MAA Problem Books

J. DOUGLAS FAIRES

FIRST STEPS FOR MATH OLYMPIANS

Using the American Mathematics Competitions

The Mathematical Association of America

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First Steps for Math Olympians

Using the American Mathematics Competitions

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Preface

A Brief History of the American Mathematics Competitions

In the last year of the second millennium, the American High School Mathematics Examination, commonly known as the AHSME, celebrated its fiftieth year. It began in 1950 as a local exam in the New York City area, but within its first decade had spread to most of the states and provinces in North America, and was being administered to over 150,000 students. A third generation of students is now taking the competitions.

The examination has expanded and developed over the years in a number of ways. Initially it was a 50-question test in three parts. Part I consisted of 15 relatively routine computational problems; Part II contained 20 problems that required a thorough knowledge of high school mathematics, and perhaps some ingenuity; those in Part III were the most difficult, although some of these seem, based on the latter problems on the modern examination, relatively straightforward. The points awarded for success increased with the parts, and totaled 150.

The exam was reduced to 40 questions in 1960 by deleting some of the more routine problems. The number of questions was reduced again, to 35, in 1968, but the number of parts was increased to four. The number of problems on the exam was finally reduced to 30, in 1974, and the division of the exam into parts with differing weights on each part was eliminated. After this time, each problem would be weighted equally. It continued in this form until the end of the century, by which time the exam was being given to over 240,000 students at over 5000 schools.

One might get the impression that with a reduction in the number of problems the examination was becoming easier over the years, but a brief look at the earlier exams (which can be found in *The Contest Problem Book,* *Volumes I through V*) will dissuade one from this view. The number of problems has been reduced, but the average level of difficulty has increased. There are no longer many routine problems on the exams, and the middle-range problems are more difficult than those in the early years.

Since 1974, students from the United States have competed in the International Mathematical Olympiad (IMO), and beginning in 1972 students with very high scores on the AHSME were invited to take the United States of America Mathematical Olympiad (USAMO). The USAMO is a very difficult essay-type exam that is designed to select the premier problem-solving students in the country. There is a vast difference between the AHSME, a multiple-choice test designed for students with a wide range of abilities, and the USAMO, a test for the most capable in the nation. As a consequence, in 1983 an intermediate exam, the American Invitational Mathematics Examination, was instituted, which the students scoring in approximately the top 5% on the AHSME were invited to take. Qualifying for the AIME, and solving even a modest number of these problems, quickly became a goal of many bright high school students, and was seen as a way to increase the chance of acceptance at some of the select colleges and universities.

The plan of the top high school problem solvers was to do well enough on the AHSME to be invited to take the AIME, solve enough of the AIME problems to be invited to take the USAMO, and then solve enough USAMO problems to be chosen to represent the United States in the International Mathematical Olympiad. Also, of course, to do well in the IMO, that is, to win a Gold Medal! But I digress, back to the history of the basic exams.

The success of the AHSME led in 1985 to the development of a parallel exam for middle school students, called the American Junior High School Mathematics Examination (AJHSME). The AJHSME was designed to help students begin their problem-solving training at an earlier age. By the end of the 20th century nearly 450,000 students were taking these exams, with representatives in each state and province in North America.

In 2000 a major change was made to the AHSME-AJHSME system. Over the years there had been a reduction in the number of problems on the AHSME with a decrease in the number of relatively elementary problems. This reduction was dictated in large part by the demands of the school systems. Schools have had a dramatic increase in the number of both curricular and extra-curricular activities, and time schedules are not as flexible as in earlier years. It was decided in 2000 to reduce the AHSME examination to 25 questions so that the exams could be given in a 75 minute period. However, this put students in the lower high school grades at an additional disadvantage, since it resulted in a further reduction of the more elementary problems. The Committee on the American Mathematics Competitions (CAMC) was particularly concerned that a capable student who had a bad experience with the exam in grades 9 or 10 might be discouraged from competing in later years. The solution was to revise the examination system by adding a competition specifically designed for students in grades 9 and 10. This resulted in three competitions, which were renamed AMC 8, AMC 10 and AMC 12. The digits following AMC indicate the highest grade level at which students are eligible to take the exam. There was no change in the AJHSME except for being renamed AMC 8, nor, except for the reduction in problems, was there a change in AHSME.

The new AMC 10 was to consist of problems that could be worked with the mathematics generally taught to students in grades 9 and lower and there would be overlap, but not more than 50%, between the AMC 10 and AMC 12 examinations. Excluded from the AMC 10 would be problems involving topics generally seen only by students in grades 11 and 12, including trigonometry, logarithms, complex numbers, functions, and some of the more advanced algebra and geometry techniques.

The AMC 10 was designed so that students taking this competition are able to qualify for the AIME, however only approximately the top 1% do so. The reason for making the qualifying score for AMC 10 students much higher than for AMC 12 students was three-fold. First, there are students in grades 9 and 10 who have the mathematical knowledge required for the AMC 12, and these students should take the AMC 12 to demonstrate their superior ability. Having to score at the 1% level on the AMC 10 is likely to be seen to be riskier for these students than having to score at the 5% level on the AMC 12. Second, the committee wanted to be reasonably sure that a student who qualified for the AIME in grades 9 or 10 would also qualify when taking the AMC 12 in grades 11 and 12. Not to do so could discourage a sensitive student. Third, the AIME can be very intimidating to students who have not prepared for this type of examination. Although there has been a concerted effort recently to make the first group of problems on the AIME less difficult, there have been years when the median score on this 15-question test was 0. It is quite possible for a clever 9th or 10th grader without additional training to do well on the AMC 10, but not be able to begin to solve an AIME problem. This, again, could discourage a sensitive student from competing in later years. The primary goal of the AMC is to promote interest in mathematics by providing a positive problem-solving experience for all students taking the exams. The AMC exam is also the first step in determining the top problem-solving high school students in the country, but that goal is decidedly secondary.

My Experience with the American Mathematics Competitions

My first formal involvement with the AMC began in 1996 when I was appointed to the CAMC as a representative from Pi Mu Epsilon, the National Honorary Mathematics Society. Simultaneously, I began writing problems for the AHSME and the AJHSME. In 1997 I joined the committee that constructs the examination for the AJHSME, based on problems submitted from a wide range of people in the United States and Canada. At the same time, I had been helping some local students in middle school prepare for the AJHSME and for the MathCounts competition, and had discovered how excited these students were even when they didn't do as well in the competitions as they had expected. The next year, when they were in 9th grade, I encouraged them to take the AHSME, since that was the only mathematical competition that was available to them. The level of difficulty on this AHSME was so much higher than the exams they were accustomed to taking that most of them were devastated by the experience. I believe that for all but two of these students this was their last competitive problem-solving experience.

