CHAPTER 9

Interest rate sensitivity

9.1 OVERVIEW

For our readers who know calculus and wish a more comprehensive treatment, this section serves as an introduction to the issues that we discuss in greater depth, using calculus, in Sections (9.2) through (9.5). For others, we hope to provide a brief survey of problems and possible solutions that face an investor who lives in a world where interest rates may change in an unpredictable fashion.

To get started, imagine that you are obligated to pay \$50,000 five years from now. To cover this liability, you decide to buy bonds now. You believe that the prices for five-year zero-coupon bonds are not as attractive as those for three-year zero-coupon bonds or seven-year zero-coupon bonds. However, you note that should you buy a five-year zero-coupon bond with maturity \$50,000, you are *certain* to meet your obligation. On the other hand, the attractively priced three-year bond that sells for the same price as the five-year \$50,000 par-value bond will mature for less than \$50,000, so you only accumulate \$50,000 at time five if you can reinvest at a sufficiently good rate. In other words, there is some **reinvestment risk**, because you are hoping to have high interest rates three years from now but could possibly have very low rates.[1] Realizing this, you wonder whether you can avoid risk caused by uncertain rates in the future by purchasing the seven-year bond, then selling it after five years so as to realize the needed \$50,000. But of course you once again have a problem, since the price your seven-year bond commands at time 5 is based on the rates of return that are available then. If interest rates have risen, you may again have

[1] Some of you may be wondering about whether you should speculate on the forward rate $f_{[3,5]}$. That is an interesting possibility, but in this motivating example, it is not what we wish to consider, nor are other possible ways of purchasing a guaranteed rate at time 3.

a problem meeting your obligation. So, under this scenario, you are hoping to see low rates at time five.

We note that if you invest for less than five years, you hope for *high* rates at reinvestment time, while if you start with an investment of more than five years, you hope to see *low* interest rates at time five. So, either low or high interest rates can cause problems. It is natural to wonder whether there might be some way that you could split your investment, buying three-year zero-coupon bonds and seven-year zero-coupon bonds, and put yourself in a good position no matter whether rates go up or down. This is the basic idea behind **immunization** strategies. Of course, you could forget this puzzle and just invest in a five-year \$50,000 zero-coupon par bond. This would be an example of **asset matching** or producing a **dedicated portfolio**. However, sometimes it pays to **immunize** your position, even though it may require you to be vigilant with respect to interest rates over time.

We now have established the idea that you may wish to follow how the price of a set of *predetermined* future cashflows changes as the interest rate fluctuates. Actually, as you know from Section (8.3), the term "the interest rate" may not be appropriate since you probably have a nonflat yield curve, but in Sections (9.2)–(9.4) we *assume that the yield curve is flat and remains so*. In Section (9.5), we will consider the problem when the cashflows are interest-sensitive (so not predetermined), and in that section we also briefly address how you might take into consideration the existence of a nonflat yield curve.

EXAMPLE 9.1.1 Immunization

Problem: As in the preceding paragraphs, suppose that you need to pay \$50,000 in five years and that you can finance this with zero-coupon bonds with terms of three years and seven years. Imagine that you buy a three-year zero-coupon bond with a \$22,675.74 redemption amount and a seven-year zero-coupon bond redeemable at maturity for \$27,562.51, each priced to yield 5% annually. Suppose also that, at the end of three years, no matter what the yield rate i (from a flat yield curve) may be, you sell the remaining bond at a purchase price to yield i, combine the proceeds with the \$22,675.74 from the redeemed bond, and use the total to buy a two-year zero-coupon bond. Illustrate that this immunizes against interest rate risk by showing that it produces the needed \$50,000 five years after your initial bond purchases if $i = 20\%$ (a high rate), $i = 5\%$ (the present moderate rate), or $i = 1\%$ (a low rate).

Solution If $i = 20\%$, then the original seven-year \$27,562.51 bond that has four years until maturity can be sold for $\frac{\$27{,}562.51}{(1.2)^4} \approx \$13{,}292.11$. Combining this with the \$22,675.74 from the redeemed three-year bond, you have $\$13{,}292.11 + \$22{,}675.74 = \$35{,}967.85$ to invest for two years at 20%, producing $\$35{,}967.85(1.2)^2 \approx \$51{,}793.70$. You have \$1,793.70 more than you needed.

On the other hand, if $i = 5\%$, then the original seven-year bond can be sold for $\frac{\$27{,}562.51}{(1.05)^4} \approx \$22{,}675.75$. Combining this with the \$22,675.74 from redemption, you have \$22,675.75 + \$22,675.74 = \$45,351.49 to invest for two years at $i = 5\%$. So, five years after your initial bond purchases, you have $\$45{,}351.49(1.05)^2 \approx \$50,000.02$. You have only two cents more than you need for your \$50,000 obligation.

Finally, if $i = 1\%$, then the original seven-year bond can be sold for $\frac{\$27{,}562.51}{(1.01)^4} \approx$ \$26,487.03. Putting this with the \$22,675.74 from redemption, you have \$26,487.03+ \$22,675.74 = \$49,162.77 to invest for two years at $i = 1\%$. in this case, five years after your initial bond purchases, you have $\$49{,}162.77(1.01)^2 \approx \$50{,}150.94$. This is \$150.94 more than you needed.

Note: Section (9.4) uses calculus to develop the tools needed to determine this immunization strategy. ■

Another important concept we can introduce here is the sensitivity of the market value of a set of cashflows — that is, how changes in the interest rate change the value, or price, of the cashflows. Fix a set of cashflows, and let $P(i)$ denote their total price (present value) using compound interest at an effective interest rate i. If all the cashflows are positive, $P(i)$ is a decreasing function of i. The graph of $P(i)$ is called the **price curve**. The price curve is discussed further in Section (9.2), and Figure (9.2.2) depicts a price curve corresponding to a set of positive cashflows.

The **duration** of a set of cashflows is a measure of how sensitive the price of the portfolio is to a shift in the interest rate. If the portfolio has duration 5, then for small h it will lose about the fraction $5h$ of its value if the annual effective interest rate goes up from i_0 to $(i_0 + h)$ and go up in value by approximately the fraction $5h$ of its value if the interest rate decreases to $(i_0 - h)$. This is only an approximation since the price curve is not a straightline, but the approximation should be good if h is small. Actually, we will define several types of duration and the above statement refers to the **modified duration** $D(i_0, 1)$ defined by (9.2.14). The above description may be interesting, but it does not tell us how to precisely calculate duration; that requires calculus.

Another form of duration is the **Macaulay duration** $D(i_0, \infty)$, which is given by $D(i_0, \infty) = (1 + i_0)D(i_0, 1)$. Historically, the concept now known as Macaulay duration predates the modified duration, and it was introduced in a 1938 paper by Frederick Macaulay concerning the volatility of Unites States bond yield rates and stock prices to changes in interest rates.[2] The Macaulay duration is a weighted average of the times of the cashflows, the weights being the associated proportion of the total present value that is attributable to the cashflow at the corresponding time. You should view $D(i_0, \infty)$ as some sort of average length of the investment. The Macaulay duration of an n-year zero-coupon bond turns out to be n, but an n-year bond with coupons has a smaller Macaulay duration.

[2] Macaulay, F. R. *Some Theoretical Problems Suggested by Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1856* New York: National Bureau of Economic Research, 1938.

Modified duration captures the rate of change of the price relative to the price. It does not, however, tell you anything beyond that about the way the price curve is shaped. More precisely, as we shall explain in Section (9.2), a duration D at i_0 can be used to approximate the price curve $P(i)$ near $i = i_0$ by a linear function as in $P(i_0 + h) \approx P(i_0)(1 - Dh)$, but this does not indicate whether the price curve is decreasing more or less rapidly near $i = i_0$ as i increases. The concept of **convexity** is introduced in Section (9.3) to address this. If the convexity is positive, the curve bends upward and decreases less rapidly as i increases, and if it is negative it bends downward and decreases more rapidly as i increases.

Asset-liability management refers to techniques for maintaining a firm's ability to meet its cashflow obligations in a world of changing interest rates. Tools used to counter interest rate risk include immunization procedures such as **Redington immunization** and **full immunization**. These are methods that involve the ideas of duration and convexity, and immunization is discussed in Section (9.4). Redington immunization and full immunization are based on a hypothetical flat yield curve, but they give a strategy for protecting a portfolio of cashflows from interest rate changes: The strategy might be useful even if the yield curve is not flat. Redington immunization focuses on small interest rate shifts while full immunization addresses arbitrary parallel shifts of a flat yield curve. For Redington immunization, you structure your holdings so that the **assets** (promised incoming cashflows) and **liabilities** (promised outgoing cashflows) have equal present values and durations while the convexity of the assets is greater than the convexity of the liabilities. The immunization in Example (9.1.1) is an example of full immunization.

We end this section with a simple example of **asset-liability matching**[3] which requires that the outflows from liabilities are exactly offset by inflows from assets. These inflows may include bond coupons and redemption payments.

EXAMPLE 9.1.2 Asset-liability matching

Problem: Sandy is obligated to pay \$10,000 in six months, \$15,000 in twelve months, and \$25,000 in eighteen months. He wishes to purchase bonds to exactly match these liabilities — that is, to provide inflows so that the net cashflow at the three payment times, as well as at all other times times, is zero. The bonds available for purchase by Sandy are of the following three types:

(a) Six-month zero-coupon bonds, sold to yield the investor 6% nominal interest convertible semiannually;

(b) Twelve-month 6% par-value bonds with semiannual coupons;

(c) Eighteen-month 5% par-value bonds with semiannual coupons.

How much of each of these should Sandy purchase? Assume that each may be purchased for any par value that Sandy would like.

[3] Asset-liability matching is also called **dedication**.

Solution Sandy should first figure out how much of the eighteen-month bonds to purchase to match his \$25,000 obligation, since the other types of bonds will not help with this liability. If he purchases face amount F_{c} of the eighteen-month bond, then Sandy has the rights to coupon payments of $(.025)F_{\mathrm{c}}$ every six months and also to the redemption payment of F_{c}. Therefore, at time eighteen months, he stands to receive $(1.025)F_{\mathrm{c}}$, the total from the time-eighteen-months coupon payment and the redemption payment. Setting this equal to \$25,000, we find $F_{\mathrm{c}} \approx \$24{,}390.2439$. We round to the nearest penny, so $F_{\mathrm{c}} = \$24,390.24$, and the coupon payments every six months are each $(.025)(\$24{,}390.24) \approx \$609.76$. Sandy wishes to receive a total of \$15,000 at time twelve months and since he is already scheduled to be paid a \$609.76 coupon, he needs to purchase a one-year bond that will pay him $\$15{,}000-\$609.76 = \$14{,}390.24$ at the end of the year. So, he should purchase face amount F_{b} of the twelve-month 6% bond where $(1.03)F_{\mathrm{b}} = \$14{,}390.24$. Rounding the face value to the nearest cent, we obtain $F_{\mathrm{c}} = \$13{,}971.11$. Since $(.03)F_{\mathrm{b}} \approx \$419.13$ and the six-month bond is bought to yield 3% per coupon period, Sandy should purchase a face amount F_{a} satisfying $(1.03)F_{\mathrm{a}} = \$10{,}000 - \$609.76 - \$419.13$. Therefore, $F_{\mathrm{a}} = \$8{,}709.82$. ■

9.2 DURATION

(calculus needed here)

We have many times looked at sequences of cashflows. The important concept of the net present value of a set of cashflows was introduced in Section (1.7). We recall that the net present value depends on the choice of a particular accumulation function and that usually we specify a compound interest accumulation function. We once again assume that accumulation is by compound interest. Moreover, we add two assumptions. Firstly, we *assume a flat term structure of interest rates*. That is, we assume that spot rates are all equal to some common value. Secondly, we *assume that if one spot rate changes, all other spot rates change by an equal amount* so that we again have a flat yield curve. This second assumption is often expressed by saying that "the term structure only experiences parallel shifts." We wish to understand how the value of a series of cashflows shifts as the interest rate (the common value of the spot rates) changes. [In Section (9.5) we will address what might be done if we drop our two assumptions about the spot rates. We recognize that our spot rate assumptions rarely jibe with reality, but our analysis will still be helpful in real-life situations.]

Fix a set of cashflows $\{C_t : t \geq 0\}$ with only finitely many of the C_t being nonzero, and a reference time zero at which we will value the set.[4] Given an interest rate i, the price of the set that will provide the buyer with a yield rate equal to i is

[4]We have fixed the cashflows. In particular, we do not allow them to be sensitive to the interest rate i. This would not, for instance, be true of the cashflows of a callable or putable bond. We will consider such interest rate-dependent cashflows in Section (9.5).

just the present value

$$P(i) = \sum_{t \geq 0} C_t (1+i)^{-t}. \tag{9.2.1}$$

We let i_0 denote the initial interest rate. If $P(i_0) > 0$, then the continuous function $P(i)$ remains positive for i close to i_0, and a typical **price curve** might look as in Figure (9.2.2).

