let $B(z) = e^{i\theta}\tilde{B}(z)$. Then B is the desired automorphism. Notice that $g \in \mathcal{F}$. Let $h = A^{-1} \circ s \circ B^{-1} : \Delta \to \Delta$, and notice that $f = h \circ g$. By construction, h is a surjective degree-2 mapping from Δ to itself, so the Schwarz lemma implies that $|h'(0)| < 1$. But $f'(p) = h'(0)g'(p)$, so $|f'(p)| < |g'(p)|$. This is a contradiction, since we assumed $|f'(p)|$ to be maximal among functions in $\mathcal{F}$. Therefore, f was surjective, and we're done.

To establish uniqueness, suppose that f and g are two isomorphisms from U to Δ with $f(p) = g(p) = 0$ and $f'(p), g'(p) > 0$. Then $g \circ f^{-1} : \Delta \to \Delta$ is an automorphisms taking 0 to 0. The Schwarz lemma implies that $g \circ f^{-1}$ is a rotation $z \mapsto e^{it}z$ for some $t \in \mathbb{R}$. But since $e^{it} = (g \circ f^{-1})'(0) = g'(p)/f'(p) > 0$, it must be that $g \circ f$ is the identity. Thus, $f = g$. $\square$

Extension to the boundary. We may wish to know when a conformal isomorphism $\phi : \Delta \to U$ extends as a homeomorphism (or at least as a continuous map) over the boundary of Δ. There is a simple answer. Recall that a Hausdorff topological space X is *locally connected* if every $x \in X$ has arbitrarily small connected open neighborhoods.

Theorem 12 (Carathéodory)**.** A conformal mapping $f : \Delta \to U$ extends as a continuous map $f : \overline{\Delta} \to \overline{U}$ if and only if ∂U is locally connected.

The proof of Theorem 12 amounts to showing that for any point $p \in S^1$ there is a point $\hat{p} \in \partial U$ such that for any path $\gamma : [0, 1) \to \Delta$ tending towards p, the image curve $\phi \circ \gamma$ tends towards $\hat{p}$. We can easily see how this may go wrong when ∂U is not locally connected. Consider the region U of Figure 2.3. Explicitly, given a sequence of numbers $1 > a_1 > a_2 >$

Figure 2.3: A poorly behaved example.

... converging to 0, let $U \subset \mathbb{C}$ be obtained from the open unit square $(0,1) \times (0,1)$ by removing the line

$$[a_n, 1] \times \{a_n\} \text{ for each odd value of } n,$$
$$[0, 1 - a_n] \times \{a_n\} \text{ for each even value of } n.$$

Evidently, ∂U is not locally connected at any point on the bottom edge. Since U is simply connected, the Riemann mapping theorem implies that there is a conformal isomorphism $\phi : \Delta \to U$. One can show that for each $z_0 \in U$, there is a unique geodesic in the Poincaré metric on U which begins at z_0 and accumulates on every point on the bottom edge of U. The preimage of this geodesic converges to a well defined point in the boundary of Δ, so this illustrates that we cannot extend ϕ over the boundary. For more details on this example, see §5 of [Mil06a].

Before we prove Theorem 12, we will establish a lemma. Let R denote the rectangle in $\mathbb{C}$ consisting of $x+iy$ with $0 \leqslant x, y \leqslant 1$. Let L_x denote the vertical line $L_x = \{x+iy \mid y \in [0,1]\}$ in R. Let $\ell(C)$ denote the length of an arc C, and let $A(R)$ denote the area of a region R.

Lemma 4. Suppose $\phi : R \to U \subset \mathbb{C}$ is a conformal isomorphism. Then there is $x \in [0,1]$ such that $\text{length}(\phi(L_x))^2 \leqslant \text{area}(U)$.

Proof. The average length of L_x satisfies

$$\begin{aligned}
\text{Avg}(\text{length}(\phi(L_x)))^2 &= \left(\int_0^1 \text{length}(\phi(L_x))\, dx \right)^2 \\
&= \left(\int_0^1 \int_0^1 |\phi'(x,y)|\, dy dx \right)^2 \\
&= \left(\int_R |\phi'(z)|\, d\mu(z) \right)^2 \\
&\leqslant \left[\int_R |\phi'(z)|\, d\mu(z) \right] \left[\int_R 1\, d\mu(z) \right] \\
&= \text{area}(U) \cdot 1,
\end{aligned}$$

by Cauchy-Schwarz, where μ denotes the Lebesgue measure on $\mathbb{C}$. Since the average is no less than the minimum, there is some x such that $\text{length}(\phi(L_x))^2 \leqslant A(U)$. □

Lemma 5. If a compact metric space X is locally connected, then for all $\epsilon > 0$, there exists $\delta > 0$ so that any two points of distance less than δ are contained in a connected subset of X with diameter less than ϵ.

Proof. For any $\epsilon > 0$, let $\{Y_\alpha\}$ be the collection of all connected open sets of diameter less than ϵ. This collection is non-empty by local connectivity of X. Consider the set $S = (X \times X) \setminus \bigcup_\alpha (Y_\alpha \times Y_\alpha)$, which is compact since it is a closed subset of $X \times X$. Let

$$\delta = \min_{(x,y) \in S} d(x,y).$$

Then δ is the minimum distance between two points which never belong to the same Y_α. If follows that for all $x, y \in X$ with $d(x, y) < \delta$, there is Y_α with $x, y \in Y_\alpha$. $\square$

We now prove Theorem 12

Proof of Theorem 12. Let $f : \Delta \to U$ be a conformal isomorphism. Without loss of generality, we have assume that U is bounded. Assume that ∂U is locally connected.

Take any $p = e^{i\theta} \in S^1$. Notice that the map $h : \Delta \to \mathbb{C}$ given by

$$h(z) = \frac{1}{\pi} \log\left(i\frac{1 - e^{-i(\theta+\pi)}z}{1 + e^{-i(\theta+\pi)}z}\right),$$

takes Δ conformally to the infinite strip $S = \{0 < \mathrm{Im}(z) < 1\}$. Moreover, notice that h extends continuously to S^1, with $h(p) = h(-p)$ at ∞. Let $g : S \to U$ be given by $g = f \circ h^{-1}$. Notice that

$$\lim_{z \to p} f(z) = \lim_{z \to p} g(h(z)) = \lim_{\mathrm{Re}(z) \to +\infty} g(z).$$

Therefore, it suffices to show that the right hand side exists.

We now tile S by squares $R_n = \{z \in S \mid n \leqslant \mathrm{Re}(z) \leqslant n{+}1\}$ for each $n \in \mathbb{N}$. Since U has finite area, the areas of $g(R_n)$ must tend to zero. For each n, Lemma 4 tells us we can find a vertical line $L_n = \{z \in S_n \mid \mathrm{Re}(z) = x_n\}$ for some x_n such that $\mathrm{length}(g(L_n))^2 \leqslant \mathrm{area}(g(R_n))$. Let $\gamma_n = g(L_n)$. Then $\mathrm{length}(\gamma_n) \to 0$ as $n \to \infty$.

In particular, for n sufficiently large, γ_n has finite length. In this case, γ_n has well defined end points on ∂U. Indeed, if $g(x_n + it)$ has two accumulation points a_1 and a_2 as $t \to 1$, then $g(x_n + it)$ traverses between ϵ neighborhoods of a_1 and a_2 infinitely many times as $t \to 1$, and therefore γ_n has infinite length. A similar argument suffices as $t \to 0$.

Let $a_n = \lim_{t\to 1} g(x_n{+}it)$ and $b_n = \lim_{t \to 0} g(x_n{+}it)$ be the end points of γ_n. Notice that ∂U is a compact, locally connected metric space under the euclidean norm. Since $\mathrm{length}(\gamma_n) \to 0$, we have that $d(a_n, b_n) \to 0$ as $n \to \infty$. Now by Lemma 5, there are connected subsets $S_n \subset \partial U$ with $\mathrm{diam}(S_n) \to 0$ as $n \to \infty$.

Let $A_n = S_n \cup \gamma_n$, which is connected, and let $D_n = g(\{\mathrm{Re}(z) > x_n\})$. Then A_n separates D_n from $U \setminus \overline{D}_n$. If not, then choose a smooth arc A' from a point $x \in D_n$ to a point $y \in U \setminus \overline{D}_n$ and is disjoint from A_n. Let A'' be another smooth arc from x to y which crosses γ_n. Then $A' \cup A''$ is a Jordan curve which separates a_n from b_n and does not intersect S_n. Hence it separates S_n, which is impossible.

Now A_n bounds D_n, and has diameter at most $\mathrm{diam}(S_n) + \mathrm{length}(\gamma_n)$. Thus, $\mathrm{diam}(D_n) \to 0$ as $n \to \infty$. If $\eta : [0, 1) \to S$ is a path with $\mathrm{Re}(\eta(t)) \to +\infty$ as $t \to 1$, then $g(\eta(t)) \in D_n$ for all t sufficiently close to 1. Therefore the accumulation set of $g \circ \eta$ is contained in $\bigcap_n \overline{D}_n$. But since $\mathrm{diam}(\overline{D}_n) \to 0$, this intersection consists of at most one point. Therefore $g \circ \eta$ has a well defined limit as $t \to 1$, and we have shown that f extends continuously over p.

Conversely, if f extends continuously over S^1, then since S^1 is locally connected and compact and $f(S^1) = \partial U$, it must be that ∂U is locally connected. $\square$

Recall that a *Jordan curve* is a topological space which is homeomorphic to a circle

Corollary 7 (Extension as a homeomorphism)**.** A conformal mapping $f : \Delta \to U$ extends as a homeomorphism $f : \overline{\Delta} \to \overline{U}$ if and only if ∂U is a Jordan curve.

Proof. If f extends as a homeomorphism $f : \overline{\Delta} \to \overline{U}$, then $f|_{S^1}$ is a homeomorphism onto ∂U. Thus, ∂U is a Jordan curve. Conversely, if ∂U is a Jordan curve, then ∂U is locally connected, so f extends continuously as a proper map $f : \overline{\Delta} \to \overline{U}$. If $f(s) = f(t)$ for $s, t \in S^1$, then $\partial U \setminus \{f(s)\}$ is disconnected, which is impossible, since ∂U is a Jordan curve. Therefore f is injective and is thus a homeomorphism. $\square$

Chapter 3

Complex Dynamics on the Riemann Sphere

Complex dynamics is the study of iterated holomorphic functions. Suppose X is a Riemann surface and $f : X \to X$ is holomorphic. For any $z \in X$, we let $f^n(z)$ denote the n-fold composition $\underbrace{f(f(\ldots f(z)))}_{n \text{ times}}$. The *orbit* of a point $z \in X$ under the map f is defined to be $\mathcal{O}_f(z) = \{f^n(z) \mid n \in \mathbb{N}_0\}$. The study of the behavior of X under f^n as n tends to infinity reveals much about the function f as well as the geometry of X. Objects of interest include periodic cycles, the points which tend towards them, and regions of X on which the dynamics are chaotic.

For example, the map $f(z) = z^2$ on $\widehat{\mathbb{C}}$ has fixed points at $z = 0$, 1, and ∞. For each $z \in \Delta$, $f^n(z) \to 0$ as $n \to \infty$. For each $z \in \widehat{\mathbb{C}} \setminus \overline{\Delta}$, $f^n(z) \to \infty$ as $n \to \infty$. On S^1, the dynamics are chaotic. The roots of unit, which are dense in S^1 have finite orbits: odd roots of unity are periodic ($f^m(p) = p$ for some $m \in \mathbb{N}$), and even roots of unity have the property that $f^n(p) = 1$ for all n sufficiently large. Points in S^1 which are not roots of unity have infinite orbits under f. We can even find points $p \in S^1$ for which $\mathcal{O}_f(p)$ is dense in S^1.

3.1 The Fatou and Julia Sets

The notion of controlled and chaotic dynamics are formalized respectively by the *Fatou set* and *Julia set* of a map $f : X \to X$. Heuristically, chaos means sensitive dependence on initial conditions: when the orbits of points which are arbitrarily close together diverge.

Definition 4 (Fatou and Julia sets)**.** The *Fatou set* of a holomorphic map $f : X \to X$ on a Riemann surface X is the union of all open sets U such that $\{f^n|_U\}_{n\in\mathbb{N}}$ is a normal family in $\mathrm{Hol}(U, X)$.

The *Julia set* $J(f)$ is the complement of the Fatou set in X.

By Proposition 7, $\{f^n\}_{n\in\mathbb{N}}$ is a normal family on connected components of the Fatou set. By definition, the Fatou set is open, and the Julia set is closed.

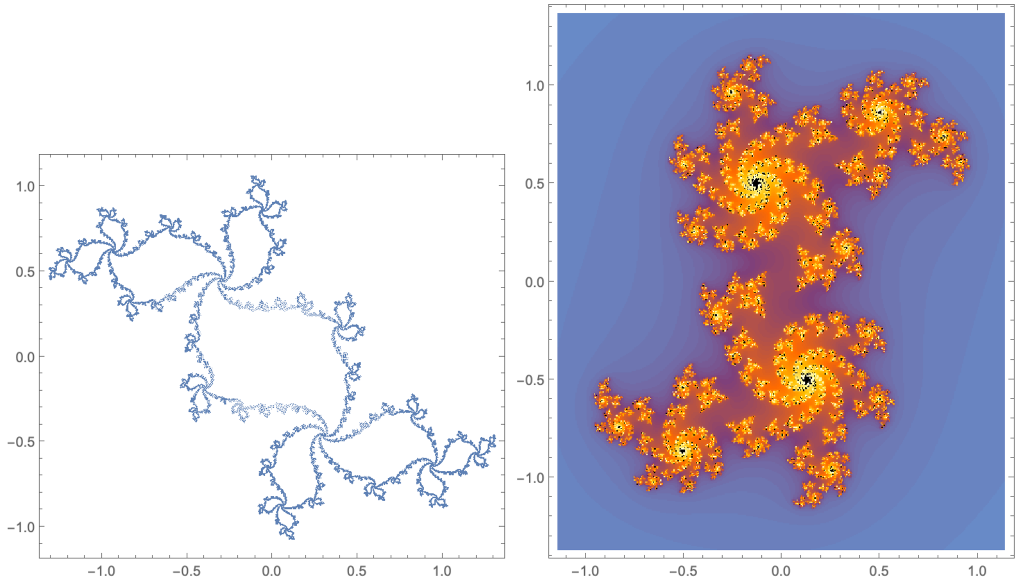

Figure 3.1: Julia sets for $f(z) = z^2 - 0.1838 + 0.7294i$ and $f(z) = z^2 + 0.365 - 0.37i$

Example 7. We again consider the map $f(z) = z^2$ on $\widehat{\mathbb{C}}$. As we have previously remarked, $\widehat{\mathbb{C}} \setminus S^1$ is contained in the Fatou set. Any open set U intersecting S^1 contains points from Δ and $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. The former points tend to zero under iteration, and the latter tend to ∞. Hence, the limit of any convergent subsequence of (f^n) on U would have a jump discontinuity as we cross S^1. Therefore $\{f^n\}$ cannot be normal on an open set intersecting S^1, so S^1 is not contained in the Fatou set. We conclude that the Fatou set is $\widehat{\mathbb{C}} \setminus S^1$, and $J(f) = S^1$. See Figure 3.2.

Basic properties. While we will eventually specialize to the case of holomorphic mappings on the Riemann sphere, we will begin with some general facts about the Fatou and Julia sets.

Lemma 6 (Invariance lemma)**.** The Julia set $J(f)$ is fully invariant under $f : X \to X$. That is, a point $z \in X$ is in $J(f)$ if and only if $f(z) \in J(f)$.

Proof. An equivalent statement is that the Fatou set is fully invariant under f. Suppose $z \in X$ lies in the Fatou set of f. Then there is an open neighborhood U of z such that $\{f^n|_U\}$ is a normal family. Since f is holomorphic, $V = f(U)$ is an open set containing $w = f(z)$. Moreover, $\{f^n|_V\}_{n\in\mathbb{N}} = \{f^{n+1}|_U\}_{n\in\mathbb{N}} \subset \{f^n|_U\}_{n\in\mathbb{N}}$, so it follows that $\{f^n|_V\}$ is a normal family. Therefore $f(z)$ is in the Fatou set.

If $f(z)$ is in the Fatou set, then there is a neighborhood V of $f(z)$ on which $\{f^n\}$ is a normal family. Let $U = f^{-1}(V)$, which is an open neighborhood of z. For sequence $(f^{n(j)}|_U)$, the sequence $(f^{n(j)-1}|_V)$, defined when $n(j) > 1$, has a locally uniformly convergent subsequence. Precomposing the elements of this sequence with f, we produce a locally uniformly convergent subsequence of $(f^{n(j)}|_U)$. Hence, z is in the Fatou set. $\square$

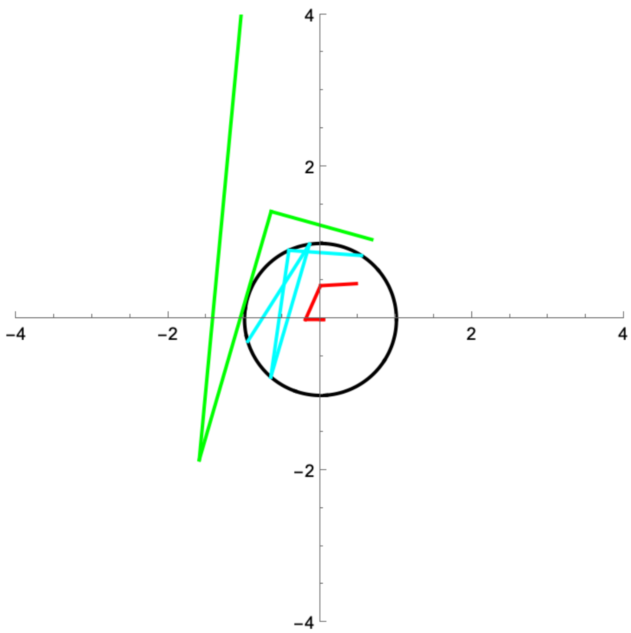

Figure 3.2: Orbits of some points under $f(z) = z^2$.

This gives intuition for why the Julia set appears to possesses self-similarity. Given points $z_2 = f(z_1)$ in $J(f)$, with $f'(z_1) \neq 0$, the map f takes a neighborhood N_1 of z_1 isomorphically onto a neighborhood N_2 of z_2, and moreover, $J(f) \cap N_1$ is mapped precisely to $J(f) \cap N_2$.

Lemma 7 (Iteration lemma)**.** For each $k \in \mathbb{N}$, we have $J(f^k) = J(f)$.

Proof. It again suffices to show that the Fatou set of f^k is equal to the Fatou set of f. If z is in the Fatou set of f, then there is a neighborhood U of z on which $\{f^n\}$ is a normal family. Any subsequence of (f^{kn}) must then have a locally uniformly convergent subsequence, so $\{f^{kn}\}$ is a normal family on U. Thus, z is in the Fatou set of f^k.

Suppose z is in the Fatou set of f^k. There is a neighborhood U of z on which $\{f^{kn}|_U\}$ is normal. Let $(f^{n_j}|_U)$ be any sequence in $\{f^n|_U\}$. For each j, we can write $n_j = m_j k + r_j$, where $m_j \in \mathbb{N}_0$ and $0 \leqslant r_j < k$. Since there are finitely many possible r_j, there is r such that $n_j = m_j k + r$ for infinitely many j. Let $\{n_{\hat{j}}\}$ be the set of such indices. Now $\{f^{n_{\hat{j}}}|_U\} = \{(f^{m_{\hat{j}}k} \circ f^r)|_U\}$. Since $(f^{m_{\hat{j}}k}|_U)$ has a locally uniformly converging subsequence, it follows that $(f^{n_{\hat{j}}}|_U)$ has a locally uniformly converging subsequence, which is a subsequence of $(f^{n_j}|_U)$. Thus, z is in the Fatou set of f. $\square$

Periodic cycles. A *periodic cycle* is a sequence of points

$$f : z_0 \mapsto z_1 \mapsto \cdots \mapsto z_p = z_0.$$

If the $z_1, \ldots, z_{p-1}$ are all distinct, then p is called the *period* of the cycle. Such a cycle is often called a *p-cycle*. A 1-cycle is a *fixed point*. The derivative $\lambda = (f^p)'(z_0)$ is called the *multiplier* of the periodic cycle. By the chain rule, this value does not depend on the choice of z_0.

The multiplier gives us useful geometric information about the map f near a periodic cycle. A cycle is called *attracting* if $|\lambda| < 1$ and *repelling* if $|\lambda| > 1$. If $|\lambda| = 1$, then the cycle is *indifferent*. An attracting cycle for which $|\lambda| = 0$ is called *superattracting*. Otherwise, the attracting cycle is *geometrically attracting*.

It will later be clear that attracting cycles are *topologically attracting*: each point z_i in the cycle has a neighborhood U_i on which $(f^{pn}|_{U_i})$ converges uniformly to z_i. Similarly, we will see that repelling cycles are *topologically repelling*: each z_i in the repelling cycle has a neighborhood U_i such that for each $z \in U$, there is n so that $f^{np}(z) \notin U$. Also see [Mil06a].

Remark 2. One must use caution when computing the derivative of $f'(\infty)$. It is not necessarily the case that $f'(\infty) = \lim_{z\to\infty} f'(z)$. If $f(\infty) \in \mathbb{C}$, then let $h(z) = f(1/z)$. Now $f'(\infty)$ is defined to be $h'(0)$. But by L'Hôpital's rule

$$h'(0) = \lim_{z\to 0} \frac{f(1/z) - f(\infty)}{z} = \lim_{z\to 0} -f'(1/z)/z^2 = \lim_{z\to\infty} -z^2 f'(z).$$

However, if $f(\infty) = \infty$, then $f'(\infty)$ is the reciprocal of $\lim_{z\to\infty} f'(z)$. To see this, note that if $g(z) = 1/f(z/1)$, the $\lambda = f'(\infty)$ is defined to be $g'(0)$. Then for z in a neighborhood of 0,

$$g(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots.$$

Now

$$\lim_{z\to\infty} \frac{1}{f'(z)} = \lim_{z\to\infty} \frac{(zg(1/z))^2}{g'(1/z)} = \lim_{z\to\infty} \frac{(\lambda + a_2/z + \dots)^2}{g'(0)} = \frac{\lambda^2}{\lambda} = \lambda = f'(\infty).$$

In particular, this emphasizes the fact that on general Riemann surfaces, f' cannot be considered as a holomorphic (or even continuous) function from its domain to $\mathbb{C}$. This makes sense, since there are no non-constant holomorphic functions from $\widehat{\mathbb{C}}$ to $\mathbb{C}$.

If $\mathcal{O} = \{z_0, \dots, z_{p-1}\}$ is an attracting periodic cycle of period p, we define the *basin of attraction* to be the set $\mathcal{A} \subset \widehat{\mathbb{C}}$ of points z with $f^n(z) \to \mathcal{O}$ as $z \to \infty$. Observe that $\mathcal{A}$ is an open set.

Lemma 8 (Attracting basins and repelling points)**.** The basin of every attracting cycle is contained in the Fatou set. Every repelling cycle is contained in the Julia set.

Proof. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic and let $\mathcal{O} = \{z_0, z_1, \dots, z_{p-1}\}$ be an attracting cycle of period p. Then $f^p(z_i) = z_i$. Let $\mathcal{A}$ be the attracting basin of $\mathcal{O}$. If $z \in \mathcal{A}$, then $\lim_{n\to\infty} f^{pn}(z) = z_i$ for some i. The particular z_i which z converges to is an open condition, so it is constant on connected components of $\mathcal{A}$.

For each i, let ϕ_i be a local coordinate, and let $w_i = \phi_i(z_i)$. Let g be the induced map $\phi \circ f^p \circ \phi^{-1}$. Then $g(w_i) = w_i$ and $g'(w_i) = \lambda$ with $|\lambda| < c < 1$. Then Taylor's theorem implies that there is a neighborhood of U_i of w_i on which $|g(w) - w_i| \leqslant c|w - w_i|$ for all $w \in U$. Then $|g^n(w) - w_i| \leqslant c^n|w - w_i|$ for all $n \in \mathbb{N}$ Therefore g converges uniformly to the constant function w_i on U_i.

Let $V_i = \phi^{-1}(U_i)$. Then $(f^{pn}|_{V_i})$ converges uniformly to z_i. Let K be any compact subset of $\widehat{\mathbb{C}}$. Then K has finitely many connected components. It suffices to show that (f^{pn}) converges uniformly on the connected components of K. Thus, assume that K is connected. Then there is N such that $f^{pN}(K) \subset V_i$ for some i. It follows that (f^{pn}) converges uniformly to z_i on K. Therefore $\{f|_{\mathcal{A}}\}$ is a normal family, so $\mathcal{A}$ lies in the Fatou set.

If z_0 lies in repelling cycle of period p, then $(f^p)'(z_0) = \lambda$. With $|\lambda| > 1$. Then $(f^{pn})'(z_0) \to \infty$ as $n \to \infty$, so (f^{pn}) cannot converge uniformly on any neighborhood of z_0. Hence, $z_0 \in J(f)$. □

The case of indifferent cycles is more complicated. A particularly important case is that of *parabolic* cycles. A periodic point $z_0 = f^p(z_0)$ is called *parabolic* if its multiplier is 1, but f^p is not identically the identity map. A parabolic point can be more generally defines as a periodic point for which the multiplier is a root of unity but no iterate of f is the identity map.

Lemma 9 (Parabolic points)**.** Every parabolic point belongs to the Julia set.