At the next meeting of the CAMC I brought my experience to the attention of the members and showed figures that demonstrated that only about 20% of the 9th grade students and less than 40% of the 10th grade students who had taken the AJHSME in grade 8 were taking the AHSME. Clearly, the majority of the 9th and 10th grade teachers had learned the lesson much earlier than I had, and were not encouraging their students to take the AHSME. At this meeting I proposed that we construct an intermediate exam for students in grades 9 and 10, one that would provide them with a better experience than the AHSME and encourage them to continue improving their problem-solving skills. As any experienced committee member knows, the person who proposes the task usually gets assigned the job. In 1999 Harold Reiter, the Chair of the AHSME, and I became joint chairs of the first AMC 10, which was first given on February 15, 2000.

Since 2001 I have been the chair of AMC 10. I work jointly with the AMC 12 chair, Dave Wells, to construct the AMC 10 and AMC 12 exams. In 2002 we began to construct two sets of exams per year, the AMC 10A and AMC 12A, to be given near the beginning of February, and the AMC

10B and AMC 12B, which are given about two weeks later. This gives a student who has a conflict or unexpected difficulty on the day that the A version of the AMC exams are given a second chance to qualify for the AIME. For the exam committee, it means, however, that instead of constructing and refining 30 problems per year, as was done in 1999 for the AHSME, we need approximately 80 problems per year, 25 for each version of the AMC 10 and AMC 12, with an overlap of approximately ten problems.

There are a number of conflicting goals associated with constructing the A and B versions of the exams. We want the versions of the exams to be comparable, but not similar, since similarity would give an advantage to the students taking the later exam. Both versions should also contain the same relative types of problems, but be different, so as not to be predictable. Additionally, the level of difficulty of the two versions should be comparable, which is what we have found most difficult to predict. We are still in the process of grappling with these problems but progress, while slow, seems to be steady.

The Basis and Reason for this Book

When I became a member of the Committee on American Competitions, I found that students in the state of Ohio had generally done well on the exams, but students in my local area were significantly less successful. By that time I had over 25 years experience working with undergraduate students at Youngstown State University and, although we had not done much with problem-solving competitions, our students had done outstanding work in undergraduate research presentations and were very competitive on the international mathematical modeling competition sponsored by COMAP.

Since most of the Youngstown State students went to high school in the local area, it appeared that their performance on the AHSME was not due to lack of ability, but rather lack of training. The mathematics and strategies required for successful problem solving is not necessarily the same as that required in general mathematical applications.

In 1997 we began to offer a series of training sessions at Youngstown State University for high school students interested in taking the AHSME, meeting each Saturday morning from 10:00 until 11:30. The sessions began at the end of October and lasted until February, when the AHSME was given. The sessions were attended by between 30 and 70 high school students. Each Saturday about three YSU faculty, a couple of very good local high school teachers, and between five and ten YSU undergraduate students presented some topics in mathematics, and then helped the high school students with a collection of exercises.

The first year we concentrated each week on a specific past examination, but this was not a successful strategy. We soon found that the variability in the material needed to solve the problems was such that we could not come close to covering a complete exam in the time we had available.

Beginning with the 1998–1999 academic year, the sessions were organized by mathematical topic. We used only past AHSME problems and found a selection in each topic area that would fairly represent the type of mathematical techniques needed to solve a wide range of problems. The AHSME was at that time a 30-question exam and we concentrated on the problem range from 6 to 25. Our logic was that a student who could solve half the problems in this range could likely do all the first five problems and thus easily qualify for the AIME. Also, the last few problems on the AHSME are generally too difficult to be accessible to the large group we were working with in the time we had available.

This book is based on the philosophy of sessions that were run at Youngstown State University. All the problems are from the past AMC (or AHSME, I will not subsequently distinguish between them) exams. However, the problems have been edited to conform with the modern mathematical practice that is used on current AMC examinations. So, the ideas and objectives of the problems are the same as those on past exams, but the phrasing, and occasionally the answer choices, have been modified. In addition, all solutions given to the Examples and the Exercises have been rewritten to conform to the material that is presented in the chapter. Sometimes this solution agrees with the official examination solution, sometimes not. Multiple solutions have occasionally been included to show students that there is generally more than one way to approach the solution to a problem.

The goal of the book is simple: To promote interest in mathematics by providing students with the tools to attack problems that occur on mathematical problem-solving exams, and specifically to level the playing field for those who do not have access to the enrichment programs that are common at the top academic high schools.

The material is written with the assumption that the topic material is not completely new to the student, but that the classroom emphasis might have been different. The book can be used either for self study or to give people who would want to help students prepare for mathematics exams easy access to topic-oriented material and samples of problems based on that material. This should be useful for teachers who want to hold special sessions for students, but it should be equally valuable for parents who have children with mathematical interest and ability. One thing that we found when running our sessions at Youngstown State was that the regularly participating students not only improved their scores on the AMC exams, but did very well on the mathematical portion of the standardized college admissions tests. (No claim is made concerning the verbal portion, I hasten to add.)

I would like to particularly emphasize that this material is not a substitute for the various volumes of *The Contest Problem Book*. Those books contain multiple approaches to solutions to the problems as well as helpful hints for why particular "foils" for the problems were constructed. My goal is different, I want to show students how a few basic mathematical topics can be used to solve a wide range of problems. I am using the AMC problems for this purpose because I find them to be the best and most accessible resource to illustrate and motivate the mathematical topics that students will find useful in many problem-solving situations.

Finally, let me make clear that the student audience for this book is perhaps the top 10–15% of an average high school class. The book is not designed to meet the needs of elite problem solvers, although it might give them an introduction that they might otherwise not be able to find. References are included in the Epilogue for more advanced material that should provide a challenge to those who are interested in pursuing problem solving at the highest level.

Structure of the Book

Each chapter begins with a discussion of the mathematical topics needed for problem solving, followed by three Examples chosen to illustrate the range of topics and difficulty. Then there are ten Exercises, generally arranged in increasing order of difficulty, all of which have been on past AMC examinations. These Exercises contain problems ranging from relatively easy to quite difficult. The Examples have detailed solutions accompanying them. The Exercises also have solutions, of course, but these are placed in a separate Solutions chapter near the end of the book. This permits a student to read the material concerning a topic, look at the Examples and their solutions, and then attempt the Exercises before looking at the solutions that I have provided.

Within the constraints of wide topic coverage, problems on the most recent examinations have been chosen. It is, I feel, important to keep in mind that a problem on an exam as recent as 1990 was written before many of our current competitors were born!

The first four chapters contain rather elementary material and the problems are not difficult. This material is intended to be accessible to students in grade 9. By the fifth chapter on triangle geometry there are some more advanced problems. However, triangle geometry is such an important subject on the examinations, that there are additional problems involving these concepts in the circle geometry and polygon chapters.