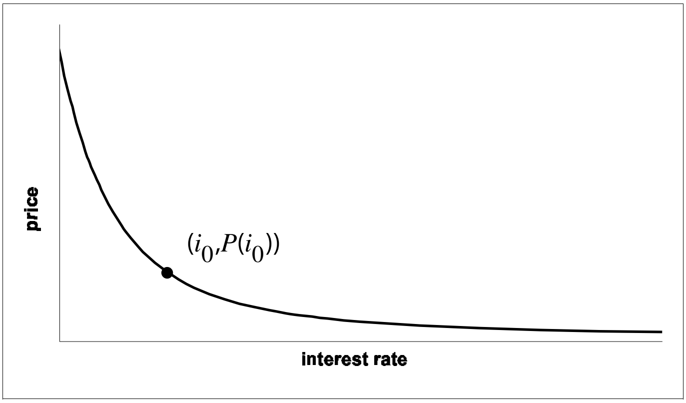

FIGURE (9.2.2)

The function $P(i)$ is an infinitely differentiable function, so it has a Taylor series

$$\sum_{n=0}^{\infty} \frac{P^{(n)}(i_0)}{n!}(i-i_0)^n = P(i_0)+P'(i_0)(i-i_0)+\frac{P''(i_0)}{2}(i-i_0)^2+\cdots. \tag{9.2.3}$$

The first two terms of this series $P(i_0) + P'(i_0)(i - i_0)$ give us the tangent-line approximation to $P(i)$ at interest rate i_0, namely

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0), \tag{9.2.4}$$

which is of course only a good approximation for i very close to i_0. The first three terms give us the second-Taylor-polynomial approximation

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0) + \frac{P''(i_0)}{2}(i - i_0)^2, \tag{9.2.5}$$

and we can expect this to be a better approximation for i close to i_0. We note that the derivatives involved in Approximations (9.2.4) and (9.2.5) are straightforward to compute; taking successive derivatives in Equation (9.2.1), we have

$$P'(i) = -\sum_{t \geq 0} C_t t (1+i)^{-t-1} \tag{9.2.6}$$

and

$$P''(i) = \sum_{t \geq 0} C_t t(t+1)(1+i)^{-t-2}. \tag{9.2.7}$$

EXAMPLE 9.2.8

Problem: Excel Finance holds loans that guarantee they receive repayments of \$1,000,000 in one year, \$2,000,000 in two years, and \$2,000,000 in six years. They also have issued bonds that require them to make payments of \$300,000 at the end of each of the next three years and \$3,000,000 in four years. Calculate Excel Finance's price function at 2%, 4%, 4.9%, 5%, 5.1%, 6%, and 10%. Then use the tangent line approximation at $i_0 = 5\%$ to estimate the prices at each of these rates, excluding 5%. Also use the second Taylor polynomial around $i_0 = 5\%$ to estimate the prices at 2%, 4%, 6%, and 10%. Discuss your findings.

Solution Denote the amount of any positive inflow at time t by A_t, and let L_t designate the amount of any liability you must pay at t. Then $A_1 = \$1{,}000{,}000$, $A_2 = A_6 = \$2{,}000{,}000$, $L_1 = L_2 = L_3 = \$300{,}000$, and $L_4 = \$3{,}000{,}000$. Therefore, the price, when figured at interest rate i, is

$$\begin{aligned}P(i) &= (\$1{,}000{,}000 - \$300{,}000)(1+i)^{-1} + (\$2{,}000{,}000 - \$300{,}000)(1+i)^{-2} \\ &\quad - \$300{,}000(1+i)^{-3} - \$3{,}000{,}000(1+i)^{-4} + \$2{,}000{,}000(1+i)^{-6}.\end{aligned}$$

Computing this net present value seven times, with i equal to .02, .04, .049, .05, .051, .06, and .10, we obtain approximately \$1,041,971.22, \$994,340.06, \$975,755.73, \$973,788.87, \$971,840.87, \$955,125.61, and \$895,835.37 respectively. [Equipped with a BA II Plus calculator, you can most easily check these values using the **Cash Flow worksheet** and $\boxed{\text{NPV}}$. The contribution registers should have non-zero entries CF0 = 0, C01 = 700,000, C02 = 1,700,000, C03 = −300,000, C04 = −3,000,000, C05 = 0, C06 = 2,000,000, and the corresponding frequency registers should all contain the numeral 1. Remember to enter the interest rates as percents after keying $\boxed{\text{NPV}}$ (Don't forget to press $\boxed{\text{ENTER}}$), and then to press $\boxed{\downarrow}$ $\boxed{\text{CPT}}$.] Note that

$$\begin{aligned}P'(i) &= -\$700{,}000(1+i)^{-2} - \$3{,}400{,}000(1+i)^{-3} + \$900{,}000(1+i)^{-4} \\ &\quad + \$12{,}000{,}000(1+i)^{-5} - \$12{,}000{,}000(1+i)^{-7},\end{aligned}$$

and therefore $P'(.05) \approx -\$1{,}957{,}398.21$. (Again, the BA II Plus calculator **Cash Flow worksheet** and $\boxed{\text{NPV}}$ allow you to check this calculation efficiently.) Therefore, the tangent line at $i = .05$ gives

$$P(i) \approx \$973{,}788.87 - \$1{,}957{,}398.21(i - .05).$$

In particular, using this repeatedly, we obtain $P(.02) \approx \$1{,}032{,}510.82$, $P(.04) \approx \$993{,}362.85$, $P(.049) \approx \$975{,}746.27$, $P(.051) \approx \$971{,}831.47$, $P(.06) \approx \$954{,}214.89$, and $P(.10) \approx \$875{,}918.96$. We note that these approximations *underestimate* the exact values by \$9,460.40 ($\approx .91\%$), \$977.21 ($\approx .098\%$), \$9.46 ($\approx .00097\%$), \$9.40 ($\approx .00097\%$), \$910.72 ($\approx .095\%$), and \$19,916.41 ($\approx 2.2\%$), respectively. So, they are all very good approximations! Of course, those for interest rates very close to 5% are spectacular.

Now for the quadratic approximations. Note that

$$\begin{aligned} P''(i) = {} & \$1{,}400{,}000(1+i)^{-3} + \$10{,}200{,}000(1+i)^{-4} \\ & - \$3{,}600{,}000(1+i)^{-5} - \$60{,}000{,}000(1+i)^{-6} \\ & + \$84{,}000{,}000(1+i)^{-8}, \end{aligned}$$

and $P''(.05) \approx \$18{,}861{,}826.29$. Therefore,

$$P(i) \approx \$973{,}788.87 - \$1{,}957{,}398.21(i - .05) + \$9{,}430{,}913.15(i - .05)^2.$$

Using this approximation, we find $P(.02) \approx \$1{,}040{,}998.64$, $P(.04) \approx \$994{,}305.94$, $P(.06) \approx \$955{,}125.98$, and $P(.10) \approx \$899{,}496.24$. The first two understate the values by \$972.58($\approx .09334\%$) and \$34.12 ($\approx .00343\%$), and the last two overstate it by \$32.37 ($\approx .00339\%$) and \$3,660.87 ($\approx .40865\%$). These are improved estimates from our previous ones. ■

In Example (9.2.8) we worked first with the tangent approximation (9.2.4), and this approximation may be rewritten as

(9.2.9) $$\boxed{\frac{P(i) - P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i - i_0).}$$

The approximation (9.2.9) allows us to estimate the **fractional price change** or **relative price change** — the price change divided by the original price — as the interest rate increases.

It is best to avoid talking about a "one-percent increase of the interest rate" i, since the phrase is ambiguous. Instead, we introduce **basis points**. If the yield increases from $i = q\% = .01q$ to $i = (q+1)\% = .01(q+1)$, we say that the yield has increased by **one hundred basis points**, and more generally an increase of $100b$ basis points means that the yield has increased from one value to another value that is numerically $(.01)b$ more. That is, 100 basis points equals $.01 = 1\%$. Substituting $i - i_0 = .01 = 100$ basis points into the approximation (9.2.9) gives $\frac{P(i)-P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(.01)$. With this new language, the approximation (9.2.9) gives us the following statement.

> If the yield is initially equal to i_0 and then it increases by one hundred basis points, the approximate relative price change is $\frac{P'(i_0)}{P(i_0)}$ percent.

Another way to write Approximation (9.2.9) is as

(9.2.10) $$\left[\frac{P(i) - P(i_0)}{P(i_0)}\right] \Big/ (i - i_0) \approx \frac{P'(i_0)}{P(i_0)}.$$

We may view the right-hand side of Approximation (9.2.10) as a normalized or relative rate of change of the price. This is reminiscent of the way we viewed $\left[\frac{a(t+h)-a(t)}{a(t)}\right] \big/ h$ when we discussed force of interest [see Section (1.12)]. By Approximation (9.2.9), multiplying the ratio $\frac{P'(i_0)}{P(i_0)}$ by the numerical change in the interest rate gives an estimate of the relative change in the price $P(i)$. You expect this to be a good estimate if i is close to i_0.

EXAMPLE 9.2.11

Problem: Use Approximation (9.2.9) to estimate the relative price change in Example (9.2.8) when i changes from .05 to .051.

Solution In Example (9.2.8), we calculated $P(.05) \approx \$973{,}788.87$, and $P(.051) \approx \$971{,}840.87$. Therefore, the true relative change is

$$\frac{P(.051) - P(.05)}{P(.05)} \approx \frac{\$971{,}840.87 - \$973{,}788.87}{\$973{,}788.87} \approx -.002000434.$$

Approximation (9.2.9) estimates this to be $\frac{P'(.05)}{P(.05}(.051 - .05)$, which from Example (9.2.8) equals

$$\left(\frac{-\$1,957,398.21}{\$973,788.87}\right)(.001) \approx -.002010085$$

rather than the true $-.002000434$. ■

The factor $\frac{P'(i_0)}{P(i_0)}$ plays a key role in both Approximations (9.2.9) and (9.2.10) and leads to our first formal definition of a duration. For any number $i > -1$, we define the **modified duration** $D(i, 1)$ by

(9.2.12) $$\boxed{D(i, 1) = -\frac{P'(i)}{P(i)}.}$$

Approximation (9.2.9) indicates that the prices of investments with larger modified durations are more sensitive to interest rate changes.

Recalling Equation (9.2.6), we see that

(9.2.13) $$D(i, 1) = \sum_{t \geq 0} C_t t (1 + i)^{-t-1} \Big/ \sum_{t \geq 0} C_t (1 + i)^{-t}.$$

The "1" in the notation $D(i, 1)$ reminds us that we were considering the price as a function of the effective interest rate, which is a nominal interest rates convertible *one* time per year. More generally, we may define a **modified duration** for a nominal interest rate $i^{(m)}$ convertible m times per year. To do this, we begin by observing that the price of a cashflow is a function of the nominal interest rate $i^{(m)}$, since i is a function of $i^{(m)}$, namely $i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1]$. We define the **modified duration** $D(i,m)$ by

$$\boxed{D(i,m) = -\frac{\frac{dP}{di^{(m)}}}{P(i)} \quad \text{where} \quad i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1].} \tag{9.2.14}$$

Since $i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$, the chain rule gives

$$\begin{aligned}\frac{dP}{di^{(m)}} &= \frac{dP}{di}\frac{di}{di^{(m)}} = P'(i)\left(1 + \frac{i^{(m)}}{m}\right)^{m-1} \\ &= P'(i)\left(\frac{\left(1+\frac{i^{(m)}}{m}\right)^m}{1+\frac{i^{(m)}}{m}}\right) = P'(i)\left(\frac{1+i}{1+\frac{i^{(m)}}{m}}\right).\end{aligned}$$

Therefore,

$$\boxed{\begin{aligned}&D(i,m) = \left(\frac{1+i}{1+\frac{i^{(m)}}{m}}\right) D(i,1) \quad \text{or equivalently} \\ &D(i,m)\left(1 + \frac{i^{(m)}}{m}\right) = D(i,1)(1+i).\end{aligned}} \tag{9.2.15}$$

It follows that

$$\boxed{D(i,m)\left(1 + \frac{i^{(m)}}{m}\right) \text{ is independent of } m.}$$

Define

$$\boxed{D(i,\infty) = -\frac{\frac{dP}{d\delta}}{P(i)} \quad \text{where} \quad \delta = \ln(1+i).} \tag{9.2.16}$$

Then, since $i = e^{\delta} - 1$,

$$\frac{dP}{d\delta} = \frac{dP}{di}\frac{di}{d\delta} = \frac{dP}{di}(1+i), \tag{9.2.17}$$

and

$$\boxed{D(i,\infty) = D(i,1)(1+i).} \tag{9.2.18}$$

Combining (9.2.15) and (9.2.18), we have

IMPORTANT FACT 9.2.19

If m is a positive integer, then

$$D(i,\infty) = D(i,m)\left(1 + \frac{i^{(m)}}{m}\right).$$

Also note that

$$D(i,\infty) = D(i,1)(1+i) = D(i,1)\left(\lim_{m\to\infty} \frac{1+i}{1+\frac{i^{(m)}}{m}}\right)$$
$$= \lim_{m\to\infty} D(i,m).$$

We call $D(i,\infty)$ the **Macaulay duration**. Important Fact (9.2.19) shows that what we call modified duration $D(i,1)$ is obtained from Macaulay duration by a simple modification. This is the origin of the term modified duration. Another name for modified duration is **volatility**, but the word "volatility" has other meanings too.