Proof. Let z_0 be a parabolic fixed point of period p, and let $w = \phi(z)$ for some local coordinate ϕ at z_0. We may assume that $\phi(z_0) = 0$. Let g be the induced map $\phi \circ f^p \circ \phi^{-1}$ in the w-plane. Then near 0,

$$g(w) = w + a_q w^q + a_{q+1} w^{q+1} + \dots,$$

where $q \geqslant 2$ and $a_q \neq 0$. Now the n-th iterate is given by

$$g^n(w) = w + n a_q w^q + \dots.$$

Therefore the q-th derivative of g^n is $q! n a_q$, which tends to ∞ as $n \to \infty$. Therefore the induced map of f^{pn} in the w-plane has qth derivative tending to ∞, so $\lim_{n\to\infty} f^{pn}$ cannot be holomorphic in any neighborhood of z_0. Hence, $z_0 \in J(f)$. □

On the Riemann sphere. We will now specialize to the case of holomorphic maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. In this case, f is a rational function. In some sense, these are the most interesting examples.

On hyperbolic surfaces, one can show that holomorphic maps have no Julia sets, and the dynamics are easily classifiable [Mil06a].

On Euclidean surfaces, the dynamics can be quite unwieldy, as functions are not necessarily proper. For example, consider the exponential function. Lyubich and Rees proved that for almost every starting point $z \in \mathbb{C}$, the set of accumulation points for the orbit of z under exp is equal to the orbit of zero [Mil06a]. However, Misiurewicz has shown that the Julia set of exp is the entire plane, so a generic orbit is dense in $\mathbb{C}$ [Mil06a]. The function sin can be considered as a holomorphic function on the cylinder $\mathbb{C}/2\pi\mathbb{Z}$. According to McMullen, the Julia set has infinite area, while the Fatou set is dense, but has finite area [Mil06a].

Therefore, the study of maps on $\widehat{\mathbb{C}}$, while still rich and difficult, is more manageable than the euclidean case.

Lemma 10 (Non-empty Julia set)**.** If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic with degree at least two, then $J(f)$ is non-empty.

Proof. Let $d(z, w)$ denote the distance between points $z, w \in \widehat{\mathbb{C}}$ with the spherical metric introduced in Corollary 5.

If $J(f) = \emptyset$, then each point in $\widehat{\mathbb{C}}$ has a neighborhood on which $\{f^n\}$ is a normal family. Then by Proposition 7, $\{f^n\}$ is normal on $\widehat{\mathbb{C}}$. Since $\widehat{\mathbb{C}}$ is compact, some subsequence (f^{n_j}) converges uniformly to a limit $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Then the degree of f^{n_j} agrees with the degree of g for sufficiently large j. Indeed, for j large enough $d(f^{n_j}(z), g(z)) < \pi$, so there is a unique shortest path between $f^{n_j}(z)$ and $g(z)$. By translating along shortest paths, we obtain a homotopy between f^{n_j} and g. Since f^{n_j} and g are homotopic, they must have the same degree.

Notice that f^n has degree d^n, where $n = \deg f$. Hence ,f^{n_j} has degree tending to infinity, while g has finite degree. This is a contradiction. □

Definition 5 (Grand orbit)**.** The *grand orbit* of a point z under $f : X \to X$ is the set of points z' for which the orbit under f of z' eventually coincides with the orbit under f of z. Specifically, there is $k \in \mathbb{Z}$ such that $f^n(z') = f^m(z)$ for some $n, m \geqslant 0$.

A point is called *exceptional* if its grand orbit is finite. For a map $f : X \to X$, the *exceptional set* $\mathcal{E}(f)$ is the set of exceptional points of f.

Lemma 11 (Exceptional points are exceptional)**.** For a holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least two, the exceptional set $\mathcal{E}(f)$ consists of at most two points. Each exceptional point must belong to a superattracting cycle of f, and hence lie in the Fatou set.

Proof. Since f maps $\widehat{\mathbb{C}}$ surjectively to itself, it must map any grand orbit surjectively to itself. Let G be the grand orbit of an exceptional point z_0. Since G is finite, f maps G bijectively to itself. Hence G consists of a single periodic orbit $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_p = z_0$. Since the degree of f is at least 2, any point $z \in \widehat{\mathbb{C}}$ has at least two preimage points unless z is a critical point. Hence each point of G belongs to a superattracting cycle.

If there were three grand orbit finite points, then the union of the grand orbits of these points would have a complement U which is a hyperbolic Riemann surface by Proposition 6. By version two of Montel's theorem (Theorem 11), $\{f^n\}$ is a normal family on U, so U lies in the Fatou set. Since exceptional points are superattracting, they lie in the Fatou set as well. Now the Fatou set is all of $\widehat{\mathbb{C}}$, which is impossible by Lemma 10. □

Theorem 13 (Transitivity)**.** Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic and let N be any neighborhood of $z_1 \in J(f)$. Then $U = \bigcup_n f^n(N)$ contains $J(f)$ and all but at most two points in $\widehat{\mathbb{C}}$. Specifically, if N is sufficiently small, then $U = \widehat{\mathbb{C}} \setminus \mathcal{E}(f)$.

Proof. Suppose that $\widehat{\mathbb{C}} \setminus U$ contains more than two points. Then U is a hyperbolic Riemann surface. Since $f(U) \subset U$ it follows by version two of Montel's theorem (Theorem 11) that U is contained in the Fatou set. This is impossible since U intersects the Julia set.

Again using the fact that $f(U) \subset U$, we can see that any preimage of a point $z \in \widehat{\mathbb{C}} \setminus U$ must also lie in $\widehat{\mathbb{C}} \setminus U$.Now some iterated preimage of z is periodic: If $V = \bigcup_n f^{-n}(z)$, then $V \in \widehat{\mathbb{C}} \setminus U$, and is thus finite. Since f maps V surjectively to itself, it must be that f maps V to itself bijectively. Therefore V contains a periodic cycle since no orbit can be infinite. In particular, z itself is periodic, and hence must be grand orbit finite. Thus $\widehat{\mathbb{C}} \setminus U \subseteq \mathcal{E}(f)$, which is disjoint from $J(f)$, so U contains J.

If we make N small enough so that it does not intersect $\mathcal{E}(f)$, then $U = \widehat{\mathbb{C}} \setminus \mathcal{E}$. ☐

This transitivity theorem gives us several important corollaries.

Corollary 8 (Julia sets with interior)**.** If $J(f)$ contains interior points, then $J(f) = \widehat{\mathbb{C}}$.

Proof. Suppose that N is an open subset of $\widehat{\mathbb{C}}$ contained in the Julia set. Then by Theorem 13, $U = \bigcup_n f(N)$ is dense in $\widehat{\mathbb{C}}$. By the invariance lemma (Lemma 6), $U \subseteq J(f)$. But since $J(f)$ is closed, $J(f) = \widehat{\mathbb{C}}$. ☐

Corollary 9 (Basin boundaries)**.** If $\mathcal{A} \subset \widehat{\mathbb{C}}$ is an attracting basin for an attracting periodic cycle, then the boundary $\partial\mathcal{A}$ is equal to the Julia set. Each connected component of the Fatou set is either a connected component of $\mathcal{A}$, or else is disjoint from $\mathcal{A}$.

Proof. Given a neighborhood N of a point in the Julia set, Theorem 13 implies that some iterate $f^n(N)$ intersects $\mathcal{A}$. Therefore N itself intersects $\mathcal{A}$. Now $J(f) \subset \overline{\mathcal{A}}$. But $J(f)$ is disjoint from $\mathcal{A}$, so $J(f) \subset \partial\mathcal{A}$.

On the other hand, if N is any neighborhood of a point in $\partial\mathcal{A}$, then the limit of any sequence of iterates $f^n|_N$ must have a jump discontinuity between $\mathcal{A}$ and $\partial\mathcal{A}$, as the points in $\partial\mathcal{A}$ do not tend to attracting cycle which $\mathcal{A}$ tends towards. Therefore $\partial\mathcal{A} \subset J(f)$.

Any connected component C of the Fatou set cannot intersect $\partial\mathcal{A}$, so it must be contained in $\mathcal{A}$ or else disjoint from $\mathcal{A}$. In the former case, since $\mathcal{A}$ is contained in the Fatou set, $C = \mathcal{A}$. ☐

Corollary 10 (Iterated preimages)**.** For any $z_0 \in J(f)$, the set $V = \bigcup_{n \geqslant 0} f^{-n}(z_0)$ of iterated preimages is everywhere dense in $J(f)$.

Proof. Since $z_0 \in J(f)$, we know $z_0 \notin \mathcal{E}(f)$. Take another point $z_1 \in J(f)$ and any neighborhood N of z_1. Then by Theorem 13, $z_0 \in f^n(N)$ for some $n \geqslant 0$. Hence there is a point $p \in N$ such that $f^n(p) = z_0$, so $p \in V$. Thus V is dense in $J(f)$. □

Remark 3. Corollary 10 gives us a helpful way of visualizing the Julia set with a computer. If we can find a single point of the Julia set, then by writing a program to calculate iterated preimages and plot them, we get a plot of the Julia set.

Corollary 11 (Isolated points)**.** If the degree of f is at least two, then $J(f)$ has no isolated points.

Proof. Note that the Julia set must contain infinitely many points: if $J(f)$ is finite, then its points are grand orbit finite, which is impossible since such points belong to the Fatou set. Therefore $J(f)$ has at least one limit point z_0, which is in $J(f)$ since $J(f)$ is closed. The iterated preimages of z_0 are dense in $J(f)$ by Corollary 10, and moreover, the iterated preimages of z_0 are not isolated. Since $J(f)$ has a dense subset of non-isolated points, there can be no isolated points in $J(f)$. □

Corollary 12 (Components of the Julia set)**.** If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic and has degree at least 2, then $J(f)$ is connected or has uncountably many connected components.

Proof. Suppose that $J = J(f)$ is not connected. Write $J = J_0 \sqcup J_1$, where J_0 and J_1 are compact and non-vacuous. By Corollary 11, J_0 and J_1 are both infinite.

For all $z \in J$, we can assign a bit sequence $(\beta_0, \beta_1, \beta_2, \dots)$, where $\beta_n = \beta_n(z) \in \{0, 1\}$ is defined by $f^n(z) \in J_{\beta_n}$. Points in the same connected component of J must have the same bit sequence. It then suffices to show that uncountably many bit sequences may be realized.

Suppose an initial finite bit sequence $(\beta_0, \beta_1, \dots, \beta_k)$ is realized by some $z' \in J$. We want to show that there exists $z'' \in J$ with the same initial bit sequence, but $\beta_n(z') \neq \beta_n(z'')$ for some $n > k$.

Let $U_\beta = \widehat{\mathbb{C}} \setminus J_{1-\beta}$, which is a hyperbolic Riemann surface. Let

$$J_{\beta_0,\dots,\beta_k} = \{z \in J \mid f^n(z) \in J_{\beta_n},\ 0 \leqslant n \leqslant k\}$$
$$U_{\beta_0,\dots,\beta_k} = \{z \in \widehat{\mathbb{C}} \mid f^n(z) \in U_{\beta_n},\ 0 \leqslant n \leqslant k\}.$$

In other words, $J_{\beta_0,\dots,b_k}$ is the set of all points in J with the same initial bit sequence as z'. Moreover, $J_{\beta_0,\dots,\beta_k} \subset U_{\beta_0,\dots,\beta_k}$. Assume for contradiction that all points of $J_{\beta_0,\dots,\beta_k}$ have the same infinite bit sequence (β_n). Then $\{\beta_n\}$ has infinitely many zeros or infinitely many ones. Assume without loss of generality that there are infinitely many zeros. Then every infinite sequence of iterates contains a subsequence (f^{n_j}) such that $f^{n_j}(U_{\beta_0,\dots,\beta_k}) \subset U_0$ for all j. But by Montel's theorem (Theorem 11), $U_{\beta_0,\dots,\beta_k}$ lies in the Fatou set. But $J_{\beta_0,\dots,\beta_k} \subset U_{\beta_0,\dots,\beta_k}$ is contained in the Julia set, so we have a contradiction.

Therefore every finite bit sequence which can be realized, can be extended in at least two ways. It follows that there are uncountably many realizable bit sequences. □

3.2 Local Fixed Point Theory

The most basic example of an attracting or repelling fixed point is the point 0 under the map $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \neq 0, 1$. Indeed, this map turns out to be the local model for all geometrically attracting and repelling fixed points. For any holomorphic map $f : X \to X$ on a Riemann surface X, we can always study the behavior of f at a given point $x_0 \in X$ be considering the map $g(x) = f(x) - x_0$, so the local properties of fixed points tells us a great deal about the local properties of holomorphic maps in general.

Theorem 14 (Kœnigs Linearization)**.** Let $f : X \to X$ be a holomorphic map on a Riemann surface X, and suppose that x_0 is a fixed point of f with multiplier λ. If $|\lambda| \neq 0, 1$, then there is a neighborhood U of x_0 and holomorphic change of coordinate $w = \phi(z)$ such that $\phi \circ f \circ \phi^{-1} = w \mapsto \lambda w$ on some neighborhood of the origin. Moreover, ϕ is unique up to multiplication by a nonzero constant.

Proof. Since we are only concerned about f in a neighborhood x_0, and all Riemann surfaces look the same locally, it suffices to consider f as a power series

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$$

defined in a neighborhood of the origin.

We first prove the uniqueness of ϕ. If ϕ and ψ are two maps with the desired property, then $\psi(\phi^{-1}(\lambda w)) = \lambda\psi(\phi^{-1}(w))$. Write $\psi \circ \phi^{-1}$ as a power series:

$$(\psi \circ \phi^{-1})(w) = b_1 w + b_2 w^2 + b_3 w^3 + \dots$$

We have

$$\begin{aligned}
(\psi \circ \phi^{-1})(\lambda w) &= \lambda b_1 w + \lambda^2 b_2 w^2 + \lambda^3 b_3 w^3 + \dots \\
\lambda(\psi \circ \phi^{-1})(w) &= \lambda b_1 w + \lambda b_2 w^2 + \lambda b_3 w^3 + \dots .
\end{aligned}$$

Now $\lambda b_n = \lambda^n b_n$, so $b_n = 0$ when $n \neq 1$. Therefore $(\psi \circ \phi^{-1})(w) = \lambda b_1 w$, so $\psi(w) = \lambda b_1 \phi(w)$.

We now prove existence when $|\lambda| < 1$. Choose a constant $c < 1$ so that $c^2 < |\lambda| < c$. By Taylor's theorem, there are constants $r_0 > 0$ and C such that $|f(z) - \lambda z| \leqslant C|z^2|$ whenever $|z| < r_0$. Choose r with $0 < r \leqslant r_0$ so that $|\lambda| + Cr < c$. Now

$$|f(z)| \leqslant |\lambda z| + C|z^2| \leqslant c|z|.$$

Therefore $|f(z)| \leqslant c|z|$ for $z \in \Delta_r$. Now f is geometrically attracting on Δ_r, so for any point $z_0 \in \Delta_r$, the forward orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots$ converges to 0. Specifically, $|z_n| \leqslant rc^n$. Moreover,

$$|z_{n+1} - \lambda z_n| \leqslant C|z_n|^2 \leqslant Cr^2 c^{2n}.$$

Let $k = Cr^2/|\lambda|$. Then if $w_n = z_n/\lambda^n$,

$$|w_{n+1} - w_n| \leqslant k(c^2/|\lambda|)^n.$$

The value $k(c^2/|\lambda|)^n$ does not depend on our starting point z_0, and moreover $k(c^2/|\lambda|)^n \to 0$ as $n \to \infty$. Therefore, the function $z_0 \mapsto w_{n+1} - w_n$ converges uniformly to 0 on Δ_r. Thus, the functions $z_0 \mapsto z_n$ converge uniformly in Δ_r to a holomorphic limit function $\phi(z_0) = \lim_{n\to\infty} z_n/\lambda^n$. Moreover,

$$\phi(f(z)) = \lim_{n\to\infty} f^{n+1}(z)/\lambda^n = \lambda \lim_{n\to\infty} f^{n+1}(z)/\lambda^{n+1} = \lambda\phi(z),$$

which is the desired identity. Finally, $z_0 \mapsto z_n/\lambda^n$ has derivative 1 at the origin for each n, so $\phi'(0) = 1$. Thus, ϕ is a univalent when r is small enough. This proves the theorem when $|\lambda| < 1$.

If $|\lambda| > 1$, then there is a neighborhood of 0 on which f is univalent, so f^{-1} is well defined. Moreover, $(f^{-1})'(0) = 1/\lambda$, which has norm less than one. We can apply the argument above to get a univalent map ϕ with $\phi(f^{-1}(z)) = (1/\lambda)\phi(z)$. Now $\phi(z) = (1/\lambda)\phi(f(z))$, so $\phi(f(z)) = \lambda\phi(z)$, and we're done. $\square$

Dependence on parameters. If we have a family of maps f_α given by

$$f_\alpha(z) = \lambda(\alpha)z + b_2(\alpha)z^2 + \dots,$$

where the coefficients depend holomorphically on α and $|\lambda(\alpha)| \neq 0, 1$, then we can show that the Kœnigs linearization $\phi_\alpha(z)$ depends holomorphically on α for each z. With c as in the proof of Theorem 2.16, suppose that $|\lambda(\alpha)|$ varies in some compact subset of (c^2, c). Then for fixed z, the convergence of $z \mapsto z_n/\lambda(\alpha)^n$ is uniform in α. Since we can freely choose c, the general case follows.

Parabolic and irrationally indifferent fixed points. Recall that a *parabolic fixed point* is a fixed point of $f : X \to X$ with multiplier a root of unity, but for which no iterated of f is the identity. As shown in Lemma 9, parabolic points always lie in the Julia set.

A fixed point z_0 of $f : X \to X$ with multiplier λ satisfying $\lambda = e^{2\pi i\theta}$ with θ irrational, is said to be *irrationally indifferent*. If there exists, as in the case of the Kœnigs linearization, a local holomorphic change of coordinates h such that

$$f(h(w)) = h(\lambda w)$$

near z_0, then f is *locally linearizable* at z_0. The following is proven in [Mil06a].

Lemma 12 (Locally linearizable points)**.** Suppose f is a rational function of degree at least two, and a fixed point z_0 with multiplier satisfying $|\lambda| = 1$. Then the following are equivalent.

- f is locally linearizable at z_0,
- z_0 belongs to the Fatou set of f,
- if U is the connected component of the Fatou set which contains z_0, then there is

a conformal isomorphism $h : U \to \Delta$ such that $h(f(z)) = \lambda h(z)$ for all $z \in U$.

If z_0 is locally linearizable, we call z_0 a *Siegel points*, and call the open set U of this lemma a *Siegel disk*. An irrationally indifferent point which is not a Siegel point is called a *Cremer point*. Evidently, Cremer points are contained in the Julia set.

Superattracting fixed points. As in the case of geometrically attracting and repelling fixed points, there is a local model for the behavior of holomorphic maps near superattracting fixed points. Locally, any such map can be expressed as

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots,$$

where $n \geqslant 2$ and $a_n \neq 0$.

Theorem 15 (Böttcher)**.** With f as above, there is a local change of coordinates $w = \phi(z)$ with $\phi(0) = 0$ and $\phi \circ f \circ \phi^{-1} = z \mapsto z^n$ in some neighborhood of the origin. The map ϕ is unique up to multiplication by an $(n-1)$st root of unity.

Proof. We begin with the proof of existence. As usual, it suffices to assume that $a_n = 1$: if $c^{n-1} = a_n$, then $cf(z/c)$ has leading coefficient 1. We may write

$$\begin{aligned} f(z) &= z^n(1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots) \\ &= z^n(1 + \eta(z)) \end{aligned}$$

Since $\eta(0) = 0$, we can find $0 < r < 1/2$ such that $|\eta(z)| < 1/2$ for all $z \in \Delta_r$. Then f maps Δ_r into itself, as

$$|f(z)| \leqslant \frac{3}{2}|z|^n \leqslant \frac{3}{4}|z|,$$

for $z \in \Delta_r$. Therefore f^k also maps Δ_r into itself, and we have $f^k(z) = z^{n^k}(1 + n^{k-1} b_1 z + \dots)$. The idea is to set

$$\phi_k(z) = \sqrt[n^k]{f^k(z)} = z(1 + n^{k-1} b_1 z + \dots) = z(1 + b_1 z/n + \dots),$$

where the right hand side gives an explicit choice of n^kth root. We can see that

$$\phi_k(f(z)) = \sqrt[n^k]{f^{k+1}(z)} = \sqrt[n^{k+1}]{f^{k+1}(z)}^{\,n} = \phi_{k+1}(f(z))^n.$$

We will extract a uniform limit $\phi : \Delta_r \to \Delta$, which will then satisfy the equation $\phi(f(z)) = \phi(z)^n$. In order to do so, consider the change of variables $z = e^Z$, with Z in the left half-plane $\mathbb{H}_r$, defined by $\mathrm{Re}(Z) < \log(r)$. Then f corresponds to a map $F(Z) = \log f(e^Z)$ from $\mathbb{H}_r$ to itself. To make this precise, we set $\eta = \eta(e^Z) = b_1 e^Z + b_2 e^{2Z} + \dots$. Still, $|\eta| < 1/2$ on $\mathbb{H}_r$. Now

$$F(Z) = \log(e^{nZ}(1 + \eta)) = nZ + \log(1 + \eta) = nZ + \eta - \eta^2/2 + \eta^3/3 - \dots,$$

using the usual power series expansion for $\log(1+z)$. This makes F into a well defined holomorphic function. Since $|\eta| < 1/2$, we have

$$|F(Z) - nZ| = |\log(1+\eta)| < \log 2 < 1. \tag{3.1}$$

We similarly define the map $\Phi_k(Z)$ corresponding to ϕ_k by

$$\Phi_k(Z) = \log \phi_k(e^Z) = F^k(Z)/n^k,$$

which is also well defined and holomorphic in $\mathbb{H}_r$. Now by (2.1),

$$|\Phi_{k+1}(Z) - \Phi_k(Z)| = |F^{k+1}(Z) - nF^k(Z)|/n^{k+1} < 1/n^{k+1}.$$

The map $Z \mapsto e^Z$ is distance reducing, so we have that

$$|\phi_{k+1}(z) - \phi_k(z)| < 1/n^{k+1}$$

when $|z| < r$. Therefore the functions ϕ_k converge uniformly to a limit $\phi : \Delta_r \to \Delta$ as $k \to \infty$, and we have $\phi(f(z)) = \phi(z)^n$.

We will now prove the uniqueness of ϕ. It suffices to assume that $f(z) = z^n$. Write $\phi(z) = c_1 z + c_k z^k + \dots$, where $c_1, c_k \neq 0$. If this map conjugates $z \mapsto z^n$ to itself, then we have

$$\phi(z^n) = c_1 z^n + c_k z^{nk} + \cdots = \phi(z)^n = c_1^n z^n + n c_1^{n-1} c_k z^{n+k-1} + \dots$$

Therefore $c_1^{n-1} = 1$, and all higher coefficients are zero. ∎

It is worth noting that the Böttcher map ϕ is given somewhat explicitly by

$$\phi(z) = \lim_{k\to\infty} (f^k(z))^{1/n^k}.$$

This will be of use as we continue with the theory of polynomial dynamics. The function ϕ can be extended holomorphically beyond just a small neighborhood of ∞. This is proven in the next few corollaries.

Corollary 13. If $f : X \to X$ has a superattracting fixed point p with basin of attraction $\mathcal{A}$, then the function $z \mapsto |\phi(z)|$, where ϕ is the Böttcher map, extends uniquely to a continuous map $|\phi| : \mathcal{A} \to [0, 1)$, where $|\phi(f(z))| = |\phi(z)|^n$. Moreover, $|\phi|$ is real analytic away from the iterated preimages of p, where $|\phi|$ is zero.

Proof. Suppose ϕ is defined and univalent on a neighborhood V of ∞. For any $z_0 \in \mathcal{A}$, there is a neighborhood U_{z_0} of z_0 in $\mathcal{A}$ and and k large enough that $f^k(z) \in V$ for all $z \in U$. Set $|\phi(z)| = |\phi(f^k(z))|^{1/n^k}$. Clearly, this is the unique map which satisfies $|\phi(z)| = |\phi(f(z))|^n$. Notice that if $k' = k + m$ for some $m \in \mathbb{N}$, then

$$|\phi(f^{k'}(z))|^{1/n^{k'}} = |\phi(f^{k+m}(z))|^{1/n^{k+m}} = |\phi(f^k(z))|^{n^m/n^{k+m}} = |\phi(f^k(z))|^{1/n^k}.$$

Therefore $|\phi|$ is well defined on overlapping neighborhoods. It is clear from the definition that $|\phi|$ is real analytic as long as $f^k(z) \neq p$ for any k. ∎

If $f : X \to X$ has a superattracting fixed point p and Böttcher map ϕ, then let ψ_ϵ be the inverse of ϕ on Δ_ϵ. Let $\mathcal{A}_0$ be the component of $\mathcal{A}(p)$ which contains p. We call $\mathcal{A}_0$ the *immediate basin* of p.

Theorem 16 (Extension of the Böttcher map)**.** There is a unique $0 < r \leqslant 1$ such that ψ_ϵ extends holomorphically to a map $\psi : \Delta_r \to \mathcal{A}_0$. If $r = 1$, then ψ is a conformal isomorphism between Δ and $\mathcal{A}_0$ and p is the only critical point of f in $\mathcal{A}_0$. If $r < 1$, then there is at least one critical point in $\mathcal{A}_0$ on the boundary of $\psi(\Delta_r)$.

Proof. A priori, by analytic continuation, there is a maximal radius $0 < r \leqslant 1$ such that $\phi = \phi_r$ can be defined holomorphically. Moreover, there are no critical points of ϕ_r in Δ_r: if $\phi'(w) = 0$, then $w \neq 0$ because ψ is univalent near zero. But since $\phi(w^n) = f(\psi(w))$, we have that w^n is also a critical point of ψ. Hence, ψ has a sequence of critical points $w, w^n, w^{n^2}, \ldots$ tending to zero, which is impossible. Therefore ψ is locally one to one.