Chapters 8 and 9 concern counting techniques and probability problems. There is no advanced material in these chapters, but some of the probability problems can be difficult. More counting and probability problems are considered in later chapters. For example, there are trigonometry and three-dimensional geometry problems that require these notions.

Chapters 10 and 11 concern problems with integer solutions. Since these problems frequently occur on the AMC, Chapter 10 is restricted to those problems that essentially deal with the Fundamental Theorem of Arithmetic, whereas Chapter 11 considers the more advanced topics of modular arithmetic and number bases. All of this material should be accessible to an interested younger student.

Chapter 12 deals with sequences and series, with an emphasis on the arithmetic and geometric sequences that often occur on the AMC. Sequences whose terms are recursive and repeat are also considered, since the AMC sequence problems that are not arithmetic or geometric are frequently of this type. This material and that in Chapter 13 that deals with statistics may not be completely familiar to younger students, but there are only a few concepts to master, and some of these problems appear on the AMC 10.

The final four chapters contain material that is not likely to be included on an AMC 10. Definitions for the basic trigonometric and logarithm functions are given in Chapters 14 and 17, respectively, but these may not be sufficient for a student who has not previously seen this material. Chapter 15 considers problems that have a three-dimensional slant, and Chapter 16 looks at functions in a somewhat abstract setting. The final chapter on complex numbers illustrates that the knowledge of just a few concepts concerning this topic is all that is generally required, even for the AMC 12.

One of the goals of the book is to permit a student to progress through the material in sequence. As problem-solving abilities improve, more difficult notions can be included, and problems presented that require greater ingenuity. When reviewing this material I hope that you will keep in mind that the intended student audience for this book is perhaps the top 10–15% of an average high school class. The more mature (think parental) audience is probably the working engineer or scientist who has not done problems of this type for many years, if ever, but enjoys a logical challenge and/or wants to help students develop problem-solving skills.

Acknowledgments

This is my first experience at writing what might be called an anthology since, although I have constructed my own solutions and study material, all the problems came from past AMC exams and were posed by many different people. I am in their debt, even though for the earlier years I do not know who they are. In more recent years, I have had the pleasure of working primarily with David Wells, and I particularly thank him for all his advice and wisdom. I would also like to thank Steve Dunbar at American Mathematics Competition headquarters for making any information I needed easily accessible, as well as Elgin Johnston, the Chair of the Committee on the American Mathematics Competitions. He and his selected reviewers, Dick Gibbs, Jerry Heuer, and Susan Wildstrom made many very valuable suggestions for improving the book, not the least of which included pointing out where I was in error.

Finally, I would like to express my sincere appreciation to Nicole Cunningham, who did much of the editorial work on this book. She has been working with me for nearly four years while a student at Youngstown State University, and will be greatly missed when she graduates this Spring.

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1
Arithmetic Ratios

1.1 Introduction

Nearly every AMC exam contains problems that require no more mathematical knowledge than the manipulation of fractions and ratios. The most difficult aspect of these problems is translating information given in sentences into an equation form.

1.2 Time and Distance Problems

Problems involving time, distance, and average rates of speed are popular because the amount of knowledge needed to solve the problem is minimal, simply that

$$\text{Distance} = \text{Rate} \cdot \text{Time}.$$

However, the particular phrasing of the problem determines how this formula should be used. Consider the following:

Problem 1 You drive for one hour at 60 mph and then drive one hour at 40 mph. What is your average speed for the trip?

First, we translate mph into units that can be balanced, that is, to miles/hour. This indicates more clearly that mph is a rate. Using the basic distance formula for the first and second rates we have

$$60 \text{ miles} = 1 \text{ hour} \cdot 60\,\frac{\text{miles}}{\text{hour}} \quad \text{and} \quad 40 \text{ miles} = 1 \text{ hour} \cdot 40\,\frac{\text{miles}}{\text{hour}}.$$

So the total distance for the trip is $60 + 40 = 100$ miles, the total time is $1 + 1 = 2$ hours, and the rate, or average speed, for the trip is

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}} = \frac{100 \text{ miles}}{2 \text{ hours}} = 50\,\frac{\text{miles}}{\text{hour}}. \qquad \square$$

You might be thinking it was obvious from the start that the answer was 50, and that there was no reason to go into all this detail. However, consider the following modification of the problem.

PROBLEM 2 You drive the first half of a 100 mile trip at 60 mph and then drive the second half at 40 mph. What is your average speed for the trip?

In this case you are driving the first 50 miles at 60 miles/hour and the second 50 miles at 40 miles/hour. We first find the times, T_1 and T_2, that each of these portions of the trip took to complete.

For the first part of the trip we have

$$50 \text{ miles} = T_1 \text{ hour} \cdot 60\,\frac{\text{miles}}{\text{hour}}, \quad \text{which implies that } T_1 = \frac{5}{6} \text{ hours.}$$

For the second part of the trip we have

$$50 \text{ miles} = T_2 \text{ hour} \cdot 40\frac{\text{miles}}{\text{hour}}, \quad \text{which implies that } T_2 = \frac{5}{4} \text{ hours.}$$

So the total time for the trip is $5/6 + 5/4 = 25/12$ hours, and the average speed for the trip is

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}} = \frac{100 \text{ miles}}{(25/12) \text{ hours}} = 48\,\frac{\text{miles}}{\text{hours}}. \qquad \square$$

The difference in the two problems is that in Problem 2 the trip takes longer because the distances at each of the rates is the same. In Problem 1 it was the times that were the same. As you can imagine, it is the second version of the problem that you are likely to see on the AMC, and the "obvious" answer of 50 miles/hour would certainly be one of the incorrect answer choices.

1.3 Least Common Multiples

The **least common multiple**, denoted **lcm**, of a collection of positive integers is the smallest integer divisible by all the numbers in the collection. Problems involving least common multiples often occur in situations where a number of events occur, each in different times and it is needed to determine when they simultaneously occur. For example, suppose students A, B, and C fail to do their homework every 3, 4, and 5 days, respectively. How frequently will they simultaneously fail to do their homework?

For each student, we have the following schedule of days for unsolved homework:

A: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 52, 54, 57, 60, ...

B: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, ...

C: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, ...

The smallest number that 3, 4, and 5 all divide is 60, so $60 = \text{lcm}\{3, 4, 5\}$ days is how frequently they will all fail to do their homework. When we look at the prime decomposition of integers in Chapter 10 we will see ways to simplify this process when the least common multiple is not so easily seen.

1.4 Ratio Problems

Often a problem that involves ratios between quantities will be posed, but the quantities themselves are not specified. In this case the problem can sometimes be simplified by assigning an arbitrary number to the quantities that preserves the given ratios. We will demonstrate this in the first Example.

Examples for Chapter 1

The first Example is number 8 from the 2000 AMC 10.