It is instructive to write a formula for $D(i,\infty)$ that does not involve derivatives or other durations. Specifically, the price of the cashflows is given by $\sum_{t\geq 0} C_t e^{-\delta t}$ and the derivative of this expression *with respect to* δ is $-\sum_{t\geq 0} C_t t e^{-\delta t}$. Therefore,

$$D(i,\infty) = -\frac{\frac{dP}{d\delta}}{P(i)} = \sum_{t\geq 0} C_t t e^{-\delta t} \Bigg/ \sum_{t\geq 0} C_t e^{-\delta t}$$
$$\text{(9.2.20)} \qquad = \sum\nolimits_{t\geq 0}\left(\frac{C_t e^{-\delta t}}{\sum_{t\geq 0} C_t e^{-\delta t}}\right)t$$
$$= \sum\nolimits_{t\geq 0}\left(\frac{C_t(1+i)^{-t}}{P(i)}\right)t.$$

We therefore have

IMPORTANT FACT 9.2.21

The Macaulay duration $D(i,\infty)$ of a set of cashflows is a weighted average of the payment times at which the cashflows occur. If the cashflow at time t is C_t, the weight given to the time t is $C_t(1+i)^{-t}/P(i)$, the proportion of the total price attributable to the time t cashflow.

We note that if the interest rate i were zero, then the Macaulay Duration $D(i, \infty)$ would be the weighted average used in the method of equated time [see Section (2.3)]. Macaulay and modified duration each provide a measure of the average length of an investment. Consequently, they are important in considering risk arising from the uncertainty of future interest rates when money needs to be reinvested. This risk is often referred to as **reinvestment risk.**

The simplest set of cashflows is a single payment — for instance, the redemption payment of a zero-coupon bond.

EXAMPLE 9.2.22 Duration of a zero-coupon bond

Problem: An N-year zero-coupon bond is purchased to provide an annual effective yield i. Find the Macaulay duration $D(i, \infty)$ and the modified duration $D(i, 1)$.

Solution The bond has a single cashflow at its redemption time, so the average of its payment times is simply N. Therefore, Important Fact (9.2.21) gives $D(i, \infty) = N$. Equation (9.2.18) tells us that $D(i, 1) = (1 + i)^{-1} D(i, \infty) = \frac{N}{1+i}$. ■

Calculating duration is more complicated for a bond with coupons, as we see in our next examples.

EXAMPLE 9.2.23 Duration of a coupon bond

Problem: Find the Macaulay duration $D\big([(1.03)^2 - 1], \infty\big)$ of a ten-year 8% \$15,000 bond with semiannual coupons and redemption amount \$16,500.

Solution This bond has twenty semiannual $\left(\frac{.08}{2}\right)$ \$15,000 = \$600 coupon payments, occurring at times $\frac{1}{2}, \frac{2}{2}, \ldots, \frac{20}{2}$, and a redemption amount of \$16,500 at time $\frac{20}{2}$. The given annual effective interest rate is $i = (1.03)^2 - 1$, so the semiannual interest rate is $(1 + i)^{\frac{1}{2}} - 1 = .03$. Therefore, the basic price formula tell us that the price of the bond is $P(.03) = \$600 a_{\overline{20|}3\%} + \$16{,}500(1.03)^{-20} \approx \$18{,}062.13486$. The Macaulay duration of the bond is, according to Important Fact (9.2.21), the weighted average of the payment times at which the cashflows occur: The weight given to time $t \in \left\{\frac{1}{2}, \frac{2}{2}, \ldots, \frac{19}{2}\right\}$ is

$$\frac{\$600(1+i)^{-t}}{P(i)} = \frac{\$600}{\$18{,}062.13486}(1.03)^{-2t},$$

and the weight given to time $\frac{20}{2}$ (the maturity time) is

$$\frac{(\$600 + \$16{,}500)(1+i)^{-t}}{P(i)} = \frac{\$600 + \$16{,}500}{\$18{,}062.13486}(1.03)^{-20}.$$

Thus, the Macaulay duration is

$$\begin{aligned}
&D\big([(1.03)^2-1],\infty\big)\\
&=\left[\left(\sum\nolimits_{k=1}^{20}\$600(1.03)^{-k}\left(\frac{k}{2}\right)\right)+\$16{,}500(1.03)^{-20}\left(\frac{20}{2}\right)\right]\Big/\$18{,}062.13486\\
&=\left[\$300\left(\sum\nolimits_{k=1}^{20}(1.03)^{-k}k\right)+\$91{,}356.49944\right]\Big/\$18{,}062.13486\\
&=\left[\$300(Ia)_{\overline{20}|3\%}+\$91{,}356.49944\right]\Big/\$18{,}062.13486.
\end{aligned}$$

We can now use Equation (3.9.6) to compute $\$300(Ia)_{\overline{20}|3\%}$. We find

$$\$300(Ia)_{\overline{20}|3\%}=\$300\left(\frac{\ddot{a}_{\overline{20}|3\%}-20(1.03)^{-20}}{.03}\right)\approx\$42{,}502.84023.$$

Therefore,

$$D\big([(1.03)^2-1],\infty\big)\approx\frac{\$42{,}502.84023+\$91{,}356.49944}{\$18{,}062.13486}\approx 7.411047515$$

for this ten-year coupon bond. ■

We next consider the Macaulay duration of a par-value bond with coupons, bought for its redemption amount. In this situation, the Macaulay duration, calculated at the bond's yield rate, turns out to be given by a simple formula.

EXAMPLE 9.2.24 Duration of a coupon bond

Problem: A noncallable par-value bond has m coupons per year and matures at the end of n years. As usual, the coupons are level. Find the Macaulay duration $D(i,\infty)$ and the modified duration $D(i,m)$ where i is the effective annual yield realized by a buyer who purchases the bond for its redemption amount and then holds it to maturity.

Solution The coupon amount is $P\big((1+i)^{\frac{1}{m}}-1\big)=P\big(\frac{i^{(m)}}{m}\big)$, because the bond is purchased for its redemption amount and the resulting effective yield on the investment of $P=C$ is i. Let $N=mn$. The cashflows provided by the bond are the level coupon payments at times $\frac{1}{m},\frac{2}{m},\ldots,\frac{N}{m}$, as well as a redemption payment $C=P$ at time $n=\frac{N}{m}$, the end of the last coupon period. Recalling that the Macaulay duration is the weighted average of the payment times, the weight at time t being the amount of the total present value contributed at time t [see Equation (9.2.20)], since $P=C$

we have

$$
\begin{aligned}
D(i,\infty) &= \left[\left[\sum_{k=1}^{N} P\left(\frac{i^{(m)}}{m}\right)(1+i)^{-\frac{k}{m}}\left(\frac{k}{m}\right)\right] + P(1+i)^{-n}n\right] \Big/ P \\
&= \left[\sum_{k=1}^{N}\left(\frac{i^{(m)}}{m}\right)(1+i)^{-\frac{k}{m}}\left(\frac{k}{m}\right)\right] + (1+i)^{-n}n \\
&= i^{(m)}\left[\sum_{k=1}^{N}(1+i)^{-\frac{k}{m}}\left(\frac{k}{m^2}\right)\right] + (1+i)^{-n}n \\
&\overset{(*)}{=} i^{(m)}\left[(I^{(m)}a)^{(m)}_{\overline{n}|i}\right] + (1+i)^{-n}n \\
&= i^{(m)}\left[\frac{\ddot{a}^{(m)}_{\overline{n}|i} - n(1+i)^{-n}}{i^{(m)}}\right] + (1+i)^{-n}n \\
&= \ddot{a}^{(m)}_{\overline{n}|i}.
\end{aligned}
$$

In obtaining the above equality marked "$\overset{(*)}{=}$", we have used the definition of $(I^{(m)}a)^{(m)}$ [see Figure (4.5.1) and Equation (4.5.2)]. To obtain the modified duration $D(i,m)$, we use Equation (9.2.19). We have,

$$
D(i,m) = \frac{\ddot{a}^{(m)}_{\overline{n}|i}}{1+\frac{i^{(m)}}{m}} = a^{(m)}_{\overline{n}|i}.
$$

■

The result of Example (9.2.24) is significant, and we record it here so that it is easy to reference, along with a helpful observation.

> **IMPORTANT FACT 9.2.25**
> The Macaulay duration of a par-value bond bought for its redemption amount, with m coupons per year and a fixed n-year term, is $\ddot{a}^{(m)}_{\overline{n}|i}$ where i is the yield rate to the purchaser. So, the duration increases with n and decreases if i or m increase.

If a bond is bought at a discount, there is a larger redemption payment than the price would indicate. The redemption payment is made at the end of the term, so this larger redemption payment results in an increased duration compared to that in Important Fact (9.2.25). On the other hand, if a bond is bought at a premium, the price exceeds the redemption amount, and the duration is decreased.

We next consider the duration for a mortgage that does not allow early repayment. How one should measure the length of time until reinvestment for a mortgage allowing repayment or for a callable or putable bond is discussed in Section (9.5).

EXAMPLE 9.2.26 Duration of an amortized loan with no early repayment option

Problem: On the first day of the year, Friendly Mortgage Company counts among its assets the rights to all future payments on a mortgage having N level payments of X remaining, the payments due at the end of each m-th of a year. Friendly wishes to determine the Macaulay duration $D(i,\infty)$ of this asset. Find the numerical value if $N = 180$, $m = 12$, and $i = 6\%$.

Solution If we let time zero be the first day of the year referred to in the statement of the problem, then payments are to occur at times $\frac{1}{m}, \frac{2}{m}, \ldots, \frac{N}{m}$, and the present value of the payment at time $\frac{k}{m}$ is $X(1+i)^{-\frac{k}{m}}$. So, the total present value is $\sum_{k=1}^{N} X(1+i)^{-\frac{k}{m}}$, and this is the outstanding loan balance if the loan was made at the annual effective rate i. Let $n = \frac{N}{m}$. The Macaulay duration is

$$\begin{aligned}
D(i,\infty) &= \sum_{k=1}^{N} \left(\frac{k}{m}\right) X(1+i)^{-\frac{k}{m}} \Big/ \sum_{k=1}^{N} X(1+i)^{-\frac{k}{m}} \\
&= \sum_{k=1}^{N} \left(\frac{k}{m^2}\right)(1+i)^{-\frac{k}{m}} \Big/ \sum_{k=1}^{N} \left(\frac{1}{m}\right)(1+i)^{-\frac{k}{m}} \\
&= (I^{(m)}a)_{\overline{n}|i}^{(m)} \big/ a_{\overline{n}|i}^{(m)} = \frac{\ddot{a}_{\overline{n}|i}^{(m)} - nv^n}{i^{(m)}} \Bigg/ \frac{1-v^n}{i^{(m)}} \\
&= \left(\frac{\ddot{a}_{\overline{n}|i}^{(m)} - nv^n}{1-v^n}\right)\left(\frac{(1+i)^n}{(1+i)^n}\right) = \frac{\ddot{s}_{\overline{n}|i}^{(m)} - n}{(1+i)^n - 1} \\
&= \frac{\ddot{s}_{\overline{n}|i}^{(m)}}{(1+i)^n - 1} - \frac{n}{(1+i)^n - 1} = \frac{1}{d^{(m)}} - \frac{n}{(1+i)^n - 1}.
\end{aligned}$$

Here we have used Equations (4.3.11) and (4.5.2). Now suppose $N = 180$, $m = 12$, and $i = 6\%$. Then $n = \frac{N}{m} = 15$, and $d^{(m)} = d^{(12)} = 12\left[1 - (1.06)^{-\frac{1}{12}}\right]$. So,

$$\begin{aligned}
D(i,\infty) &= \frac{1}{d^{(m)}} - \frac{n}{(1+i)^n - 1} = \frac{1}{12\left[1-(1.06)^{-\frac{1}{12}}\right]} - \frac{15}{(1.06)^{15} - 1} \\
&\approx 6.462820597.
\end{aligned}$$

The mortgage lasts for 15 years, but the Macaulay duration is less than half the length of the mortgage. ■

We next consider the duration of a portfolio of assets, each with known Macaulay duration at interest rate i. We suppose that the portfolio consists of s assets and that the k-th asset has present value $P^{[k]}(i)$ and Macaulay duration $D^{[k]}(i,\infty)$ when figured using compound interest at effective interest rate i or the equivalent continuously compounded rate δ. Note that

$$P^{[k]}(i) = \sum_t C_t^{[k]}(1+i)^{-t},$$

where $C_t^{[k]}$ is the cashflow of the k-th asset at time t. In addition,

$$D^{[k]}(i,\infty) = \sum_t \left(\frac{C_t^{[k]}(1+i)^{-t}}{P^{[k]}(i)}\right) t.$$

Then, with $P(i)$ as in Equation (9.2.1), the Macaulay duration of the whole portfolio $D^{\text{portfolio}}(i,\infty)$ is given by

$$\boxed{\begin{aligned} D^{\text{portfolio}}(i,\infty) &= \sum_t \left(\sum_{k=1}^{s} C_t^{[k]}(1+i)^{-t} t\right) / P(i) \\ &= \sum_{k=1}^{s} \left(\sum_t C_t^{[k]}(1+i)^{-t} t\right) / P(i) \\ &= \sum_{k=1}^{s} D^{[k]}(i,\infty) \frac{P^{[k]}(i)}{P(i)}. \end{aligned}} \tag{9.2.27}$$

Therefore,

IMPORTANT FACT 9.2.28

The Macaulay duration of a portfolio is a weighted average of the Macaulay durations of the individual assets, the weight given to the duration of an individual asset being the proportion of the total price attributable to that asset. Moreover, Important Fact (9.2.19) guarantees that you may replace “Macaulay” with “modified” in this statement.