To show that ψ is globally injective on Δ_r, we will use Corollary 7. Since $\phi(\psi(z)) = z$ near zero, this remains true on Δ_r by analytic continuation. Therefore $|\phi(\psi(z))| = |z|$. Suppose that $\psi(w_1) = \psi(w_2)$ and $w_1 \neq w_2$. If we apply $|\phi|$ to both sides, we have $|w_1| = |w_2|$. Choose $|w_1| = |w_2|$ minimal. Because ψ is an open mapping, we can take w_1' sufficiently close to w_1 and w_2' close to w_2 such that $\psi(w_1') = \psi(w_2')$. But we can choose $|w_1'| < |w_1|$, so we contradict the minimality of $|w_1|$. Hence, ψ is injective.

If $r = 1$, then we claim ψ maps Δ onto $\mathcal{A}$. If not, then $U = \psi(\Delta)$ has a boundary point $z_0 \in \mathcal{A}$. Approximate z_0 with a sequence $\psi(w_i)$. We have $|\phi(\psi(w_i))| = |w_i| \to 1$, so it must be that $|\phi(z_0)| = 1$. But this is impossible since $\phi(z_0) \in \Delta$.

Now suppose that $r < 1$. For any $w_0 \in \partial\Delta_r$, choose an accumulation point z_0 of the curve $t \mapsto \psi(tw_0)$. If z_0 is not a critical point of f, then there is a local inverse g of f, with $g(f(z_0)) = z_0$ in a neighborhood of z_0. Now in a neighborhood of w_0, let $\phi(z) = g(\phi(z^n))$, which is holomorphic. If f has no critical points in the boundary of $\psi(\Delta_r)$, then in this way, we can extend ψ to a strictly larger radius circle, a contradiction. $\square$

3.3 Polynomial Dynamics

Most relevant to this text is the case of a polynomial

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0,$$

where $a_d \neq 0$. We will generally assume that f is *monic*; that is, $a_d = 1$. Indeed, since $a_d \neq 0$, we can find non-zero constant c such that $c^{d-1} = a_d$. Now the conjugate polynomial $cf(z/c)$ is monic.

Any polynomial of degree at least two has a superattracting fixed point at ∞. We often hear that for sufficiently large z, $f(z)$ "looks like" $z \mapsto z^d$. The Böttcher conjugacy allows us to make this precise.

The filled Julia set. First, we define the *filled Julia* $K(f) = \{z \in \mathbb{C} \mid \sup_n |f^n(z)| < \infty\}$ as the set of points in $\mathbb{C}$ with bounded orbits under f. Evidently $K(f) \subset \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$. By Böttcher's theorem, there is a neighborhood U of ∞ on which every $z \in U$ tends to ∞ under iteration. If $z_0 \in \mathbb{C}$ has an unbounded orbit, then some iterate of z_0 enters U. Therefore $K(f) = \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$.

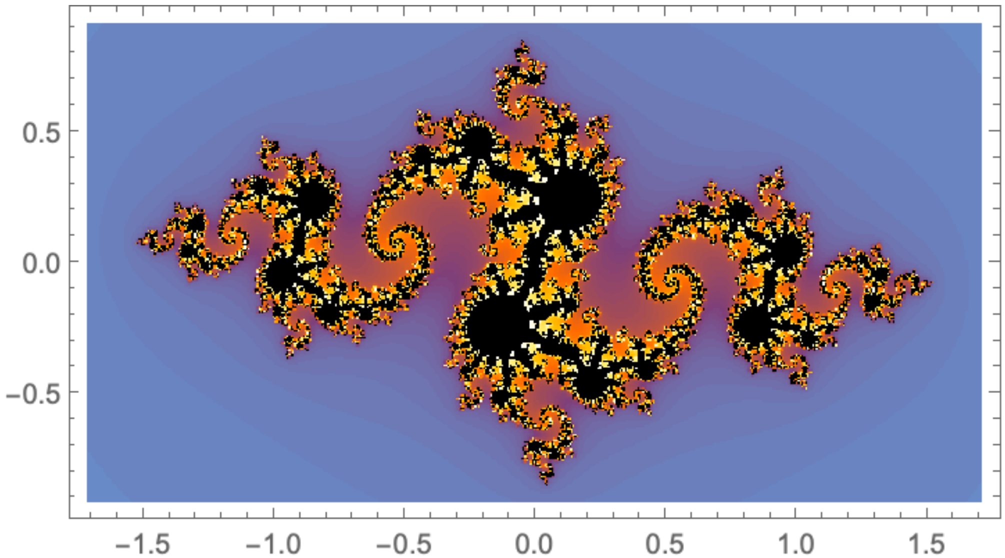

Figure 3.3: The filled Julia set (in black) for $f(z) = z^2 - 0.8 + 0.156i$.

Recall that a compact subset of $\mathbb{C}$ is called *full* if its complement is connected. Any compact subset of $\mathbb{C}$ can be *filled* by adjoining all bounded components of its complement. The name "filled" Julia set suggests the following lemma.

Lemma 13. For any polynomial of degree at least 2, the filled Julia set $J \subset \mathbb{C}$ is compact and full. Its topological boundary ∂K is equal to the Julia set. Hence, $K(f)$ is the union of $J(f)$ and all bounded Fatou components. Each bounded Fatou component is necessarily simply connected.

Proof. Since $K(f) = \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$, $K(f)$ is closed, and is bounded by definition. Therefore $K(f)$ is compact. By Corollary 9, $J(f) = \partial\mathcal{A}(\infty) = \partial K(f)$.

To show that $K(f)$ is full, we must show that $\mathcal{A}(\infty)$ is connected. As stated before, Böttcher's theorem implies that there is $R \gg 0$ such that $D = \mathbb{C} \setminus \overline{\Delta}_R$ is contained in $\mathcal{A}(\infty)$. Let U be a bounded component of the Fatou set. Then $|f^n(z_0)| \leqslant R$ for all $z_0 \in U$: if not, then the maximum modulus principle implies that $|f^n(\hat{z})| > R$ for some $\hat{z} \in \partial U \subset J(f)$. This is impossible since $J(f)$ does not intersect $\mathcal{A}(\infty)$.

Therefore every bounded component of the Fatou set is contained in $K(f)$. If there were two unbounded Fatou components U and V, then they both intersect U. Since U is path connected, there is a path from U to V in U. But such a path must cross $\partial U \subset J(f)$. This is impossible since $J(f)$ does not intersect U. Therefore there is a unique unbounded component, which is thus equal to $\mathbb{C} \setminus K = \mathbb{C} \cap \mathcal{A}(\infty)$. Hence, $K(f)$ is full.

If Γ is a simple closed curve in a bounded Fatou component U, let V be the bounded component of $\mathbb{C} \setminus \Gamma$. Again by the maximum modulus principal, $V \subset K$. In particular, Γ cannot contain points of $J(f)$, so $V \subset U$. It follows that U is simply connected. $\square$

Upon seeing pictures of polynomial Julia sets, one may remark that the Filled Julia set appears to be a union of disk-like blobs strung together by the Julia set. The lemma makes this notion precise.

We now prove a theorem which will be useful in our discussion of the Mandelbrot set.

Theorem 17 (Connected Julia sets and bounded critical orbits)**.** Let f be a polynomial of degree $d \geqslant 2$. If $K(f)$ contains all finite critical points, then $K(f)$ and its boundary $J(f)$ are connected. Moreover, there is a conformal isomorphism

$$\phi : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \overline{\Delta}$$

Such that $\phi \circ f \circ \phi^{-1} = z \mapsto z^d$. If at least one critical point is not in $K(f)$, then $K(f)$ and $J(f)$ have uncountably many connected components.

If we make the change of coordinates $z \mapsto 1/z$, then we can examine $F(\zeta) = 1/f(1/\zeta)$, which has a superattracting fixed point at $z = 0$. Thus we have the corresponding Böttcher map ϕ

$$\phi(\zeta) = \lim_{k\to\infty} (F^k(\zeta))^{1/n^k},$$

which is defined and biholomorphic on a small neighborhood of zero, and conjugating $1/f(1/z)$ to $z \mapsto z^d$. We can assume that f is monic, so $\phi'(0) = 1$. Now define $\hat{\phi}(z) = 1/\phi(1/z)$, which conjugates f to z^d on a neighborhood of ∞.

This theorem says that if all critical points are contained in $K(f)$, then $\hat{\phi}$ is defined on all of $\mathcal{A}(\infty)$. Hence $\hat{\phi}$ is the Riemann map from $\widehat{\mathbb{C}} \setminus K(f)$ to $\widehat{\mathbb{C}} \setminus \overline{\Delta}$.

Proof of Theorem 17. Assume that there are no critical points other than ∞ in $\mathcal{A} = \mathcal{A}(\infty)$. It then follows from Theorem 16 that $\hat{\phi}$ extends to a conformal isomorphism $\mathbb{C}\setminus K(f) \to \mathbb{C}\setminus\overline{\Delta}$. Let $\hat{\psi} = \hat{\phi}^{-1}$. For each annulus

$$\mathbb{A}_{1+\epsilon} = \{z \in \mathbb{C} \mid 1 < |z| < 1 + \epsilon\}$$

maps under $\hat{\psi}$ to a connected set in $\mathbb{C} \setminus K$. Then $\overline{\hat{\psi}(\mathbb{A}_{1+\epsilon})}$ is compact and connected. and contains the Julia set. Indeed,

$$J(f) = \bigcap_{\epsilon>0} \overline{\hat{\psi}(\mathbb{A}_{1+\epsilon})},$$

which must then be connected. It follows from Lemma 13 that $K(f)$ is connected.

Now assume there is at least one critical point in $\mathbb{C}\setminus K(f)$. The Theorem 16 implies that there is a smallest $r > 1$ such that the inverse of $\hat{\phi}$ extends univalently to a map $\hat{\psi} : \mathbb{C}\setminus\overline{\Delta}_r \to \mathbb{C}\setminus K$. Let $U = \hat{\psi}(\mathbb{C} \setminus \overline{\Delta}_r)$. Then ∂U is compact and contains at least one critical point of f.

We will show that $\overline{U}$ separates $\mathbb{C}$ into two parts, each of which contains points of $J = J(f)$. Let c be a critical point of f in ∂U, and let $v = f(c)$ be the critical value. Evidently, $v \in U$, since $|\hat{\phi}(v)| = r^d > r$. Let R be the ray

$$R = \{t\hat{\phi}(v) \mid t \geqslant 1\} \subset \mathbb{C} \setminus \Delta_r,$$

and let $R' = \hat{\psi}(R) \subset U$, which has ending point v. Consider $f^{-1}(R') \subset \overline{U}$. The intersection $U \cap f^{-1}(R')$ consists of d distinct rays, which correspond to the d rays $\sqrt[d]{R} \subset \mathbb{C} \setminus \Delta_r$. The rays in $U \cap f^{-1}(R')$ end at the d points (with multiplicity) of $f^{-1}(v)$. Two of these rays, R_1' and R_2', must then end at c.

Now $R_1' \cup R_2' \subset \overline{U}$ splits $\mathbb{C}$ into two disjoint, connected open sets V_0 and V_1. We claim that $f(V_0)$ and $f(V_1)$ both contain all of $\mathbb{C} \setminus R'$. Indeed, take $\hat{z} \in \partial(f(V_k))$. Then there is a sequence $z_j \in V_k$ such that $f(z_j) \to \hat{z}$. Evidently the z_j are bounded, so there is a subsequence converging to a point $z' \in V_k$. Then $f(z') = \hat{z}$. Since f is an open map, z' lies in $\partial V_k = R_1' \cup R_2'$, which implies that $\hat{z} \in R'$, and we have $\partial(f(V_k)) \subset R'$. Moreover, since $\mathbb{C} \setminus R'$ is connected,

$$f(V_k) \supset \mathbb{C} \setminus R' \supset K(f).$$

Let $J_0 = J \cap V_0$ and $J_1 = J \cap V_1$. Since J is not empty and J is backwards invariant under f, J_0 and J_1 are non-empty. Since J does not intersect $R_1' \cup R_2'$, J_0 and J_1 disconnected J. Corollary 12 then implies that there are uncountably many connected components of J. An analogous argument applies to $K(f)$. □

External rays. In the proof of Theorem 17, we made use of rays which were the images of radial lines under the Böttcher map. This is a generally useful technique, and is made formal with *external rays*. Suppose that $K(f)$ is connected, so that the Böttcher map $\phi : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \overline{\Delta}$ is a conformal isomorphism.

Definition 6 (External rays)**.** For any $t \in \mathbb{R}/\mathbb{Z}$, let

$$\mathcal{R}_t = \{\phi^{-1}(re^{2\pi it}) \mid r > 1\}$$

Then $\mathcal{R}_t$ is called the *external ray* of external angle t.

Observe that

$$f(\phi^{-1}(re^{2\pi it})) = \phi^{-1}((re^{2\pi it})^d) = \phi^{-1}(r^d e^{2\pi idt})$$

Therefore f maps $\mathcal{R}_t$ bijectively to $\mathcal{R}_{dt}$.

We say an external ray $\mathcal{R}_t$ *lands* at a point $\hat{z} \in \partial K(f) = J(f)$ if $\hat{z} = \lim_{s \to 1} \phi^{-1}(se^{2\pi it})$. If such a $\hat{z}$ exists for the ray $\mathcal{R}_t$, then we simply say $\mathcal{R}_t$ lands. Theorem 12 implies that if $J(f)$ is locally connected, then every ray lands. In any case, we have the following.

Theorem 18 (Rational rays)**.** For any polynomial f with $K(f)$ connected, every rational ray lands; that is, if $t \in \mathbb{Q}$, then $\mathcal{R}_t$ lands.

Details are discussed in §18 of [Mil06a].

A ray is *periodic* if $f^n(\mathcal{R}_t) = \mathcal{R}_t$ for some $n \in \mathbb{N}$. Equivalently, $tdn \equiv t \bmod 1$. Then $tdn = t + m$ for some $m \in \mathbb{Z}$, so $t = m/(dn - 1)$ and t is rational. It also easy to see that every rational ray is eventually periodic; that is $f^k(\mathcal{R}_t)$ is periodic for some $k \in \mathbb{N}_0$. Such rays are called *pre-periodic* Hubbard proves the following in [Hub93].

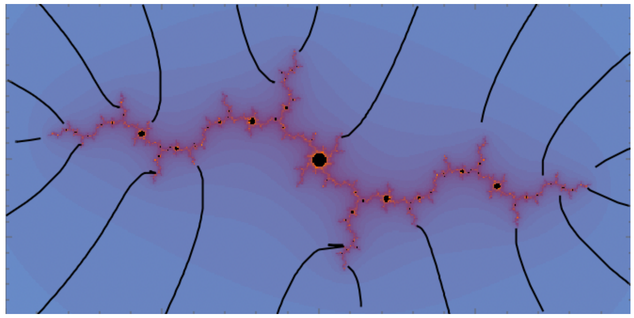

Figure 3.4: External rays for the the map $f(z) = z^2 - 1.2564 + 0.3803i$.

Theorem 19 (Landing of (pre)periodic rays)**.** If f is a polynomial with $K(f)$ connected, then every periodic external ray lands at repelling or parabolic periodic point of f. Every pre-periodic external ray lands at a repelling or parabolic pre-periodic point of f. Every repelling or parabolic periodic or pre-periodic point is the landing point of at least one, and at most finitely many rational rays.

Green's function and equipotentials. If f is a polynomial with Böttcher map ϕ defined in a neighborhood of ∞, then the map $|\phi|$ extends continuously over all of $\mathcal{A}(\infty) = \mathbb{C} \setminus K(f)$. The *Green's function*, sometimes called the *canonical potential function*, associated with $K(f)$ is the function $G : \mathbb{C} \to [0, \infty)$ which is identically 0 on $K(f)$ and is given by

$$G(z) = \log |\phi(z)|$$

for $z \in \mathbb{C} \setminus K(f)$. It is easy to check that G is continuous on $\mathbb{C}$, and harmonic away from $K(f)$. The level sets $\mathcal{C}_r = G^{-1}(S^1_r)$ for $r > 1$ are called *equipotentials*. Notice that

$$G(f(z)) = \log |\phi(f(z))| = \log |\phi(z)^d| = d \log |\phi(z)| = dG(z).$$

Therefore f maps equipotentials to equipotentials. Douady and Hubbard show in [DH09] that the properties

1. G is harmonic,
2. $G(z)/\log |z| \to 1$ as $|z| \to \infty$,
3. $G(z) \to 0$ as $d(z, K(f)) \to 0$,
4. $G(f(z)) = d \cdot G(z)$,

as sufficient to characterize G. Indeed, they imply that

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log |f^n(z)|.$$

Polynomial-like Mappings. We now introduce *polynomial-like mappings*, which generalize the topological and dynamical properties of polynomials, and will be useful going forward. The notion of Polynomial-like mappings was first introduced by Douady and Hubbard [DH85], and is of great importance in understanding the structure of the Mandelbrot set.

Definition 7 (Polynomial-like Mapping)**.** Let $U, V \subsetneq \mathbb{C}$ be simply connected open sets with $\overline{U} \subset V$. The triple (f, U, V) called is a polynomial-like mapping of degree d if $f : U \to V$ is proper and of degree d.

We are only interested in the case where $d \geqslant 2$. Note that f is a branched cover with $d-1$ critical points in U when counted with multiplicity.

While this definition does not require U and V to have analytic boundaries, we can generally assume this to be true. If $U', V' \subsetneq \mathbb{C}$ are simply connected open sets with $\overline{U'} \subset V'$ and $f : U' \to V'$ is holomorphic and proper of degree d, then we can always find a simply connected open set V with analytic boundary so that $\overline{V} \subset V'$ and $\overline{U'} \subset V$. Defining $U = f^{-1}(V)$, it follows that $f : U \to V$ is a polynomial-like mapping of degree d.

As with polynomials, we can define the *filled Julia set* of a polynomial-like mapping $f : U \to V$ to be

$$K(f) = \bigcap_{n>0} f^{-n}(V),$$

i.e. the set of points which remain in U under iteration of f. On $K(f)$, the map f is a well defined dynamical system. The *Julia set* $J(f)$ is defined to be the boundary of $K(f)$.

Example 8. If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, let V be a large disk and $U = f^{-1}(V)$, then (f, U, V) is polynomial-like. Evidently $f : U \to V$ is proper and degree d.

Chapter 4

Quasiconformal Geometry

Complex analysis and topology are greatly intertwined. The Riemann mapping theorem, for instance, gives us an equivalence between a topological class (simply connected proper open subsets of $\mathbb{C}$) and a complex analytic class (surfaces conformally isomorphic to Δ).

The fundamental mappings in the study of topology are continuous functions, while the fundamental mappings in complex analysis are holomorphic functions. We will define *quasiconformal* mappings, which lie in between: they are homeomorphisms with added regularity. These maps will be a useful tool in the analysis of polynomial dynamics and the Mandelbrot set.

Recall that a *conformal mapping* $\phi : A \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic map with non-vanishing derivative. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then an equivalent definition is that at each $x \in A$,

$$d\phi_x = r \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

where $r > 0$. Hence, $d\phi_x$ is a rotation composed with a scaling, and preserves angles in the tangent space. Another equivalent definition is that at each $x \in A$, the differential $d\phi_x$ is orientation preserving and $(d\phi_x)^{-1}(S^1) = S^1_{1/r}$; that is, $d\phi_x$ pulls back the unit circle in the tangents space to the circle of radius r.

The intuition for a quasiconformal mapping ϕ is that it distorts the conformal geometry of its domain to a bounded extent. Heuristically, at each x in its domain, $d\phi_x$ pulls back the unit circle to an ellipse with eccentricity bounded over the domain of ϕ. We will make this precise in the following sections.

4.1 Almost Complex Structures and Pullbacks

For a more thorough treatment of the following material, we recommend [BF14].

Linear algebra. Let $\mathbb{C}_\mathbb{R}$ denote the set $\mathbb{C}$, considered as real two-dimensional vector space with basis $\{1, i\}$. We will use the standard coordinates (x, y) on $\mathbb{C}_\mathbb{R}$, as well as the coordinates $(z, \bar{z})$, related to x and y by the change of variables $z = x + iy$ and $\bar{z} = x - iy$.

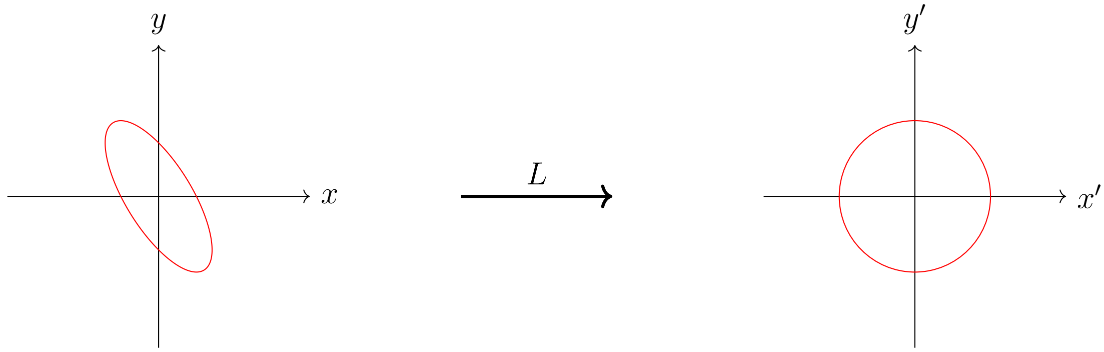

Figure 4.1: The pullback of $E(L)$ of S^1 under L.

Any linear transformation $L : \mathbb{C}_\mathbb{R} \to \mathbb{C}_\mathbb{R}$ may be written as $L(z) = az + b\bar{z}$, with $a, b \in \mathbb{C}$. We will only consider *orientation preserving* transformations, which are precisely when $|a| > |b|$. We define the *Beltrami coefficient* of L to be $\mu(L) := b/a$.

Notice that $\mu(L) \in \Delta$ whenever L is orientation preserving. Moreover, $\mu(L) = 0$ if and only if $b = 0$, which occurs if and only if L is holomorphic.

Let $E(L) = L^{-1}(S^1)$. Then $E(L)$ is an ellipse, and a circle if and only if $\mu(L) = 0$. One may compute that the half-major and half-minor axes of $E(L)$ are $\frac{1}{|a|(1-|\mu(L)|)}$ and $\frac{1}{|a|(1+|\mu(L)|)}$ respectively. We define the *dilatation* $K(L)$ of L to be the ratio of the major axis to the minor axis:

$$K(L) = \frac{1 + |\mu(L)|}{1 - |\mu(L)|} = \frac{|a| + |b|}{|a| - |b|}$$

We will often call $\mu(L)$ the *complex dilatation* of L. The dilatation determines the shape of $E(L)$ up to complex scaling, but not the position of the axes. The Beltrami coefficient determines the shape and position of the ellipse up to real scaling. Conversely, if we are given an ellipse E, the Beltrami coefficient is determined by $\mu(E) = \frac{M-m}{M+m} e^{2i\theta}$, where M and m are the major and minor axes respectively, and θ is the argument of the direction of the minor axis, chosen in $[0, \pi)$. One may compute that

$$K(L^{-1}) = K(L), \qquad K(L_1 \circ L_2) \leqslant K(L_1)K(L_2).$$

Complexification. As it stands, $\mathbb{C}_\mathbb{R}$ is 2-dimensional vector space over $\mathbb{R}$. We are, of course, more familiar with $\mathbb{C}$ considered as a 1-dimensional complex vector space. To convert $\mathbb{C}$ into a vector space over $\mathbb{C}$, we wish to extend the action of $\mathbb{R}$ on $\mathbb{C}$, given by $r(x, y) = (rx, ry)$, to an action of $\mathbb{C}$ on $\mathbb{C}$ which gives $\mathbb{C}$ the structure of a complex vector space. This process is called *complexification*.

The most familiar way of doing this is to consider the linear transformation $I : \mathbb{C}_\mathbb{R} \to \mathbb{C}_\mathbb{R}$ given in (x, y) coordinates by

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which rotates the plane 90 degrees counterclockwise. Then given a complex number $a+bi \in \mathbb{C}$ and a vector $z \in \mathbb{C}_{\mathbb{R}}$, we define

$$(a+bi)z := az + bI(z).$$

One can check with this $\mathbb{C}$ action, $\mathbb{C}$ obtains the structure of a complex vector space. Moreover, its dimension is 1, since it is spanned by 1. We call the pair $(\mathbb{C}, I)$ *the standard conformal structure* on $\mathbb{C}$, and denote this pair σ_0. This conformal structure extends the real structure of $\mathbb{C}_{\mathbb{R}}$ in that multiplication by a real number in σ_0 agrees with multiplication in $\mathbb{C}_{\mathbb{R}}$.

This complexification of $\mathbb{C}_{\mathbb{R}}$ is by no means unique. Any invertible $\mathbb{R}$-linear map $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ can be used to define a new conformal structure $\sigma(L)$ on $\mathbb{C}_{\mathbb{R}}$ which extends its real structure. The goal is to define 'multiplication by i': we choose an $\mathbb{R}$-linear transformation $J : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ with $J^2 = -\mathrm{Id}$, and let $(a+ib) \cdot z = az + bJ(z)$ for any $c, z \in \mathbb{C}$.