Example 1 At Olympic High School, $2/5$ of the freshmen and $4/5$ of the sophomores took the AMC 10. The number of freshmen and sophomore contestants was the same. Which of the following must be true?

(A) There are five times as many sophomores as freshmen.

(B) There are twice as many sophomores as freshmen.

(C) There are as many freshmen as sophomores.

(D) There are twice as many freshmen as sophomores.

(E) There are five times as many freshmen as sophomores.

Answer (D) Suppose that we arbitrarily assume that there are 100 freshmen in the school. The problem then indicates that 40 freshmen and 40 sophomores took the exam, and that the 40 sophomores is 4/5 of the total sophomores in the school. So

$$40 = \frac{4}{5} \text{ sophomores} \quad \text{and} \quad \text{sophomores} = \frac{5}{4}(40) = 50.$$

Since there are 100 freshmen and 50 sophomores, the answer is (D). □

It was convenient to assign a specific value for the number of freshmen, but this is not necessary. We could simply have assumed that there were, say, F freshmen and conclude that there were $F/2$ sophomores. However, you can make the problem more concrete if it helps to see the solution. We chose to use 100 for the number of freshmen, but any value would do. It simplifies the situation, of course, if the number chosen results in an integer for all the calculations in the problem. So 100 is a good choice when the fraction has a denominator of 5, but would be a poor choice if there were denominators of 6 or 7.

The second Example is number 17 from the 2004 AMC 10A and number 15 from the 2004 AMC 12A.

EXAMPLE 2 Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. Each girl runs at a constant speed. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. What is the length of the track in meters?

(A) 250 **(B)** 300 **(C)** 350 **(D)** 400 **(E)** 500

Answer (C) This is a Distance = Rate · Time problem, but it will require some care in its solution. First we set some variables so that we can transform the problem into a form that we can solve algebraically. Let

- L be the length of the track.
- R_s be the rate at which Sally runs.
- R_b be the rate at which Brenda runs.
- T_1 be the time it takes them to first meet.
- T_2 be the time after they first meet until they again meet.

They start at opposite sides of the track and run in opposite directions, so they first meet when their combined distance run is $L/2$. We are told that Brenda has run 100 meters during this time T_1, so

$$100 = T_1 \cdot R_b.$$

When they meet again they have together run the full length, L, of the track since their first meeting. Since their speeds are constant and they ran together $L/2$ in time T_1, we have $T_2 = 2T_1$. Also, Sally has run 150 meters during this time T_2, so

$$L = T_2 \cdot R_s + T_2 \cdot R_b = 150 + 2T_1 \cdot R_b = 150 + 2 \cdot 100 = 350 \text{ meters.} \quad \square$$

The final Example is number 21 from the AMC 2002 10B and number 17 from the 2002 AMC 12B.

EXAMPLE 3 Andy's lawn has twice as much area as Beth's lawn and three times as much area as Carlos' lawn. Carlos' lawn mower cuts half as fast as Beth's mower and one third as fast as Andy's mower. They all start to mow their lawns at the same time. Who will finish first?

(A) Andy **(B)** Beth **(C)** Carlos
(D) Andy and Carlos tie for first. **(E)** They all tie.

Answer (B) As in Example 2, we first define some variables so that we can more easily express the problem mathematically. Let

- A_a, R_a, and T_a be the area, rate, and time for Andy.
- A_b, R_b, and T_b be the area, rate, and time for Beth.
- A_c, R_c, and T_c be the area, rate, and time for Carlos.

Then the problem tells us that

$$A_b = \frac{1}{2}A_a, \quad A_c = \frac{1}{3}A_a, \quad R_b = 2R_c, \quad \text{and} \quad R_a = 3R_c.$$

To solve the problem we express the times for each worker using a common base. We have chosen to use the fraction A_a/R_c, but any combination of A/R is possible. This is similar to a Distance = Rate · Time problem, with

$$T_a = \frac{A_a}{R_a} = \frac{A_a}{3R_c} = \frac{1}{3}\frac{A_a}{R_c},$$
$$T_b = \frac{A_b}{R_b} = \frac{(1/2)A_a}{2R_b} = \frac{1}{4}\frac{A_a}{R_c},$$

and

$$T_c = \frac{A_c}{R_c} = \frac{(1/3)A_a}{R_c} = \frac{1}{3}\frac{A_a}{R_c}.$$

So Beth finishes first, with Andy and Carlos taking the same amount of time.

OR

We can also solve the problem by making it more concrete, as we did in Example 1. Since the problem contains multiples of 2 and 3, suppose that we arbitrarily assume that for Andy we have:

Andy: 6 Acres of lawn and 1 hour to cut, so he cuts at 6 $\dfrac{\text{Acres}}{\text{Hour}}$.

Since Andy's lawn is three times that of Carlos, and Andy's mower cuts three times faster, we have:

Carlos: 2 Acres of lawn and cuts at 2 $\dfrac{\text{Acres}}{\text{Hour}}$, so 1 hour to cut.

Finally, Beth's lawn is half the size of Andy's and her mower cuts at twice Carlos' rate, so for Beth we have:

Beth: 3 Acres of lawn and cuts at 4 $\dfrac{\text{Acres}}{\text{Hour}}$, so $\dfrac{3}{4}$ hours to cut.

Hence Beth finishes first, with Andy and Carlos taking the same amount of time. □

Exercises for Chapter 1

Exercise 1 Each day Jenny ate 20% of the jellybeans that were in her jar at the beginning of that day. At the end of the second day, 32 remained. How many jellybeans were in the jar originally?

(A) 40 **(B)** 50 **(C)** 55 **(D)** 60 **(E)** 75

Exercise 2 Wanda, Darren, Beatrice, and Chi are tutors in the school math lab. Their schedules are as follows: Darren works every third school day, Wanda works every fourth school day, Beatrice works every sixth school day, and Chi works every seventh school day. Today they are all working in the math lab. In how many school days from today will they next be together tutoring in the lab?

(A) 42 **(B)** 84 **(C)** 126 **(D)** 178 **(E)** 252

Exercise 3 Suppose hops, skips, and jumps are specific units of length. We know that b hops equal c skips, d jumps equal e hops, and f jumps equal g meters. How many skips are equal to one meter?

(A) $\frac{bdg}{cef}$ **(B)** $\frac{cdf}{beg}$ **(C)** $\frac{cdg}{bef}$ **(D)** $\frac{cef}{bdg}$ **(E)** $\frac{ceg}{bdf}$

Exercise 4 A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. Suppose that the diameter of the jars is increased by 25% without altering the volume. By what percent must the height be decreased?

(A) 10 **(B)** 25 **(C)** 36 **(D)** 50 **(E)** 60

Exercise 5 Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

(A) 45 **(B)** 48 **(C)** 50 **(D)** 55 **(E)** 58

Exercise 6 An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius?