EXAMPLE 9.2.29 Duration of a portfolio of bonds

Problem: Julian Bradley’s bond portfolio consists of two bonds. Specifically, there is a \$10,000 five-year zero-coupon bond and a \$2,000 6% par-value two-year bond with semiannual coupons. Compute the Macaulay duration $D^{\text{portfolio}}(.045,\infty)$ for the portfolio.

Solution Unfortunately, we can not just use Important Fact (9.2.25), because the nominal coupon rate, namely 6%, is not equivalent to the interest rate $i = 4.5\%$ at which we want to calculate the Macaulay duration. However, we can determine the individual Macaulay durations $D^{\text{two}}(.045,\infty)$ and $D^{\text{five}}(.045,\infty)$ of the two-year and five-year bonds respectively, and then use Important Fact (9.2.28) to find the Macaulay duration $D^{\text{portfolio}}(.045,\infty)$ of the portfolio of bonds.

The two-year bond has semiannual coupons that are each for an amount $\$2{,}000\left(\frac{.06}{2}\right) = \$60$ and an additional redemption payment of \$2,000 at the end of the second year. So, calculated at 4.5%, its price is

$$\$60(1.045)^{-.5} + \$60(1.045)^{-1} + \$60(1.045)^{-1.5} + \$2{,}060(1.045)^{-2}$$
$$\approx \$2{,}058.680315,$$

and by Important Fact (9.2.21),

$$D^{\text{two}}(.045,\infty)$$
$$= \frac{1}{\$2{,}058.680315}\left(\$60\frac{.5}{(1.045)^{.5}} + \$60\frac{1}{(1.045)^{1}} + \$60\frac{1.5}{(1.045)^{1.5}} + \$2{,}060\frac{2}{(1.045)^{2}}\right)$$
$$\approx 1.915703104.$$

This is a bit less than 2, the Macaulay duration of a two-year zero-coupon bond.

The price of the five-year zero-coupon bond to yield 4.5% is the present value of the redemption amount $\$10{,}000(1.045)^{-5} \approx \$8{,}024.510465$. The bond has a single payment at the end of five years, and therefore $D^{\text{five}}(.045,\infty) = 5$.

The price of the portfolio, to yield 4.5%, is the sum of the prices of the individual bonds, namely $\$2{,}058.680315 + \$8{,}024.510465 \approx \$10{,}083.19078$. So, using Important Fact (9.2.28) or Equation (9.2.27), we calculate

$$D^{\text{portfolio}}(.045,\infty) \approx \left(\frac{\$2{,}058.680315}{\$10{,}083.19078}\right)(1.915703104) + \left(\frac{\$8{,}024.510465}{\$10{,}083.19078}\right)(5)$$
$$\approx 4.370280555.$$

■

We end this section by noting that (Macaulay or modified) duration is a measure of the length of an investment. However, the duration does not always decrease proportionally to the elapsed time. Rather, the way it decreases depends on the type of investment. To see this, consider Examples (9.2.24) and (9.2.26). This observation will be important in connection with immunization in Section (9.4), since it implies that immunized portfolios do not generally stay immunized without active management.

9.3 CONVEXITY

(calculus needed here)

Modified duration $D(i,1)$ was introduced when we discussed how the tangent line could be used to estimate the value of the price function $P(i)$ as interest rates change. As we witnessed in Example (9.2.8), such estimation tends to work extremely well for very *small* changes in the interest rate, but using the quadratic approximation (9.2.5) in place of the tangent-line approximation (9.2.4) may give you a significantly better estimate for larger interest rate changes. The quadratic approximation (9.2.5) may be rewritten as

$$\text{(9.3.1)} \qquad \frac{P(i)-P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i-i_0) + \frac{P''(i_0)}{2P(i_0)}(i-i_0)^2.$$

Define the **modified convexity** by

$$\text{(9.3.2)} \qquad \boxed{C(i,1) = \frac{P''(i)}{P(i)}.}$$

(We recall that the sign of the second derivative of a function governs whether it is concave upward or concave downward. But a function being concave is equivalent to a certain associated region of the xy-plane being convex,[5] and this is why the term "convexity" is used.) Using Definitions (9.2.12) and (9.3.2), we may rewrite the approximation (9.3.1) as

$$\text{(9.3.3)} \qquad \boxed{\frac{P(i)-P(i_0)}{P(i_0)} \approx -D(i_0,1)(i-i_0) + C(i_0,1)\frac{(i-i_0)^2}{2}.}$$

Approximation (9.3.3) indicates that if two investments have the same modified duration, then the one whose convexity $C(i_0,1)$ has the larger absolute value is apt to have the price that is more sensitive to interest rate changes.

The sign of the modified convexity $C(i_0,1)$ is positive if $P(i_0)$ and $P''(i_0)$ are both positive. Recalling Equation (9.2.7), we see that this is the case for a set of nonnegative cashflows, at least one of which is positive — for example, for the set of cashflows obtained by the holder of a bond. Comparing Approximations (9.2.9) and (9.3.3), if the convexity is positive, we would guess that the estimate of the relative price change given by Approximation (9.2.9) is too low.

[5] A subset S of the xy-plane is **convex** if whenever points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ lie in S, the line segment joining them consists entirely of points from S.

Let $a < b$ and suppose $f(x)$ is a function that is differentiable on (a,b) and continuous on $[a,b]$. Further suppose that $S_{[a,b]}$ is the subset of the xy-plane bounded by the graph $\{(x, f(x))|a \le x \le b\}$ along with the line segment joining the points $(a, f(a))$ and $(b, f(b))$. The function $f(x)$ is **concave upward** on the interval $[a,b]$ if its graph $\{(x, f(x))|a \le x \le b\}$ curves in a counterclockwise direction and it is **concave downward** on the interval $[a,b]$ if its graph $\{(x, f(x))|a \le x \le b\}$ curves in a clockwise direction. The function $f(x)$ is concave, either upward or downward, on $[a,b]$ if and only if the subset $S_{[a,b]}$ is convex.

It is a standard fact of calculus that $f(x)$ is concave upward on $[a,b]$ exactly when the derivative $f'(x)$ is an increasing function on $[a,b]$, and the condition for $f(x)$ to be concave downward on $[a,b]$ is that the derivative $f'(x)$ be a decreasing function on $[a,b]$. Therefore, for a twice differentiable function on (a,b), the second derivative $f''(x)$ being positive on (a,b) is equivalent to the function being concave upward on $[a,b]$, while $f''(x)$ being negative on (a,b) is exactly what is needed for the function to be concave downward.

EXAMPLE 9.3.4

Problem: A five-year zero-coupon bond redeemable at C is purchased to provide an annual effective yield of 6%. Find the modified convexity $C(i,1)$. Use this, along with the modified duration $D(i,1)$, to estimate the relative price change if the interest rate goes up by 100 basis points.

Solution The bond provides a single cashflow C at time 5, so has price function $P(i) = C(1+i)^{-5}$. Therefore, $P'(i) = -5C(1+i)^{-6}$, $P''(i) = 30C(1+i)^{-7}$,

$$D(i,1) = -\left[\frac{-5C(1+i)^{-6}}{C(1+i)^{-5}}\right] = \frac{5}{1+i} \text{ and } C(i,1) = \frac{30C(1+i)^{-7}}{C(1+i)^{-5}} = \frac{30}{(1+i)^2}.$$

Approximation (9.3.3) thus gives us

$$\begin{aligned}\frac{P(i)-P(.06)}{P(.06)} &\approx -D(.06,1)(i-.06) + C(.06,1)\frac{(i-.06)^2}{2} \\ &= -\frac{5}{1.06}(i-.06) + \frac{30}{(1.06)^2}\left[\frac{(i-.06)^2}{2}\right].\end{aligned}$$

Thus, if the interest rate goes up by 100 basis points, so that it is 7%, the relative price change is estimated by (9.3.3) to be

$$-\frac{5}{1.06}(.01) + \frac{30}{(1.06)^2}\left[\frac{(.01)^2}{2}\right] \approx -.047169811 + .001334995 \approx -.045834817.$$

Approximation (9.2.9) would yield the estimate $-.047169811$. In fact, the issue price of the bond to yield 6% is $C/(1.06)^5$, and the issue price to yield 7% is $C/(1.07)^5$. So, the exact relative price change is

$$\frac{C/(1.07)^5 - C/(1.06)^5}{C/(1.06)^5} = \left(\frac{1.06}{1.07}\right)^5 - 1 \approx -.045863658.$$

Approximation (9.3.3) does indeed give a better approximation than (9.2.9). ■

So far, we have introduced the modified convexity $C(i_0,1)$, which arose naturally as we considered the price as a function $P(i)$ of the annual effective interest rate, and the second Taylor polynomial of $P(i)$. Of course, for any positive number m, the price of a fixed set of cashflows may also be thought of as a function of the nominal interest rate $i^{(m)}$ or of the force of interest δ. For any positive number m, we define the **modified convexity** $C(i,m)$ and **Macaulay convexity** $C(i,\infty)$ by

(9.3.5) $$\boxed{C(i,m) = \frac{\frac{d^2P}{di^{(m)2}}}{P(i)},}$$

and

$$C(i,\infty) = \frac{\frac{d^2 P}{d\delta^2}}{P(i)}. \tag{9.3.6}$$

Since $P(i) = \sum_{t\geq 0} C_t(1+i)^{-t} = \sum_{t\geq 0} C_t e^{-\delta t}$, the Macaulay convexity is given by

$$\begin{aligned} C(i,\infty) &= \frac{\frac{d^2 P}{d\delta^2}}{P(i)} = \frac{\sum_{t\geq 0} C_t t^2 e^{-\delta t}}{\sum_{t\geq 0} C_t e^{-\delta t}} \\ &= \sum_{t\geq 0} \left(\frac{C_t e^{-\delta t}}{\sum_{t\geq 0} C_t e^{-\delta t}} \right) t^2 \\ &= \sum_{t\geq 0} \left(\frac{C_t (1+i)^{-t}}{P(i)} \right) t^2. \end{aligned} \tag{9.3.7}$$

So, just as we can view the Macaulay duration as a weighted sum of the times of the cashflows [see Important Fact (9.2.21)], we see the Macaulay convexity as a weighted sum of the *squares* of the times of the cashflows. Specifically, we have

IMPORTANT FACT (9.3.8)

The Macaulay convexity $C(i,\infty)$ of a set of cashflows is a weighted average of the squares of the times at which the cashflows occur. If the cashflow at time t is C_t, the weight given to t^2 is $C_t(1+i)^{-t}/P(i)$, the proportion of the total price attributable to the time t cashflow — the same weight as is used to compute the Macaulay duration.

We now know that the Macaulay convexity is easy to compute, as is the Macaulay duration. Moreover, you can quickly find the modified convexity $C(i,m)$ if you know both the Macaulay convexity and the Macaulay duration. In fact,

$$C(i,m) = \frac{C(i,\infty) + \frac{1}{m} D(i,\infty)}{\left(1 + \frac{i^{(m)}}{m}\right)^2}. \tag{9.3.9}$$

The derivation of Equation (9.3.9) is a calculus exercise, and Problem (9.3.5) guides you through it.