We do this in the following way: let I be as in σ_0. We then let $J = L^{-1} \circ I \circ L$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "L"] \arrow[d, dashed, "J"'] & \mathbb{C} \arrow[d, "I"] \\
\mathbb{C} & \mathbb{C} \arrow[l, "L^{-1}"']
\end{tikzcd}$$

We can easily verify that the conformal structure $\sigma(L) = (\mathbb{C}, J)$ makes $\mathbb{C}$ a complex vector space, and that L defines an isomorphism from $\sigma(L)$ to σ_0:

$$L((a+bi)z) = L(az + bJ(z)) = aL(z) + bL(J(z)) = aL(z) + bI(L(z)) = (a+bi)L(z).$$

By definition, the unit circle S^1 pulls back under L to the ellipse $E(L)$. The transformation J rotates vectors around $E(L)$, just as I rotates vectors around S^1. The Beltrami coefficient $\mu(L)$ determines the conformal structure $\mu(L)$, as if $\mu(L_1) = \mu(L_2)$, then $L_1^{-1} \circ I \circ L_1 = L_2^{-1} \circ I \circ L_2$. To see this, suppose that $L_1(z) = a_1 z + b_1 \bar{z}$ and $L_2(z) = a_2 z + b_2 \bar{z}$. Then $L_2 = \frac{a_2}{a_1} L_1$. We then the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "L_1"] \arrow[d, dashed, "J_1"'] & \mathbb{C} \arrow[r, "a_2 z/a_1"] \arrow[d, "I"] & \mathbb{C} \arrow[d, "I"] \\
\mathbb{C} \arrow[r, "L_1"] & \mathbb{C} \arrow[r, "a_1 z/a_1"] & \mathbb{C}
\end{tikzcd}$$

But the map $(z \mapsto a_2 z/a_1) \circ L_1$ is just L_2, so the map $L_2^{-1} \circ I \circ L_2$ is just J_1.

Almost complex structures. Let $U \subset \mathbb{C}$ be an open set, and let $TU = \bigcup_{u \in U} T_u U$ be the tangent bundle over U. Each tangent space is viewed as a copy of $\mathbb{C}_{\mathbb{R}}$.

Definition 8 (Almost complex structure)**.** An *almost complex structure* on σ on U is a choice of conformal structure $\sigma(u)$ on T_uU for almost every u in U, such that the map $u \mapsto \mu(u) \in \Delta$ from u to the complex dilatation of $\sigma(u)$ is Lebesgue measurable.

We can also think of an almost complex structure as a measurable field of infinitesimal ellipses $\mathcal{E} \subset TU$. By this, we mean an ellipse $E_u \subset T_uU$ defined up to real scaling for almost every $u \in U$, such that the map $u \mapsto \mu(u)$ is Lebesgue measurable, and where $\mu(u) = \mu(E_u)$ denotes the Beltrami coefficient of E_u.

We define the dilatation of an almost complex structure σ as

$$K(\sigma) = \operatorname*{ess\,sup}_{u\in U} K(u), \qquad \text{where} \quad K(u) = \frac{1+|\mu(u)|}{1-|\mu(u)|}.$$

Notice that $K(\sigma) \in [1, \infty]$. Moreover, any measurable function $\mu : U \to \Delta$ defines an almost complex structure.

We may also obtain almost complex structures from certain maps. Suppose $U, V \subset \mathbb{C}$ are open sets and let $D^+(U, V)$ denote the class of continuous, orientation preserving functions $f : U \to V$ which are $\mathbb{R}$-differentiable almost everywhere, and with a non-singular derivative $df_u : T_uU \to T_{f(u)}V$ almost everywhere, depending measurably on u. We use the infinitesimal coordinates dz and $d\bar{z}$ on the tangent spaces, so that

$$df_u = \partial_z f(u)dz + \partial_{\bar{z}} f(u)d\bar{z}$$

where

$$\partial_z f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \partial_{\bar{z}} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Whenever df_u is non-singular, it defines an ellipse in T_uU with Beltrami coefficient

$$\mu_f(u) = \frac{\partial_{\bar{z}} f(u)}{\partial_z f(u)}.$$

Thus, f defines a new conformal structure on $T_u(U)$. The dilatation is written as

$$K_f(u) := K(df_u) = \frac{1+|\mu_f(u)|}{1-|\mu_f(u)|}.$$

Observe that $K_f(u) = 1$ if and only if the Cauchy-Riemann equation $\partial_{\bar{z}} f(u) = 0$ is satisfied, which occurs if and only if the ellipse is a circle.

If we do the same for each $u \in U$ for which f is differentiable and with non-singular derivative, we obtain an almost complex structure σ_f on U, with Beltrami coefficient μ_f. We say that σ_f is the *pullback* of σ_0 by f, where σ_0 is the *standard complex structure* with Beltrami coefficient $\mu_0 \equiv 0$. Equivalently, we may say that μ_f is the pullback of μ_0 by f. We write for almost every u,

$$\mu_f(u) = f^*\mu_0(u), \quad \text{or} \quad \sigma_f(u) = f^*\sigma_0(u).$$

The dilatation of this complex structure is $K_f = \operatorname{ess\,sup}_{u\in U} K_f(u)$.

We may generalize the notion of pullback and consider the pullback of any almost complex structure σ, not necessarily σ_0. In order to do so, we need slightly stronger assumptions on f. We require f to be *absolutely continuous with respect to Lebesgue measure*, meaning that the preimage of any measure zero set is of measure zero. Let $D_0^+(U,V) \subset D^+(U,V)$ be the subclass of functions with this property. Let $\mu : V \to \Delta$ be the Beltrami coefficient corresponding to an almost complex structure σ on V.

Let E_v be the infinitesimal ellipse in T_vV defined by σ for almost every $v \in V$. By the pullback of the infinitesimal ellipse field $\mathcal{E}$, we mean the field of infinitesimal ellipses $\mathcal{E}'$ defined by $E'_u = (D_uf)^{-1}(E_{f(u)})$. This exists and is well defined as long as D_uf exists and is non-singular. Since f is absolutely continuous, this holds for almost all $u \in U$. Therefore $\mathcal{E}'$ defines an almost complex structure on U. We write

$$(U, \mu_1) \xrightarrow{f} (V, \mu_2)$$

to denote that f maps from U to V and $\mu_1 = f^*\mu_2$. If μ is given by another map $g \in D^+(V,W)$, so $\mu = \mu_g$, then

$$f^*\mu_g = f^*(g^*\mu_0) = (g \circ f)^*\mu_0 = \mu_{g\circ f}.$$

If then follows that $K_{g\circ f} \leqslant K_f \cdot K_g$. We can compute the pullback more explicitly. We have

$$f^*\mu(u) = \frac{\partial_{\bar z} f(u) + \mu(f(u))\overline{\partial_z f(u)}}{\partial_z f(u) + \mu(f(u))\overline{\partial_{\bar z} f(u)}}.$$

If f is holomorphic, so that $\partial_{\bar z} f \equiv 0$, this reduces to

$$f^*\mu(u) = \mu(f(u))\frac{\overline{\partial_z f(u)}}{\partial_z f(u)}.$$

A particularly relevant class of almost complex structures on U are those which are preserved by a function $f : U \to U$.

> **Definition 9.** (f-invariant almost complex structure) Let U be an open subset of $\mathbb{C}$ and $f : U \to U$ a map in $D_0^+(U,U)$. Let σ be an almost complex structure on U with Beltrami coefficient μ. We say that μ (or σ) is f-invariant if $f^*\mu(u) = \mu(u)$ for almost every $u \in U$. We also write $f^*\sigma = \sigma$. In this case, we say that f is holomorphic with respect to μ or σ.

One property of note is that if $F \in D_0^+(V,V)$, $G \in D_0^+(U,U)$, and $f \in D_0^+(U,V)$ with $f \circ G = F \circ f$, then if σ is an F-invariant structure on V, the pullback $f^*\mu$ is a G-invariant structure on U, as

$$G^*(f^*\mu) = (f \circ G)^*\mu = (F \circ f)^*\mu = f^*(F^*\mu) = f^*\mu.$$

The orientation reversing case. Up until this point, we have only defined the pullback of an almost complex structure in the case of an orientation preserving map. One can also formulate a definition for orientation reversing maps.

Let $U, V \subset \mathbb{C}$ and let $D_0^-(U,V)$ be the defined analogously to $D_0^+(U,V)$, but instead comprising orientation reversing maps. Let μ denote a Beltrami coefficient in V.

Definition 10 (Pullback of an orientation reversing map)**.** Suppose $f \in D_0^-(U,V)$. We define the *pullback of μ under f* as

$$f^{\circledast}\mu := \overline{f}^{*}\overline{\mu}.$$

In particular,

$$f^{\circledast}\mu_0 = \mu_{\overline{f}} \quad \text{and} \quad K_f = K_{\overline{f}}.$$

With this definition, we can now define what it means for an almost complex structure to be symmetric with respect to some curve. Notice that $c(z) = \bar{z}$ and $\tau(z) = 1/\bar{z}$ have $\mathbb{R}$ and S^1 respectively as axes of symmetry. We say a Beltrami coefficient μ is *symmetric with respect to* $\mathbb{R}$ if and only if $c^{\circledast}\mu = \mu$. Similarly, we say μ is *symmetric with respect to* S^1 if and only if $\tau^{\circledast}\mu = \mu$.

4.2 Quasiconformal Mappings

There are several equivalent definitions of a conformal mapping. The intuition for such mappings comes from the differentiable case. The general idea is that a quasiconformal mapping between domains U and V in $\mathbb{C}$ is a homeomorphism $\phi : U \to V$ such that $\|\mu_\phi\|_\infty < 1$. In other words, $d\phi$ pulls back infinitesimal circles in TV to infinitesimal circles of bounded eccentricity. Equivalently, the dilatation K_ϕ is finite.

However, we wish to formulate a definition of quasiconformal which applies to a wider class of homeomorphisms, not just those which are differentiable. We will begin with the most geometric definition of quasiconformal.

4.2.1 Geometric Definition of Quasiconformal Mappings

An *annulus* $A \subset \mathbb{C}$ is an open, doubly connected domain in $\mathbb{C}$, meaning that $\mathbb{C} \setminus A$ comprises two connected components. As an open subset of $\mathbb{C}$, A inherits the structure of a Riemann surface. In general, we can also consider abstract Riemann surfaces which are conformally isomorphic to annuli in $\mathbb{C}$. The most basic example of an annulus is $\mathbb{A}_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\}$, for $0 \leqslant r < R < \infty$. We have the following classification of annuli.

Proposition 8 (Classification of Annuli)**.** Every annulus $A \subset \mathbb{C}$ is conformally isomorphic to a standard annulus $\mathbb{A}_{r,R}$. Two standard annuli $\mathbb{A}_{r_1,R_1}$ and $\mathbb{A}_{r_2,R_2}$ are isomorphic if and only if $R_1/r_1 = R_2/r_2$.

We will not give a proof here, but one is given in §4 of [McM23]. Given any annulus A which is isomorphic to $\mathbb{A}_{r,R}$, we define the *modulus* of A to be $\text{mod}(A) = \frac{1}{2\pi}\log(R/r)$. It follows from Proposition 7 that $\text{mod}(A)$ is well defined and is a *conformal invariant* of A: two annuli A and A' are conformally isomorphic if and only if $\text{mod}(A) = \text{mod}(A')$.

We can now state the first definition of quasiconformal mappings.

Definition 11 (Geometric definition of quasiconformal mappings)**.** Suppose U and V are open domains in $\mathbb{C}$ and $\phi : U \to V$ is an orientation preserving homeomorphism. Let $K \geqslant 1$ be given. We say ϕ is K-quasiconformal if

$$\frac{\text{mod}(A)}{K} \leqslant \text{mod}(\phi(A)) \leqslant K\text{mod}(A),$$

for each annulus A with $\overline{A} \subset U$.

A conformal homeomorphism must be 1-quasiconformal by Proposition 8. In fact, any 1-quasiconformal map must be conformal (implying that it has partial derivatives everywhere in U). This is known as Weyl's lemma, and will be stated in more detail later.

This definition makes it clear that quasiconformal mappings are those which distort the conformal geometry of their domain to a bounded extent. While this definition of quasiconformal is intuitive, it is not always the easiest to work with. We will next state the most modern definition of a quasiconformal mapping.

4.2.2 Analytic Definitions of Quasiconformal Mappings

A *test function* $h : U \to \mathbb{C}$ is a C^∞ function with compact support in U. Let $C_c^\infty(U)$ denote the set of test functions on U. We can give $C_c^\infty(U)$ the *canonical LF-topology*. With this topology, C_c^∞ is a topological vector space, and the *distributions* are precisely the continuous linear functionals $L : C_c^\infty(U) \to \mathbb{C}$. For example, if $\phi \in L^1_{\text{loc}}(U)$, then ϕ defines a distribution

$$L_\phi(h) := \int_U \phi \cdot h \, dx$$

where the integral is taken with respect to Lebesgue measure. If h is a test function, then $\partial_z h$ and $\partial_{\bar{z}} h$ are also test functions. Hence, if L is a distribution, then $h \mapsto L(\partial_z h)$ and $h \mapsto L(\partial_{\bar{z}} h)$ are also distributions.

Definition 12 (Distributional derivatives)**.** If L is a distribution, we define its distributional derivatives to be

$$(\partial L)(h) := -L(\partial_z h), \quad (\overline{\partial} L)(h) := -L(\partial_{\bar{z}} h).$$

Notice that if ϕ is C^1, $\partial L_\phi = L_{\partial_z \phi}$ and $\overline{\partial} L_\phi = L_{\partial_{\bar{z}} \phi}$. Indeed, using integration by parts and the fact that h is compactly supported in U, we have

$$\partial L_\phi(h) = -\int_U \phi \cdot \partial_z h \, dx = -\int_{\partial U} \phi \cdot h \, dx + \int_U \partial_z \phi \cdot h \, dx = L_{\partial_z \phi}(h).$$

An identical argument shows that $\overline{\partial} L_\phi = L_{\partial_{\bar{z}} \phi}$.

A priori, ϕ need not be C^1, but if there happen to be functions ϕ_1 and ϕ_2 such that $\partial L_\phi = L_{\phi_1}$ and $\overline{\partial} L_\phi = L_{\phi_2}$, then we denote $\phi_1 = \partial \phi$ and $\phi_2 = \overline{\partial} \phi$ and call them the *distributional derivatives of* ϕ in $L^1_{\text{loc}}(U)$. Precisely, ϕ_1 and ϕ_2 must satisfy

$$\int_U \phi_1 \cdot h \, dx = -\int_U \phi \cdot \partial_z h \, dx \quad \text{and} \quad \int_U \phi_2 \cdot h \, dx = -\int_U \phi \cdot \partial_{\bar{z}} h \, dx$$

for any test function h. More generally, if $\phi_1, \phi_2 \in L^p_{\text{loc}}(U)$, then we say ϕ has distributional derivatives in $L^p_{\text{loc}}(U)$. The space of functions in $L^p_{\text{loc}}(U)$ having distributional derivatives in $L^p_{\text{loc}}(U)$ is called *Sobolev space* $W^{1,p}_{\text{loc}}(U)$.

If $\phi \in C^1$, then its distributional derivatives are just its derivatives in the ordinary sense. We can now state the first analytic definition of a quasiconformal mapping.

Definition 13 (First analytic definition of a K-quasiconformal mapping)**.** Let U and V be domains in $\mathbb{C}$, and let $K \geqslant 1$ be given. Set $k = (K-1)/(K+1)$. Then $\phi : U \to V$ is K*-quasiconformal* if and only if

1. ϕ is a homeomorphism,
2. ϕ has distributional derivatives $\partial \phi$ and $\overline{\partial} \phi$ in L^2_{loc} almost everywhere,
3. $\left\|\overline{\partial} \phi\right\|_{2,\text{loc}} \leqslant k \left\|\partial \phi\right\|_{2,\text{loc}}$.

Notice that if ϕ is a C^1 diffeomorphism, the ϕ is quasiconformal if $|\partial_z \phi| \leqslant k |\partial_{\bar{z}} \phi|$ on U for some $k < 1$. Moreover, if ϕ is a C^1 diffeomorphism between compact sets, then ϕ is quasiconformal.

Second analytic definition. We may also state an equivalent definition which relates more to previous discussion of almost complex structures.

Definition 14 (Absolute continuity on an interval)**.** A continuous complex valued function f defined on an interval $I \subset \mathbb{R}$ is said to be *absolutely continuous* if for every ϵ, there exists δ such that $\sum_j |f(b_j) - f(a_j)| < \epsilon$ for every finite sequence of non-intersecting intervals (a_j, b_j) whose closure is contained in I with $\sum_j |b_j - a_j|. < \delta$.

If f is absolutely continuous on I, then f is of bounded variation, and hence f is differentiable almost everywhere. We may generalize this notion to domains $U \subset \mathbb{C}$ as follows.

Definition 15 (ACL, absolute continuity on lines)**.** A continuous function $f : U \to \mathbb{C}$ is said to be *absolutely continuous on lines* if for any family of parallel lines in any disk compactly contained in U, f is absolutely continuous on almost all of them.

It now follows that if f is ACL, then f has partial derivatives in the ordinary sense almost everywhere. While the existence of partial derivatives at a point does not a priori imply differentiability, we habe the following result of Gehring and Lehto, which is proven in [Ahl66] pages 17-18.

Theorem 20 (Differentiability almost everywhere)**.** Let $f : U \to V$ be a continuous mapping. If f has partial derivatives f_x and f_y almost everywhere, then f is differentiable almost everywhere.

We are now ready to state the second analytic definition of quasiconformal.

Definition 16 (Second analytic definition of a K-quasiconformal mapping)**.** A mapping $\phi : U \to V$ between domains in $\mathbb{C}$ is K-quasiconformal if and only if

1. ϕ is a homeomorphism,
2. ϕ is ACL,
3. $|\partial_{\bar{z}} f| \leqslant k|\partial_z f|$, where $k := \frac{1-K}{1+K}$.

For a proof that the definitions of quasiconformal we have given so far are equivalent, see [BF14]. However, we will state the following lemma, which relates the ACL property to our first analytic definition.

Lemma 14 (ACL characterization of Sobolev spaces)**.** A continuous function $f : U \to V$ is in $W^{1,1}_{\text{loc}}$ if and only if f is ACL and its ordinary partial derivatives are in $L^1_{\text{loc}}(U)$. When this is the case, the distributional and ordinary derivatives coincide.

This is proven in [Ahl66].

4.2.3 Properties of Quasiconformal Mappings

If $\phi : U \to V$ is K-quasiconformal, then ϕ has partial derivatives $\partial_z \phi$ and $\partial_{\bar{z}} \phi$ almost everywhere, and we have the following theorem from [Ahl66].

Theorem 21. If $\phi : U \to V$ is quasiconformal, then $\phi(E)$ is Lebesgue measure zero if and only if E is measure zero. Moreover the Jacobian determinant $\mathrm{Jac}(\phi)$ satisfies

$$m(\phi(E)) = \int_E \mathrm{Jac}(\phi)\, dx,$$

It then follows that $\mathrm{Jac}(\phi) > 0$ almost everywhere, so ϕ is orientation preserving. We needed to requires this in the geometric definition of quasiconformal, but it is a consequence of the analytic definition.

Hence we have that $\phi \in D_0^+(U, V)$. Therefore the pullback $\phi^*\mu$ of any measurable Beltrami coefficient on V is defined. In particular, if $k = (1-K)/(1+K)$, then $\mu_\phi := \phi^*\mu_0 = \partial_{\bar{z}}\phi/\partial_z\phi$ satisfies $\|\mu_\phi\|_\infty \leqslant k$. Hence, a quasiconformal mapping distorts the almost complex structure of its domain by at most a factor of k, and pulls back circles in the tangent bundle of V to ellipses of globally bounded eccentricity in TU.

We shall now state a few basic properties. Proofs can be found in [Ahl66].

- If ϕ is K-quasiconformal, then ϕ^{-1} is K-quasiconformal.
- If ϕ is K-quasiconformal, then any composition on the left or right with a conformal mapping is K-quasiconformal.
- The composition of a K_1-quasiconformal and a K_2-quasiconformal mapping is K_1K_2-quasiconformal.
- A homeomorphism ϕ is K-quasiconformal if and only if it is locally K-quasiconformal.
- If ϕ is K-quasiconformal and of class C^1, then the dilitation of the infinitesimal ellipse E_z in T_zU is bounded by K for all $z \in U$.
- If ϕ is K-quasiconformal, the ϕ satisfies the Hölder condition

  $$|\phi(z_1) - \phi(z_2)| \leqslant M|z_1 - z_2|^{1/K}$$

  on every compact subset of U.
- If ϕ is bilipschitz, i.e., there exists $L > 0$ such that

  $$L^{-1}|z_1 - z_2| \leqslant L|\phi(z_1) - \phi(z_2)| < L|z_1 - z_2|$$

  then ϕ is quasiconformal.

As alluded to earlier, there is Weyl's lemma, which we will make use of. See [Ahl66] page 16.

Theorem 22 (Weyl's lemma)**.** If ϕ is 1-quasiconformal, then ϕ is conformal. In other words, if ϕ is quasiconformal and $\partial_{\bar{z}}\phi = 0$ almost everywhere, then ϕ is conformal.

It is often useful to define mappings piecewise by gluing together quasiconformal mappings along well behaved arcs. This process is called quasiconformal surgery, and the hope is that the resulting map is also quasiconformal. We have the following, which is proven in [Hub06].

Theorem 23 (Removability of quasiarcs)**.** Suppose that $\Gamma \subset U$ is the image of a straight line under a quasiconformal mapping (such an arc is called a quasiarc). If $\phi : U \to V$ is K-quasiconformal on $U \setminus \Gamma$, then ϕ is K-quasiconformal on U. Hence Γ is quasiconformally removable.

Examples of quasiconformally removable sets then include points and smooth curves. As a result, we can glue quasiconformal maps along quasiarcs, and the resulting map is quasiconformal.

In our study of the local connectivity of the Mandelbrot set, we will need to extract quasiconformal limits of quasiconformal mappings. The following theorem is thus quite useful.

Theorem 24 (Compactness)**.** The space of K-quasiconformal mappings $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ is compact in the topology of local uniform convergence.

A proof is given on page 131 of [Hub06], but the general idea is to use the Arzela-Ascoli theorem to prove that this family is compact in $C^0(U, \mathbb{C})$, and then to show that it equals its closure. This amounts to showing that all limits are homeomorphisms with distributional derivatives in L^2_{loc} satisfying $\left\|\overline{\partial}\phi\right\|_{2,\text{loc}} \leqslant k \left\|\partial\phi\right\|_{2,\text{loc}}$, where $k = (K-1)/(K+1)$.

4.3 The Measurable Riemann Mapping Theorem

We have established that if $\phi : U \to V$ is quasiconformal, the pullback $\phi^*\sigma_0$ defines an almost complex structure on U. We now pose the converse: given an almost complex structure μ on U, when is there a quasiconformal mapping $\phi : U \to V$ with $\phi^*\mu_0 = \mu$? The Measurable Riemann mapping theorem gives us an answer. This theorem is foundational in the study of quasiconformal geometry and in complex dynamics.

Theorem 25 (Measurable Riemann mapping theorem)**.** Let $U \subseteq \mathbb{C}$ be a simply connected open set. Let σ be an almost complex structure on U with Beltrami coefficient μ. Suppose the dilatation of σ is uniformly bounded, that is $K(\sigma) < \infty$, or equivalently the essential supremum of $|\mu|$ on U is

$$\|\mu\|_\infty = k < 1.$$

Then μ is *integrable*, that is there exists a quasiconformal mapping $\phi : U \to \Delta$ if $U \subsetneq \mathbb{C}$ or $\phi : U \to \mathbb{C}$ if $U = \mathbb{C}$ such that $\phi^*\mu_0(z) = \mu(z)$ for almost all $z \in U$. Moreover, ϕ is unique up to post-composition with automorphisms Δ or $\mathbb{C}$.

Notice that this theorem generalizes the classical Riemann mapping theorem in the setting of quasiconformal mappings, for if we let $\mu = \mu_0$, then the statement is the familiar Riemann mapping theorem. Notice that the condition $\phi^*\mu_0(z) = \mu(z)$ is equivalent to the Beltrami equation

$$\frac{\partial \phi}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z},$$

so this theorem is fundamentally about solving a partial differential equation.

To prove the theorem, we first have the following lemma.

Lemma 15 (The case when μ is real analytic)**.** If μ is real analytic, then every $z \in U$ has a neighborhood V such that there is a real analytic function $f : V \to \mathbb{C}$ that is a homeomorphism onto its image and satisfies the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}.$$

Moreover, if $f_1 : V_1 \to \mathbb{C}$ and $f_2 : V_2 \to \mathbb{C}$ are two such functions, there exists an analytic homeomorphism

$$h : f_1(V_1 \cap V_2) \to f_2(V_1 \cap V_2)$$

such that $f_2 = h \circ f_1$ on $V_1 \cap V_2$.

