# Math 223
# Assignment 7: Examples of Groups

Quinn Neumiiller

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**Question Q1.** Perform the following calculations in $\mathbb{C}$:

**Question 1a.**

$$\begin{aligned}(2-3i)+(-1+4i) &= (2-3i)+(-1+4i)\\ &= 2-3i-1+4i\\ &= 1+i\end{aligned}$$

**Question 1b.**

$$\begin{aligned}(2-11i)(-4+3i) &= -8+6i+44i-33i^2\\ &= 50i+33-8\\ &= 50i+25\end{aligned}$$

**Question 1c.**

$$\begin{aligned}(4+3i)^{-1} &= \frac{4-3i}{4^2+3^3}\\ &= \frac{4-3i}{16+9}\\ &= \frac{4-3i}{25}\end{aligned}$$

**Question 1d.**

$$\begin{aligned}(\sqrt{2}i)^8 &= (\sqrt{2})^8(i)^8\\ &= ((\sqrt{2})^2)^4((i)^2)^4\\ &= (2)^4(-1)^4\\ &= 16\end{aligned}$$

**Question 1e.**

$$\begin{aligned}(2+3i)\overline{(3+4i)} &= (2+3i)(3-4i)\\ &= 6-8i+9i-12i^2\\ &= 6+i+12\\ &= 18+i\end{aligned}$$

**Question 1f.** Let $x=-2$ and $y=2\sqrt{3}$

$$\begin{aligned}r &= \sqrt{x^2+y^2}\\ &= \sqrt{(-2)^2+(2\sqrt{3})^2}\\ &= \sqrt{4+(4*3)}\\ &= \sqrt{4+12}\\ &= \sqrt{16}\\ &= 4\end{aligned}$$

$$\begin{aligned}\theta &= tan^{-1}(\frac{2\sqrt{3}}{-2})\\ &= tan^{-1}(\sqrt{-3})\\ &= -\frac{\pi}{3}\end{aligned}$$

$$\begin{aligned}(2\sqrt{3}i-2)^6 &= (re^{i\theta})^6\\ &= ((4)e^{i(-\frac{\pi}{3})})^6\\ &= (4^6)e^{6i(-\frac{\pi}{3})}\\ &= 4096e^{-2\pi i}\\ &= 4096(\cos(-2\pi)+i\sin(-2\pi))\\ &= 4096(1+i0)\\ &= 4096\end{aligned}$$

**Question 1g.** Let $x=1$ and $y=-1$

$$\begin{aligned}r &= \sqrt{(1)^2+(-1)^2}\\ &= \sqrt{1+1}\\ &= \sqrt{2}\end{aligned}$$

$$
\begin{aligned}
\theta &= tan^{-1}(-\sqrt{3}) \\
&= tan^{-1}(-\sqrt{3}) \\
&= -\frac{\pi}{3}
\end{aligned}
$$

$$
\begin{aligned}
(1-i)^4 &= (re^{i\theta})^4 \\
&= ((\sqrt{2})e^{i(-\frac{\pi}{4})})^4 \\
&= (\sqrt{2})^4 e^{4i(-\frac{\pi}{4})} \\
&= 4e^{-4i\pi} \\
&= 4(\cos(-\pi) + i\sin(-\pi)) \\
&= 4(-1 + i0) \\
&= -4
\end{aligned}
$$

**Question 1h.** Express in polar form Let $x = 1$ and $y = -\sqrt{3}$

$$
\begin{aligned}
\theta &= tan^{-1}(-\sqrt{3}) \\
&= tan^{-1}(-\sqrt{3}) \\
&= -\frac{\pi}{3}
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{1^2 + (\sqrt{3})^2} \\
&= \sqrt{1+3} \\
&= \sqrt{4} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
(\frac{1}{4\sqrt{2}} - \frac{\sqrt{3i}}{4\sqrt{2}})^9 &= (\frac{1}{4\sqrt{2}}(1-\sqrt{3i}))^9 \\
&= (\frac{1}{4\sqrt{2}}(re^{i\theta}))^9 \\
&= (\frac{1}{4\sqrt{2}}(2)e^{i(-\frac{\pi}{3})}))^9 \\
&= (\frac{1}{2\sqrt{2}}e^{i(-\frac{\pi}{3})}))^9 \\
&= (\frac{1}{2\sqrt{2}})^9 e^{-3\pi i} \\
&= \frac{1}{8192\sqrt{2}}e^{-3\pi i} \\
&= \frac{1}{8192\sqrt{2}}(\cos(-3\pi) + i\sin(-3\pi)) \\
&= \frac{1}{8192\sqrt{2}}(-1 + i0) \\
&= -\frac{1}{8192\sqrt{2}}
\end{aligned}
$$

**Question 1i.**

$$(re^{i\theta})^{-1} = \frac{e^{-i\theta}}{r}$$

**Question 1j.** Express in standard $x + yi$ form

$$
\begin{aligned}
(\sqrt{5}e^{\frac{2\pi i}{5}})(2\sqrt{5}e^{\frac{3\pi i}{5}}) &= 2(\sqrt{5})^2 e^{\pi i} \\
&= 10(-1) \\
&= -10
\end{aligned}
$$

**Question Q2.** Solve the following equations. Find all solutions for $z$ in $\mathbb{C}$. Express your answer in $x + yi$ form when convenient, otherwise leave them in polar form.