(A) 2 : 1 **(B)** 3 : 1 **(C)** 4 : 1 **(D)** 16 : 3 **(E)** 6 : 1

Exercise 7 Cassandra sets her watch to the correct time at noon. At the actual time of 1:00 PM, she notices that her watch reads 12:57 and 36 seconds. Assume that her watch loses time at a constant rate. What will be the actual time when her watch first reads 10:00 PM?

(A) 10:22 PM and 24 seconds **(B)** 10:24 PM **(C)** 10:25 PM
(D) 10:27 PM **(E)** 10:30 PM

Exercise 8 Jack and Jill run 10 kilometers. They start at the same place, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10-minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill. How far from the top of the hill are they when they pass going in opposite directions?

(A) $\frac{5}{4}$ km **(B)** $\frac{35}{27}$ km **(C)** $\frac{27}{20}$ km **(D)** $\frac{7}{3}$ km **(E)** $\frac{28}{9}$ km

Exercise 9 Sarah places four ounces of coffee into an eight-ounce cup and four ounces of cream into a second cup of the same size. She then pours half the coffee from the first cup to the second and, after stirring thoroughly, pours half the liquid in the second cup back to the first. What fraction of the liquid in the first cup is now cream?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

Exercise 10 In an h-meter race, Sam is exactly d meters ahead of Walt when Sam finishes the race. The next time they race, Sam sportingly starts d meters behind Walt, who is at the original starting line. Both runners run at the same constant speed as they did in the first race. How many meters ahead is the winner of the second race when the winner crosses the finish line?

(A) $\frac{d}{h}$ **(B)** 0 **(C)** $\frac{d^2}{h}$ **(D)** $\frac{h^2}{d}$ **(E)** $\frac{d^2}{h-d}$

2

Polynomials and their Zeros

2.1 Introduction

The most basic set of numbers is the integers. In a similar manner, the most basic set of functions is the polynomials. Because polynomials have so many applications and are relatively easy to manipulate, they appear in many of the problems on the AMC.

DEFINITION 1   A **polynomial of degree *n*** has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some collection of constants $a_0, a_1, \ldots, a_n$, with *leading coefficient* $a_n \neq 0$. We will assume these constants are all real numbers.

One of the most commonly needed features of a polynomial is the location of those values of x such that $P(x) = 0$.

DEFINITION 2   A **zero** r of a polynomial $P(x)$ is a number with $P(r) = 0$. A zero of a polynomial is also called a **root** of the equation $P(x) = 0$.

A number r is a zero of a polynomial $P(x)$ if and only if $P(x)$ has a factor of the form $(x - r)$. The number of such factors gives the multiplicity of the zero.

DEFINITION 3   A zero r of the polynomial $P(x)$ is said to have **multiplicity** m if there is a polynomial $Q(x)$ with

$$P(x) = (x - r)^m Q(x) \quad \text{and} \quad Q(r) \neq 0.$$

A zero of multiplicity $m = 1$ is called a **simple** zero.

For example, the number 2 is a simple zero of

$$P(x) = x^2 - 4 = (x-2)(x+2),$$

but is a zero of multiplicity 3 of the polynomial

$$P(x) = x^4 - 6x^3 + 12x^2 - 8x = (x-2)^3 \cdot x.$$

In section 2.4 we will consider a number of results concerning general polynomials and their zeros, but we will first look at polynomials with degree 1 and 2, since these are seen on nearly every AMC 10 and AMC 12.

2.2 Lines

The most elementary polynomials are those described by **linear equations**, those whose graphs are straight lines. The equation of a non-vertical line is completely determined by a point (x_1, y_1) on the line and its **slope** m, as

$$y - y_1 = m(x - x_1) \quad \text{or as} \quad y = mx + b, \text{ where } b = y_1 - mx_1.$$

The form $y = mx + b$ is called the *slope-intercept* form of the line because b tells where the line intersects the y-axis. Any two distinct points (x_1, y_1) and (x_2, y_2) can be used to find the slope of the line as

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since lines are likely to be familiar to any reader of this material, we will not review linear equations except to remark that:

- Two non-vertical lines are parallel if and only if their slopes are the same.
- Two non-vertical lines are perpendicular if and only if the product of their slopes is -1.

2.3 Quadratic Polynomials

There are some results about **quadratic polynomials**, those with the form $P(x) = ax^2 + bx + c$, where $a \neq 0$, that can frequently be used to solve AMC problems. The first of these describes how the zeros of these equations relate to the values of the coefficients.

Result 1 Zero-Coefficient Relationship for Quadratic Polynomials: Suppose that r_1 and r_2 are zeros of a quadratic polynomial of the form $P(x) = x^2 + bx + c$. Then

$$x^2 + bx + c = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2,$$

so $c = r_1 r_2$ and $b = -(r_1 + r_2)$.

Result 2 Completing the Square: Completing the square of a quadratic polynomial permits us to write the quadratic in a form that eliminates the linear term. This is a valuable technique for many applications in mathematics, so problem posers often construct problems that incorporate this method.

Suppose that $P(x) = ax^2 + bx + c$. Then we can write

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c.$$

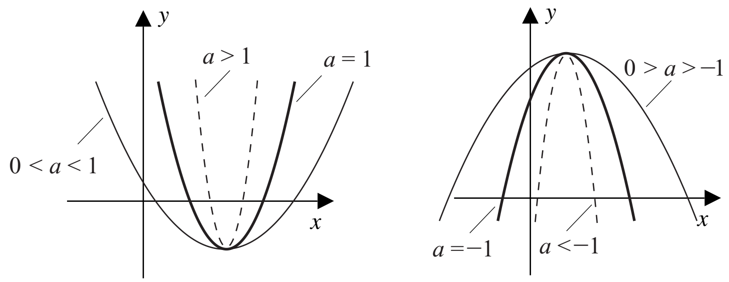
If we add the term $(b/(2a))^2$ inside the parentheses, we have a perfect square. We must, of course, compensate by subtracting $a(b/(2a))^2$ outside the parentheses. This gives

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}. \end{aligned}$$

The graph of the quadratic equation is a parabola whose vertex is at the value of x that makes the squared term zero, that is, at

$$x = -\frac{b}{2a} \quad \text{with} \quad P\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}.$$

- When $a > 0$, the parabola opens upward and the value of P at the vertex gives the minimal value of $P(x)$.
- When $a < 0$, the parabola opens downward and the value of P at the vertex gives the maximum value of $P(x)$.
- The larger the magnitude of a the narrower the graph of the parabola.