Proof of Lemma 13. For any $z \in U$, write $z := (x, y)$. The trick is to consider x and y as complex variables, so that $\mathbb{C}$ is identified with the real locus $\mathbb{R}^2 \subset \mathbb{C}^2$. Take any $z_0 = (x_0, y_0) \in U$. Since μ is real analytic at z_0, its power series must converge in a neighborhood W of z_0 in $\mathbb{C}^2$. We can rewrite the Beltrami equation as

$$(1 - \mu(x, y))\frac{\partial f}{\partial x} + i(1 + \mu(x, y))\frac{\partial f}{\partial y} = 0. \tag{4.1}$$

Solutions of (3.1) are precisely the functions which are constant on the solutions of

$$\frac{dy}{dx} = i\frac{1 + \mu}{1 - \mu}. \tag{4.2}$$

which are guaranteed by the Picard–Lindelöf theorem. Indeed, if $y(x)$ defines a curve satisfying (3.2), and $f(x, y)$ is a function constant on $(x, y(x))$, let $g(x) = f(x, y(x))$. Then we have

$$\begin{aligned} g'(x) &= \frac{\partial f}{\partial x} + \frac{dy}{dx}\frac{\partial f}{\partial y} \\ 0 &= \frac{\partial f}{\partial x} + i\frac{1 + \mu(x, y)}{1 - \mu(x, y)}\frac{\partial f}{\partial y} \\ &= (1 - \mu(x, y))\frac{\partial f}{\partial x} + i(1 + \mu(x, y))\frac{\partial f}{\partial y}, \end{aligned}$$

and hence (3.1) is satisfied. Moreover, if (3.1) is satisfied and $y(x)$ satisfies (3.2), then we can reverse this process. Again let $g(x) = f(x, y(x))$. We have

$$\begin{aligned}
0 &= (1-\mu(x,y))\frac{\partial f}{\partial x} + i(1+\mu(x,y))\frac{\partial f}{\partial y} \\
&= \frac{\partial f}{\partial x} + i\frac{1+\mu(x,y)}{1-\mu(x,y)}\frac{\partial f}{\partial y} \\
&= \frac{\partial f}{\partial x} + \frac{dy}{dx}\frac{\partial f}{\partial y} \\
&= g'(x),
\end{aligned}$$

so f vanishes on $y(x)$.

Therefore solutions of (3.1) are determined by their values on any subspace transversal to the solution on (3.2) at (x_0, y_0). For instance, take the line $x = x_0$ intersected with W. Let f be the solution of (3.1) which takes the value y on this line. Then of course $\frac{\partial f}{\partial y}(x_0, y_0) = 1$, so

$$\frac{\partial f}{\partial x}(x_0, y_0) = -i\frac{1+\mu(x_0,y_0)}{1-\mu(x_0,y_0)},$$

which cannot be real. Therefore $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ span $\mathbb{C}$, so f induces a local homeomorphism between $W \cap \mathbb{R}^2$ near (x_0, y_0) and $\mathbb{C}$, as desired.

Moreover, if g is another solution to (3.1) at (x_0, y_0), then let $h(y) = g(x_0, y)$, the restriction of g to the line $x = x_0$, which is holomorphic. Then we have $g = h \circ f$. $\square$

We will now prove Theorem 25.

Proof of Theorem 25. It suffices to assume that μ has support in Δ. If $U \neq \mathbb{C}$, then there is a conformal isomorphism $\phi : \Delta \to U$. If we solve the Beltrami equation for $\mu_\phi = \phi^*\mu$, then we precompose with ϕ^{-1} and we obtain the desired map on U. If $U = \mathbb{C}$, then we solve the Beltrami equation $\partial_{\bar{z}} f_1 = \mu\chi_\Delta \cdot \partial_z f$ on Δ_ρ for some $\rho > 1$ and produce a K_1-quasiconformal map $f_1 : \Delta_\rho \to \Delta_\rho$ which is holomorphic outside of $\overline{\Delta}$. Then $f_1|_{S^1}$ is an analytic homeomorphism, so by the Beurling-Ahlfors quasiconformal extension [BA56], f_1 extends to a quasiconformal map $\mathbb{C} \to \mathbb{C}$ which is holomorphic outside of $\overline{\Delta}$.

Since $\widehat{\mathbb{C}} \setminus f_1(\overline{\Delta})$ is isomorphic to Δ, we can use the same method to find a K_2-quasiconformal mapping $f_2 : \mathbb{C} \to \mathbb{C}$ with $\partial_{\bar{z}} f_2 = (f_1^{-1})^*\mu\partial_z f_2$ on $\widehat{\mathbb{C}} \setminus f_1(\overline{\Delta})$ and f_2 holomorphic on $f_1(\Delta)$. The composite $f = f_2 \circ f_1$ is $\max\{K_1, K_2\}$-quasiconformal and the desired map.

Now assume that $\mu \in L^\infty(\mathbb{C})$ with support in Δ and $\|\mu\|_\infty = k < 1$. Let

$$\eta(z) := \frac{1}{\pi}e^{-|z|^2}, \qquad \eta_\epsilon(z) := \frac{1}{\epsilon^2}\eta(z/\epsilon), \quad \epsilon > 0.$$

Let μ_ϵ be the convolution $\mu_\epsilon := \mu * \eta_\epsilon$. Then μ_ϵ is a sequence of real analytic functions in L^∞ with $\|\mu_\epsilon\|_\infty \leqslant k$. Since, μ_ϵ converges to μ almost everywhere and $|\mu_\epsilon| \leqslant \|\mu\|_\infty$, the dominated convergence theorem tells us that μ_ϵ converges to μ in L^1.

The injective real analytic solutions of

$$\frac{\partial f}{\partial \bar{z}} = \mu_\epsilon \frac{\partial f}{\partial z},$$

defined in open subsets of $\mathbb{C}$ from Lemma 15 form an atlas for a new Riemann surface structure on $\mathbb{C}$. Call this Riemann surface $\mathbb{C}_{\mu_\epsilon}$. Then $\Delta \subset \mathbb{C}_{\mu_\epsilon}$ is a new Riemann surface Δ_{μ_ϵ}, which must be isomorphic to $\mathbb{C}$ or Δ by the uniformization theorem. Since it is compactly contained in $\mathbb{C}_{\mu_\epsilon}$, Δ_{μ_ϵ} must be isomorphic to Δ.

For each ϵ, choose an isomorphism $f_\epsilon : \Delta_{\mu_\epsilon} \to \Delta$ with $f_\epsilon(0) = 0$. Then each f_ϵ is K-quasiconformal with $K = (1+k)/(1-k)$. The set of K-quasiconformal mappings $f : \Delta \to \Delta$ with $f(0) = 0$ is compact in the topology of uniform convergence on compact sets, so we can extract a subsequence (f_n) that converges uniformly on compact subsets of Δ to a K-quasiconformal mapping $f : \Delta \to \Delta$. The derivatives of the f_n then converge weakly to the derivatives of f in L^2. We will now make use of the following lemma.

Lemma 16. Suppose that (v_n) and (u_n) are sequences converging weakly to v and u respectively in L^2, and (μ_n) is a bounded sequence in L^∞ which converges to μ in L^1. If $u_n = \mu_n v_n$ for all n, then $u = \mu v$.

Proof. For any fixed test function ϕ, we have

$$\langle u_n, \phi\rangle = \langle \mu_n v_n, \phi\rangle = \langle \mu v_n, \phi\rangle + \langle(\mu_n - \mu)v_n, \phi\rangle = \langle v_n, \mu\phi\rangle + \langle(\mu_n - \mu)v_n, \phi\rangle.$$

Since (u_n) converges weakly to u, left hand side is just $\langle u, \phi\rangle$. On the right hand side, $\langle v_n, \mu\phi\rangle$ converges to $\langle v, \mu\phi\rangle = \langle v\mu, \phi\rangle$. and $\langle(\mu_n - \mu)v_n, \phi\rangle$ converges to zero, as the $\|v_n\|$ are bounded. Hence, we have $\langle u, \phi\rangle = \langle v\mu, \phi\rangle$. Now $\langle u - v\mu, \phi\rangle = 0$, so $u - v\mu = 0$ in L^1. Hence, $u = v\mu$. □

Now since $\frac{\partial f_n}{\partial \bar{z}}$ and $\frac{\partial f_n}{\partial z}$ converge weakly to $\frac{\partial f}{\partial \bar{z}}$ and $\frac{\partial f}{\partial z}$ in L^1, and μ_n is bounded sequence in L^∞ converging to μ in L^1, with $\frac{\partial f_n}{\partial \bar{z}} = \mu_n \frac{\partial f_n}{\partial z}$ for all n, the lemma tells us that

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z},$$

as desired. □

Dependence on parameters. We will frequently deal with families of Beltrami coefficients μ_λ, indexed by a complex parameter λ. In this case, we wish to know when the integrating maps given by the measurable Riemann mapping theorem *depend holomorphically* on the parameter λ.

The integrating map is uniquely determined if it is sufficiently normalized. For instance, given a Beltrami coefficient μ on $U \simeq \Delta$ with $\|\mu\|_\infty \leqslant k < 1$, the integrating map $\phi : U \to \Delta$ is uniquely determined if we take two points $u_1, u_2 \in U$ and require that $\phi(u_1) = 0$ and $\phi(u_2) > 0 \in \mathbb{R}$. Similarly, if $U \simeq \mathbb{C}$, then $\phi : U \to \mathbb{C}$ is uniquely determined if we require that $\phi(s_1) = 0$ and $\phi(s_2) = 1$.

The following theorem is proven in [Hub06].

Theorem 26 (Dependence on parameters)**.** Let Λ be an open subset of $\mathbb{C}$.

1. Suppose μ_λ is a family of Beltrami forms on $\mathbb{C}$, depending on $\lambda \in \Lambda$, and that $\lambda \mapsto \mu_\lambda(z)$ is holomorphic in λ. Moreover, suppose there is $k < 1$ such that $\|\mu_\lambda\|_\infty < k$ for all λ. Let $\phi_\lambda : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal integrating map with $\phi(0) = 0$ and $\phi(1) = 1$. Then the map $\lambda \mapsto \phi_\lambda(z)$ is holomorphic in λ for each $z \in \mathbb{C}$.

2. If μ_λ is a family of Beltrami forms on $U \simeq \Delta$, satisfying the same conditions as before, choose $u_1, u_2 \in U$ and let $\phi_\lambda : U \to \Delta$ be the unique integrating map with $\phi(u_1) = 0$ and $\phi(u_2) > 0 \in \mathbb{R}$. Then the map $\lambda \mapsto \phi_\lambda(u)$ is holomorphic in λ for each $u \in U$.

4.4 Application to Polynomials

Recall that if U and V are simply connected open sets in $\mathbb{C}$ with $\overline{U} \subset V$, a map $f : U \to V$ is polynomial-like of degree d if f is proper and degree d.

Definition 17 (Hybrid Equivalence)**.** Two polynomial-like mappings f and g are hybrid equivalent if there exist neighborhoods U_f and U_g of $K(f)$ and $K(g)$ and a quasiconformal conjugacy $\phi : U_f \to U_g$ between f and g satisfying $\overline{\partial}\phi = 0$ almost everywhere on $K(f)$. In other words, the following diagram commutes.

$$\begin{tikzcd} f^{-1}(U_f) \arrow[r, "\phi"] \arrow[d, "f"] & g^{-1}(U_g) \arrow[d, "g"] \\ U_f \arrow[r, "\phi"] & U_g \end{tikzcd}$$

Note that by Weyl's lemma, ϕ is holomorphic on the interior of $K(f)$, if it is not empty. The name 'polynomial-like' suggests a relationship between polynomial-like maps and ordinary polynomials. This is the main theorem of this section.

Theorem 27 (Straightening Theorem)**.** Suppose (f, U, V) is a polynomial-like mapping of degree d.

1. The map f is hybrid equivalent to a polynomial P of degree d

2. If $K(f)$ is connected, then P is unique up to affine conjugation.

Proof. We may assume that U and V have analytic boundaries. For any $\rho > 1$, let $R : \widehat{\mathbb{C}} \setminus \overline{V} \to \widehat{\mathbb{C}} \setminus \overline{\Delta}_{\rho^d}$ be the Riemann map fixing ∞. Then R extends continuously to the boundaries as an analytic map, say $\psi_1 : \partial V \to S^1_{\rho^d}$. Lift ψ_1 to an analytic map $\psi_2 : \partial U \to S^1_\rho$ such that $\psi_1(f(z)) = \psi_2(z)^d$.

$$\begin{tikzcd}
\partial U \arrow[r, dashed, "\psi_2"] \arrow[d, "f"'] & S^1_\rho \arrow[d, "z^d"] \\
\partial V \arrow[r, "\psi_1"] & S^1_{\rho^d}
\end{tikzcd}$$

Let $A_0 = V/U$, a half open annulus, and $\mathbb{A}_{\rho,\rho^d} = \{z \in \mathbb{C} \mid \rho < |z| < \rho^d\}$. Then since ψ_1 and ψ_2 are analytic, they extend to a map $\psi : \overline{A}_0 \to \overline{\mathbb{A}}_{\rho,\rho^d}$ which is quasiconformal in $\mathrm{Int}(A_0)$ (see [BF14]). We will now extend f to a degree d map on $\mathbb{C}$. Define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & z \in U \\ R^{-1}(\psi(z))^d & z \in V \setminus U \\ R^{-1}(R(z)^d) & z \in \mathbb{C} \setminus V. \end{cases}$$

Notice that F is *quasiregular*, meaning it is the composition of holomorphic and quasiconformal mappings. We define a Beltrami coefficient μ on $\mathbb{C}$ as follows. On A_0, let $\mu = \psi^*\mu_0$. On $\mathbb{C} \setminus V$, let $\mu = \mu_0 = R^*\mu_0$. Notice that μ is F-invariant where defined.

We extend μ to the rest of $\mathbb{C}$ using the dynamics of F. Observe that A_0 is a fundamental domain for F, that is each orbit passes through A_0 at most once. Therefore, the sets $A_n = \{z \in U \mid f^n(z) \in A_0\}$ are disjoint for different values of n. Now, we set

$$\mu(z) = \begin{cases} \psi^*\mu_0(z) & z \in A_0 \\ (f^n)^*\mu(z) & z \in A_n \\ \mu_0(z) & \text{elsewhere.} \end{cases}$$

By construction, μ is F-invariant. Moreover, $\|\mu\|_\infty$ is equal to $\|\psi^*\mu_0\|_\infty$, as all other pullbacks are holomorphic. Since ψ is quasiconformal, we have that $\|\mu\|_\infty = k < \infty$. Therefore the measurable Riemann mapping theorem gives a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi^*\mu_0 = \mu$. Moreover, since $\mu = \mu_0$ on $K(f)$, we know $\overline{\partial}\phi = 0$ on $K(f)$.

Set $P := \psi \circ F \circ \phi^{-1}$. Clearly P is holomorphic of degree d on $\mathbb{C}$, so P is a polynomial. Now ϕ is a hybrid equivalence between f and P.

Suppose that $K(f)$ is connected. If f is hybrid equivalent to two polynomials P_1 and P_2, then P_1 and P_2 are themselves hybrid equivalent and with connected Julia sets. But then P_1 and P_2 are equivalent up to affine conjugation. $\square$

Chapter 5

The Mandelbrot Set

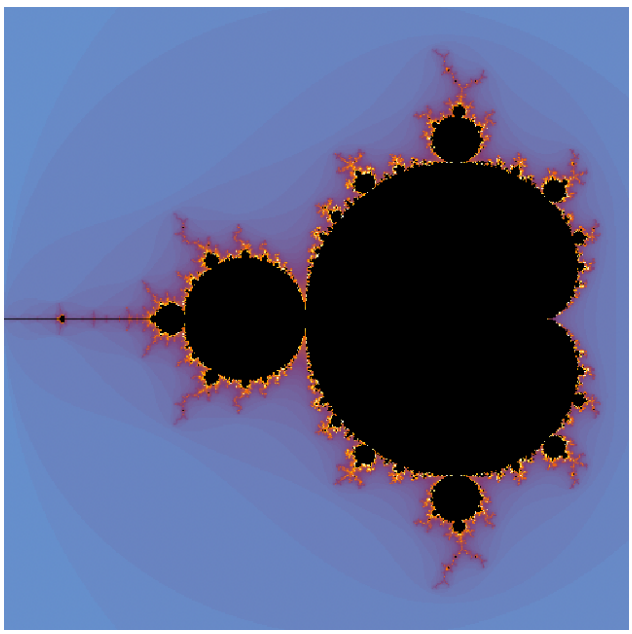

Figure 5.1: The Mandelbrot set.

We will now restrict to the study of quadratic polynomials.

Definition 18 (Conformal equivalence)**.** Let X be a Riemann surface, and let f and g be holomorphic self-maps of X. We say that f and g are *conformally conjugate* if there is an automorphism $\phi : X \to X$ such that $\phi(f(z)) = g(\phi(z))$ for all $z \in X$. In other words, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{g} & X \end{array}$$

If f and g are conformally conjugate, then they are the same from the point of view of complex analysis and dynamics: any analytic or dynamical property of f can be translated to an analytic or dynamical property of g. For example, any periodic cycle $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_p = z_0$ of f maps to a periodic cycle $\phi(z_0) \mapsto \phi(z_1) \mapsto \cdots \mapsto \phi(z_p) = \phi(z_0)$ of g with the same period. Moreover, the multiplier remains the same, as $f'(z) = g'(\phi(z))$ for all $z \in X$.

In the case of the Riemann sphere, $\mathrm{Aut}(\widehat{\mathbb{C}}) \simeq \mathrm{PSL}_2(\mathbb{C})$ is the group of Möbius transformations and the holomorphic self-maps are rational functions, of which polynomials are a special subset.

Definition 19 (Moduli space of rational functions)**.** The moduli space of rational functions of degree d is defined to be

$$R_d = \{f \in \mathbb{C}(z) \mid \deg(f) = d\}/\mathrm{PSL}_2(\mathbb{C}),$$

where $f \sim g$ if and only if f and g are conformally conjugate. Similarly, we define the moduli space of polynomials of degree d to be

$$M_d = \{f \in \mathbb{C}[z] \mid \deg(f) = d\}/\mathrm{Aut}(\mathbb{C}),$$

where $f \sim g$ if and only if f and g are conformally conjugate in $\mathbb{C}$.

Notice that any degree 2 polynomial $f(z) = az^2+bz+c$ has a unique critical point $z_0 = -b/2a$. Consider the map $\phi(z) = (2z-b)/2a$. We choose ϕ so that the conjugate map $g = \phi \circ f \circ \phi^{-1}$ has its critical point at $z = 0$ and is monic. Indeed, we calculate that

$$g(z) = z^2 + ac - \frac{b^2}{4} + \frac{b}{2}.$$

Therefore each polynomial $f \in \mathbb{C}[z]$ is conformally conjugate to a polynomial f_c of the form $f_c(z) = z^2$. Moreover, notice that if $f_c \sim f_{c'}$, then by definition there is an automorphism ϕ of $\mathbb{C}$ with $f_{c'} = \phi \circ f_c \circ \phi^{-1}$. Hence, $\phi(0) = 0$, since ϕ maps the critical point of f_c to $f_{c'}$. Now $\phi(z) = \lambda z$ for some $\lambda \in \mathbb{C}^*$. But now

$$z^2 + c' = \lambda(z^2/\lambda^2 + c) = z^2/\lambda + \lambda c.$$

The critical value of the left hand side is c', while the critical value of the right hand side is λc, so we have $c' = \lambda c$. Subtracting this quantity from both sides, we see that $z^2 = z^2/\lambda$, so it must be that $\lambda = 1$. Thus, we see that $f_c \sim f_{c'}$ if and only if $c = c'$.

Let $[f]$ denote the equivalence class of f in M_2. The following theorem follows from this discussion.

Theorem 28. The space M_2 is naturally parametrized by $\mathbb{C}$ via the map

$$c \in \mathbb{C} \mapsto [f_c]$$

We will want to differentiate between $\mathbb{C}$ as the space of quadratic polynomials, and $\mathbb{C}$ as the domain of a particular quadratic polynomial. We call the former the *parameter plane*, and the latter *dynamical plane*. An element c in the parameter plane, corresponding to a function f_c, is called a *parameter*. With this context in mind, we define the Mandelbrot set.

Definition 20 (Mandelbrot set definition 1)**.** Let $f_c(z) = z^2 + c$. The Mandelbrot set, denoted M, is the set of all $c \in \mathbb{C}$ for which the filled Julia set $K(f_c)$ is connected.

This is to say that M is the *connectivity locus* in the space of quadratic polynomials. In Theorem 17, we proved that $K(f)$ is connected if and only if all critical points lie in $K(f)$. Since f_c as a single critical point at $z = 0$, we have the following equivalent formulation.

Definition 21 (Mandelbrot set definition 2)**.** The Mandelbrot set is the set of all $c \in \mathbb{C}$ for which the orbit of 0 is bounded under $f_c(z) = z^2 + c$.

5.1 Basic Properties of the Mandelbrot Set

From the picture of the Mandelbrot set given in Figure 5.1, it is clear that the Mandelbrot set is a complicated object. We will begin with a few basic properties which allow us to make sense of this object.

Proposition 9. The Mandelbrot set is compact and full.

Proof. We must show that M is closed and bounded. Let $z_n = f_c^n(0)$. We claim that if $|z_N| > 2$ for some N, then $|z_n| \to \infty$. If suffices to show that if $|c| = |z_1| > 2$, then $|z_n| \to \infty$. If $|c| > 2$, we claim that $|z_n| \geqslant 2^{n-1}|c|$. As a base case, notice that $|z_1| = |c| = 2^{n-1}|c|$. Now assume that $|z_n| \geqslant 2^{n-1}|c|$. Then

$$|z_{n+1}| = |z_n^2 + c| \geqslant |z_n|^2 - |c| \geqslant 2^{2n-2}|c|^2 - |c| \geqslant (2^{2n-2} - 1/2)|c|^2 \geqslant 2^{2n-3}|c|^2 \geqslant 2^n|c|,$$

as desired. It follows that $M \subset \overline{\Delta}_2$. Moreover, if $c \notin M$, then $|z_N| > 2$ for some N, so we have that $c \in M$ if and only if no iterate of 0 leaves $\overline{\Delta}_2$. Hence,

$$M = \bigcap_{n \in \mathbb{N}} \{c \in \mathbb{C} \mid |f_c^n(0)| \leqslant 2\}.$$

The map $c \mapsto f_c^n(0)$ for fixed n is a polynomial in c, which we denote $p_n(c)$. Hence the map $g_n(c) = |f_c^n(0)|$ is continuous.

$$M = \bigcap_{n \in \mathbb{N}} g_n^{-1}([0, 2]),$$

which is an intersection of closed sets. Hence M is closed and bounded.

We will now show that M is full. It suffices to show that $\mathbb{C} \setminus M$ has no bounded component. Suppose for contradiction that U is such a component. Then there is p_N such that $|p_N(z)| > 2$ for some $z \in U$. By p_N is holomorphic, so by the maximum modulus principle, $|p_N(z_0)| > 2$ for some $z_0 \in \partial U \subset M$. This is a contradiction. $\square$

Uniformization and Green's function. Since M is compact and full, its complement is connected and open. We will show that $\mathbb{C} \setminus M$ is in fact simply connected, by constructing the Riemann map $\Phi : \mathbb{C} \setminus \overline{\Delta} \to \mathbb{C} \setminus M$ with $\Phi'(\infty) > 0$. Let ϕ_c denote the Böttcher map for f_c, defined in a neighborhood of ∞. Let G_c denote the Green's function $G_c(z) = \log |\phi_c(z)|$. Let $h_0(c) = e^{G_c(0)}$. Then by Theorem 16, there is a univalent map $\psi : \mathbb{C} \setminus \overline{\Delta}_{h_0(c)} \to \mathbb{C}$, which inverts ϕ_c. Therefore the Böttcher map extends holomorphically to a map

$$\phi_c : \psi(\overline{\Delta}_{h_0(c)}) \to \mathbb{C} \setminus \overline{\Delta}_{h_0(c)}.$$

Suppose that $c \notin M$. Since $G_c(c) = G_c(f(0)) = 2G_c(0) > G_c(0)$, we see that ϕ_c is defined and holomorphic at c. Let

$$\Phi(c) = \phi_c(c) = \lim_{k \to \infty} (f_c^k(c))^{1/2^k}.$$

Evidently, the map Φ is defined on all of $\mathbb{C} \setminus M$, and maps to $\mathbb{C} \setminus \Delta$.

Theorem 29. The map $\Phi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \Delta$ is a conformal isomorphism.

Proof. Since Φ is the uniform limit of the maps $c \mapsto (f_c^k(c))^{1/n^k}$, which are well defined (compare the proof of Theorem 15) and holomorphic in c. In the region $|c| \geqslant 4$, we can write $\Phi(c)/c$ as a uniformly convergent infinite product

$$\frac{\Phi(c)}{c} = \left(1 + \frac{c}{c^2}\right)^{1/2} \cdot \left(1 + \frac{c}{(c^2+c)^2}\right)^{1/4} \cdots \left(1 + \frac{c}{(f^n(c))^2}\right)^{1/2^{n+1}} \cdots.$$

All factors tend to 1 as $c \to \infty$, so $\Phi(c) \to \infty$ as $c \to \infty$. We can thus extend Φ to the Riemann sphere, setting $\Phi(\infty) = \infty$.

This map is proper, as if $c \to c_0 \in \partial M$, $G_c(c) \to G_{c_0}(c_0) = 0$, so $|\Phi(c)| \to \partial\Delta$. It now suffices to show that the degree of Φ is 1. Since $\Phi^{-1}(\infty) = \{\infty\}$ with multiplicity 1, it follows that Φ is an isomorphism. □

Corollary 14. The Mandelbrot set is connected.

Proof. By Theorem 29, the complement of M is simply connected in $\widehat{\mathbb{C}}$. □

That M is connected is not obvious when looking at pictures. For example, Figure 5.2 shows a small copy of the Mandelbrot set which is contained in M, and almost appears to be isolated. Kahn also gives a topological proof of Corollary 14, in [Kah01].

Equipotentials and parameter rays. Theorem 29 also allows us to define the Green's function in the parameter plane.

