Time Series Econometrics: Homework assignment 1

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1 Problem 1

Consider the second-order difference equation (p = 2)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

Using direct multiplication, show that

1.1 the effect on y_{t+3} of a one-unit increase in w_t is

$$\phi_1^3 + 2\phi_1\phi_2 \tag{1}$$

Answer

Using recursive substitution, we can expand y_{t+1}:

$$\begin{aligned}
y_{t+1} &= \phi_1 y_t + \phi_2 y_{t-1} + w_{t+1} \\
&= \phi_1(\phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t) + \phi_2 y_{t-1} + w_{t+1} \\
&= \phi_1^2 y_{t-1} + \phi_1\phi_2 y_{t-2} + \phi_1 w_t + \phi_2 y_{t-1} + w_{t+1}
\end{aligned}$$

Likewise, for y_{t+2}, and y_{t+3} we have:

$$\begin{aligned}
y_{t+2} &= \phi_1 y_{t+1} + \phi_2 y_t + w_{t+2} \\
&= \phi_1(\phi_1^2 y_{t-1} + \phi_1\phi_2 y_{t-2} + \phi_1 w_t + \phi_2 y_{t-1} + w_{t+1}) + \phi_2(\phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t) + w_{t+2} \\
&= \phi_1^3 y_{t-1} + \phi_1^2\phi_2 y_{t-2} + \phi_1^2 w_t + \phi_1\phi_2 y_{t-1} + \phi_1 w_{t+1} + \phi_1\phi_2 y_{t-1} + \phi_2^2 y_{t-2} + \phi_2 w_t + w_{t+2}
\end{aligned}$$

$$\begin{aligned}
y_{t+3} &= \phi_1 y_{t+2} + \phi_2 y_{t+1} + w_{t+3} \\
&= \phi_1(\phi_1^3 y_{t-1} + \phi_1^2\phi_2 y_{t-2} + \phi_1^2 w_t + \phi_1\phi_2 y_{t-1} + \phi_1 w_{t+1} + \phi_1\phi_2 y_{t-1} + \phi_2^2 y_{t-2} + \phi_2 w_t + w_{t+2}) + \\
&\phi_2(\phi_1^2 y_{t-1} + \phi_1\phi_2 y_{t-2} + \phi_1 w_t + \phi_2 y_{t-1} + w_{t+1}) + w_{t+3} \\
&= \phi_1^4 y_{t-1} + \phi_1^3\phi_2 y_{t-2} + \phi_1^3 w_t + \phi_1^2\phi_2 y_{t-1} + \phi_1^2 w_{t+1} + \phi_1^2\phi_2 y_{t-1} + \phi_1\phi_2^2 y_{t-2} + \phi_1\phi_2 w_t + \phi_1 w_{t+2} + \\
&\phi_1^2\phi_2 y_{t-1} + \phi_1\phi_2^2 y_{t-2} + \phi_1\phi_2 w_t + \phi_2^2 y_{t-1} + \phi_2 w_{t+1} + w_{t+3}
\end{aligned}$$

Then:

$$\frac{\partial y_{t+3}}{\partial w_t} = \phi_1^3 + \phi_1\phi_2 + \phi_1\phi_2 = \phi_1^3 + 2\phi_1\phi_2$$

1.2 the effect on y_{t+4} of a one-unit increase in w_t is

$$\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2 \tag{2}$$

Answer

Following similar steps as in 1.1

$$\begin{aligned}
y_{t+4} &= \phi_1 y_{t+3} + \phi_2 y_{t+2} + w_{t+4} \\
&= \phi_1(\phi_1^4 y_{t-1} + \phi_1^3\phi_2 y_{t-2} + \phi_1^3 w_t + \phi_1^2\phi_2 y_{t-1} + \phi_1^2 w_{t+1} + \phi_1^2\phi_2 y_{t-1} + \phi_1\phi_2^2 y_{t-2} + \phi_1\phi_2 w_t + \phi_1 w_{t+2} + \\
&\phi_1^2\phi_2 y_{t-1} + \phi_1\phi_2^2 y_{t-2} + \phi_1\phi_2 w_t + \phi_2^2 y_{t-1} + \phi_2 w_{t+1} + w_{t+3}) + \\
&\phi_2(\phi_1^3 y_{t-1} + \phi_1^2\phi_2 y_{t-2} + \phi_1^2 w_t + \phi_1\phi_2 y_{t-1} + \phi_1 w_{t+1} + \phi_1\phi_2 y_{t-1} + \phi_2^2 y_{t-2} + \phi_2 w_t + w_{t+2}) + w_{t+4}
\end{aligned}$$

$$\frac{\partial y_{t+4}}{\partial w_t} = \phi_1^4 + \phi_1^2\phi_2 + \phi_1^2\phi_2 + \phi_1^2\phi_2 + \phi_2^2 = \phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2$$

2 Problem 2

Consider the same difference equation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t$$

where $\phi_1 = 3/4$ and $\phi_2 = -1/8$.