Result 3 The Quadratic Formula: For the general quadratic polynomial of the form

$$P(x) = ax^2 + bx + c,$$

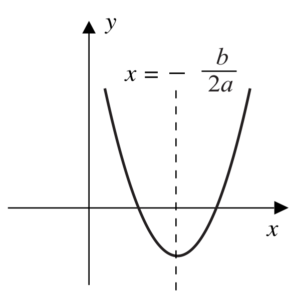
completing the square provides us with the **Quadratic Formula**, which states that if $P(x) = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The **discriminant** $b^2 - 4ac$ tells us the character of the zeros.

- If $b^2 - 4ac > 0$, there are two distinct real zeros.
- If $b^2 - 4ac = 0$, there is one (double) real zero.
- If $b^2 - 4ac < 0$, there are two complex zeros, which are complex conjugates of one another.

Finally, the graph of the quadratic polynomial $P(x) = ax^2 + bx + c$ is symmetric about the line $x = -b/(2a)$, as shown in the figure.



2.4 General Polynomials

For general polynomials we have a number of useful results:

Result 1 The Linear Factor Theorem: If $P(x)$ has degree n and is divided by the linear factor $(x - c)$, then

$$P(x) = (x - c)Q(x) + P(c)$$

for some polynomial $Q(x)$ of degree $n - 1$. The linear term $(x - c)$ is a factor of the polynomial $P(x)$ if and only if $P(c) = 0$.

Result 2 A General Factor Theorem: If $P(x)$ has degree n and is divided by a polynomial $D(x)$ of degree $m < n$ with

$$P(x) = D(x) \cdot Q(x) + R(x),$$

then the quotient $Q(x)$ is a polynomial of degree $n - m$ and the remainder $R(x)$ is a polynomial of degree less than m.

Result 3 The Rational Zero Test: Suppose that $a_0, a_1, \ldots, a_n$ are integers, $a_n \neq 0$, and p/q is a rational zero of

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then p divides a_0 and q divides a_n.

Result 4 Zeros-Coefficient Relationship for General Polynomials: General polynomials of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

have a zero-coefficient relationship similar to that of quadratics. Specifically:

If the n zeros of $P(x)$ are $r_1, r_2, \ldots, r_n$, then

$$x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = (x - r_1)(x - r_2)\cdots(x - r_n).$$

By equating the powers on both sides of the equation we have

$$\begin{aligned} -a_{n-1} &= r_1 + r_2 + \cdots + r_n = \text{(the sum of all the zeros)}, \\ a_{n-2} &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n \\ &= \text{(the sum of the zeros taken two at a time)}, \end{aligned}$$

and

$$(-1)^n a_0 = r_1 r_2 \cdots r_n = \text{(the product of all the zeros)}.$$

These are the most frequently used formulas, but there is also a general result that holds for each $i = 0, 1, \ldots, n$:

$$\begin{aligned}(-1)^i a_{n-i} &= r_1 r_2 r_3 \cdots r_i + r_1 r_2 r_4 \cdots r_{i+1} + \cdots + r_{n-i+1} \cdots r_{n-2} r_{n-1} r_n \\ &= \text{(the sum of the zeros taken } i \text{ at a time)}.\end{aligned}$$

Result 5 Some other useful facts about polynomials $P(x)$ are that

- $P(0)$ is the constant term of $P(x)$:

$$P(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \cdots + a_n \cdot 0^n = a_0.$$

- $P(1)$ is the sum of the coefficients of $P(x)$:

$$P(1) = a_0 + a_1 + a_2 + \cdots + a_n.$$

- $P(-1)$ is the alternating sum of the coefficients of $P(x)$:

$$P(-1) = a_0 - a_1 + a_2 + \cdots + (-1)^n a_n.$$

Result 6 Descarte's Rule of Signs: Let $P(x)$ be a polynomial with real coefficients. Then

- the number of positive zeros of $P(x)$ is either equal to the number of variations in sign of $P(x)$ or less than this by an even number;
- the number of negative real zeros of $P(x)$ is either equal to the number of variations in sign of $P(-x)$ or less than this by an even number.

For example, the polynomial $P(x) = x^5 + 4x^4 - x^3 - 2x^2 + 3x - 1$ has either 3 or 1 positive real zeros, and since $P(-x) = -x^5 + 4x^4 + x^3 - 2x^2 - 3x - 1$ it has either 2 or 0 negative real zeros.

The final result concerning zeros of polynomials was first proved by Carl Fredrich Gauss, one of the greatest of all mathematicians. In this chapter we will only be concerned with the zeros that are real numbers. In Chapter 18 we will reconsider this result in the case that the zeros are complex numbers.

Result 7 The Fundamental Theorem of Algebra: Suppose that

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial of degree $n > 0$ with real or complex coefficients. Then $P(x)$ has at least one real or complex zero.

In fact,

- $P(x)$ has precisely n zeros, when a zero of multiplicity m is counted m times.
- If the coefficients of

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  are all real numbers and z is a complex zero of $P(x)$ with multiplicity m, then its complex conjugate $\overline{z}$ is also a zero of $P(x)$ with multiplicity m.

Examples for Chapter 2

The first Example is number 5 from the 1988 AHSME.

EXAMPLE 1 Suppose that b and c are constants and

$$(x + 2)(x + b) = x^2 + cx + 6.$$

What is c?

(A) -5 **(B)** -3 **(C)** -1 **(D)** 3 **(E)** 5

Answer (E) The factored form of the polynomial implies that its zeros are -2 and $-b$. By Result 1 of section 2.3, the product of the zeros, $2b$, is the constant term of the polynomial, which is 6. Hence $b = 3$. In addition, the linear term, c, is the negative of the sum of the zeros. Thus

$$c = -(-2 - b) = 2 + b = 2 + 3 = 5.$$

The second Example is number 13 from the 1986 AHSME.

EXAMPLE 2 A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$, and $(2, 0)$ is on the graph of the parabola. What is abc?

(A) -12 **(B)** -6 **(C)** 0 **(D)** 6 **(E)** 12

Answer (E) We will look at two solutions to this problem.

For the first approach, we use the fact that the vertex is at $(4, 2)$, and complete the square of the quadratic to give

$$y = a(x-4)^2 + 2.$$

Since $y = 0$ when $x = 2$, we have

$$0 = a(2-4)^2 + 2 = 4a + 2 \quad \text{so} \quad a = -\frac{1}{2}.$$

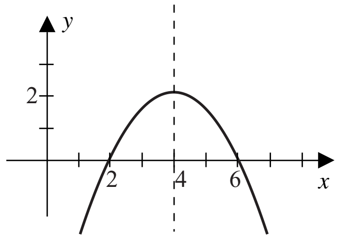
Hence

$$y = -\frac{1}{2}(x-4)^2 + 2 = -\frac{1}{2}x^2 + 4x - 6,$$

so $a = -1/2$, $b = 4$, $c = -6$, and $abc = 12$.