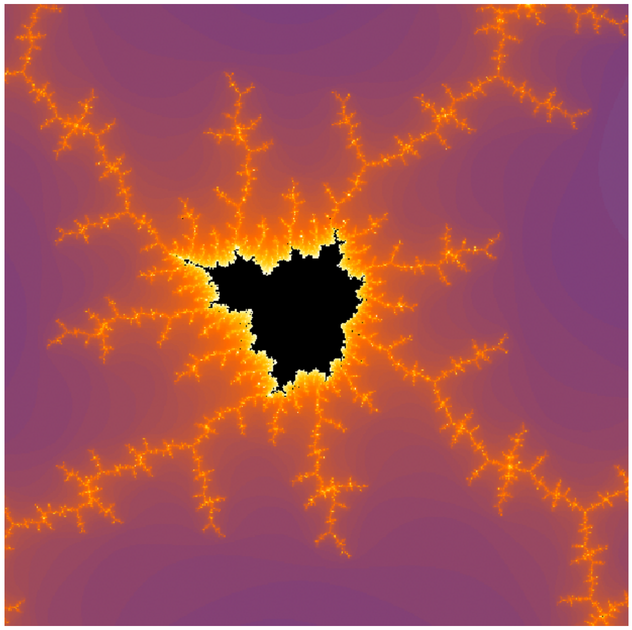

Figure 5.2: A small copy of the Mandelbrot set

Definition 22 (Green's function for M)**.** Let $G_M : \mathbb{C} \to [0, +\infty)$ be the continuous map given by

$$G_M(c) = G_c(c) = \log |\Phi(c)|$$

for $c \notin M$ and $G_M(c) = 0$ for $c \in M$. We call the level sets $G_M^{-1}(r)$ *equipotentials*, denoted $\mathcal{C}_r$.

Continuity comes from the fact that $|\Phi(c)| \to 1$ as $c \to \partial M$. We will also define external rays in the context of M.

Definition 23 (Parameter Rays)**.** For any $t \in \mathbb{R}/\mathbb{Z}$, we define

$$\mathcal{R}_M(t) = \{\Phi^{-1}(re^{2\pi it}) \mid r > 1\}$$

We call $\mathcal{R}_M(t)$ the *parameter ray of external angle* t.

To differentiate external rays in the parameter plane from those in the dynamical plane, we will use the notation

$$\mathcal{R}_c(t) = \{\phi_c^{-1}(re^{2\pi it}) \mid r > 1\}$$

which we call *dynamical rays*. An equivalent definition of the parameter ray of external angle t is $\mathcal{R}_M(t) = \{c \in \mathbb{C} \mid c \in \mathcal{R}_c(t)\}$. Indeed, $c \in \mathcal{R}_M(t)$ if and only if $\Phi(c) = \phi_c(c) = re^{2\pi it}$ for some $r > 1$.

A parameter ray *lands* at a point $c_0 \in \partial M$, if z_0 is the only accumulation point of $\mathcal{R}_M(t)$ in ∂M. Douady and Hubbard prove in [DH85] a landing criterion analogous to that of periodic and preperiodic rays given in Theorem 19.

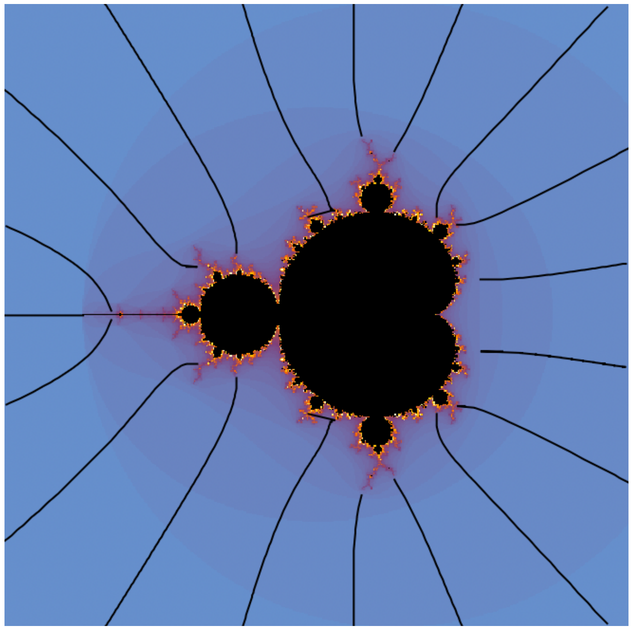

Figure 5.3: Parameter rays.

We call a parameter c *parabolic* if f_c has a parabolic periodic orbit. A *Misiurewicz* parameter, is a $c \in \mathbb{C}$ for which 0 is *strictly* pre-periodic for f_c; that is 0 is not periodic, but $f^k(0)$ is periodic for some $k > 0$.

Theorem 30 (Landing of rational rays)**.** For each $t \in \mathbb{Q}/\mathbb{Z}$, the parameter ray $\mathcal{R}_M(t)$ lands at a parabolic or Misiurewicz parameter, depending on whether t is preperiodic or periodic respectively under the doubling map $t \mapsto 2t$. Conversely, every parabolic or Misiurewicz parameter is the landing point of at least one, and at most finitely many parameter rays.

Hyperbolic parameters. *Hyperbolic* parameters are a particularly nice class of parameters which we will study at length.

Definition 24 (Hyperbolicity)**.** A rational map f is *dynamically hyperbolic* if f is *expanding* on its Julia set: there is a conformal metric ρ defined on some neighborhood of $J(f)$, such that for each $z \in J(f)$, the derivative df_z satisfies

$$\|df_z(v)\|_\rho > \|v\|_\rho$$

for each non-zero tangent vector $v \in T_z\widehat{\mathbb{C}}$.

Since $\widehat{\mathbb{C}}$ is compact, it follows that there is an *expansion constant* $k > 1$ such that

$$\|df_z(v)\|_\rho \geqslant k \|v\|_\rho$$

for all z in some neighborhood of $J(f)$. Then for any smooth path γ in this neighborhood, $\text{length}(f \circ \gamma) \geqslant k \cdot \text{length}(\gamma)$. Therefore, each $z \in J(f)$ has a neighborhood N_z such that

$$d_\rho(f(x), f(y) \geqslant k \cdot d_\rho(x, y)$$

for all $x, y \in N_z$.

Let P_f denote the *postcritical set* of f; that is

$$P_f = \{f^k(p) \mid f'(p) = 0,\ k \in \mathbb{N}\},$$

which is the set of all forward iterates of critical points. In the case of quadratic polynomials f_c, this coincides with $\mathcal{O}_f(0)$, the forward orbit of zero.

Theorem 31 (Hyperbolic maps)**.** A rational map of degree $d \geqslant 2$ is hyperbolic if and only if $\overline{P}_f$ is contained in the Fatou set, or equivalently, if and only if the iterates of every critical point converge to an attracting periodic orbit.

Theorem 32. If the Julia set of a hyperbolic rational map is connected, then it is locally connected.

We leave the proof of both theorems to §19 of [Mil06a].

Proposition 10 (Hyperbolic components)**.** If f_c is hyperbolic, then $f_{c'}$ is also hyperbolic for all c' in some neighborhood of c. A connected component of the set of hyperbolic parameters is called a *hyperbolic component*.

Proof. Suppose that $f_{\hat{c}}$ is hyperbolic. Then there is a periodic cycle $P = \{p_0 \mapsto p_1 \mapsto \cdots \mapsto p_k = p_0\}$ with multiplier $|\lambda| < 1$, such that $d_{\widehat{\mathbb{C}}}(f_{\hat{c}}^n(0), P) \to 0$. Assume without loss of generality that $\lim_{n\to\infty} f_{\hat{c}}^{kn}(0) = p_0$. Consider function $g : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by $g(c, z) = f_c^k(z) - z$, which is holomorphic in c for each fixed z, and in z for each fixed c.

Notice that $g(\hat{c}, p_0) = 0$, and

$$\frac{dg}{dz}(\hat{c}, p_0) = \lambda - 1 \neq 0.$$

Then by the implicit function theorem, there is a function $z = p(c)$, which is holomorphic in c and defined in a neighborhood of $\hat{c}$, with $p(\hat{c}) = p_0$ and $g(c, p(c)) = 0$. It follows that $p(c)$ is an attracting periodic point of period k. Since $(f_c^k)'(p(c))$ depends continuously on c, it follows that the multiplier λ_c is less than one for c sufficiently close to $\hat{c}$. Therefore $p(c)$ is attracting.

Let N be a neighborhood of $\hat{c}$ with compact closure on which $p(c)$ is an attracting periodic point of f_c. Then there is $\epsilon > 0$ such that $\Delta_\epsilon(p(c))$ is contained in the attracting basin of $p(c)$ under f_c^k for all $c \in N$. For some $\delta < \epsilon$, let $N' \subset N$ be a neighborhood of $\hat{c}$ small enough that $\Delta_\delta(p_0) \subset \Delta_\epsilon(p(c))$ for all $c \in N'$. Evidently, there is M such that $f_{\hat{c}}^M(0) \in \Delta_\delta(p_0)$. Since $c \mapsto f_c^{kM}(0)$ is continuous, we have that $f_c^{kM}(0) \in \Delta_\delta(p_0) \subset \Delta_\epsilon(p(c))$ for all c in a neighborhood $N'' \subset N'$ of $\hat{c}$. Now for all $c \in N''$, the orbit of zero under f^{kM} eventually lands in the attracting basin of $p(c)$. Thus, zero is in the attracting basin of $p(c)$, and we're done. $\square$

It follows from the proof that period of the cycle which attracts zero under f_c is locally constant in an open set of hyperbolic parameters. Therefore, this period is globally constant in a hyperbolic component.

It also follows from the proof that the map $\lambda(c)$, mapping c in a hyperbolic component H to its multiplier in Δ is holomorphic and proper. Hence $\lambda(c)$ is surjective, there is a trajectory in H for which the multipliers accumulate on any root of unity. It follows that there are infinitely many parabolic parameters on the boundary of a hyperbolic component.

The most prominent hyperbolic component is the *main cardioid*, which is the set of all $c \in M$ for which f_c has an attracting fixed point in $\mathbb{C}$.

Proposition 11 (The main cardioid)**.** Consider the curve

$$\Gamma = \left\{ e^{it} \frac{2 - e^{it}}{4} \mid t \in \mathbb{R} \right\},$$

which defines a cardioid in $\mathbb{C}$. Let C be the bounded component of $\mathbb{C} \setminus \Gamma$. Then C is the *period one hyperbolic component*; that is, the set of all $c \in \mathbb{C}$ for which zero tends to an attracting fixed point.

Proof. The fixed points of f_c are the solutions of $z = z^2 + c$, which are given by

$$z_* = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$

If z_* is attracting, then $|f_c'(z_*)| < 1$, so $|z_*| < 1/2$. Suppose that $|z_*| = 1/2$, meaning the multiplier at z_* is 1. If write $z_* = e^{it}/2$, then we solve

$$\frac{e^{it}}{2} = \frac{1 \pm \sqrt{1 - 4c}}{2},$$

and find that

$$c = e^{it} \frac{\pm 2 - e^{it}}{4}.$$

Notice that

$$e^{it} \frac{-2 - e^{it}}{4} = e^{i(t+\pi)} \frac{2 - e^{i(t+\pi)}}{4},$$

so both solutions lie in Γ. Now since the multiplier of z_* depends continuously on c, all parameters in C have an attracting fixed point, or none of them do. Since $f_0(z) = z^2$ has attracting fixed point $z = 0$, we see that all parameters in C have an attracting fixed point. Similarly, $f_1(z) = z^2 + 1$ does not have an attracting fixed point, so no parameters outside of C have an attracting fixed point. Since every basin of attraction contains a critical point, zero must tend towards the attracting fixed point of f_c for every $c \in C$. □

Another easily computed hyperbolic component is the *period two bulb*, consisting of all parameters in $\mathbb{C}$ with an attracting period two cycle.

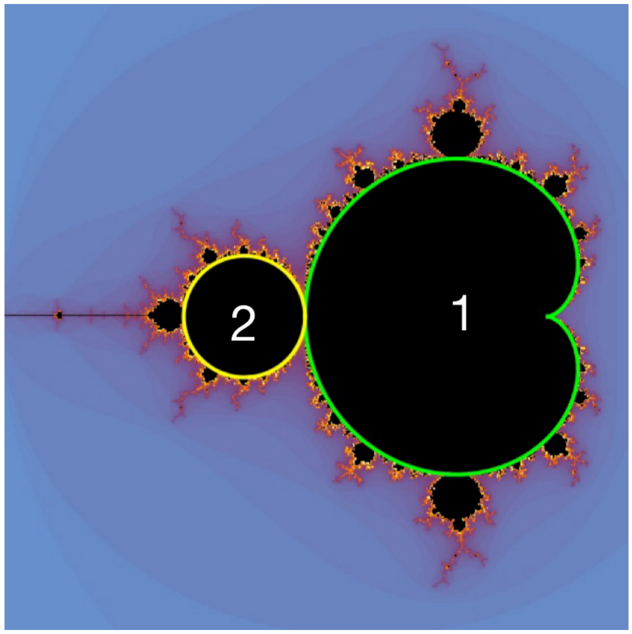

Figure 5.4: The main cardioid and the period two bulb.

Proposition 12 (Period two bulb)**.** The set $D = \Delta_{1/4}(-1)$ is the set of all $c \in \mathbb{C}$ for which $f_c^n(0)$ converges to a period two attracting cycle.

The proof proceeds similarly to Proposition 11, and we leave the details to the reader. See Figure 4 for a visualization of the main cardioid and the period two bulb. We will see that for each $n \in \mathbb{N}$, there is at least one *period n bulb* in the Mandelbrot set, which is a hyperbolic component containing parameters with an attracting cycle of period n.

Non-hyperbolic parameters in M have filled Julia sets with no interior and yet with positive measure. Occasionally, they have non-locally connected Julia sets. As we will see, it is conjectured that such parameters are rare.

5.2 Self-similarity and Renormalization

Computer graphics reveal that the Mandelbrot set appears to bare some self-similarity. For instance, each bulb attached to the main cardioid looks similar to the entire Mandelbrot set. Small, distorted copies of the Mandelbrot set are found floating at various points around M.

We can explain this self-similarity in terms of *renormalization*. Ideas in this section are explained in detail in [McM94a].

Definition 25 (Renormalizable polynomial)**.** Let f be a polynomial of degree $d \geqslant 2$. We say f^n is *renormalizable* if there are simply connected domains $U \Subset V$ such that $f^n : U \to V$ is polynomial-like of degree d with connected filled Julia set, and has a critical point in U.

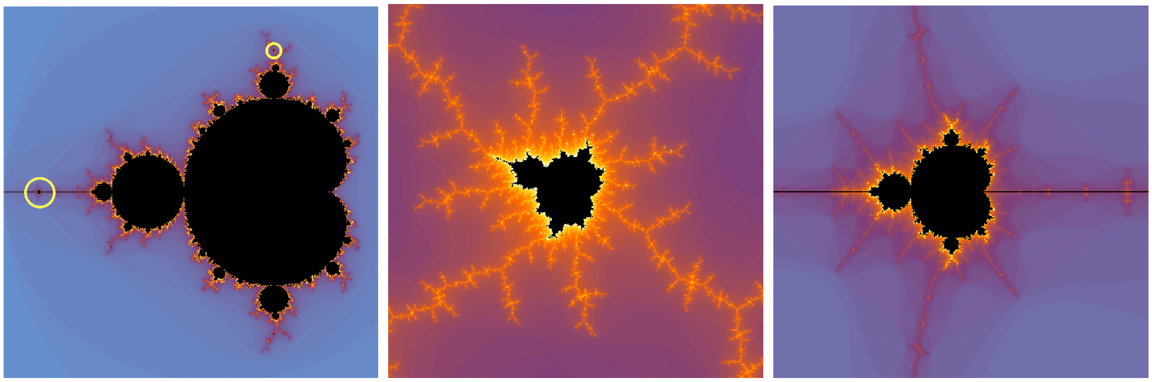

Figure 5.5: Small copies of the Mandelbrot set.

The pair (U, V) is a *renormalization* of f^n. The filled Julia set of $f^n : U \to V$ is called the *small Julia set*.

We will often say that a polynomial f is *renormalizable* if some iterate f^n is renormalizable. A renormalizable polynomial is either of *primitive* or *satellite* type. In the satellite case, the small Julia set J of the renormalization $f^n|_U$, and its forward iterates $f^i(J)$ all touch at a repelling fixed point α, which has some rotation number p/q. The map f permutes the $f^i(J)$ according to this rotation number. In the satellite case, the small Julia sets $f^i(J)$ are disjoint.

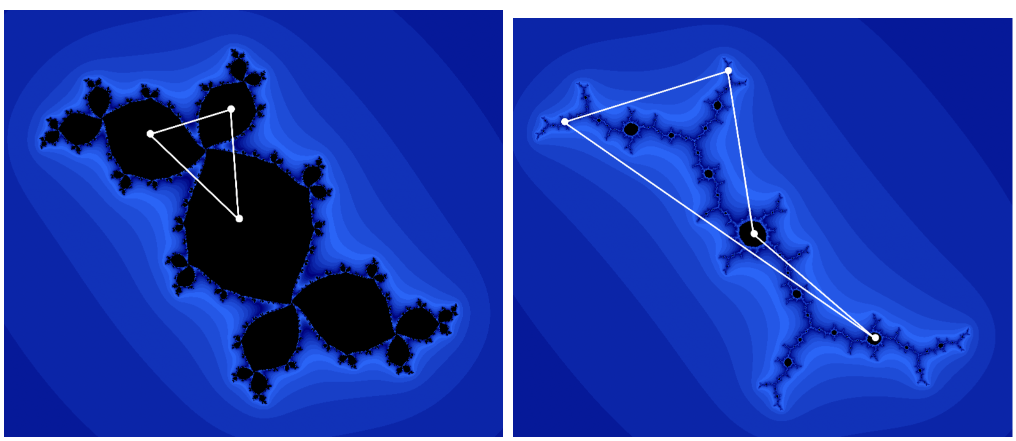

Figure 5.6: Filled Julia sets of satellite type (left) and primitive type (right) renormalizable polynomials.

Let

$$\mathcal{R}(f) = \{n \geqslant 1 \mid f^n \text{ is renormalizable}\}.$$

If f^n is renormalizable, then the straightening theorem implies that $f^n : U \to V$ is hybrid equivalent to an ordinary polynomial g of degree d, which is unique up to an automorphism of $\mathbb{C}$. It is possible that g is also renormalizable, and this procedure can be repeated. If this can be done k times, then we say f is *k-times renormalizable*. Here $k = |\mathcal{R}(f)|$. It is also possible that f is *infinitely renormalizable*, in the case that $|\mathcal{R}(f)| = \infty$.

Theorem 33 (Uniqueness of renormalization)**.** Any two renormalizations of f^n have the same filled Julia set.

We leave the proof to [McM94a]. It follows that any two renormalizations of f^n are hybrid equivalent, and we can choose a well defined representative $f^n : U \to V$ up to hybrid equivalence for each $n \in \mathcal{R}(f)$. In the case of *quadratic-like maps*, polynomial-like maps of degree 2, we can choose a representative of the form $f_c(z) = z^2 + c$ by the straightening theorem.

Remark 4 (Renormalization operator)**.** Let S_d denote the space of polynomial-like mappings of degree d up to hybrid equivalence. If $\mathcal{R}(f)$ is not empty for some $f \in S_d$, then we can find a smallest $n \in \mathcal{R}(f)$, and map f to $f^n : U \to V$, which is unique up to hybrid equivalence. The map $R : S_d \dashrightarrow S_d$, (partially defined) is called the *renormalization* operator. This operator is a well-defined dynamical system on the set of infinitely renormalizable parameters.

Example 9 (Feigenbaum polynomial)**.** Consider the map $f(z) = z^2 - 1.401155\ldots$, known as the Feigenbaum polynomial. This map lies in the Mandelbrot set, and is the limit point of a sequence of hyperbolic components H_n corresponding to attracting periodic cycles of period 2^n. The Feigenbaum polynomial is infinitely renormalizable: each f^{2^n} is satellite renormalizable and is hybrid equivalent to f itself. Thus, f is a fixed point of the renormalization operator on S_2. See Figure 5.7.

The case of infinitely renormalizable polynomials in particularly interesting and exhibit pathological behavior. The following theorem is proven in [McM94a].

Theorem 34 (Infinitely renormalizable polynomials)**.** Suppose f is infinitely renormalizable. Then

1. All periodic cycles of f are repelling.
2. The filled Julia set $K(f)$ has no interior.
3. The intersection of the infinitely many Julia sets contains no periodic points.
4. The postcritical set P_f is finite (we say f is *postcritically finite* is finite and contains no periodic points.

The rounded Mandelbrot set. The theory of renormalization can be used to explain self-similarity in the Mandelbrot set. First, we must adjust our setup. We make reference to [Mil06b].

Every quadratic map $f_c(z) = z^2 + c$ has two fixed points α and β, with $\alpha + \beta = 1$, $\alpha\beta = c$, and $\mathrm{Re}(\alpha) \leqslant \mathrm{Re}(\beta)$. We can thus use the multiplier $\lambda = f_c'(\alpha) = 2\alpha$ as an alternative parameter to describe M_2. Indeed, c is uniquely determined by λ, and vice versa, since

$$c = \lambda(1 - \lambda/2)/2, \quad \lambda = 1 - \sqrt{1 - 4c},$$

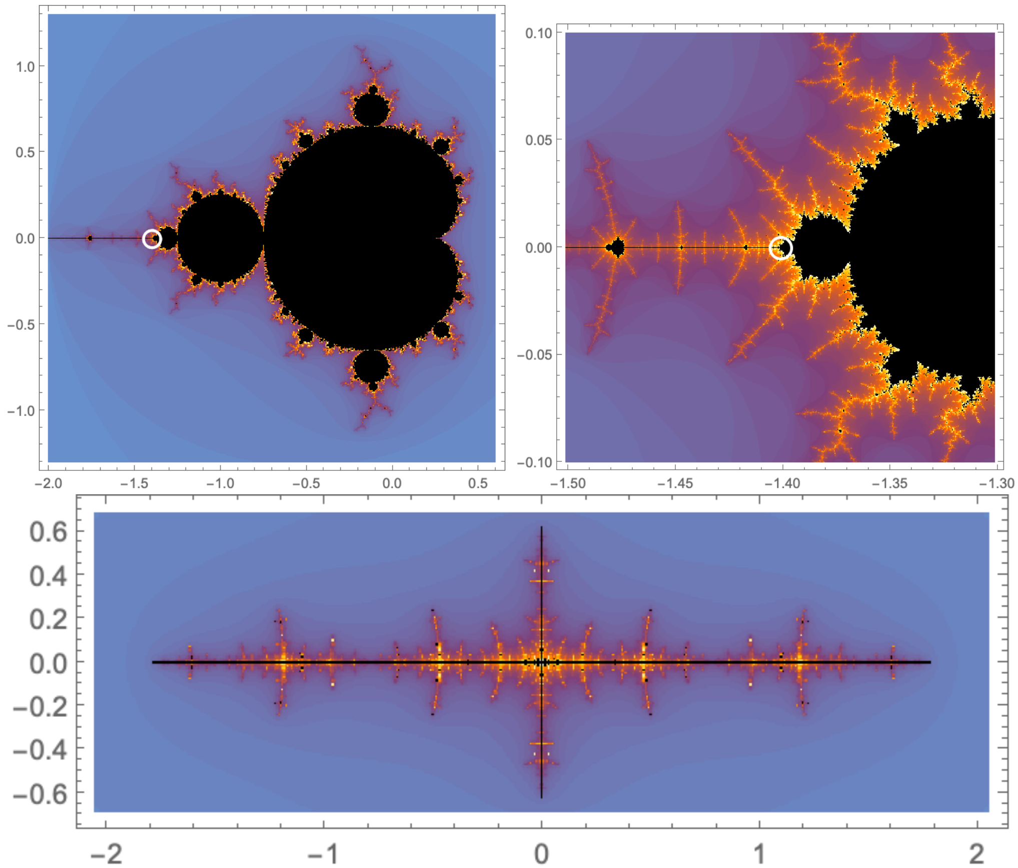

Figure 5.7: The filled Julia set of the Feigenbaum polynomial (bottom) and its location in M (top).

when $\mathrm{Re}(\lambda) \leqslant 1$. When c and λ correspond to the same quadratic map, write $c = c(\lambda)$ and $\lambda = \lambda(c)$. We will write $f_{c(\lambda)}(z) = \hat{f}_\lambda(z) = z^2 + c(\lambda)$. The subset of parameters $\lambda \in \{\mathrm{Re}(\lambda) \leqslant 1\}$ such that $f_{c(\lambda)} \in M$ will be called the *rounded Mandelbrot set* $\widehat{M}$. The change of parameters $\lambda \to c(\lambda)$ gives a homeomorphism between M and $\widehat{M}$.

Evidently, the unit disk $\Delta \subset \widehat{M}$ is the *period one hyperbolic component*, set of all parameters λ for which $\hat{f}_\lambda$ has an attracting fixed point. This is the component which corresponds with the main cardioid in M.

At each p-th root of unity $e^{2\pi i n/p}$, there is a *satellite* hyperbolic component $H(n/p)$ of period p attached to Δ [Mil06b]. Similarly, there are satellite hyperbolic components $H(n/p) \rhd H(n'/p')$ of period pp' attached to $H(n/p)$ along its boundary points, and so on.

