Kinematics

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The study of motion without consideration for the study of this motion.
Motion is: the change of an object's position over an arbitrary time period.
To study motion, we require a dependable description of position. Such a description must have:

- a constant and mutually-agreed fiducial point (origin)
- a constant and mutually-agreed set of axes

The assembly of the fiducial point and these axes is called a **Frame of Reference** (FoR or simply "frame").
To locate an object in a frame requires a distance and a direction from the fiducial.

1 Vectors

Any variable which requires a magnitude and a direction to make sense is called a vector. Example:

- Force

A quantity that has only magnitude is called a scalar. For example:

- mass
- temperature
- money

We will denote a vector x with an arrow above the letter: $\vec{x}$.

1.1 Stating Vectors

An example: Let us say we have a vector going on a 2D cartesian plane going from the origin to (3, 4). We have two basic coordinate systems to choose from:

1. Polar, in which we have an angle θ and a radius (magnitude) r.
2. Cartesian or rectangular, in which we have an ordered pair (x, y).

For polar form, **all** $\theta' s$ are referenced from the $+x$ axis. (Counterclockwise). As long as θ is referenced from the $+x$ axis, your x, y components will always be correct. In polar form, the magnitude is **always** positive.

1.1.1 Other angle issues

Bearings Some of our kinematics work will involve directions. A compass direction is called a bearing (γ) and is measured clockwise from $+y$. However, components of such vectors require a θ reference counter-clockwise from $+x$. Conversion:

$$(\theta = 450 - \gamma) \in [0, 360)$$

Types

- Absolute
  Bearing is an angle clockwise from $+y \in [0, 360)$
  Very typical for professional navigation.
- Relative
  Suppose $\gamma = 128°$

  We can also produce a bearing like this:

  1. Choose the cardinal direction (N, E, S, W) to which this γ most closest points.

  $$S42°E$$

  2. Correct is so many degrees on either side of the chosen cardinal. For N or S, it is E or W, and vice-versa.

  Two basic rules

  1. If correction is $45°$ use NE, NW, etc
  2. If correction is $< 45°$, use trig and common sense.

1.2 $\mathbb{R}^3$ - Spatial Motion

Can use (x, y, z), or (r, θ, ϕ), spherical, or (r, θ, h), cylindrical.
$\theta \in [0, 360)$ or $[-180, 180)$
$\phi \in [-90, 90]$

1.3 $\mathbb{R}^3$

Above $\mathbb{R}^3$ only n-tuples are defined. We cannot visualize them and must trust the theory.

1.4 Lines

For $y = mx$, which passes through the origin, $\theta =$ constant.

For $y = mx + b$, $r = \frac{b}{\sin\theta - m\cos\theta}$ (derivation will be left as an exercise to the reader.)

2 Position Vectors($\vec{r}$)

To analyze motion, it is sessential to locate objects in a frame. A position vector begins at the origin and ends at the object. If the object moves the $\vec{r}$ instantaneously moves to follow it.

3 Displacement($\vec{s}$)

If an object moves, there is a straight vector that points from the start to the finish. This is called **displacement**. It is the net change of position.

In addition or subtraction, vector variable algebra is identical to $\mathbb{R}$.

- Commutivity. $a + b = b + a$
- Associativity. $(a + b) + c = a + (b + c)$

If we move to point A, then to point B, we can simply move to point B.

Unit vectors are vectors with magnitudes of 1. We can represent vectors as a scalar multiplied by a unit vector: $\vec{a} = a\hat{a}$.

There are three special unit vectors: $\hat{\imath}, \hat{\jmath}, \hat{k}$, which represent vectors in x, y, z respectively. Therefore, a rectangular coordinate in three dimensions (x, y, z) can be represented as $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

3.1 Distance (d)

The length of the actual path taken is called distance. Since distance is a scalar and is accumulative, the distance is always greater than or equal to zero.

3.2 Vector "Locations"

A vector is defined only by its magnitude & direction, never its position. As such, parallel vectors are not only identical and indistinguishable. The direction of a vector can be mathematically described as the slope.

$$y = mx + b$$

So y is a family of all lines of slope m.

4 Rates

A rate is a comparison of a variable and time, typically standardized as **unit time**.