OR

The second approach uses the symmetry of quadratics about the vertical line through the vertex. Since $(2, 0)$ is on the graph and $x = 4$ passes through the vertex, the point $(6, 0)$ is also on the graph.



The zeros are 2 and 6 so the quadratic has the form

$$y = a(x-2)(x-6).$$

Since $y = 2$ when $x = 4$ we have

$$2 = a(4-2)(4-6) \quad \text{and} \quad a = -\frac{1}{2}.$$

Hence

$$y = -\frac{1}{2}(x-2)(x-6) = -\frac{1}{2}x^2 + 4x - 6,$$

and $abc = -\frac{1}{2}(4)(-6) = 12$. $\square$

The final Example is number 15 from the 1988 AHSME.

EXAMPLE 3 Suppose that a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$. What is b?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Answer (A) Dividing $ax^3 + bx^2 + 1$ by $x^2 - x - 1$ gives

$$ax^3 + bx^2 + 1 = (ax + (a + b))(x^2 - x - 1) + (2a + b)x + (a + b + 1),$$

so the remainder is

$$(2a + b)x + (a + b + 1).$$

Since $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, this remainder is zero, so the linear and constant terms of the remainder must both be zero, that is,

$$2a + b = 0 \quad \text{and} \quad a + b + 1 = 0.$$

So $b = -2a$ and $0 = a - 2a + 1 = 1 - a$, which implies that

$$a = 1 \quad \text{and} \quad b = -2(1) = -2. \qquad \square$$

Exercises for Chapter 2

Exercise 1 Let $P(x)$ be a linear polynomial with $P(6) - P(2) = 12$. What is $P(12) - P(2)$?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 36

Exercise 2 Let $x_1 \neq x_2$ be such that $3x_1^2 - hx_1 = b$ and $3x_2^2 - hx_2 = b$. What is $x_1 + x_2$?

(A) $-\frac{h}{3}$ **(B)** $\frac{h}{3}$ **(C)** $\frac{b}{3}$ **(D)** $2b$ **(E)** $-\frac{b}{3}$

Exercise 3 What is the remainder when $x^{51} + 51$ is divided by $x + 1$?

(A) 0 **(B)** 1 **(C)** 49 **(D)** 50 **(E)** 51

Exercise 4 What is the maximum number of points of intersection of the graphs of two different fourth-degree polynomial functions $y = P(x)$ and $y = Q(x)$, each with leading coefficient 1?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

Exercise 5 The parabola with equation $y(x) = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $y_r(x) = dx^2 + ex + f$. Which of the following equals $a+b+c+d+e+f$?

(A) $2b$ **(B)** $2c$ **(C)** $2a + 2b$ **(D)** $2h$ **(E)** $2k$

Exercise 6 Let $P(x)$ be a polynomial which when divided by $x - 19$ has the remainder 99, and when divided by $x - 99$ has the remainder 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

(A) $-x + 80$ **(B)** $x + 80$ **(C)** $-x + 118$ **(D)** $x + 118$ **(E)** 0

Exercise 7 The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the average of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The y-intercept of the graph of $y = P(x)$ is 2. What is b?

(A) -11 **(B)** -10 **(C)** -9 **(D)** 1 **(E)** 5

Exercise 8 Suppose that $P(x/3) = x^2 + x + 1$. What is the sum of all values of x for which $P(3x) = 7$?

(A) $-\dfrac{1}{3}$ **(B)** $-\dfrac{1}{9}$ **(C)** 0 **(D)** $\dfrac{5}{9}$ **(E)** $\dfrac{5}{3}$

Exercise 9 For how many values of the coefficient a do the equations

$$0 = x^2 + ax + 1 \quad \text{and} \quad 0 = x^2 - x - a$$

have a common real solution?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

Exercise 10 The solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$. Which of the following must be true?

(A) $p = m^3 + 3mn$ **(B)** $p = m^3 - 3mn$ **(C)** $p = 3mn - m^3$

(D) $p + q = m^3$ **(E)** $\left(\dfrac{m}{n}\right)^3 = \dfrac{p}{q}$

3
Exponentials and Radicals

3.1 Introduction

There is little to learn in order to work the exponential problems on the AMC. It is mainly a matter of knowing and applying the relevant definitions and arithmetic properties. However, this topic always seems to cause some students difficulty. As a consequence, most AMC exams include problems that use this knowledge.

3.2 Exponents and Bases

The first step is to develop working definitions of exponentials.

- If n is a positive integer and a is a real number, then the symbol

  $$a^n = \overbrace{a \cdot a \cdots a}^{n\text{-times}}$$

  represents the product of n factors of a. In the expression a^n, the number a is called the **base** and n is called the **exponent** or power.

- If n is a positive integer, then the **principal *n*th root** of the real number a, written $a^{1/n} = \sqrt[n]{a}$, is the largest real number b such that $a = b^n$, provided such a number exists. Such a number exists for all positive integers n when $a \geq 0$. But when $a < 0$, it exists only for odd positive integers n.

  The symbol $\sqrt[n]{a}$ is called a **radical** with the *index* n and *radicand* a. If the index $n = 2$ we write $\sqrt[2]{a}$ simply as $\sqrt{a}$, the square root of a. If the index $n = 3$, we call $\sqrt[3]{a}$ the cube root of a.

- If p and q are positive integers with no common factors and $n = p/q$, then we define

$$a^n = a^{p/q} = (\sqrt[q]{a})^p, \quad \text{provided that } \sqrt[q]{a} \text{ is defined.}$$

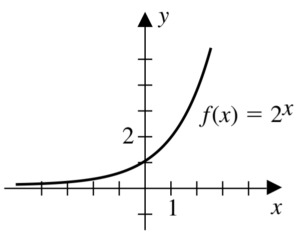
- When n is a positive rational number, and a^n is defined and nonzero, we define

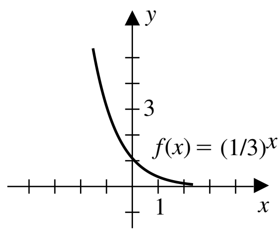
$$a^{-n} = \frac{1}{a^n}.$$

3.3 Exponential Functions

To define an **exponential function** we need to extend the exponent concept to permit the exponent to take on any real number value. A complete discussion of this extension requires calculus to do completely, so we will simply assume that it can be satisfactorily completed in a manner that is consistent with the definitions we have already given. Because there is difficulty with taking roots of negative numbers, the base a of an exponential function is always assumed to be positive, and to simplify matters we also specify that $a \neq 1$.

This produces exponential functions that have graphs similar to those shown below. On the left we have the situation when $a > 1$, the graph of $f(x) = 2^x$, and on the right when $a < 1$, the graph of $f(x) = 3^{-x} = (1/3)^x$.