2.1 Using the eigenvalues of the matrix $\boldsymbol{F}$, show that

$$\frac{\partial y_{t+j}}{\partial w_t} = \left(\frac{1}{2}\right)^{j-1} - \left(\frac{1}{4}\right)^{j} \tag{3}$$

Answer

Let:

$$\boldsymbol{\xi_t} = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{8} \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{v_t} = \begin{bmatrix} w_t \\ 0 \end{bmatrix},$$

Consider the following first-order vector difference equation:

$$\boldsymbol{\xi_t} = \boldsymbol{F}\boldsymbol{\xi_{t_1}} + \boldsymbol{v_t}$$

Recall that the eigenvalues of a matrix $\boldsymbol{F}$ are those numbers λ for which

$$\mid \boldsymbol{F} - \boldsymbol{\lambda I_2} \mid = \begin{vmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{vmatrix} = 0$$

The two eigenvalues of $\boldsymbol{F}$ for the second-order difference equation are thus given by

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} = \frac{1}{2}$$
$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} = \frac{1}{4}$$

we can characterize $\boldsymbol{F^j}$ in terms of the eigenvalues of $\boldsymbol{F}$ as

$$\boldsymbol{F^j} = \boldsymbol{T\Lambda^j T^{-1}}$$

therefore:

$$\boldsymbol{F^j} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{bmatrix} \begin{bmatrix} t^{11} & t^{12} \\ t^{21} & t^{22} \end{bmatrix}$$

from which the (1, 1) element of $\boldsymbol{F^j}$ is given by

$$f_{11}^{(j)} = \underbrace{[t_{11}t^{11}]}_{c_1} \lambda_1^j + \underbrace{[t_{12}t^{21}]}_{c_2} \lambda_2^j$$

Using Proposition 1.2 on pp.12 in Hamilton

$$c_1 = \frac{\lambda_1}{(\lambda_1 - \lambda_2)} = \frac{\frac{1}{2}}{(\frac{1}{2} - \frac{1}{4})} = 2, \quad c_2 = \frac{\lambda_2}{(\lambda_2 - \lambda_1)} = \frac{\frac{1}{4}}{(\frac{1}{4} - \frac{1}{2})} = -1$$

The dynamic multiplier is given by:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1\lambda_1^j + c_2\lambda_2^j = 2\left(\frac{1}{2}\right)^j + (-1)\left(\frac{1}{4}\right)^j = \left(\frac{1}{2}\right)^{j-1} - \left(\frac{1}{4}\right)^j$$

2.2 For j = 3 and j = 4, verify that (1) and (2) produce the same results as (3)

Answer

Yes, (1) and (2) produce the same results as (3).

$$\frac{\partial y_{t+3}}{\partial w_t} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^3 = \frac{15}{64}$$
$$\frac{\partial y_{t+4}}{\partial w_t} = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{4}\right)^4 = \frac{31}{256}$$

2.3 Is the system stable? Motivate your answer.

Answer

Given that the two eigenvalues of $\boldsymbol{F}$ are real (i.e., $\phi_1^2 + 4\phi_2 = \frac{1}{16} > 0$), and the two eigenvalues are less than 1 in absolute value, then the system is stable.

3 Problem 3

Let $\{y_t\}_t^\infty = \infty$ be given by

$$z_s = \begin{pmatrix} y_{2s-1} \\ y_{2s} \end{pmatrix}, \quad s = 0, \pm 1, \pm 1, \ldots$$

where z_s is iid $N(\mathbf{0}, \mathbf{\Sigma})$, with

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$$

Using standard results for the multivariate normal distribution,

3.1 Verify that $y_t \sim N(0,1)$ for all $t = 0, \pm 1, \pm 2, \ldots$

Answer

We know that if $\boldsymbol{Y} = \boldsymbol{c} + \boldsymbol{BX}$ is an affine transformation of $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then:

$$\mathbf{Y} \sim N\left(\mathbf{c} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^{\mathrm{T}}\right).$$

In particular, any subset of the Xi has a marginal distribution that is also multivariate normal. Therefore Since z_s is iid and $N(\mathbf{0}, \boldsymbol{\Sigma})$

$$\begin{aligned} y_t = \mathbf{B}\mathbf{z}_{t=2s} &= (0,1)\begin{pmatrix} y_{t-1} \\ y_t \end{pmatrix} = y_t \\ \mathbf{B}\boldsymbol{\mu} &= (0,1)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \\ \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T &= (0,1)\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \\ y_t &\sim N(0,1)\ \forall t = \pm 1, \pm 2, \ldots \end{aligned}$$