The Douady-Hubbard tuning. As shown in [DH85], for each hyperbolic component $H \subset \widehat{M}$, there is a homeomorphism

$$H \rhd \colon \widehat{M} \to (H \rhd \widehat{M}) \subset \widehat{M}$$

from $\widehat{M}$ to a small copy of itself. The tuning $H \rhd$ takes hyperbolic components to hyperbolic components, with $\mathrm{per}(H \rhd H') = \mathrm{per}(H)\mathrm{per}(H')$. Each $H \rhd \colon \overline{\Delta} \to \overline{H}$ is holomorphic, and

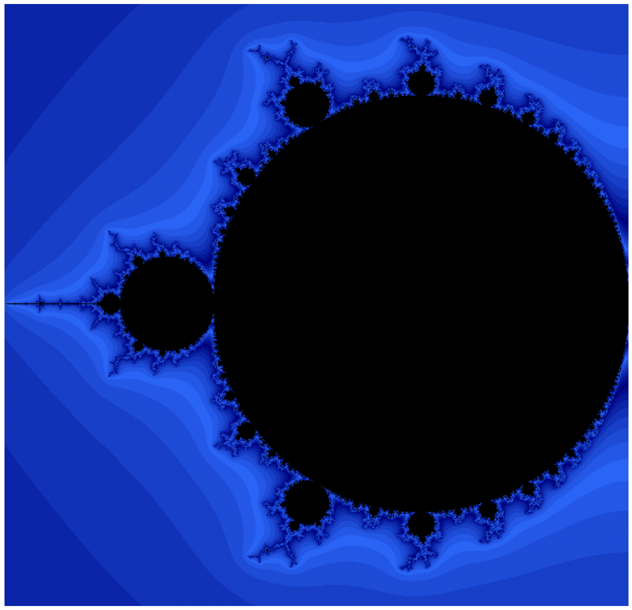

Figure 5.8: The rounded Mandelbrot set.

gives us the *Douady-Hubbard parametrization* of H: for all $\lambda \in \Delta$, the attracting periodic orbit for $\hat{f}_{H \rhd \lambda}$ has multiplier λ. The point $H \rhd 1 \in \partial H$ is the *root point* of H, which must be parabolic.

The Douady-Hubbard tuning corresponds with renormalization. Suppose that λ lies in a hyperbolic component H of period p. Then $\widehat{f}_\lambda^p$ is renormalizable [DH85]. Moreover, if $H \rhd \lambda' = \lambda$, then $\widehat{f}_\lambda$ is hybrid equivalent to a quadratic-like renormalization of $\widehat{f}_\lambda^p$. If $H = H' \rhd \widehat{M}$, for some other hyperbolic component H' of period greater than 1, then $\widehat{f}_\lambda^p$ is twice renormalizable.

It follows that the infinitely renormalizable parameters are those which lie in an infinite sequence of small Mandelbrot sets

$$\widehat{M} \lhd H_1 \lhd H_2 \lhd H_3 \lhd \dots$$

where the H_i are hyperbolic components of period greater than 1. Evidently, the Feigenbaum point lies in $\widehat{M} \lhd H(1/2) \lhd H(1/2) \lhd H(1/2) \lhd \dots$.

Parabolic wakes. Everything we have done so far with $\widehat{M}$ transfers to M, and we will

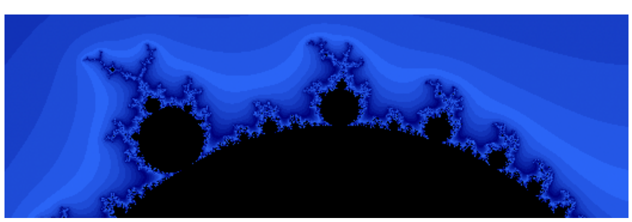

Figure 5.9: Satellite hyperbolic components along Δ.

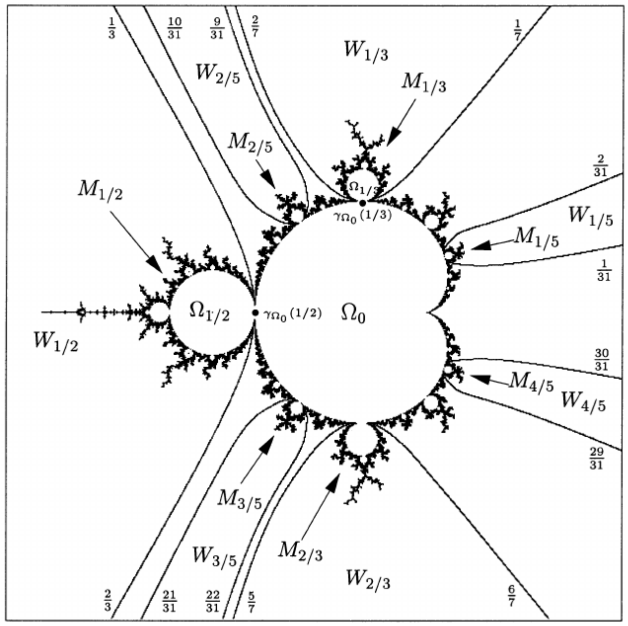

Figure 5.10: Parabolic Wakes in the Mandelbrot set (Image courtesy of [BF01])

freely use the Douady-Hubbard tuning in the case of M. We have the following theorem of [DH85].

> **Theorem 35** (Parabolic wakes)**.** Every root point $H \rhd 1 \in \widehat{M}$ is the landing point of exactly two external rays of external angles $0 \leqslant t_- < t_+ \leqslant 1$. Conversely, any cycle of periodic dynamic rays has two *characteristic angles* $0 \leqslant t_- < t_+ \leqslant 1$, such that the corresponding parameter rays land together at a parabolic parameter.
>
> These rays partition the parameter plane in two halves. One half contains the main cardioid, and the other is the *parabolic wake* W_{t_+,t_-}. Two dynamic rays of angles t_+ and t_- land together for a map f_c if and only if $c \in W_{t_+,t_-}$.

The interval $[t_-, t_+]$ is called the *characteristic interval* for the wake W_{t_-,t_+}. All parameter rays $\mathcal{R}_M(t)$ for $t \in [t_-, t_+]$ lie in the wake W_{t_-,t_+}. Let H_{t_-,t_+} denote the hyperbolic component with root point that is the landing point of $\mathcal{R}_M(t_\pm)$. Then for each $c \in H_{t_-,t_+}$, the dynamical rays $\mathcal{R}_c(t_\pm)$ land together at a common repelling periodic point [Mil06b].

Aside: universality of the Mandelbrot set. We have now seen that there are infinitely many homeomorphic copies of the Mandelbrot set contained in itself. This is made even stronger by a theorem of McMullen.

> **Definition 26** (Holomorphic family of rational maps)**.** A *holomorphic family of rational maps* is a holomorphic function $f : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where X is a connected complex manifold and $z \mapsto f_t(z)$ has degree $d \geqslant 2$ for each $t \in X$.

Definition 27 (Bifurcation locus)**.** The bifurcation locus of a holomorphic family of rational maps $f : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the set $B(f) \subset X$ of t such that

1. The number of attracting cycles of f_t is not locally constant as we vary t,
2. The period of the attracting cycles of f_t is locally unbounded, or
3. The Julia set $J(f_t)$ does not move continuously in the Hausdorff-Gromov topology over any neighborhood of t.

As an example, the map $f : \mathbb{C} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $f_t(z) = z^2 + t$ is a holomorphic family of rational maps, and its bifurcation locus is the boundary of the Mandelbrot set. By a *small Mandelbrot set*, we mean the image of M under a quasiconformal mapping.

Theorem 36 (The Mandelbrot set is universal - McMullen)**.** For any holomorphic family f of rational maps of degree 2, small Mandelbrot sets are dense in $B(f)$.

In particular, small Mandelbrot sets are dense in M itself. All details are given in [McM00].

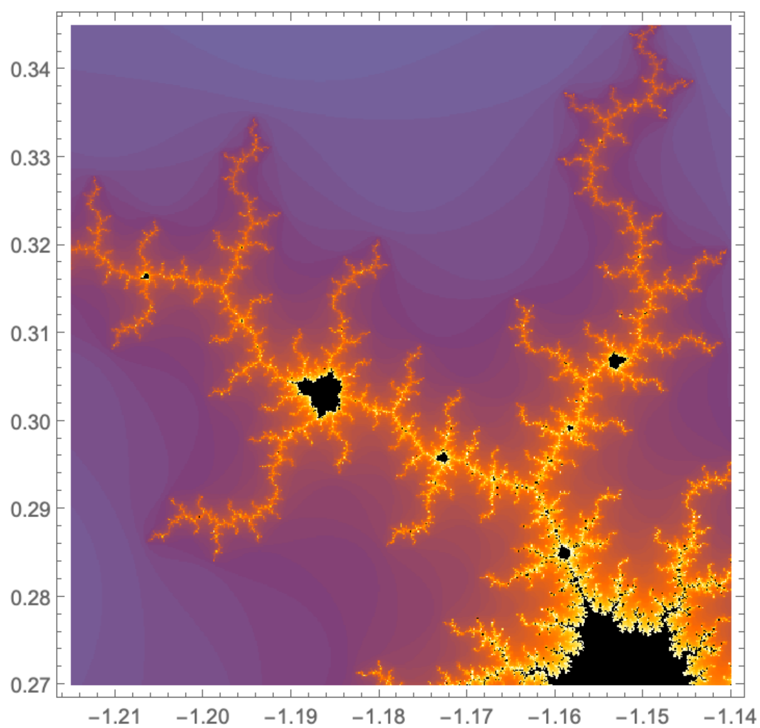

Figure 5.11: Small Mandelbrot sets in M.

5.3 MLC

Conjecture 2 (MLC)**.** The Mandelbrot set is locally connected.

This is arguably the most open important open problem in the study of quadratic dynamics, both for its direct implications and for the theorems which it implies. From our discussion of locally connectivity in Chapter 2, it follows that M is locally connected if and only if ∂M is locally connected, if and only if every parameter ray $\mathcal{R}_M(t)$ lands on ∂M.

There are several theorems which are consequences of MLC. We will describe each of these theorems, and its relationship to MLC.

5.3.1 Density of Hyperbolicity & No Invariant Line Fields

We have previously discussed hyperbolic parameters and hyperbolic components. MLC implies the following conjecture.

Conjecture 3 (Density of hyperbolicity (DHC))**.** Hyperbolic parameters are dense in $\mathbb{C}$.

For any $c \notin M$, the critical point $z = 0$ lies in the attracting basin of ∞, so c is a hyperbolic parameter. Therefore it suffices to show that hyperbolic parameters are dense in M. Conjecture 3 is particularly important because it implies that the interior of the Mandelbrot set consists only of hyperbolic components. Hence, we would have a good understanding of the topology of M and the possible dynamical behaviors.We call a hypothetical connected open set of non-hyperbolic parameters a *ghost component*. It is hard to say what the possible dynamical behavior would be in such components. We will prove the following in Section 5.3.2.

Theorem 37 (MLC $\Rightarrow$ DHC)**.** Conjecture 2 implies Conjecture 3.

No invariant line fields. Let f be a polynomial. We say that f *admits an invariant line field* if there is a Beltrami differential μ such that $f^*\mu = \mu$ almost everywhere on $\mathbb{C}$, $|\mu| = 1$ on a set X of positive measure, and $|\mu| = 0$ elsewhere.

We call μ a 'line field' because when $|\mu| = 1$, μ defines a slope. We are particularly interested in the case when $X \subseteq J(f)$. Julia sets of measure zero cannot support line fields.

Conjecture 4 (No invariant line fields (NILF))**.** Suppose that $f \in M$. Then f supports no invariant line fields on its Julia set.

Theorem 38 (NILF $\iff$ DHC)**.** Conjecture 4 is equivalent to Conjecture 3.

Holomorphic motions and J-stability. In order to prove Theorem 38, we must introduce some machinery. We will define a *holomorphic motion*, which makes precise the idea of a entire set changing holomorphically.

Definition 28 (Holomorphic motion)**.** Let Λ be a complex manifold and $\lambda_0 \in \Lambda$. A *holomorphic motion* of a set $X \subset \mathbb{C}$ over Λ with center at λ_0 is a function $f : \Lambda \times X \to \mathbb{C}$ such that

1. The map $\lambda \mapsto f(\lambda, z)$ is holomorphic for each $z \in X$;
2. The map $z \mapsto f(\lambda, z)$ is injective for each $\lambda \in \Lambda$; and
3. $z \mapsto f(\lambda_0, z)$ is the identity.

We will often write f_λ for the map $z \mapsto f(\lambda, z)$. For our purposes, Λ will usually be an open subset of parameter space, and X will be a subset of the dynamical plane. Miraculously, we have the following lemma, which is proven in [MSS82].

Lemma 17 (λ-lemma)**.** Suppose $f : \Lambda \times X \to \mathbb{C}$ is a holomorphic motion. Then f can be extended uniquely to a holomorphic motion $f : \Lambda \times \overline{X} \to \mathbb{C}$, and the map $\phi_\lambda : \overline{X} \to \mathbb{C}$ is quasiconformal for any $\lambda \in \Lambda$.

Definition 29 (J-stability)**.** In a parameter space of rational functions, we say that a parameter c_0 is *J-stable* if the Julia set $J(f_{c_0})$ *moves holomorphically* at c_0, meaning there is a neighborhood Λ of c_0 and a homeomorphism $\psi_\lambda : J(f_0) \to J(f_\lambda)$ such that the following diagram commutes:

$$\begin{array}{ccc} J(f_0) & \xrightarrow{f_0} & J(f_0) \\ \downarrow{\scriptstyle \psi_\lambda} & & \downarrow{\scriptstyle \psi_\lambda} \\ J(f_\lambda) & \xrightarrow{f_\lambda} & J(f_\lambda) \end{array}$$

and moreover, $\psi(\lambda, z) = \psi_\lambda(z)$ is a holomorphic motion of $J(f_0)$.

The following theorem is a result of Mane, Sad, and Sullivan, and is proven in [MSS82].

Theorem 39 (Density of J-stability)**.** J-stable parameters are dense in $\mathbb{C}$.

We are ready to prove Theorem 38.

Proof of Theorem 38. First suppose that there is a non-hyperbolic parameter c belonging to a ghost component U of M. We will show that the map f_c has Julia set of positive measure and admits an invariant line field. By Sullivan's classification of Fatou components (see §16 of [Mil06a]), the Fatou set of f_λ coincides with the attracting basin of infinity for all $\lambda \in U$. Let $\mathcal{A}_\lambda$ denote the attracting basin $\mathcal{A}_{f_\lambda}(\infty)$ intersected with $\mathbb{C}$. Hence $\overline{\mathcal{A}_\lambda} = \mathbb{C}$ for each $\lambda \in \Lambda$ because $K(f_c)$ has no interior.

For each $\lambda \in U$, we have the corresponding Böttcher map $\phi_\lambda : \mathcal{A}_\lambda \to \mathbb{C} \setminus \overline{\Delta}$, which is given by

$$\phi_\lambda(z) = \lim_{k\to\infty} (f_\lambda^k(z))^{1/2^k}.$$

conjugating f_λ to $z \mapsto z^2$. For any $z \in \mathcal{A}_\lambda$, let $\psi_\lambda(z) = (\phi_\lambda^{-1} \circ \phi_c)(z)$. This gives a holomorphic motion of $I(f_c)$ over U with center at c. (That ϕ_λ is holomorphic in λ is clear from the construction of ϕ_λ shown above.) Moreover, the following diagram commutes.

$$\begin{tikzcd} \mathcal{A}_c \arrow[r, "f_c"] \arrow[d, "\psi_\lambda"] & \mathcal{A}_c \arrow[d, "\psi_\lambda"] \\ \mathcal{A}_\lambda \arrow[r, "f_\lambda"] & \mathcal{A}_\lambda \end{tikzcd}$$

By the λ-lemma, ψ_λ extends to a holomorphic motion of $\overline{\mathcal{A}_c} = \mathbb{C}$, with $\psi_\lambda : \mathbb{C} \to \mathbb{C}$ quasiconformal. Moreover, ψ_λ conjugates f_c and f_λ on all of $\mathbb{C}$. For $\lambda \neq c$, ψ_λ cannot be conformal on all of $\mathbb{C}$ because f_c and f_λ are not conformally conjugate. However, ψ_λ is conformal on $\mathcal{A}_c$, so the complex dilatation $\mu_\lambda = \psi_\lambda^* \mu_0$ is supported on $J(f_c)$, which must have positive measure. Otherwise, ψ_λ is conformal by Theorem 22 (Weyl's lemma).

Since ψ_λ conjugates f_c and f_λ we calculate that $\mu := \mu_\lambda/|\mu_\lambda|$ is an invariant line field for f_c:

$$f^*\mu_\lambda = f^*(\psi^*\mu_0) = (\psi_\lambda \circ f)^*\mu_0 = (f_\lambda \circ \psi_\lambda)^*\mu_0 = \psi_\lambda^*(f_c^*\mu_0) = \psi_\lambda^*\mu_0 = \mu_\lambda.$$

Now let μ be an invariant line field for f_c. We will show that f_c belongs to a non-hyperbolic component. The measurable Riemann mapping theorem tells us that there exist quasiconformal mappings $\phi_t : \mathbb{C} \to \mathbb{C}$ satisfying $\phi_t^*\mu_0 = t\mu$ for each $t \in \Delta$. Since μ is f_c invariant, we deduce that $g_t := \phi_t \circ f_c \circ \phi_t^{-1}$ is holomorphic for each $t \in \Delta$:

$$g_t^*\mu_0 = \phi_t^{-1^*}(f_c^*(\phi_t^*\mu_0)) = \phi_t^{-1^*}(f_c^*(t\mu)) = \phi_t^{-1^*}(t\mu) = \mu_0.$$

Since ϕ_t is a homeomorphism, we find that g_t is proper and of degree 2. Hence, we may write $g_t(z) = z^2 + c(t)$, where $c(t)$ is holomorphic in t by Theorem 26. Now $U = \{c(t)\}$ is an open set of parameters which are quasiconformally-conjugate to f_c. But each $g(t)$ must have a Julia set of positive measure, so U is contained in a non-hyperbolic component (hyperbolic maps always have Julia set of measure zero). $\square$

Aside: Topological and quasiconformal rigidity. We say that two maps $f, g : X \to X$ in some family $\mathcal{F}$ are *topologically* conjugate, if there is a homeomorphism $\phi : X \to X$ with $f \circ \phi = \phi \circ g$. If we can choose ϕ to be quasiconformal, then we say f and g are *quasiconformally* conjugate. The topological class of f is the set of all maps in $\mathcal{F}$ which are topologically conjugate to f. We analogously define the quasiconformal class of f.

Definition 30 (Topological and quasiconformal rigidity)**.** We say a map $f \in \mathcal{F}$ is *topologically* (resp. *quasiconformally*) *rigid* if there is a neighborhood U of f in $\mathcal{F}$ such that f is not topologically (resp. quasiconformally) conjugate to any $g \in U$.

Conjecture 5 (Topological rigidity)**.** If f_c is non-hyperbolic, then f_c is topologically rigid in $\mathbb{C}$.

From the proof of Theorem 38, we established that in a non-hyperbolic component, all parameters are topologically conjugate. Therefore Conjecture 5 implies DHC.

Conjecture 6 (Quasiconformal Rigidity)**.** If f_c is non-hyperbolic, then f_c is quasiconformally rigid in $\mathbb{C}$.

The following proposition is proven in [Ben18]

Proposition 13 (Quasiconformal classes of hyperbolic parameters)**.** Let H be a hyperbolic component of M. If $c, c' \in H$, then f_c and $f_{c'}$ are quasiconformally conjugate.

We will later see in Theorem 41 that quasiconformal classes or singletons. Therefore a non-hyperbolic parameter which is not quasiconformally rigid belongs to a non-hyperbolic component. Hence, DHC implies Conjecture 6.

5.3.2 Combinatorial Rigidity

We will state a few more conjectures which are equivalent to MLC, and can be used to show that MLC implies density of hyperbolicity.

Geodesic laminations. We will give a combinatorial description of polynomials using *geodesic laminations*, as described in [McM95]. In general, a *geodesic lamination* is an equivalence relation $\lambda \subset S^1 \times S^1$ on S^1 such that the convex hulls of equivalence classes in the Poincaré metric on Δ are disjoint. In other words, if $t_1 \sim t_2$ and $t_3 \sim t_4$, then the unique geodesic with endpoints t_1 and t_2 does not cross the geodesic with end points t_3 and t_4.

If f is a polynomial with connected filled Julia set and $J(f)$ locally connected, then $J(f)$ is homeomorphic to S^1/λ for some lamination λ. We define λ by setting $t_1 \sim t_2$ if $\mathcal{R}_f(t_1)$ lands at the same point as $\mathcal{R}_f(t_2)$. With some work, we can show that if geodesics connecting points in an equivalence class cross, then the corresponding dynamical rays cross, which is impossible. Hence, this defines a geodesic lamination.

In general, we cannot guarantee that every ray lands. The *rational lamination* $\lambda_{\mathbb{Q}}(f)$ of a polynomial f with connected filled Julia set is defined by $t_1 \sim t_2$ if and only if $\mathcal{R}_f(t_1)$ and $\mathcal{R}_f(t_2)$ have the same landing point. The set of equivalence classes in $\lambda_{\mathbb{Q}}(f)$ is called the *orbit portrait* O_f.

Suppose that $\lambda_{\mathbb{Q}}(f_c) = \lambda_{\mathbb{Q}}(f_{c'})$. A parabolic wake W_{t_+,t_-} contains c if and only if $(t_+, t_-) \in \lambda_{\mathbb{Q}}(f_c) = \lambda_{\mathbb{Q}}(f_{c'})$, which is the case if and only if $c' \in W_{t_+,t_-}$. Conversely, if $\lambda_{\mathbb{Q}}(f_c) \neq \lambda_{\mathbb{Q}}(f_{c'})$, then we may assume there are characteristic angles $(t_+, t_-) \in \lambda_{\mathbb{Q}}(f_c)$ which are not in $\lambda_{\mathbb{Q}}(f_{c'})$. Therefore, $c \in W_{t_+,t_-}$ and $c' \notin W_{t_+,t_-}$. We then have the following.

Lemma 18 (Parabolic wakes and orbit portraits)**.** Orbit portraits are created and destroyed when crossing parabolic wakes: $\lambda_{\mathbb{Q}}(f_c) = \lambda_{\mathbb{Q}}(f_{c'})$ if and only if c and c' belong to an identical sequence of parabolic wakes.

Therefore the orbit portrait of an infinitely renormalizable parameter c specifies the same data as the nested sequence of small Mandelbrot sets $M \lhd H_1 \lhd H_2 \lhd H_3 \lhd \ldots$ which contain c.

Definition 31 (Combinatorial rigidity)**.** A map f with all periodic cycles repelling is *combinatorially rigid* if $\lambda_{\mathbb{Q}}(f) = \lambda_{\mathbb{Q}}(g)$ only when f and g are quasiconformally conjugate on their Julia sets. More precisely, if $\lambda_{\mathbb{Q}}(f) = \lambda_{\mathbb{Q}}(g)$, then the composition of Böttcher maps

$$\phi_f \circ \phi_g^{-1} : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus K(g)$$

extends quasiconformally over $K(f)$.

Note that when all periodic cycles are repelling, the filled Julia set is equal to the Julia set. If f and g, as above, are quasiconformally conjugate but not conformally conjugate, then they must have Julia sets of positive measure.

Conjecture 7 (Combinatorial rigidity)**.** If $c \in M$ has all periodic cycles repelling, then f_c is combinatorially rigid.

Theorem 40 (Combinatorial rigidity $\Rightarrow$ DHC)**.** Suppose that Conjecture 7 holds. Then hyperbolic maps are dense in $\mathbb{C}$.

Proof. If there is a non-hyperbolic component U, then we have seen in the proof of Theorem 38 that U is a maximal component of parameters which are quasiconformally conjugate, and each $f \in U$ supports an invariant line field on its Julia set. Moreover parabolic parameters lie in ∂M, so each element of U has all periodic cycles repelling.

The set of points $B_n \subset \partial U$ with an indifferent cycle of period n are nowhere dense, so there is $g \in \partial U$ with no indifferent cycle. But the rational lamination $\lambda_{\mathbb{Q}}(f)$ changes only when f develops a parabolic cycle, so $\lambda_{\mathbb{Q}}(g) = \lambda_{\mathbb{Q}}(f)$ for each $f \in U$. By combinatorial rigidity, g is quasiconformally conjugate on its Julia set to each $f \in U$, so g supports an invariant line field on its Julia set. As we saw in the proof of Theorem 38, this means that g belongs to a ghost component, a contradiction since $g \in \partial U$. $\square$

The open-closed argument. Let $\mathrm{Comb}(f_c)$ denote the combinatorial class of f_c, and $\mathrm{Qc}(f_c)$ denote the quasiconformal class of f_c. Evidently, $\mathrm{Qc}(f_c) \subset \mathrm{Comb}(f_c)$. If f_c and $f_{c'}$

are quasiconformally equivalent, then the quasiconformal conjugacy maps dynamical rays to dynamical rays, and preserves the period and multiplier of fixed points. Conjecture 7 states that $\mathrm{Comb}(f_c) \subset \mathrm{Qc}(f_c)$ when f_c has all periodic cycles repelling. Hence, for such parameters, $\mathrm{Qc}(f_c) = \mathrm{Comb}(f_c)$. By construction, $\mathrm{Comb}(f_c)$ is closed in $\mathbb{C}$. We will argue that quasiconformal classes are either open or singletons. This allows us to conclude the following.

Theorem 41 (Combinatorial rigidity 2)**.** Conjecture 7 holds if and only if for each f_c with all periodic points repelling, $\mathrm{Comb}(f) = \{f_c\}$.

Proof. As stated previously, Conjecture 7 holds if and only if $\mathrm{Comb}(f_c) = \mathrm{Qc}(f_c)$ for each f_c with all periodic cycles repelling. Since $\mathrm{Comb}(f_c)$ is closed, we will argue that $\mathrm{Qc}(f_c)$ is open or a singleton. Since both classes are proper subsets of $\mathbb{C}$, we then conclude that $\mathrm{Comb}(f_c)$ is a singleton.