EXAMPLE 9.3.10

Problem: As in Example (9.3.4), a five-year zero-coupon bond is purchased to provide an annual effective yield of 6%. Find the Macaulay convexity $C(i, \infty)$, and verify that in this case, Equation (9.3.9) with $m = 1$ is correct.

Solution There is a single cashflow made at $t = 5$. So, according to Important Fact (9.3.8), we have Macaulay convexity $C(.06, \infty) = 5^2 = 25$. The Macaulay duration $D(.06, \infty) = 5$, the time of the lone payment. Thus, equation (9.3.9) gives

$$C(i, 1) = \frac{25 + 5}{(1.06)^2},$$

the same value for the modified convexity as we found in Example (9.3.4). ■

It is worth noting conditions that guarantee that the Macaulay duration $D(i, \infty)$ is a decreasing function of i. The reason for our interest is that, if $D(i, \infty)$ is a decreasing function of i, when the interest rate increases, the average length of the investment decreases. This may be important if you are trying to match the duration of your assets to the duration of your liabilities, in an effort to protect yourself from interest rate risk. Remember, not all durations change with an interest rate shift: In particular, the duration of a zero-coupon bond is equal to the time until the single redemption payment, no matter what the interest rate may be.

We note the following assertion that gives conditions for $D(i, \infty)$ being a decreasing function of i.

CLAIM 9.3.12

If all the cashflows are *positive* and there is *more* than one time at which there is a nonzero cashflow, then the Macaulay duration $D(i, \infty)$ [and hence each of the modified durations $D(i, m)$] is a decreasing function of the interest rate i. When one increases the interest rate, the present values of all of the cashflows are affected. However, there is a greater change in the later cashflows and consequently the duration decreases.

Our proof of Claim (9.3.12) will use a measure of how spread out the payment times are around the Macaulay duration. Define the **dispersion**

$$V(i) = \frac{1}{P(i)} \sum_{t \geq 0} \left[t - D(i, \infty)\right]^2 C_t e^{-\delta t}. \tag{9.3.11}$$

Note that $\left[t - D(i, \infty)\right]^2$ is always nonnegative, so there is no chance that values less than $D(i, \infty)$ compensate for values greater than $D(i, \infty)$, as there would be if we dropped the exponent 2.

Proof of Claim (9.3.12): Let's take a closer look at the dispersion. Note that expansion of the squared term in Equation (9.3.11), the definition of $V(i)$, gives

$$V(i) = \frac{1}{P(i)}\left(\sum_{t\geq 0} t^2 C_t e^{-\delta t} - 2D(i,\infty) t C_t e^{-\delta t} + \left[D(i,\infty)\right]^2 C_t e^{-\delta t}\right)$$

$$= \left(\frac{1}{P(i)}\sum_{t\geq 0} t^2 C_t e^{-\delta t}\right) - 2D(i,\infty)\left(\frac{1}{P(i)}\sum_{t\geq 0} t C_t e^{-\delta t}\right)$$

$$+ \left[D(i,\infty)\right]^2 \left(\frac{1}{P(i)}\sum_{t\geq 0} C_t e^{-\delta t}\right).$$

But Equations (9.3.7) and (9.2.19), respectively, tell us that $\frac{1}{P(i)}\sum_{t\geq 0} t^2 C_t e^{-\delta t} = C(i,\infty)$ and $\frac{1}{P(i)}\sum_{t\geq 0} t C_t e^{-\delta t} = D(i,\infty)$, and of course by Equation (9.2.1) $\frac{1}{P(i)}\sum_{t\geq 0} C_t e^{-\delta t} = \frac{1}{P(i)}\sum_{t\geq 0} C_t (1+i)^{-t} = 1$, so we have

$$V(i) = C(i,\infty) - \left[D(i,\infty)\right]^2. \tag{9.3.13}$$

Now compute the derivative of $D(i,\infty)$ with respect to the force of interest δ. Using the product rule and chain rule, from Equation (9.2.16) we obtain

$$\begin{aligned} \frac{d}{d\delta} D(i,\infty) &= \frac{d}{d\delta}\left(-(P(i))^{-1}\frac{dP}{d\delta}\right) \\ &= \left((P(i))^{-2}\frac{dP}{d\delta}\right)\frac{dP}{d\delta} - (P(i))^{-1}\frac{d^2P}{d\delta^2} \\ &= \left[D(i,\infty)\right]^2 - C(i,\infty) = -V(i). \end{aligned} \tag{9.3.14}$$

Equation (9.3.14) tells us that the derivative $\frac{d}{d\delta}D(i,\infty)$ is negative when the dispersion $V(i)$ is positive. By Definition (9.3.11) this happens when there are cashflows at more than one time and all the cashflows are positive.

We end the section with an example of convexities of a bond with coupons, followed by a discussion of the Macaulay convexity of a portfolio of bonds.

EXAMPLE 9.3.15 Convexity of a bond

Problem: An N-year zero-coupon bond has the same duration as a two-year 8% bond with annual coupons, when figured at the current yield rate of 5%. Calculate the Macaulay convexity of the two-year bond and of the N-year zero-coupon bond. Which is more sensitive if the interest rate increases from 5% to 6%?

Solution We begin by calculating the Macaulay duration and the Macaulay convexity of the two-year bond. To do so, we note that if the two-year bond has face value F, then it provides the holder of the bond with a payment of $.08F$ at the end of one year and a payment of $1.08F$ (coupon and redemption amount) at the end of two years. Therefore, at the current interest rate of 5%, the two-year bond sells for $.08F(1.05)^{-1} + 1.08F(1.05)^{-2} \approx 1.055782313F$. Therefore, applying Important Fact (9.2.21), we obtain the following Macaulay duration for the two-year bond at the current interest rate:

$$D(.05, \infty) \approx 1\left(\frac{.08F(1.05)^{-1}}{1.055782313F}\right) + 2\left(\frac{1.08F(1.05)^{-2}}{1.055782313F}\right) \approx 1.927835052.$$

In addition, Important Fact (9.3.8) tells us that the Macaulay convexity of the two-year bond is

$$C(.05, \infty) \approx 1^2\left(\frac{.08F(1.05)^{-1}}{1.055782313F}\right) + 2^2\left(\frac{1.08F(1.05)^{-2}}{1.055782313F}\right) \approx 3.783505155.$$

Since the duration of a zero-coupon bond is its term until maturity, and we require that the N-year zero-coupon bond has the same duration as the two-year bond, we must have $N \approx 1.927835052$. But then, the zero-coupon bond has Macaulay convexity $C(.05, \infty) = N^2 \approx (1.927835052)^2 \approx 3.716547986$ which is *slightly* less than the Macaulay convexity of the two-year bond. The convexities are very close, so we would expect that the sensitivity to small interest rate shifts would also be close.

We next wish to determine which bond's price is more sensitive to an interest rate shift from 5% to 6%. If the interest rate goes up to 6%, then the price of the two-year bond is $.08F(1.06)^{-1} + 1.08F(1.06)^{-2} \approx 1.036667853F$, and the relative price change is about $\frac{1.036667853F - 1.055782313F}{1.055782313F} \approx -.018104546$. On the other hand, the relative price change of the zero-coupon bond is

$$\frac{(1.06)^{-1.927835052} - (1.05)^{-1.927835052}}{(1.05)^{-1.927835052}} \approx -.018107508.$$

The values of the relative price changes are very close, as we might expect from the fact that the convexities are close. It is worth noting that convexity of the zero-coupon bond is slightly smaller, but the relative price change of the zero-coupon bond has absolute value that was slightly larger; this alerts us to the fact that the quadratic approximation does not tell the whole story, and that this would require the whole Taylor series! ■

We have previously noted that by stripping off the coupons of a bond with coupons, you may view a bond with coupons as a portfolio of zero-coupon bonds. Important Fact (9.2.21) explained how to find the duration of a portfolio of cashflows from the duration of the component cashflows, and there is a similar statement for how one might compute the convexity of a portfolio of cashflows from the individual convexities.

IMPORTANT FACT 9.3.16

The Macaulay convexity of a portfolio is a weighted average of the Macaulay convexities of the individual assets, the weight given to the convexity of an individual asset being the proportion of the total price attributable to that asset.

The verification of Fact (9.3.16) is left as an exercise [Problem (9.3.6)].

EXAMPLE 9.3.17 Convexity of a portfolio of bonds

Problem: As in Example (9.2.29), Julian Bradley's bond portfolio consists of a \$10,000 five-year zero-coupon bond and a \$2,000 par-value two-year 6% bond with semiannual coupons. Compute the Macaulay convexity $C^{\text{portfolio}}(.045, \infty)$ for the portfolio.

Solution We first find the individual Macaulay convexities $C^{\text{two}}(.045, \infty)$ and $C^{\text{five}}(.045, \infty)$ of the two-year and five-year bonds respectively, and then use Important Fact (9.3.16) to find the Macaulay convexity $C^{\text{portfolio}}(.045, \infty)$ of the portfolio of bonds. For our solution, it is useful to recall the prices, to yield 4.5%, that were found in the solution to Example (9.2.29): The two-year bond has price \$2,058.680315, the five-year bond has price \$8,024.510465, and the portfolio has as its price the sum of these individual prices, namely \$10,083.19078.

The two-year bond has semiannual \$60 coupons and an additional \$2,000 redemption payment at the end of two years, so Important Fact (9.3.8) tells us that

$$\begin{aligned} &C^{\text{two}}(.045, \infty) \\ &= \frac{1}{\$2{,}058.680315}\left(\$60\frac{(.5)^2}{(1.045)^{.5}} + \$60\frac{1^2}{(1.045)^1} + \$60\frac{(1.5)^2}{(1.045)^{1.5}} + \$2{,}060\frac{2^2}{(1.045)^2}\right) \\ &\approx 3.761671472, \end{aligned}$$

which is a little less than the convexity $2^2 = 4$ of a two-year zero-coupon bond. The Macaulay convexity of the five-year zero-coupon bond is 25, the square of the term of the bond. Therefore, Important Fact (9.3.16) tells us that the Macaulay convexity of the portfolio is

$$\begin{aligned} C^{\text{portfolio}}(.045, \infty) &\approx \left(\frac{\$2{,}058.680315}{\$10{,}083.19078}\right)(3.761671472) \\ &\quad + \left(\frac{\$8{,}024.510465}{\$10{,}083.19078}\right)(25) \approx 20.66378046. \end{aligned}$$

■

9.4 IMMUNIZATION

(calculus needed here)

A set of cashflows may be broken up into inflows and outflows. The promised inflows are *assets* and the required outflows are *liabilities*. The total inflow at time t will be denoted by A_t and the total outflow at time t will be called L_t; so $A_t \geq 0$ and $L_t \geq 0$.[6] Throughout this section we assume that the cashflows *do not depend on the interest rate* i. There may be offsetting inflows and outflows, and the net cashflow inward at time t is the difference $A_t - L_t$. We denote the **surplus** valued at interest rate i by $S(i)$. That is,

$$(9.4.1) \quad S(i) = \sum_{t \geq 0} (A_t - L_t)(1+i)^{-t} = \sum_{t \geq 0} A_t(1+i)^{-t} - \sum_{t \geq 0} L_t(1+i)^{-t}.$$

Note that $S(i)$ is just the price function $P(i)$ of the portfolio that consists of the assets *and* the liabilities.

> A nonnegative surplus means that the time 0 value of the assets is greater than the time 0 value of the liabilities; you are *ahead* by an amount $S(i)$. If you have a negative surplus, you are *behind* and this may have significant consequences.

We will next show that if a portfolio of cashflows is such that $S(i_0) = 0$, $S'(i_0) = 0$, and $S''(i_0) \geq 0$ where i_0 is the current interest rate, then *for sufficiently small* changes in the interest rate, the surplus is positive; that is, if i_0 is your current yield rate, making $S(i_0) = 0$, and you also have $S'(i_0) = 0$, and $S''(i_0) \geq 0$, then *nearby interest rates would make you no worse off.*[7]

Note that the surplus is zero when the value of the set of assets is equal to the value of the set of liabilities. Moreover, by Definition (9.2.12), the condition that the derivative $S'(i)$ must be zero is equivalent to the duration of the assets equaling the duration of the liabilities. (Here it doesn't matter whether we use modified duration or Macaulay duration, but we do need to use the same kind of duration for the assets as we do for the liabilities.) Further note that by Equation (9.3.2), the second derivative $S''(i)$ is nonnegative precisely when the convexity of the assets is at least as large as the convexity of the liabilities.

Assume that we have the following conditions:

$$(9.4.2) \qquad \boxed{S(i_0) = 0,\ S'(i_0) = 0, \quad \text{and} \quad S''(i_0) \geq 0.}$$

[6] You have already seen this notation used in Example (9.2.8).