Table 1: SI Base Units

	mks	cgs
Length	m	cm
Time	s	s
Mass	kg	g
Temperature	K	K
Force	N	dynes
Energy	J	ergs

4.1 Velocity ($\vec{v}$)

The rate of change of $\vec{s}$ is defined as the velocity:

$$\vec{v}_{ANG} = \frac{\Delta \vec{s}}{\Delta t}$$

If $\Delta t \to 0$ we obtain a limit

$$\vec{v}_{INST} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

$$\vec{v} = \frac{d\vec{s}}{dt}$$

4.2 Acceleration($\vec{a}$)

The rate of change of $\vec{v}$ is called *acceleration*.

$$\vec{a}_{AVG} = \frac{\Delta \vec{v}}{\Delta t}$$

If $\Delta t \to 0$ then another limit,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

4.3 Higher rates

1. Surge or jerk, in the UK and US respectively
2. Snap
3. Crackle
4. Pop

5 Mathematical Relationships in Kinematics

The interrelations between kinematics variables are firstly derived from definitions and then subject to mathematical rigour. Recall:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Hence,

$$\mathrm{d}v = a\mathrm{d}t$$
$$\int \mathrm{d}v = \int a\mathrm{d}t$$
$$v = at + c$$

The c represents $v(0)$. It is practise in physics to write $v(0) = v_0$ seen as an intitial condition. We tend to write polynomials with their terms oriented in order of increasing exponents. This is because it is typical of a Taylor series and they are used extensively to approximate more involved functions. Therefore, our equation becomes

$$v = v_0 + at \tag{1}$$

Our second equation comes from another definition.

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$
$$\mathrm{d}s = v\mathrm{d}t$$
$$\mathrm{d}s = (v_0 + at)\mathrm{d}t$$
$$= v_0\mathrm{d}t + at\mathrm{d}t$$

We arrive at:

$$s = s_0 + v_0t + \frac{1}{2}at^2 \tag{2}$$

Some constraints:

1. a is assumed constant over an interval.
2. Time intervals must be changed if:
   - $\vec{a}$ changes
   - $\vec{v}$ changes sign
   - $\vec{v}$ changes arbitrarily (magic)
   - Question dictates it.

In (1), $t = \frac{v-v_0}{a}$ and substitute this into (2).

$$s = s_0 + v_0\left(\frac{v-v_0}{a}\right) + \frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2$$
$$\vdots$$
$$v^2 = v_0^2 + 2as \tag{3}$$

5.1 Reminder

(3) is only valid iff:

- v has a constant sign.
- Single time interval.

5.2 Caution

$\vec{v}_{AVG}$ is defined as

$$\vec{v}_{AVG} = \frac{\sum \vec{s}}{\sum t}$$

and

$$v_{AVG} = \frac{\sum d}{\sum t}$$

But:

$$v_{AVG} \neq \frac{\sum v_k}{\sum t}$$

6 Graphing kinematics equations

An object starting from rest accelerates at a constant rate of $2\text{ms}^{-}2$ until the object has moved 4m. It then stops for 1s. It then goes at -2ms^{-1} for 2s and finally $\vec{a}$ at 1ms^{-2} for 2s. An exercise to the reader: Create a table.

7 2D Motion

When we move to $\mathbb{R}^2$, we have additional types of motion.

- curvilinear motion
- rotational motion, which will be covered in grade 12

2D motion is represented using polar vectors or ordered pairs. The components of each axis must be kept separate from the other. Our calculations will be effected using components exclusively (i.e. rectangular form). You are not permitted to use sine or cosine law.

Many questions use bearings instead of θ. All angles to be referenced from the $+x$ axis, counterclockwise.

General algorithm for problems:

1. $\gamma \to \theta$ angles
2. components
3. set up system of equations
4. solve it
5. put answer in acceptable form.

8 Projectile Motion

To study the flight of a projectile that is launched with velocity $\vec{v_0}$, but thereafter is affected only by gravity is called **ballistics**. The solutions to the differential equations that describe such paths are known as **ballistic trajectories**.

Most of these require advanced forms of calculus and/or a computer to solve. We can generate a special case by making some rather bold assumptions.

- No air
- The earth is flat
- Object must land at same elevation

8.1 The Situation

Since F_g is the only active force and g is constant (i.e. the acceleration of gravity: $-9.8ms^-2$).

Taking components:

$$v_x = v_0 \cos\theta, v_y = v_0 \sin\theta$$

g acts only on the y so v_x will be constant and x motion shall continue at a constant rate until contact. The y motion acts as a timer and controls the duration of the flight.

$$r = v_x t = (v_0 \cos\theta)t$$

$$v_y = v_{0y} + a_y t$$

Time to max height will be equal to the fall time. So compute the time to max height and double it:

$$t = \frac{-v_0 \sin\theta}{g}$$

$$r = \frac{-v_0^2 \sin 2\theta}{g}$$

Unfortunately, many books take $g > 0$ and produce

$$r = \frac{v_0^2 \sin 2\theta}{g} \tag{4}$$