3.2 show that if $\gamma \neq 0$, then $\{y_t\}_{t=\infty}^{\infty}$ is neither strictly stationary nor covariance stationary

Answer

We know that

$$\mathbf{z_s} \sim N\left(\mathbf{0}, \boldsymbol{\Sigma}\right).$$

therefore

$$\mathbf{z_1} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N\left(\mathbf{0}, \boldsymbol{\Sigma}\right).$$

then we can compute the covariance of y_1 and y_2

$$cov\left(y_1, y_2\right) = \gamma$$

Using $(y_2, y_3)'$ we can also get the covariance of y_2 and y_3

$$\begin{aligned} cov\left(y_2, y_3\right) &= E[y_2 y_3] - E[y_2]E[y_3] \\ E[y_2] &= 0, \quad E[y_3] = 0 \end{aligned}$$

then by independence we have

$$cov\left(y_2, y_3\right) = E[y_2 y_3] = E[y_2]E[y_3] = 0$$

Then since $cov\left(y_2, y_3\right) = 0 \neq \gamma = cov\left(y_1, y_2\right)$, y_t is not stationary if $\gamma \neq 0$.

4 Problem 4

Let $\{\epsilon_t\}_{t=\infty}^{\infty}$ be a white noise process and $\theta \neq 0$. Consider the two $MA(1)$ processes $\{y_t\}_{t=\infty}^{\infty}$ and $\{\tilde{y}_t\}_{t=\infty}^{\infty}$ given by

$$y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

and

$$\tilde{y}_t = \mu + \tilde{\epsilon}_t + \tilde{\theta}\tilde{\epsilon}_{t-1}$$

respectively, where $\tilde{\epsilon}_t = \theta\epsilon$ and $\tilde{\theta} = 1/\theta$.

4.1 Verify that $E(y_t) = E(\tilde{y}_t) = \mu$

Answer

$$\begin{aligned} E[y_t] &= E[\mu + \epsilon_t + \theta\epsilon_{t-1}] \\ &= E[\mu] + \underbrace{E[\epsilon_t]}_{0} + \theta\underbrace{E[\epsilon_{t-1}]}_{0} \\ &= \mu \end{aligned}$$

$$\begin{aligned} E[\tilde{y}_t] &= E[\mu + \tilde{\epsilon}_t + \tilde{\theta}\tilde{\epsilon}_{t-1}] \\ &= E[\mu + \theta\epsilon_t + \frac{1}{\theta}\theta\epsilon_{t-1}] \\ &= E[\mu] + \theta\underbrace{E[\epsilon_t]}_{0} + \frac{\theta}{\theta}\underbrace{E[\epsilon_{t-1}]}_{0} \\ &= \mu \end{aligned}$$

It follows that

$$E[y_t] = E[\tilde{y}_t] = \mu$$

4.2 Verify that $E(y_t - \mu)(y_{t-j} - \mu) = E(\tilde{y}_t - \mu)(\tilde{y}_{t-j} - \mu)$, for $j = 0, 1, 2, \ldots$

Answer

$$\begin{aligned} E(y_t - \mu)(y_{t-j} - \mu) &= cov(y_t, y_{t-j}) \\ &= cov(\mu + \epsilon_t + \theta\epsilon_{t-1}, \mu + \epsilon_{t-j} + \theta\epsilon_{t-j-1}) \\ &= cov(\epsilon_t + \theta\epsilon_{t-1}, \epsilon_{t-j} + \theta\epsilon_{t-j-1}) \\ &= \underbrace{cov(\epsilon_t, \epsilon_{t-j} + \theta\epsilon_{t-j-1})}_{=\begin{cases}\sigma^2 & j=0\\ 0 & j\neq 0\end{cases}} + \underbrace{cov(\theta\epsilon_{t-1}, \epsilon_{t-j} + \theta\epsilon_{t-j-1})}_{=\begin{cases}\theta^2\sigma^2 & j=0\\ \theta\sigma^2 & j=1\\ 0 & o.c.\end{cases}} \end{aligned}$$

$$cov(y_t, y_{t-j}) = \begin{cases} (1+\theta^2)\sigma^2 & j = 0 \\ \theta\sigma^2 & j = 1 \\ 0 & o.c. \end{cases}$$