[7] We remind you that we continue to assume that there is a flat yield curve and that any shift in the yield curve results in a new parallel yield curve. You may encounter the term "C3 risk" or "C-3 risk." This refers to risk that is due to change in the shape of the yield curve as well as to the risk due to parallel shifts that we allow. In addition, "C3 risk" must deal with the fact that many cashflows are interest rate sensitive, a serious complication that we are not permitting.

Then, if h is very small, the first three terms of the Taylor series expansion for $S(i)$ around i_0 give us the approximation

$$S(i_0 + h) \approx S(i_0) + S'(i_0)h + \frac{S''(i_0)}{2}h^2 \approx \frac{S''(i_0)}{2}h^2 \geq 0,$$

which is just Equation (9.2.5) with $i = i_0 + h$. Thus,

IMPORTANT FACT 9.4.3 (Redington immunization)
• *If* Condition (9.4.2) holds, if the yield curve is flat, and if all shifts in the yield curve are parallel, *then* $S(i_0 + h) \geq 0$ for h sufficiently small.
• Condition (9.4.2) is equivalent to the assets and liabilities having equal present values and equal Macaulay durations, and the Macaulay convexity of the assets being at least as large as the Macaulay convexity of the liabilities.

While Redington immunization (9.4.3) does not tell you how small h needs to be for you to be certain to have a surplus, it is still quite useful.

EXAMPLE 9.4.4 Redington immunization

Problem: Alan and Peabody Insurance is obligated to make a payment of \$120,000 in exactly four years. In order to provide for this obligation, their financial officer decides to purchase a combination of two-year zero-coupon bonds and five-year zero-coupon bonds. Each of these is sold to yield an annual effective yield of 4.5%. How much of each type of bond should be purchased so that, together with the \$120,000 outflow in four years, the zero-coupon bonds constitute a portfolio satisfying Redington immunization Condition (9.4.2) at an annual effective interest rate of 4.5%? How does this immunization help?

Solution Let a denote the amount spent on two-year bonds and b the amount spent on five-year bonds. Then, the present value of the assets, when figured using the annual effective interest rate 4.5% is $a + b$, and Important Fact (9.4.3) says that for Redington immunization this must equal the present value of the \$120,000 liability to be paid in four years, again figured at 4.5%. Therefore,

$$a + b = \frac{\$120{,}000}{(1.045)^4} \approx \$100{,}627.3612.$$

We will settle for $a + b = \$100{,}627.36$ because, as usual, there is no way to pay fractional numbers of cents.

Next, according to Important Fact (9.4.3) again, we need the Macaulay durations of the set of two assets to equal 4, the Macaulay duration of the single liability at time 4. Recalling Important Fact (9.2.21), the portfolio of assets has Macaulay duration $2\left(\frac{a}{a+b}\right) + 5\left(\frac{b}{a+b}\right)$, and therefore the duration condition for Redington immunization is

$$\frac{2a}{a+b} + \frac{5b}{a+b} = 4.$$

Multiplying this condition by $a+b$ we have $2a+5b = 4a+4b$. Thus, we must have $b = 2a$ as well as $a + b = \$100{,}627.36$. Therefore, we want $3a = \$100{,}627.36$. So $a \approx \$33{,}542.4533$ and $b = \$67{,}084.90667$. Since these purchase amounts must have whole numbers of cents, we take $a = \$33{,}542.45$ and $b = \$67{,}084.91$.

Now, let us compute the Macaulay convexity of the portfolio of assets and see whether it exceeds $4^2 = 16$, which is the Macaulay convexity of the single liability to be paid at time 4. Using Important Fact (9.3.8), we see that the set of assets has Macaulay convexity

$$\left(\frac{\$33{,}542.45}{\$100{,}627.36.}\right)2^2 + \left(\frac{\$67{,}084.91}{\$100{,}627.36.}\right)5^2 \approx 18.$$

This is greater than 16, so the portfolio satisfies the Redington immunization Condition (9.4.2) at interest rate 4.5%.

Let us now consider how paying \$33,542.45 for two-year bonds and \$67,084.91 for five-year bonds might help you if these bonds were purchased to yield 4.5%. The two-year bonds will provide $\$33{,}542.45(1.045)^2 \approx \$36{,}629.19$ in two years, and the five-year bonds produce a redemption payment of $\$67{,}084.91(1.045)^5 \approx \$83{,}600.00$ in five years. If interest rates rise 50 basis points to 5%, at time two years, then the five-year bond, with three years before it may be redeemed for \$83,600, may be sold for $\$83{,}600(1.05)^{-3} \approx \$72{,}216.82$. Combine this with the \$36,629.19 redemption from the two-year bond to obtain $\$36{,}629.19+\$72{,}216.82 = \$108{,}846.01$. You may invest this total for two years at 5%, giving you $\$108{,}846.01(1.05)^2 \approx \$120{,}002.73$, \$2.73 above the amount you need to cover your liability. On the other hand, if interest rates go down by 50 basis points to 4%, you should sell the five-year bonds at the end of two years for $\$83{,}600.00(1.04)^{-3} \approx \$74{,}320.10$. Once again, combine the money obtained from the sale of the five-year bond with the redemption amount \$36,629.19, this time realizing a total of $\$36{,}629.19 + \$74{,}320.10 = \$110{,}949.29$, and invest this for two years. This will result in $\$110{,}949.29(1.04)^2 \approx \$120{,}002.75$. Once again, you have a little money left after you take care of the \$120,000 obligation: This time you are left with \$2.75. You immunized yourself against small interest rate changes, either up or down. ■

We now modify the above example so that the two-year bond has coupons.

EXAMPLE 9.4.5 Redington immunization

Problem: Alan and Peabody Insurance is still obligated to make a payment of \$120,000 in exactly four years. In order to provide for this obligation, their financial officer now decides it is better to purchase a combination of two-year par-value 6%

bonds *with semiannual coupons* and five-year zero-coupon bonds. Each of these is sold to yield an annual effective yield of 4.5%. How much of each type of bond should be purchased so that, together with the \$120,000 outflow in four years, the zero-coupon bonds constitute a portfolio satisfying Redington immunization condition (9.4.2) at interest rate 4.5%?

Solution We begin, just as we did in Example (9.4.4), by letting a denote the amount spent on two-year bonds and b the amount spent on five-year bonds. Once again, the fact that the present value of the assets must equal the present value of the liabilities implies that $a + b = \$100{,}627.36$.

Observe that we have seen such two-year and five-year bonds in Examples (9.2.29) and (9.3.17), and that we calculated that the two-year bond has Macaulay duration equal to 1.915703104 and its Macaulay convexity is 3.761671472. The zero-coupon bond has Macaualy duration equal to 5 and Macaulay convexity 25.

Using Important Fact (9.2.21), we find that the Macaulay duration of the assets is

$$(1.915703104)\left(\frac{a}{a+b}\right) + 5\left(\frac{b}{a+b}\right).$$

But this must equal the Macaulay duration of the liability, namely 4. Therefore,

$$(1.915703104)\left(\frac{a}{a+b}\right) + 5\left(\frac{b}{a+b}\right) = 4,$$

and so $1.915703104a + 5b = 4a + 4b$, and $b = 2.084296896a$.

The two-year coupon bond has slightly lower duration than the zero-coupon bond of Example (9.4.4). To accommodate this in our Redington immunization duration condition, we need to devote a little less of our total spending to the two-year bond. Then $a + b = \$100{,}627.36$ gives us $3.084296896a = \$100{,}627.36$. So, $a \approx \$32{,}625.70479$ and $b \approx \$68{,}001.65521$. The purchase amounts a and b must each be a whole number of cents, so $a = \$32{,}625.70$ and $b = \$68{,}001.66$. Finally, we need to check the convexity condition. The liability once again has Macaulay convexity 16. To determine the Macaulay convexity $C(4.5\%, \infty)$ of the portfolio of assets, we use Important Fact (9.3.16), along with the convexities taken from Example (9.3.17). Then, the portfolio of assets has Macaulay convexity that is approximately

$$\left(\frac{\$32{,}625.70}{\$100{,}627.36.}\right)(3.761671472) + \left(\frac{\$68{,}001.66}{\$100{,}627.36.}\right)5^2 \approx 18.114.$$

Again this is greater than 16, so we have an immunized portfolio. We note that the calculation with coupon bonds is significantly longer. Sometimes, it may be a good idea to first use zero-coupon bonds to get a basic feel for your situation. ■

Let us now consider how you might try to structure a portfolio of cashflows so that you are protected from *large* interest rate changes as well as small ones. Not surprisingly, this is not generally easy to implement.

To begin with, suppose that you have a set of three cashflows consisting of a single outflow of amount L to be paid at time T and inflows U due at times $T-u$ and W due at time $T+w$. This was the situation in our Redington immunized portfolio of Example (9.4.4).

CASHFLOW:		U	$-L$	W
TIME:	0	$T-u$	T	$T+w$

Let

$$S(\delta) = Ue^{-\delta(T-u)} - Le^{-\delta T} + We^{-\delta(T+w)} = e^{-\delta T}\left(Ue^{\delta u} - L + We^{-\delta w}\right),$$

the net present value (or surplus) of the set of cashflows. By δ_0 we indicate the current force of interest, and we stipulate that $S(\delta_0) = 0$ and $S'(\delta_0) = 0$. It may be established by an argument given in the optional technical note at the end of this section that these two assumptions are enough to force $S(\delta)$ to be positive for any $\delta \neq \delta_0$. So, the pair of conditions $S(\delta_0) = 0$ and $S'(\delta_0) = 0$ are enough to guarantee that a change in the interest rate from the current rate $S(\delta_0)$ will *increase* the value of the surplus.

IMPORTANT FACT 9.4.6 (Full immunization)
Let δ_0 denote the current force of interest. Suppose a portfolio consists of a single liability L, to be paid at time T, along with a pair of assets U and and W, to be paid at times $T-u$ and $T+w$, respectively, where $0 < u < T$ and $w > 0$. Further suppose that the net present value $S(\delta_0)$ of the portfolio is 0, as is the derivative $S'(\delta_0)$. Then, if $\delta \neq \delta_0$, the net present value $S(\delta)$ is positive.

We now know that the portfolio we found in Example (9.4.4) is *fully* immunized since Important Fact (9.4.6) applies!

EXAMPLE 9.4.7 Full immunization

Problem: Consider again the situation of Example (9.4.4). As previously considered, Alan and Peabody Insurance is obligated to make a payment of \$120,000 in exactly four years and purchase \$33,542.45 of two-year zero-coupon bonds and \$67,084.91 of five-year zero-coupon bonds, priced to yield an annual effective yield of 4.5%. Suppose that we have a rather large change in the interest rate, say up to 10% or down to 1%. Illustrate that Alan and Peabody still can cover the \$120,000 liability, as is guaranteed by full immunization.

Solution If interest rates are now 10% at the end of two years, the amount Alan and Peabody get by selling the five-year bond at time two years is only $\$83{,}600.00(1.10)^{-3} \approx \$62{,}809.92$. So, at time two years, they only have \$36,629.19 + \$62,809.92 = \$99,439.11 to invest. But, thanks to the relatively high 10% interest rate, the money will still accumulate to more than the \$120,000 needed at time four — specifically to $\$99{,}439.11(1.10)^2 \approx \$120{,}321.32$.

Should the interest rate at time two years be a modest 1%, then at time two years the five-year bond, with redemption in three years, will fetch $\$83{,}600.00(1.01)^{-3} \approx \$81{,}141.34$ so Alan and Peabody would have a combined \$36,629.19+\$81,141.34 = \$117,770.53 to invest for two years at 1%. This would give them $\$117{,}770.53(1.01)^2 \approx \$120{,}137.72$ so, again, Alan and Peabody can take care of the \$120,000 liability. ■

Having a fully immunized portfolio seems to be rather fantastic, so we wish to further analyze the pair of constraint equations $S(\delta_0) = 0$ and $S'(\delta_0) = 0$. Since $e^{-\delta_0 T} \neq 0$, the condition $S(\delta_0) = 0$ is equivalent to

$$Ue^{\delta_0 u} - L + We^{-\delta_0 w} = 0. \tag{9.4.8}$$

Moreover, using Equation (9.4.8), we find

$$\begin{aligned}
0 = S'(\delta_0) &= -(T-u)Ue^{-\delta_0(T-u)} + TLe^{-\delta_0 T} - (T+w)We^{-\delta_0(T+w)} \\
&= \left(uUe^{-\delta_0(T-u)} - wWe^{-\delta_0(T+w)}\right) - T\left(Ue^{-\delta_0(T-u)} - L + We^{-\delta_0(T+w)}\right) \\
&= \left(uUe^{-\delta_0(T-u)} - wWe^{-\delta_0(T+w)}\right) - Te^{-\delta_0 T}\left(Ue^{\delta_0 u} - L + We^{-\delta_0 w}\right) \\
&= uUe^{-\delta_0(T-u)} - wWe^{-\delta_0(T+w)} = e^{-\delta_0 T}\left(uUe^{\delta_0 u} - wWe^{-\delta_0 w}\right).
\end{aligned}$$

Therefore, since $e^{-\delta_0 T} \neq 0$, we have

$$uUe^{\delta_0 u} - wWe^{-\delta_0 w} = 0. \tag{9.4.9}$$

FACT 9.4.10

A portfolio consisting of a single outflow of amount L at time T and inflows U at time $T - u$ and W at time $T + w$ is fully immunized. If the system of equations

$$\begin{cases} Ue^{\delta_0 u} + We^{-\delta_0 w} = L \\ uUe^{\delta_0 u} - wWe^{-\delta_0 w} = 0 \end{cases}$$

is satisfied, the portfolio will *never* force default.