Suppose $c' \neq c \in \mathrm{Qc}(f_c)$. Then we have a quasiconformal mapping $h : \mathbb{C} \to \mathbb{C}$ such that $f_{c'} = h \circ f_c \circ h^{-1}$. Let $\mu = \partial_{\bar{z}} h / \partial_z h$ be the Beltrami coefficient of h. Then μ is f_c-invariant, as $f_c^* \mu = h^*(f_{c'}^* \mu_0) = h^* \mu_0 = \mu$. Moreover, for $|\mu| < \|\mu\|_\infty$, $\mu_\lambda := \lambda \mu$ is f_c-invariant. Indeed,

$$f_c^* \mu_\lambda = \lambda f_c^* \mu = \lambda \mu = \mu_\lambda.$$

By the measurable Riemann mapping theorem, there is a quasiconformal mapping $h_\lambda : \mathbb{C} \to \mathbb{C}$ with Beltrami coefficient μ_λ. Let $g_\lambda = h_\lambda \circ f_c \circ h_\lambda^{-1}$. Then g_λ is holomorphic and proper of degree 2. Up to affine conjugacy, we may write $g_\lambda(z) = z^2 + c(\lambda)$. Then $c(0) = c$ and $c(1) = c'$. Moreover $c(\lambda)$ is holomorphic in λ by dependence on parameters (Theorem 26). Now $\lambda \mapsto c(\lambda)$ is an open mapping, so c' belongs an open set contained in $\mathrm{Qc}(c)$, and we conclude that $\mathrm{Qc}(c)$ is open. $\square$

5.3.3 Triviality of Fibers

So far, we have shown that combinatorial rigidity implies density of hyperbolicity. In this section we will prove that combinatorial rigidity is equivalent to MLC. To prove local connectivity at a point $c \in M$, we wish to construct a basis of connected neighborhoods of c in M, whose intersection is just the point c. We will make this precise with *parameter fibers*.

Definition 32 (Separation arc)**.** A *separation arc* is a Jordan arc in $\mathbb{C}$ which tends to ∞ in both directions, and contains only hyperbolic and finitely many parabolic parameters.

We say two points are *separated* by a separation arc γ if they lie in different connected components of $\mathbb{C} \setminus \gamma$.

Definition 33 (Fibers)**.** The *extended fiber* of a parameter c is the set of all parameters which cannot be separated from c by a separation arc. If $c \in M$, the *(parameter) fiber* of c is the intersection of the extended fiber with M.

Either type of fiber is *trivial* if it just contains the point c.

Conjecture 8 (Triviality of fibers)**.** All fibers are trivial.

Since separation arcs cannot cross non-hyperbolic components, any such component must be contained in a fiber. Thus, Conjecture 8 implies DHC.

Proposition 14 (Fibers in hyperbolic components)**.** If c lies in the closure of a hyperbolic component, then the fiber of c is trivial.

Proof. Suppose $c \in \overline{H}$, for some hyperbolic component H, and take another $c' \in \mathbb{C}$. If $c \in H$, then let $p_1, p_2 \neq c'$ be two distinct parabolic parameters on ∂U. We can choose a curve η in H with end points at p_1 and p_2. This curve can be perturbed to separate c from c' if $c' \in H$. Consider the union γ of η, p_1, p_2, and their corresponding parameter rays. We can always choose p_1 and p_2 so that c and c' lie in different components of $\mathbb{C} \setminus \gamma$.

Suppose $c \in \partial H$. Notice that ∂H is ordered: If H if the elements of H have an attracting cycle of order k, then the modulus of the multiplier of this cycle converges to 1 as $c \to \partial H$. Thus, ∂H inherits the ordering of S^1. Pick parabolic parameters p_1, p_2 with $c < p_1 < c'$ and $c' < p_2 < c$. Then the parameter rays landing at p_1 and p_2, together with p_1, p_2, and an arc connecting p_1 and p_2 in H, form a separation arc which separates c from c'. □

Therefore, Conjecture 8 holds if and only if the fibers of non-hyperbolic parameters are trivial.

Theorem 42 (Triviality of fibers $\iff$ Combinatorial rigidity)**.** If $c' \in M$ has all cycles repelling, then c' belongs to the fiber $F(c)$ of another parameter c if and only if f_c and $f_{c'}$ have the same combinatorics. In particular Conjecture 7 and Conjecture 8 are equivalent.

Proof. For parameters c and c', Lemma 18 implies that $c' \in \mathrm{Comb}(c)$ if and only if every parabolic wake containing c also contains c'. If $c' \in F(c)$, then c and c' cannot be separated by the parameter rays corresponding to any wake. Therefore f_c and $f_{c'}$ have the same combinatorics. Assume f_c and $f_{c'}$ have all cycles repelling. Then there are no rational parameter rays which land at c or c' and neither c nor c' lie in a hyperbolic component. If f_c and $f_{c'}$ have the same combinatorics, then c and c' cannot be separated by rational parameter rays, so either $c' \in F(c)$ or c and c' lie in the same non-hyperbolic component, in which case we still have $c' \in F(c)$.

If all fibers are trivial, then the intersection of any infinite sequence of nested wakes contains at most one parameter. If c has all cycles repelling, then c belongs to an infinite sequence of wakes. Since no sequence of wakes containing c can contain another parameter, we have $\mathrm{Comb}(c) = \{c\}$. By Theorem 41, we have combinatorial rigidity at c.

On the other hand, suppose that combinatorial rigidity holds. Parameters with indifferent or attracting cycles always have trivial fibers. If c has all periodic cycles repelling, then every $c' \neq c$ can be separated from c by a rational parameter by combinatorial rigidity and Lemma 18. Hence, we have triviality of fibers. □

We have shown that combinatorial rigidity is equivalent to triviality of fibers and implies density of hyperbolicity. The next theorem completes the web of implications. For more details, see [Lyu02].

Theorem 43 (MLC $\iff$ combinatorial rigidity)**.** The MLC conjecture is equivalent to Conjecture 7 and Conjecture 8.

Sketch of Proof. Assume triviality of fibers, and take $c \in M$, with a nested sequence of wakes W_n which contain c. Since the fiber of c is trivial, $\bigcap W_n \cap M = \{c\}$. Thus, if we cut the wakes by a sequence of equipotentials converging towards ∂M, we obtain a basis of connected neighborhoods at c. Hence, M is locally connected at c.

Now take $c \in M$, and let $F(c)$ be its fiber. If M is locally connected, then parameter accumulating on $c' \in F(c)$ lands at c'. Fibers are either trivial or uncountable, (they are full sets), and can only contain only contain the accumulation points of finitely many parameter rays (see Theorem 11 of [RS08]), we conclude that $F(c)$ is trivial. $\square$

To summarize, we have the following implications.

$$
\begin{array}{rcccl}
\text{Triviality of Fibers} & \iff & \text{MLC} & \iff & \text{Combinatorial Rigidity} \\
 & & \Downarrow & & \\
\text{NIL} & \iff & \text{DHC} & \impliedby & \text{Topological Rigidity} \\
 & & \Downarrow & & \\
 & & \text{Quasiconformal Rigidity} & &
\end{array}
$$

Chapter 6

Developments in MLC

To conclude, we will preview some partial results towards MLC as well as the general proof strategy. Any parameter with an attracting cycle in $\mathbb{C}$ lies in the interior of the Mandelbrot set, so we have locally connectivity at all such parameters. Hence, it suffices to show that M is connected at parabolic parameters, and parameters with all cycles repelling.

6.1 The Yoccoz Puzzle

Any quadratic polynomial has two fixed points with multiplicity. The dynamical ray $\mathcal{R}_c(0)$ is periodic under f with period one, and thus lands at a fixed point β, which we call *the β fixed point* of f. The other fixed point α is called *the α fixed point* of f. The α and β fixed points are distinct, except in the case of the cusp $c = 1/4$, where $\alpha = \beta = 1/2$.

If c does not lie in the main cardioid, then α is parabolic or repelling. Therefore there is an q-tuple of periodic rays $\mathcal{R}_c(t_0), \ldots \mathcal{R}_c(t_{q-1})$ which land at α. Because 0 is the only fixed point of $t \mapsto 2t$ in $\mathbb{R}/\mathbb{Z}$, we have $q > 1$. The equipotential $\mathcal{C}_c(2)$ bounds a topological disk in $\mathbb{C}$, which contains $K(f_c)$. The rays $\mathcal{R}_c(t_i)$ divide this disk into q closed disks.

The critical point $c_0 = 0$ lies in one of these disks $P_0(c_0)$. Since f maps $\mathcal{R}_c(t_0) \mapsto \mathcal{R}_c(t_1) \mapsto \cdots \mapsto \mathcal{R}_c(t_{q-1})$, f cyclically permutes the disks intersected with $K(f_c)$. Hence, there is a unique $c_i = f^i(0)$ in each disk for $0 \leqslant i \leqslant q-1$. We denote $P_0(c_i)$ for the disk containing c_i.

We call $P_0(c_0) \ldots P_0(c_{q-1})$ the *puzzle pieces* of depth zero. For $d \geqslant 0$, the puzzle pieces of depth $d+1$ are the preimages $f^{-1}(P)$, where P is any puzzle piece of depth d. The puzzle pieces cover the filled Julia set, and have disjoint interiors.

Each puzzle piece of depth $d+1$ is contained in a puzzle piece of depth d. The puzzle pieces $P_d(c_k)$, which contain $c_k = f^k(0)$ are form the *critical tableau*, which is *periodic* if $P_d(c_n) = P_d(0)$. for some $n > 0$ Evidently, $n > 1$, as the depth zero pieces are distinct.

The Yoccoz puzzle has been used to great effect in proving special cases of MLC. For each puzzle piece P, the intersection $P \cap M$ is connected. If the intersection of all puzzle pieces containing a point $p \in J(f_c)$, is just $\{p\}$, then we establish local connectivity of $J(f_c)$ at c. The idea is to prove local connectivity of $J(f_c)$, and argue that the M asymptotically

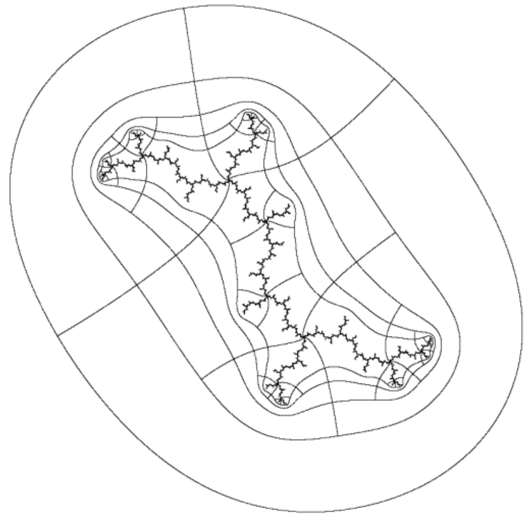

Figure 6.1: The Yoccoz Puzzle (image courtesy of [McM94b])

resembles $J(f_c)$ near c. The following theorem of Yoccoz is proven in [Hub93].

Theorem 44 (Yoccoz)**.** Suppose $c \in M$. Then c is infinitely renormalizable or the Mandelbrot set is locally connected at c.

Skeleton of Proof. Let $\mathcal{P}(c)$ denote the set of puzzle pieces for the Julia set $J(f_c)$. Then $J(f_c)$ is locally connected if $\operatorname{diam}(P_d) \to 0$ for any nested sequence of pieces P_d of depth d. This follows if

$$\sum \operatorname{mod}(P_i \setminus P_{i+1}) = \infty.$$

Here, $\operatorname{mod}(P_i \setminus P_{i+1})$ refers to the modulus of the annulus formed by removing P_{i+1} from the interior of P_i. In the degenerate case, when P_i and P_{i+1} share a boundary component, we set $\operatorname{mod}(P_i \setminus P_{i+1}) = 0$.

In the general setting of a domain U where f is one to one and holomorphic, f is conformal, so $\operatorname{mod}(f(A)) = \operatorname{mod}(A)$ for each annulus. However, if f is a d-fold covering, then $\operatorname{mod}(A) = \operatorname{mod}(f(A))/d$. The map f_c restricted to $P_i \setminus P_{i+1}$ is conformal if and only if the pieces do not contain zero, and a degree 2 covering otherwise.

Suppose f_c is finitely renormalizable with $m = |\mathcal{R}(f_c)|$. Take any $P_d \setminus P_{d+1}$, where P_d has depth d. Then $f_c^d(P_d \setminus P_{d+1}) = P_0 \setminus P_1$ for pieces P_0 and P_1 of depth 0 and 1 respectively. If k is the number of times $0 \in f_c^n(P_d)$, for $0 \leqslant n < d-1$,

$$\operatorname{mod}(P_d \setminus P_{d+1}) \geqslant \operatorname{mod}(P_0 \setminus P_1)/2^k.$$

But $k \leqslant m$, so $\operatorname{mod}(P_d \setminus P_{d+1})$ is uniformly bounded below. Therefore it is always the case that $\sum \operatorname{mod}(P_i \setminus P_{i+1}) = \infty$. We conclude that $J(f_c)$ is locally connected.

Yoccoz proves that at such parameters, the structure of the Julia set reflects the structure of $J(f_c)$. But adapting the proof above in this case, we conclude that M is locally connected at c. $\square$

Since infinitely renormalizable parameters have all cycles repelling, it follows that the Mandelbrot set is locally connected at parabolic parameters. Moreover, a small neighborhood of zero never returns to itself when c is a Misiurewicz parameter, so M is locally connected at the landing points of all rational rays.

6.2 Infinitely Renormalizable Parameters

All that is left is to verify MLC at infinitely renormalizable parameters. In many cases, the strategy is prove combinatorial rigidity. If two infinitely renormalizable maps f_c and $f_{c'}$ have the same combinatorics, then in certain circumstances, a quasiconformal conjugacy can be built.

For a polynomial f, let $f^n : U \to V$ be a renormalization. We call $V \setminus \overline{U}$ a *fundamental annulus*. This is not a canonical choice, as different fundamental annuli can have different moduli. Therefore, we define

$$\operatorname{mod}(f^n) = \sup \operatorname{mod}(A),$$

where A ranges over the possible fundamental annuli.

Definition 34 (A priori bounds)**.** Suppose that f is infinitely renormalizable. We say f has *a priori bounds* if there exists $\epsilon > 0$ such that $\operatorname{mod}(f^n) \geqslant \epsilon$ for each $n \in \mathcal{R}(f)$.

Given a wake $W_{t_\pm}$ with root point r, the *limb* L_r is the intersection of M and $W_{t_\pm}$, together with r. By the Douady-Hubbard tuning, this limb is homeomorphic to M. The limbs attached to the main cardioid are called *primary*. If H is a hyperbolic component attached to the main cardioid, and L_r is a limb attached to H, then L_r is *secondary*. A *truncated limb* is obtained from a limb L_r by removing a neighborhood of its root.

Definition 35 (Secondary limbs condition)**.** A polynomial f_c satisfies the *secondary limbs condition* if there are finitely many truncated secondary limbs L_i such that the hybrid class of f_c^n belongs to $\bigcup L_i$ for each renormalization of f_c^n.

Lyubich reduces to problem of MLC, in many cases, to the problem of finding *a priori* bounds. The following is proven in [Lyu95].

Theorem 45 (Lyubich)**.** If f_c has *a priori* bounds and satisfies the secondary limbs condition, then f_c is combinatorially rigid.

Hence, we have MLC at c.

Recall that f^n is *primitively renormalizable* if the small Julia sets of a renormalization of f^n are disjoint. If f^n is primitively renormalizable for infinitely many n, then f is *infinitely primitively renormalizable*.

Definition 36 (Infinitely primitively renormalizable of bounded type)**.** We say f is infinitely primitively renormalizable of bounded type if there is an infinite sequence $p_0 < p_1 < \dots$ such that f^{p_n} is primitively renormalizable and there is a constant B such that $p_{k+1}/p_k \leqslant B$.

Theorem 46 (Kahn)**.** If f_c is infinitely primitively renormalizable of bounded type, then f_c has *a priori* bounds, and M is locally connected at c.

Kahn proves this in [Kah23]. We will outline of the proof to give some insight into the kind of mathematics involved.

Skeleton of Proof of Theorem 46. Suppose $f = f_c$ satisfies the assumptions of Theorem 46. Let K_n be the filled Julia set of the renormalization $f^{p_n} : U_n \to V_n$. Let

$$\mathcal{K}_n = \bigcup_{i=0}^{p_n-1} f^i(K_n).$$

Then $\mathbb{C}\backslash\mathcal{K}_n$ is a hyperbolic Riemann surface. Let γ_n be the peripheral geodesic going around K_n. Then *a priori* bounds are equivalent to the hyperbolic lengths of the γ_n being bounded. The strategy of the proof is referred to as the *life improvement principle*: bad now, worse before. In this case, we wish to show that if γ_n gets long, then it was even longer previously. Specifically, there are $M > 0$ and $\ell > 0$ such that

$$|\gamma_n| > M \implies |\gamma_{n-\ell}| > 2M.$$

The difficulty arises from the fact that the number of components of $\mathcal{K}_n$ grows exponentially with n. This is overcome due to the boundedness assumption. Even though n is unbounded, ℓ is bounded. Since $p_{n+1}/p_k \leqslant B$, the number of components of $\mathcal{K}_n$ which are enclosed by $\gamma_{n-\ell}$ is at most B^ℓ. Moreover, if $f_{n-\ell} = f^{p_{n-\ell}} : U_{n-\ell} \to V_{n-\ell}$ is a renormalization, then $f_{n-\ell}$ is renormalizable of period $p_n/p_{n-\ell}$. The small Julia sets for this renormalization are the components of $\mathcal{K}_n$ surrounded by $\gamma_{n-\ell}$. ☐

The next two theorems from [KL07a] and [KL07b] will show MLC at parameters of *definitely primitive type*, that is infinitely primitively renormalizable parameters for which the renormalizations stay away from satellite renormalizable parameters.

Just as there are characteristic rays which land together at a parabolic parameter and bound a wake, two rational rays landing together at a Misiurewicz parameter bound a *decoration* of the Mandelbrot set $\mathcal{L}$.

Definition 37 (Decoration condition)**.** An infinitely primitively renormalizable $c \in M$ satisfies the *decoration condition* if there is a finite set of decorations $\mathcal{L}_i$ such that the hybrid class of f_c^n belongs to $\bigcup \mathcal{L}_i$ for each primitive renormalization of f_c^n.

Theorem 47 (Kahn-Lyubich)**.** If $c \in M$ satisfies the decoration condition, then f_c has *a priori* bounds and the Mandelbrot set is locally connected at c.

The *molecule* $\mathcal{M}$ is the subset of M consisting of parameters which are:

- finitely many times renormalizable, with all renormalizations satellite and the last renormalization with a non-repelling cycle;
- or infinitely renormalizable with all renormalizations satellite.

We again consider infinitely primitively renormalizable parameters, with primitive renormalization periods $1 = p_0 < p_1 < p_2 < \dots$, where p_i is a multiple of p_{i-1}. Let $\tilde{p}_i = p_i/p_{i-1}$ be the *relative period*. There is then a Mandelbrot copy $\tilde{M}_i$ which encodes the combinatorics of the renormalization of $\mathcal{R}^i f_c$ as the renormalization of $\mathcal{R}^{i-1} f_c$.

We say f_c satisfies the *molecule condition* if the relative Mandelbrot copies $\tilde{M}_i$ stay bounded away from $\mathcal{M}$. A more precise definition using the combinatorics of f_c is given in [KL07b].

Theorem 48 (Molecule condition)**.** Infinitely renormalizable parameters $c \in M$ satisfying the molecule condition have *a priori* bounds and M is locally connected at c.

While Kahn proved MLC at infinitely primitively renormalizable parameters of bounded type, little progress was made on satellite type parameters until work of Lyubich and Dudko in [DL23a], giving *a priori* bounds for a class of infinitely satellite renormalizable parameters. This work resulted in a proof of MLC at all infinitely renormalizable parameters of *bounded type* in [DL23b].

As in the primitive case, *bounded type* refers to all infinitely renormalizable parameters for which the relative periods of successive renormalizations are bounded above. Such parameters are also called *Feigenbaum maps*, as the Feigenbaum polynomial is infinitely renormalizable with relative period identically two.

Theorem 49 (Feigenbaum parameters)**.** Any Feigenbaum map f_c has *a priori* bounds and M is locally connected at c.

6.3 The Thurston Pullback Argument

For a map with a priori bounds, a standard outline is used to establish combinatorial rigidity, and hence MLC. It suffices to show that if f and $\tilde{f}$ are combinatorially equivalent quadratic-like maps with *a priori* bounds, then f and $\tilde{f}$ are hybrid equivalent.

The proof is split into three steps. We show successively that:

1. f and $\tilde{f}$ are topologically equivalent;
2. f and $\tilde{f}$ are quasiconformally equivalent;

3. f and $\tilde{f}$ are hybrid equivalent.

Lyubich proves in [Lyu95] that quadratic-like maps with *a priori* bounds have locally connected Julia set. Therefore $J(f)$ and $J(\tilde{f})$ are homeomorphic to their combinatorial models as a geodesic lamination, and hence to each other. McMullen's rigidity theorem in [McM94a] allows us to go from quasiconformal equivalence to hybrid equivalence, so all that is left is to argue that topological equivalence implies quasiconformal equivalence.

This is broken down into two parts. The first part is to upgrade a topological equivalence to a *Thurston equivalence*. We then convert a Thurston equivalence into the desired quasiconformal equivalence.

Both parts use some version of the Thurston pullback argument: we start with a a map that is not a conjugacy between f and $\tilde{f}$, but respects some kind of combinatorial data. We then successively lift this map to create a sequence which limits to a conjugacy. If we are careful, the limiting map has the desired properties.

We will illustrate the second part of step two. First, we define some terminology. Recall that $P(f)$ denotes the *postcritical set*, which is the forward orbit of zero.

Definition 38 (Thurston equivalence)**.** Let $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ be topologically equivalent quadratic-like maps. We say f and $\tilde{f}$ are *Thurston equivalent* if for appropriate choices of domains U, V, $\tilde{U}$, and $\tilde{V}$, this is a quasiconformal map $h : (U, V, P(f)) \to (\tilde{U}, \tilde{V}, P(\tilde{f}))$ which is homotopic to a conjugacy $\psi : (U, V, P(f)) \to (\tilde{U}, \tilde{V}, P(\tilde{f}))$ relative $(\partial U, \partial V, P(f))$. The map h is called a *Thurston conjugacy.*

Note that in this case, h conjugates $f : P(f) \cup \partial U \to P(f) \cup \partial V$ and $\tilde{f} : P(\tilde{f}) \cup \partial\tilde{U} \to P(\tilde{f}) \cup \partial\tilde{V}$.

Proposition 15 (Thurston equivalence $\Longrightarrow$ QC equivalence)**.** If two quadratic-like maps f and g are infinitely renormalizable and Thurston equivalent, then they are quasiconformally equivalent.

Proof. Suppose that $f : U_1 \to U_0$ and $g : V_1 \to V_0$ are Thurston equivalent. Let $h : U_0 \to V_0$ be a Thurston conjugacy which is K-quasiconformal and homotopic to a conjugacy ψ in the sense of Definition 38. Let c_0 be the critical point of f and c_0' the critical point of g. Let $c_{n+1} = f(c_n)$, and $c_{n+1}' = g(c_n')$.

We have the restrictions

$$\begin{tikzcd} U_1 \setminus \{c_0\} \arrow[d, "f"] & V_1 \setminus \{c_0'\} \arrow[d, "g"] \\ U_0 \setminus \{c_1\} \arrow[r, "h"] & V_0 \setminus \{c_1'\} \end{tikzcd}$$

on which f and g are covering maps. If we identify the fundamental group of each space above with $\mathbb{Z}$, then $f_*(\mathbb{Z}) = 2\mathbb{Z}$. The generator of $\pi_1(U_0 \setminus \{c_1\})$ is a simple closed curve winding once around c_1. Since $h(c_1) = h(c_1')$, this loop maps to a generator of $\pi_1(V_0 \setminus \{c_1'\})$, so $h_* = \mathrm{Id}$.

Therefore $(h \circ f)_*(\mathbb{Z}) = 2\mathbb{Z} = g_*(\mathbb{Z})$, so we can lift $h \circ f$ to a map $h_1 : U_1 \setminus \{c_0\} \to V_1 \setminus \{c_0'\}$. We can continuously extend h_1 to a map $h : U_1 \to V_1$ by setting $h(c_0) = c_0'$.

The resulting map h_1 is still K-quasiconformal because f and g are holomorphic. Set $U_n = f^{-n}(U_0)$ and $V_n = g^{-n}(V_0)$. It is not hard to show that h_1 is homotopic to ψ rel $(\partial U_1, \partial U_2, P(f))$. Therefore, $h_1 = h$ on ∂U_1, and we can extend h_1 to $U_0 \setminus U_1$ as h. By the removability of quasiarcs (Theorem 23), h_1 still has dilatation K. Moreover, h_1 is homotopic to ψ rel $(P(f), \partial U_1 \cup \partial U_2)$, and conjugates $f : P(f) \cup (U_1 \setminus U_2) \to P(f) \cup (U_0 \setminus U_1)$ to the corresponding restriction of g.

We replace h with h_1 and continue this procedure, obtaining a sequence of K-quasiconformal maps h_n which are homotopic to ψ rel $(P(f) \bigcup_{1 \leqslant k \leqslant n+1} \partial U_k)$ and conjugating $f : P(f) \cup (U_1 \setminus U_{n+1}) \to P(f) \cup (U_0 \setminus U_n)$ to the corresponding g restriction.

By the compactness of the space of quasiconformal mappings respecting a marked point (in this case mapping c_0 to c_0') from Theorem 24, we see that the limit map $\hat{h}$ is K-quasiconformal and conjugates f and g on the closures of their escape loci. Since $K(f)$ and $K(g)$ have no interior, $\hat{h}$ is a conjugacy everywhere. $\square$

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