**Question 2a.** $z^2 = 5$

$r = \sqrt{5}$
$\theta = 0$
$2\theta \in [0, 4\pi)$
$2\theta \in \{0, 2\pi\}$
$\theta \in \{0, \pi\}$
$z = \{\sqrt{5}e^{i0}, \sqrt{5}e^{i\pi}\}$
$z = \{\sqrt{5}, \sqrt{5}(\cos\pi + i\sin\pi)\}$
$z = \{\sqrt{5}, \sqrt{5}(-1 + i0\}$
$z = \{\sqrt{5}, -\sqrt{5}\}$

**Question 2b.** $z^2 + 2z + 3 = 0$

Apply the quadratic equation

$$\begin{aligned}\frac{-2 \pm \sqrt{4-4(3)}}{2} &= -1 \pm \frac{\sqrt{-8}}{2} \\ &= -1 \pm \frac{2i\sqrt{2}}{2} \\ &= -1 \pm i\sqrt{2}\end{aligned}$$

**Question 2c.** $z^3 = -1$

$r = \sqrt[3]{-1}$
$\theta = 0$
$3\theta \in [0, 6\pi)$
$3\theta \in \{0, 2\pi, 4\pi\}$
$\theta \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$
$z = \{\sqrt[3]{-1}e^{i0}, \sqrt[3]{-1}e^{i\frac{2\pi}{3}}, \sqrt[3]{-1}e^{i\frac{4\pi}{3}}\}$
$z = \{\sqrt[3]{-1}(\cos(0) + i\sin(0)), \sqrt[3]{-1}e^{i\frac{2\pi}{3}}, \sqrt[3]{-1}e^{i\frac{4\pi}{3}}\}$
$z = \{\sqrt[3]{-1}, \sqrt[3]{-1}e^{i\frac{2\pi}{3}}, \sqrt[3]{-1}e^{i\frac{4\pi}{3}}\}$

**Question 2d.** $z^3 + 6z^2 + 12z + 3 = 0$ (Hint: $(z+2)^3 = z^3 + 6z^2 + 12z + 8$)

$z^3 + 6z^2 + 12z + 8 = 5$
$(z+2)^3 = 5$
Let $\overline{z} = z + 2$
$\overline{z}^3 = 5$
$r = \sqrt[3]{5}$
$\theta = 0$
$3\theta \in [0, 6\pi)$
$3\theta \in \{0, 2\pi, 4\pi\}$
$\theta \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$
$\overline{z} = \{\sqrt[3]{5}e^{i0}, \sqrt[3]{5}e^{i\frac{2\pi}{3}}, \sqrt[3]{5}e^{i\frac{4\pi}{3}}\}$
$\overline{z} = \{\sqrt[3]{5}, \sqrt[3]{5}e^{i\frac{2\pi}{3}}, \sqrt[3]{5}e^{i\frac{4\pi}{3}}\}$
$(z+2) = \{\sqrt[3]{5}, \sqrt[3]{5}e^{i\frac{2\pi}{3}}, \sqrt[3]{5}e^{i\frac{4\pi}{3}}\}$
$z = \{\sqrt[3]{5} - 2, \sqrt[3]{5}e^{i\frac{2\pi}{3}} - 2, \sqrt[3]{5}e^{i\frac{4\pi}{3}} - 2\}$

**Question 2e.** $z^4 = 16 - 16i$

$\theta = tan^{-1}(-1) = -\frac{\pi}{4}$
$4\theta = [0, 8\pi)$
$4\theta = \{-\frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}, 6\pi - \frac{\pi}{4}\}$
$4\theta = \{-\frac{\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}, \frac{23\pi}{4}\}$
$\theta = \{-\frac{\pi}{16}, \frac{7\pi}{16}, \frac{15\pi}{16}, \frac{23\pi}{16}\}$
$r = (\sqrt{16^2 + (-16)^2})^{1/4}$
$r = (\sqrt{512})^{1/4}$
$r = (16\sqrt{2})^{1/4}$
$r = 2(2)^{1/4}$
$z = \{2(2)^{1/4}e^{i(-\frac{\pi}{16})}, 2(2)^{1/4}e^{i(\frac{7\pi}{16})}, 2(2)^{1/4}e^{i(\frac{15\pi}{16})}, 2(2)^{1/4}e^{i(\frac{23\pi}{16})}\}$