$$
\begin{aligned}
E(\tilde{y}_t - \mu)(\tilde{y}_{t-j} - \mu) &= cov(\tilde{y}_t, \tilde{y}_{t-j}) \\
&= cov(\mu + \tilde{\epsilon}_t + \tilde{\theta}\tilde{\epsilon}_{t-1}, \mu + \tilde{\epsilon}_{t-j} + \tilde{\theta}\tilde{\epsilon}_{t-j-1}) \\
&= cov(\theta\epsilon_t + \epsilon_{t-1}, \theta\epsilon_{t-j} + \epsilon_{t-j-1}) \\
&= \underbrace{cov(\theta\epsilon_t, \theta\epsilon_{t-j} + \epsilon_{t-j-1})}_{=\begin{cases}\theta^2\sigma^2 & j=0 \\ 0 & j \neq 0\end{cases}} + \underbrace{cov(\epsilon_{t-1}, \theta\epsilon_{t-j} + \epsilon_{t-j-1})}_{=\begin{cases}\sigma^2 & j=0 \\ \theta\sigma^2 & j=1 \\ 0 & o.c.\end{cases}}
\end{aligned}
$$

$$
cov(\tilde{y}_t, \tilde{y}_{t-j}) = \begin{cases} (1+\theta^2)\sigma^2 & j=0 \\ \theta\sigma^2 & j=1 \\ 0 & o.c. \end{cases}
$$

it follows that:

$$
\begin{aligned}
cov(y_t, y_{t-j}) &= cov(\tilde{y}_t, \tilde{y}_{t-j}) \\
E(y_t - \mu)(y_{t-j} - \mu) &= E(\tilde{y}_t - \mu)(\tilde{y}_{t-j} - \mu)
\end{aligned}
$$

5 Problem 5

Consider the simple $AR(1)$ process

$$
(1 - \phi L)y_t = \epsilon_t
$$

where $\{\epsilon_t\}_{t=\infty}^{\infty}$ is a white noise process.

5.1 Show, by recursive substitution, that

$$
y_{t+s} = \theta^s y_t + \sum_{i=0}^{s-1} \theta^i \epsilon_{t+s-i}
$$

Answer Using the properties of the lag operator, let's rewrite the expression $(1 - \phi L)y_t = \epsilon_t$ as

$$
y_t - \phi L y_t = \epsilon_t
$$

then using recursive substitution

$$
\begin{aligned}
y_t &= y_t \phi L + \epsilon_t \\
&= \phi y_{t-1} + \epsilon_t
\end{aligned}
$$

$$
\begin{aligned}
y_{t+1} &= \phi y_t + \epsilon_{t+1} \\
&= \phi(\phi y_{t-1} + \epsilon_t) + \epsilon_{t+1} \\
&= \phi^2 y_{t-1} + \phi\epsilon_t + \epsilon_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
y_{t+2} &= \phi y_{t+1} + \epsilon_{t+2} \\
&= \phi(\phi^2 y_{t-1} + \phi\epsilon_t + \epsilon_{t+1}) + \epsilon_{t+2} \\
&= \phi^3 y_{t-1} + \phi^2\epsilon_t + \theta\epsilon_{t+1} + \epsilon_{t+2}
\end{aligned}
$$

$$
\begin{aligned}
y_{t+2} &= \phi y_{t+1} + \epsilon_{t+2} \\
&= \phi(\phi(\phi y_{t-1} + \epsilon_t) + \epsilon_{t+1}) + \epsilon_{t+2} \\
&= \phi(\phi y_t + \epsilon_{t+1}) + \epsilon_{t+2} \\
&= \phi^2 y_t + \phi\epsilon_{t+1} + \epsilon_{t+2}
\end{aligned}
$$

Then is easy to see the patern and get the general form:

$$
y_{t+s} = \phi^s y_t + \sum_{i=0}^{s-1} \theta^i \epsilon_{t+s-i}
$$

5.2 Use the above formula to compute the conditional expectation $E(y_{t+s}|I_t)$, where I_t is the information set available at time t.

Answer

$$
\begin{aligned}
E(y_{t+s}|I_t) &= E(\phi^s y_t + \sum_{i=0}^{s-1} \theta^i \epsilon_{t+s-i}|I_t) \\
&= E(\phi^s y_t|I_t) + E(\sum_{i=0}^{s-1} \theta^i \epsilon_{t+s-i}|I_t) \\
&= E(\phi^s y_t|I_t) + \sum_{i=0}^{s-1} \theta^i E(\epsilon_{t+s-i}|I_t) \\
&= \phi^s y_t + \sum_{i=0}^{s-1} \theta^i \underbrace{E(\epsilon_{t+s-i}|I_t)}_{=0} \\
&= \phi^s y_t
\end{aligned}
$$