Of course you may wish to fully immunize a portfolio with multiple liabilities rather than a single one. You may adopt the strategy of selecting a pair of assets to immunize each of the liabilities. Just as was true for Redington immunization, you will need to frequently rebalance your portfolio to maintain an immunized status, because the derivative condition $S'(i) = 0$ will not continue to hold with the passing of time.

The assumption that you have a flat yield curve that remains flat is not a realistic one. In fact, a consequence of Redington immunization is that if assets and liabilities can be structured so as to make condition (9.4.2) hold, then there is an arbitrage opportunity. You could short-sell the liabilities and use the profits to purchase the assets. When the interest rate had a slight shift, you could sell your assets, cover the short sale, and make a profit. Similarly, full immunization is contrary to the market being essentially arbitrage-free. These arbitrage opportunities result from our unrealistic hypothesis that the yield rate curve is flat. We explained in Section (8.3) that the term structure of interest rates should be considered on the premise that there is no arbitrage. This points toward more advanced immunization strategies in which many different spot rates are used. Asset management remains an active area of research.

Optional technical note:

We end this section with the promised somewhat technical argument establishing Important Fact (9.4.6) that the net present value $S(\delta)$ will be positive if $\delta \neq \delta_0$. [$S(\delta) \geq 0$ is the important condition that you can cover your liability.] We begin by noting that Equation (9.4.9) may be rewritten as

$$We^{-\delta_0 w} = \frac{u}{w} U e^{\delta_0 u}. \tag{9.4.11}$$

Equations (9.4.8) and (9.4.11), derived from the conditions $S(\delta_0) = 0$ and $S'(\delta_0) = 0$ respectively, together give us

$$Ue^{\delta_0 u} - L + \frac{u}{w} U e^{\delta_0 u} = 0. \tag{9.4.12}$$

Now, if δ is any force of interest, we wish to show that $S(\delta)$ is positive. With this goal in mind, we note that

$$\begin{aligned}
S(\delta) &= e^{-\delta T}\left(Ue^{\delta u} - L + We^{-\delta w}\right) \\
&= e^{-\delta T}\left[\left(Ue^{\delta u} - L + We^{-\delta w}\right) - \left(Ue^{\delta_0 u} - L + \frac{u}{w}Ue^{\delta_0 u}\right)\right] && \text{by (9.4.12)} \\
&= e^{-\delta T}\left(Ue^{\delta u} + We^{-\delta w} - Ue^{\delta_0 u} - \frac{u}{w}Ue^{\delta_0 u}\right) \\
&= e^{-\delta T}\left(Ue^{\delta u} + \frac{u}{w}Ue^{\delta_0 (u+w)}e^{-\delta w} - Ue^{\delta_0 u} - \frac{u}{w}Ue^{\delta_0 u}\right) && \text{by (9.4.11)} \\
&= e^{\delta_0 u - \delta T} U\left(e^{(\delta-\delta_0)u} + \frac{u}{w}e^{(\delta_0-\delta)w} - 1 - \frac{u}{w}\right).
\end{aligned}$$

So, our objective is to prove that $e^{(\delta-\delta_0)u} + \frac{u}{w}e^{(\delta_0-\delta)w} - 1 - \frac{u}{w}$ is positive. To accomplish this, we introduce the function

$$f(x) = e^{ux} + \frac{u}{w}e^{-wx} - 1 - \frac{u}{w}.$$

What we must show is that $f(\delta - \delta_0) > 0$ for $\delta \neq \delta_0$, and because $f(0) = 0$, this is equivalent to the function $f(x)$ having a minimum at 0. It is therefore enough to show that the derivative $f'(x)$ has the same sign as x, that is $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$, a fact which is immediate from

$$f'(x) = ue^{ux} - ue^{-wx} = u(e^{ux} - e^{-wx}),$$

since the constants u and w are positive.

9.5 OTHER TYPES OF DURATION

(calculus needed here)

In Section (9.2) we introduced Macaulay duration and modified duration for a set of cashflows. To do so, we postulated that the cashflows were *not* interest rate sensitive, and we also supposed that we had a flat yield curve at all times. Of course, we know that these assumptions are not necessarily appropriate in real life applications. We conclude our discussion with a brief look at what might be done if these assumptions are dropped. This is still an active area of work with competing suggestions: You may even find "duration derbies" where competing types of durations face off, using actual real life examples, to see how well the models predict the price response of a bond or other financial instrument to changes in interest rates.[8]

Consider investments with interest-sensitive cashflows. Examples of such cashflows include the payments to be received by the holder of a callable bond or a mortgage-backed security. If interest rates fall sufficiently, then a callable bond is likely to be called early and, thanks to refinancing options, a mortgage is more apt to be repaid early.

A set of interest-sensitive cashflows may not have an everywhere differentiable price function, and since the definitions of durations use the derivative of P, the durations we have considered so far may not be defined. To illustrate why we might have interest rates i_0 at which the price function $P(i_0)$ is not differentiable, let us consider a par-value bond, with an American call option and no lockout period, that is purchased at a discount. Recall that such a bond gives the buyer a yield rate that exceeds the coupon rate, assuming that the bond is held to maturity. If, immediately after the bond is purchased, the interest rate drops *below* the coupon rate, the bond's current price would be above the redemption amount. So, from the issuer's viewpoint, it is advantageous to call in the bond and reissue it: The new purchase price,

[8]If you want an example, see Allen, D. E., Thomas, L. C., and Zheng, H. "The Duration Derby: A Comparison of Duration Strategies in Asset Liability Management," *J. Bond Trading and Management* 1, 371–380, 2003.

yielding the new buyer a lower interest rate, allows the issuer to recoup the price paid out to the holder of the old bond and also to keep the premium. But, the bond being called results in a new cashflow (the redemption payment) and cancels the old ones. This *sudden* change makes the price function have an interest rate at which it fails to be differentiable, namely the interest rate numerically equal to the coupon rate. Here we have considered an interest rate shift *immediately* after purchase; however, subsequent changes can also lead to the bond being called, and the price function failing to be differentiable at particular interest rates.

Now, recall that the modified duration was used to estimate the price change that would result from a small shift in the interest rate, and we had the basic definition

$$D(i_0, 1) = -\frac{P'(i_0)}{P(i_0)}.$$

How might we approximate this if $P'(i_0)$ were not necessarily defined? Well, it is standard to use the slope of a secant line in place of the slope of a tangent line. Specifically, we define

$$m_h(i_0) = \frac{P(i_0 + h) - P(i_0 - h)}{2h}, \tag{9.5.1}$$

which is the slope of the line containing the points $(i_0 - h, P(i_0 - h))$ and $(i_0 + h, P(i_0 + h))$,[9] and we set

$$E_h(i_0, 1) = -\frac{m_h(i_0)}{P(i_0)}. \tag{9.5.2}$$

Note that if $D(i_0, 1)$ is defined, if h is small, and if the cashflows are fixed, then, since the slope of the secant line $m_h(i_0)$ approximates $P'(i_0)$, the **effective duration** $E_h(i_0, 1)$ approximates $D(i_0, 1)$.

> The **effective duration**
>
> $$E_h(i_0, 1) = -\frac{m_h(i_0)}{P(i_0)}$$
>
> may be used in place of the modified duration. Do this when you have interest-sensitive cashflows, using a natural choice of h suggested by the problem. *Unlike modified duration, effective duration takes into account the fact that the interest rate may influence cashflows*. If the yield increases by 100 basis points and cashflows are sensitive to interest rate changes, the approximate relative price change is $-E_h(i_0, 1)$ percent.

[9] Quite likely you have considered secant lines through $(i_0, P(i_0))$ and $(i_0 + h, P(i_0 + h))$, having slope $\frac{P(i_0+h)-P(i_0)}{h}$, but then you are focusing on what happens to the right of i_0. We are concerned with shifts in the interest rate in either direction, hence adopt this two-sided approach.

EXAMPLE 9.5.3 Using effective duration to estimate prices

Problem: The price of a callable bond, purchased when interest rates are 7%, is \$970.00. The bond has effective duration $E_{.01}(.07, 1)$ that turns out to equal 4.4. Estimate the price if interest rates drop to 6%. Also, estimate the price if there is a more modest drop to 6.5%.

Solution If the interest rate decreases from 7% to 6%, we have a 100 basis point decrease. The price then increases by about 4.4%. This is an increase of about $.044 \times \$970 = \$42.68$, so we estimate a price of $\$970 + \$42.68 = \$1{,}012.68$. Should we have a rate increase of 50 basis points (to 6.5%), the price increase is estimated to be half as large, hence \$21.34. So, the estimated price if the interest rate falls to 6.5% is $\$970 + \$21.34 = \$991.34$. ■

Let's now look at an example where we calculate the effective duration.

EXAMPLE 9.5.4 Modified duration and effective duration, bond purchased at a discount

Problem: Consider a three-year \$1,000 5% par-value bond with annual coupons whose current price is \$986.51. You are given that the bond has an American call option with no lockout period, so it may be called at any time. Calculate the investor's yield j if the bond is held to maturity, and then find $E_{.01}(j, 1)$. Also, for comparison, calculate the modified duration $D(j, 1)$ and effective duration $E_{.01}(j, 1)$ of a three-year \$1,000 5% par-value noncallable bond with annual coupons whose price at issue is \$986.51.

Solution The bonds have annual \$50 coupons and a \$1,000 redemption amount. You may check that the basic price formula $\$986.51 = \$50a_{\overline{3}|j} + \$1{,}000(1 + j)^{-3}$ forces $j \approx 5.500012463\%$. If interest rates go up 100 basis points to 6.5%, since the coupon rate is only 5%, it is unlikely the bond will be called. We therefore figure the price as if the bond were noncallable. The basic price formula gives

$$\begin{aligned} P(.065) &= \$50a_{\overline{3}|j+.01} + \$1{,}000(1 + (j + .01))^{-3} \\ &\approx \$50a_{\overline{3}|6.500012463\%} + \$1{,}000(1.06500012463)^{-3} \\ &\approx \$960.2725464. \end{aligned}$$

On the other hand, should the interest rate fall by 1%, the bond would most likely be called for \$1,000. The reason for this is that the interest rate is then less than the coupon rate, so the bond should be priced at a premium. So, calling in the bond at its coupon rate gives the issuer a bargain. Therefore, with the bond able to be called in at \$1,000, we figure $P(j - .01) = \$1{,}000$. (Knowing that the bond is apt to be called in, a buyer would not want to pay more than the redemption amount!) Recalling Equations (9.5.1) and (9.5.2), we calculate that

$$m_{.01}(j) = (\$960.2725464 - \$1{,}000)/2(.01) \approx -1{,}986.372682$$

and

$$E_{.01}(j, 1) \approx \frac{1,986.372682}{986.51} \approx 2.0135353.$$

If we have a noncallable three-year \$1,000 5% par-value bond with annual coupons, then Example (9.2.24) tells us that the modified duration $D(j, 1)$ is $D(j, 1) = a_{\overline{3}|j} \approx 2.697932752$. The noncallable bond, having modest coupons, has a modified duration that is reasonably close to its term. As for the calculation of the effective duration $E_{.01}(j, 1)$, once again $P(j + .01) \approx \$960.2725464$. However, if the interest rate drops to $j - .01$, with no call option the price rises to \$1,013.744476. The effective duration is then $(\$1{,}013.744476 - \$960.2725464)/[2(.01)(\$986.51)] \approx 2.71015649$. We note that this is quite close to the value 2.697932752 just calculated for $D(j, 1)$.

The values of the durations are higher for the noncallable bond than the effective duration was for the callable bond. This will always be true! ■

The idea behind the definition of effective duration was to replace the first derivative, which gives the slope of the tangent line to the price curve at i_0, by the slope of a secant line. We replaced

$$\left.\frac{dP}{di}\right|_{i=i_0} \quad \text{by} \quad \frac{P(i_0 + h) - P(i_0 - h)}{h^2}.$$

Similarly, you may define **effective convexity** by replacing the second derivative

$$\left.\frac{d^2P}{di^2}\right|_{i=i_0} \quad \text{by} \quad \frac{P(i_0 + h) + P(i_0 - h) - 2P(i_0)}{2h}$$

[see Problem (9.5.4)]. Like effective duration, effective convexity may be useful when the cashflows are interest-sensitive.