**Question 2f.** $z^6 = -1 - \sqrt{3}i$

$\theta = tan^{-1}(\sqrt(3)) = \frac{\pi}{3}$
$6\theta = [0, 12\pi]$
$6\theta = \{\frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi + \frac{\pi}{3}, 6\pi + \frac{\pi}{3}, 8\pi + \frac{\pi}{3}, 10\pi + \frac{\pi}{3}\}$
$6\theta = \{\frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \frac{19\pi}{3}, \frac{25\pi}{3}, \frac{31\pi}{3}\}$
$\theta = \{\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}\}$
$r = (\sqrt{(-1)^2 + (-\sqrt{3})^2})^{1/6}$
$r = (\sqrt{1+3})^{1/6}$
$r = (2)^{1/6}$
$z = \{(2)^{1/6}e^{i\frac{\pi}{18}}, (2)^{1/6}e^{i\frac{7\pi}{18}}, (2)^{1/6}e^{i\frac{13\pi}{18}}, (2)^{1/6}e^{i\frac{19\pi}{18}}, (2)^{1/6}e^{i\frac{25\pi}{18}}, (2)^{1/6}e^{i\frac{31\pi}{18}}\}$

**Question 2g.** $z^2 + (4 - 4\sqrt{3}i)z - 16 = 0$

Let $b = (4 - 4\sqrt{3}i)$
$z^2 + bz - 16 = 0$

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4(-16)}}{2} &= \frac{-b}{2} \pm \frac{\sqrt{b^2+64}}{2} \\ &= \frac{-(4-4\sqrt{3}i)}{2} \pm \frac{\sqrt{(4-4\sqrt{3}i)^2 + 64}}{2} \\ &= 2 + 2\sqrt{3}i \pm \frac{\sqrt{(-32 - 32\sqrt{(3)}i) + 64}}{2} \\ &= 2 + 2\sqrt{3}i \pm \frac{\sqrt{32 - 32\sqrt{(3)}i}}{2}\end{aligned}$$

**Question 3.** Suppose $(D, +, x)$ is an integral domain that is not a field.

**Question 3a.** Prove that the ring $(D[x], +, x)$ of polynomials in the variable $x$ with coeffcients in $D$ is an integral domain. (For this question, you should assume that if $R$ is a commutative ring, then $R[x]$ is a commuative ring.)

*Proof.* $D[x]$ is a integral domain iff, $(D[x], +, x)$ is a commutative ring and $(D[x]^x, x)$ is closed.

Given that $D[x]$ is an assumed commuative ring.
When $f(x)g(x) \in D[x] \setminus \{0\} \implies$ deg $f(x) = n \in \mathbb{N}$ and deg $g(x) = m \in \mathbb{N}$
By Lemma 6.1 in the notes, deg$(f(x)g(x))$= $n + m \in \mathbb{N} \implies f(x)g(x) \neq 0$.
Therefore, $D[x]$ is closed.
Because $D[x]$ is closed and an assumed commuative ring, $D[x]$ is a integral domain.

□

**Question 3b.** Let $F$ be a field. Explain why the polynomial ring $F[x, y]$ is an integral domain

*Proof.* A similar proof given in 3a can be used to prove $F[x, y]$.

Given that F is a field, it inherits the commutative ring property.

When $f(x, y)g(x, y) \in F[x, y] \setminus \{0\} \implies \deg f(x, y) = n \in \mathbb{N}$ and $\deg g(x, y) = m \in \mathbb{N}$

By Lemma 6.1 in the notes, $\deg(f(x, y)g(x, y)) = n + m \in \mathbb{N} \implies f(x, y)g(x, y) \neq 0$.

Therefore, $F[x, y]$ is closed.

Because $F[x, y]$ is closed and a commuative ring, $F[x, y]$ is a integral domain. $\square$

**Question 4.** For each of the following, find the unique quotient $q(x)$ and remainder r(x) in the given polynomial ring when $g(x)$ is divided by $f(x)$.

**Question 4a.** $g(x) = x^4 + x^3 + x^2 - 1$ and $f(x) = x^2 + x + 2$ in $\mathbb{Z}_3[x]$

**Question 4b.** $g(x) = x^6 + 3x^5 + 4x^2 + 2x + 2$ and $f(x) = x^2 + 2x + 2$ in $\mathbb{Z}_5[x]$

**Question 4c.** $g(x) = x^6 + 3x^5 + 4x^2 + 2x + 2$ and $f(x) = x^2 + 2x + 2$ in $\mathbb{Z}_{11}[x]$

**Question 5.** Prove the *Remainder Theorem*. Let $F$ be a field. Let $f(x) \in F[x]$ and let $c \in F$. Prove that the remainder of $f(x)$ divided by $(x - c)$ is $f(c)$.

*Proof.* Let $q(x)$ be the quotient, and $r(x)$ the remainder.

By division algorithm, $f(x) = q(x)(x - c) + r(x)$

Let $x = c$

$f(c) = q(c)(c - c) + r(c)$

$f(c) = q(c)0 + r(c)$

$f(c) = r(c)$

Hence $f(c)$ is the remainder, $r(c)$.

□