As we remarked in Section (9.2), there are problems with the durations we have defined so far even if we have fixed cashflows because it is unrealistic to think that the yield curve will be flat. **Keyrate duration** or **partial duration** is defined using several different spot rates. In fact, it is common to use eleven different spot rates for Treasury securities, although any finite set of spot rates does not determine the yield curve. Partial derivatives with respect to the spot rates take the place of the derivatives we have discussed.

Recall that the motivation for considering duration is to manage your portfoilo so as to be reasonably protected from interest rate volatility. As you might now guess, no one type of duration has been found that clearly does this best for all classes of portfolios.

9.6 PROBLEMS, CHAPTER 9

(9.0) Chapter 9 writing problems

(1) [following Section (9.1)] C. L. Trowbridge classified risks as C1, C2, C3, and C4 risks when he was president of the Society of Actuaries. C3 risks are addressed by asset-liability management. Describe Trowbridge's classification. Include several examples of each type of risk.

(2) Write a short essay on the history of cash-flow management and immunization. Names you might look for in your research include Tjalling C. Koopmans and Frank M. Redington.

(9.1) Overview

(1) Leland Price must pay \$5,000 six months from now and \$10,000 one year from now. He wishes to purchase bonds so that together they form a portfolio of assets that exactly match his liabilities. Available bonds are a six-month zero-coupon \$1,000 bond that has a 3.0225% annual yield, and a one-year \$1,000 par-value 6% bond with semiannual coupons and a 4% nominal yield (convertible semiannually). How much must Leland pay to purchase the bonds? Assume that he may buy any quantity he wishes of each bond.

(2) Suppose that you need to pay \$35,000 in three years and that you can finance this with zero-coupon bonds yielding 5.5% with terms of two years and six years. Imagine that you spend \$22,354.86 purchasing a two-year bond and \$7,451.62 for a six-year bond, and these are each priced to yield 5.5%. Suppose also that, at the end of two years, no matter what the yield rate i may then be, you sell the remaining bond at a purchase price to yield i, combine the proceeds with the \$24,881.52 from the redeemed bond, and use the total to buy a one-year zero-coupon bond. Illustrate that this immunizes against interest rate risk by showing that it produces the needed \$35,000 three years after your initial bond purchases if $i = 18\%$ (a high rate) or $i = 1\%$ (a low rate).

(9.2) Macaulay and modified duration

(1) Compute the Macaulay duration of a ten-year 6% \$1,000 bond having annual coupons and a redemption of \$1,200 if the yield to maturity is 8%.

(2) The current price of a bond having annual coupons is \$1,312. The derivative of the price function of the bond with respect to the yield to maturity is $-\$7{,}443.81$ when evaluated at the current annual yield, which is 7%. Calculate the Macaulay duration $D(.07, \infty)$ and the modified duration $D(.07, 1)$ of the bond.

(3) A zero-coupon bond matures in eight years. It is sold to yield 5% annually. Find the modified duration $D(.05, 1)$.

(4) The current price of a noncallable bond with annual coupons is \$1,120.58, and the current annual yield is 4.25%. The modified duration $D(.0425, 2)$ is 3.58. Estimate the price of the bond if the yield increases to 4.4%. About what would the price be if the discount rate decreases to an annual effective rate of 3.7%?

(5) Calculate the Macaulay duration $D(.05, \infty)$ and the modified duration $D(.05, 2)$ of a preferred stock that pays dividends forever of \$50 each six months, with the next dividend in exactly six months.

(6) Calculate the Macaulay duration $D(.06, \infty)$ and the modified duration $D(.06, 1)$ of a stock that pays annual dividends forever, assuming that the first dividend, payable in exactly one year, is \$100 and then, each subsequent dividend is 2% more than the previous one.

(7) On January 1, 1978, Jonathan Linden is obligated to pay annual level payments for sixteen years, beginning with a payment on January 1, 1979. His financial advisor tells him that this liability has Macaulay duration 7.39. Determine the annual effective interest rate that the advisor used to figure the duration.

(8) Santosh inherits a perpetuity with annual payments. The first payment is \$1,400 and payments increase by 3% each year. Obtain a formula for the Macaulay duration $D(.05, \infty)$ of the remaining payments an instant after the k-th payment.

(9) Mustafa's portfolio consists of an annuity with monthly payments of \$1,000 each month for five years and a \$20,000 8% eight-year par-value bond bearing semiannual coupons. Calculate the Macaulay duration of the portfolio at 9%.

(10) A thirty-year mortgage with no early repayment option is repaid with level monthly payments. The interest rate on the mortgage is 6.8% nominal convertible monthly. Calculate the Macaulay duration on the mortgage at the equivalent annual effective interest rate.

(9.3) Convexity

(1) Calculate the Macaulay convexity of a ten-year 6% \$1,000 bond having annual coupons and a redemption of \$1,200 if the yield to maturity is 8%. [This is the same bond considered in Problem (9.2.1).]

(2) A \$2,000 7% three-year bond has semiannual coupons and redemption amount \$2,225. It is purchased to provide the investor with a 5% annual effective yield. Find the Macaulay convexities $C(4\%, \infty)$, $C(5\%, \infty)$, and $C(6\%, \infty)$. Also compute the modified convexity $C(5\%, 2)$.

(3) A bond has Macaulay duration $D(i, \infty) = 5.8$ and Macaulay convexity $C(i, \infty) = 1.2$. Determine $C(i, 4)$ as a function of i.

(4) A perpetuity has a level payment at the end of each year. The annual effective interest rate is 5.2%. Calculate the Macaulay duration and the Macaulay convexity. Use these to estimate the percentage decline in the market value of the perpetuity if the annual effective interest rate increases by 50 basis points.

(5) This problem guides you through a derivation of Equation (9.3.9).

(a) Show that

$$\frac{d\delta}{di^{(m)}} = \left(1 + \frac{i^{(m)}}{m}\right)^{-1}.$$

(b) Show that

$$\frac{d^2\delta}{d(i^{(m)})^2} = -\frac{1}{m}\left(1 + \frac{i^{(m)}}{m}\right)^{-2}.$$

(c) Use the chain rule and the product rule to establish

$$\frac{d^2P}{d(i^{(m)})^2} = \frac{d}{di^{(m)}}\left(\frac{dP}{d\delta}\right) \cdot \frac{d\delta}{di^{(m)}} + \frac{dP}{d\delta} \cdot \frac{d^2\delta}{d(i^{(m)})^2}.$$

(d) Show that

$$\frac{d^2P}{d(i^{(m)})^2} = \frac{d^2P}{d\delta^2} \cdot \left(1 + \frac{i^{(m)}}{m}\right)^{-2} - \frac{dP}{d\delta} \cdot \frac{1}{m}\left(1 + \frac{i^{(m)}}{m}\right)^{-2}.$$

(e) Use the result of (d) to demonstrate the following;

$$C(i,m) = \left(C(i,\infty) + \frac{1}{m}D(i,\infty)\right) \Big/ \left(1 + \frac{i^{(m)}}{m}\right)^{2}.$$

(6) Verify Fact (9.3.16).

(9.4) Immunization

(1) Providence Health Care is obligated to make a payment of \$300,000 in exactly three years. In order to provide for this obligation, their financial officer decides to purchase a combination of one-year zero-coupon bonds and four-year zero-coupon bonds. Each of these is sold to yield an annual effective yield of 4%. How much of each type of bond should be purchased so that the present value and duration conditions of Redington immunization are satisfied? Is the convexity condition also satisfied at $i = 4\%$?

(2) A court has ordered Security Enterprises to pay \$200,000 in two years and \$500,000 in five years. In order to meet this important liability, they wish to invest in a combination of two-year 10% par-value bonds with annual coupons and five-year zero-coupon bonds. Each of these is sold to yield an annual effective yield of 4%. How much of each type of bond should be purchased so that the present value and duration conditions of Redington immunization are satisfied at the current 4% rate? Is the convexity condition also satisfied?

(3) Tomorrow Financial Associates is required to make a \$500,000 payment in exactly four years. To cover this liability, they have purchased a two-year zero-coupon bond with redemption amount \$343,398.73 and a ten-year zero-coupon bond with redemption amount \$162,782.52. We assume that there is a flat yield curve and that the current interest rate is 4.5%. Is this portfolio fully immunized? In one year, will it be fully immunized? What will the surplus be if interest rates rise immediately to 8%? What will the surplus be in one year if interest rates are 8%?

(9.5) Other types of duration

(1) A four-year callable bond's current market value is \$95.40. If the interest rate increases by 100 basis points, the market value is anticipated to be \$92.50 and if the interest rate decreases by 100 basis points, the price is expected to be \$96.60. Compute the effective duration.

(2) A 4% \$2,000 six-year par-value bond with annual coupons is purchased for \$1,895.26. It may be called at any time. If interest rates fall by 1%, the bond should be called, while if they rise, it will not be called. Calculate the original yield i that an investor would receive if the bond were held to maturity, and then calculate the effective durations $E_{.01}(i,1)$ and $E_{.02}(i,1)$.

(3) Simeon Burnitz holds a callable bond with cashflows of \$2,500 in one year and again in two years and \$54,200 in three years. If the annual yield is 5%, then the price is expected to be \$49,200 (because of the call feature). The current yield is 6%, and the bond's price is \$48,392.75. Should the interest rate rise to 7%, the price will be \$47,181.90. Calculate the effective duration $E_{.01}(.06,1)$ of the bond and use it to approximate the price if the interest rate falls to 5.4%. What if it falls to 4%? Compare your answers to the actual prices, and also to the estimated price obtained using the modified duration $D(.06,1)$, of a noncallable bond with cashflows of \$2,500 in one year and \$54,200 in three years.

(4) Let $P(i)$ denote the market value of a set of cashflows if it is purchased to yield i. Define the effective convexity $F_h(i_0,1)$ by

$$F_h(i_0,1) = \frac{P(i_0+h) + P(i_0-h) - 2P(i_0)}{h^2 P(i_0)}.$$

Discuss how $F_h(i_0,1)$ and $C(i_0,1)$ are related. In particular, show that $F_h(i_0,1) \approx C(i_0,1)$ if the cashflows are not interest rate sensitive and h is sufficiently small.

(5) An eight-year 7% bond pays annual coupons. Its current price is \$102.43, and its current yield is 6.6%. If interest rates rise to an effective rate of 6.8%, then its price will fall to \$101.20. If interest rates fall to an effective rate of

6.4%, then its price will rise to \$103.21. Find the bond's effective duration and effective convexity, and use the calculated values to estimate the price if interest rates rise to an effective rate of 7.2%.

Chapter 9 review problems

(1) A fifteen-year 7.5% par-value bond has annual coupons. Compute the modified durations $D(9\%, 1)$ and $D(7.5\%, 1)$.

(2) The current price of a noncallable bond with semiannual coupons is \$1,020.60, and the current annual yield is 5.35%. The Macaulay duration $D(5.35\%, \infty) = 6.26$. Estimate the price of the bond if the annual effective interest rate increases to 5.6%. Approximate the bond's price if the annual effective interest rate decreases to 4%. How should these estimates be refined if you know that the Macaulay convexity $C(5.35\%, \infty)$ is equal to 1.64?

(3) A portfolio consists of three \$1,000 par-value bonds. The first is a two-year zero-coupon bond, the second is a three-year zero-coupon bond, and the last is a five-year 6% bond with annual coupons. At interest rate $i = 5\%$, calculate the Macaulay duration of each bond and of the portfolio. Also calculate the Macaulay convexities.

(4) Calculate the Macaulay duration $D(.06, \infty)$ on January 1 of a preferred stock that pays dividends forever of \$40 each June 30th and \$60 each December 31st . What is the modified duration $D(.06, 2)$? — Use the 30/360 method of counting time.

(5) National Reliance Insurance (NRI) is obligated to make a payment of \$250,000 in exactly five years. Its managers wish to fund this liability with a combination of two-year zero-coupon bonds and seven-year zero-coupon bonds, purchased today. The current yield rate for bonds is 5%. How much should the insurance company invest in each type of bond in order to immunize its portfolio? Assuming NRI allocates its resources so as to immunize its position, what is the time zero value of the portfolio of asset and liabilities if yield rates fall from 5% to 4%?

(6) A 5% \$10,000 five-year par-value bond with annual coupons is purchased for \$9,736.11. It may be called at any time. If interest rates fall by 1%, the bond should be called, while if the interest rate rises, it will not be called. Calculate the original yield i that an investor would receive if the bond were held to maturity, and then calculate the effective duration $E_{.01}(i, 1)$.