3.4 Basic Rules of Exponents

The rules of exponents are often used in the solution of AMC problems, since some problems that are initially rather intimidating can be reduced to simple equations using these properties. These properties have parallel relationships with logarithms, a topic that is considered in Chapter 17.

Result 1 Suppose that $a > 0$ and $b > 0$, then for every pair of real numbers x and y:

- $a^0 = 1$
- $a^{-x} = \dfrac{1}{a^x}$
- $a^x \cdot a^y = a^{x+y}$
- $(a^x)^y = a^{xy}$
- $\dfrac{a^x}{a^y} = a^{x-y}$
- $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

The following properties of radicals follow directly from the exponential properties.

Result 2 If $a > 0$ and $b > 0$, then for each positive integer n we have

- $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
- $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
- $\sqrt[n]{a^m} = a^{m/n}$

There are certain special situations that require care when working with exponentials and radicals.

- When $a = 0$ and $x > 0$ we define $a^x = 0$. However, when a and x are both 0, we do not define a^x. The reasons for this will be discussed in calculus, but you can see that there must be a difficulty by noting that for every positive real number x we have both $0^x = 0$ but $x^0 = 1$.
- When $a < 0$ we can sometimes define a^x as a real number, but not always. If x is an integer, a^x is always a real number. For rational values of x, whether a^x is a real number depends on the value of the denominator of x. Specifically, if $x = p/q$, where p and q are nonzero integers with no common factors, then

$$a^{p/q} = \begin{cases} \sqrt[q]{a^p} = (\sqrt[q]{a})^p, & \text{if } q \text{ is odd;} \\ \text{undefined,} & \text{if } q \text{ is even.} \end{cases}$$

So, for example, $(-8)^{2/3} = \left(\sqrt[3]{(-8)}\right)^2 = (-2)^2 = 4$, but $(-8)^{3/2}$ is not defined as a real number because it would require taking the square root of a negative number. In Chapter 18 we will consider the set of complex numbers, and then these restrictions can be relaxed. Until that time we will assume that we are considering only real numbers.

One final note is needed regarding terminology. It is a notational convention that

$$a^{b^c} = a^{(b^c)}.$$

Hence

$$2^{3^4} = 2^{(3^4)} = 2^{81}, \quad \text{whereas} \quad \left(2^3\right)^4 = 2^{3\cdot 4} = 2^{12}.$$

Also, when square roots occur in quotients it is historical practice to rewrite the quotient so the denominator contains no square roots. This process is called *rationalizing the denominator.*

For example, we have

$$\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{3+\sqrt{6}}{3-2} = 3+\sqrt{6},$$

and

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a+2\sqrt{ab}+b}{a-b}.$$

There are algebraic rules that can be used for rationalizing the denominator in the case of other roots as well, but these are seldom applied. We will give just one illustration of the process. Since

$$(a+b)(a^2-ab+b^2) = a^3+b^3,$$

we could perform the following "simplification":

$$\begin{aligned}
\frac{\sqrt[3]{2}}{\sqrt[3]{2}+\sqrt[3]{3}} &= \frac{\sqrt[3]{2}}{\sqrt[3]{2}+\sqrt[3]{3}} \cdot \frac{(\sqrt[3]{2})^2 - \sqrt[3]{2}\sqrt[3]{3} + (\sqrt[3]{3})^2}{(\sqrt[3]{2})^2 - \sqrt[3]{2}\sqrt[3]{3} + (\sqrt[3]{3})^2} \\
&= \frac{(\sqrt[3]{2})^3 - (\sqrt[3]{2})^2\sqrt[3]{3} + \sqrt[3]{2}(\sqrt[3]{3})^2}{(\sqrt[3]{2})^3 + (\sqrt[3]{3})^3} \\
&= \frac{2-\sqrt[3]{12}+\sqrt[3]{18}}{2+3} = \frac{2-\sqrt[3]{12}+\sqrt[3]{18}}{5},
\end{aligned}$$

but it is seldom worth it.

Problems involving exponentials are generally constructed so that it is difficult to determine the solution simply by substituting numbers for the variables. And it is not our intention to give techniques that defeat the objectives of the exam, which is to promote the use of mathematics in problem solving. That said, it is not a bad idea to check your answer by verifying that your result is true for some easily substituted numbers.

3.5 The Binomial Theorem

The final section in this chapter concerns the expansion of sums of terms to a positive integer power.

Result 1 The Binomial Theorem: For each pair of real numbers a and b and each positive integer n we have

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}a^0 b^n,$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

is called a *binomial coefficient*. It describes the number of distinct ways to choose k objects from a collection of n objects. Expanding and simplifying gives

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots \\ &\quad + \frac{n!}{k!\,(n-k)!}a^{n-k}b^k + \cdots + b^n.\end{aligned}$$

For example, when $n = 3$ the Binomial Theorem implies that

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 \\ &= \frac{3!}{3!\cdot 0!}\,a^3 + \frac{3!}{2!\cdot 1!}\,a^2b + \frac{3!}{1!\cdot 2!}\,ab^2 + \frac{3!}{0!\cdot 3!}\,b^3 \\ &= \frac{6}{6\cdot 1}\,a^3 + \frac{6}{2\cdot 1}\,a^2b + \frac{6}{1\cdot 2}\,ab^2 + \frac{6}{1\cdot 6}\,b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

This expansion can be used in the solution to Exercise 10 in Chapter 2.

It is interesting to see why the Binomial Theorem is true, since it forms a basis for the binomial notation. Suppose that we write n products of $(a+b)$ in factored form as

$$(a+b)^n = \overbrace{(a+b)(a+b)(a+b)\cdots(a+b)}^{n\text{-times}}.$$

The multiplication will eventually result in a summation with a term for each possible way that we can choose an a or b from each of the terms in the factored form. For example, if we choose a from each factor (so we choose 0 of the b's) then we have the term a^n. If we choose 1 of the b's from a single factor and a's from the rest, then we will have n terms in the summation of the form $a^{n-1}b$. In a similar manner, the number of terms of the form $a^{n-k}b^k$ is the number of ways to choose precisely k of the b's from n of the factors. The binomial notation gives the quantities of these terms.

The binomial coefficients satisfy a number of interesting relationships. From the definition we can see that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}.$$

Also, when $k < n$ we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

This result follows from the fact that

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-k-1)!} \\
&= \frac{n!\,((k+1)+(n-k))}{(k+1)!\,(n-k)!} \\
&= \frac{n!\,(n+1)}{(k+1)!\,((n+1)-(k+1))!} \\
&= \frac{(n+1)!}{(k+1)!\,((n+1)-(k+1))!} = \binom{n+1}{k+1}.
\end{aligned}$$

This relationship gives us the familiar **Pascal's triangle**. The kth line in the triangle give us the coefficients of a and b in the expansion of $(a+b)^k$:

$$\begin{array}{lccccccccc}
(a+b)^0: & & & & & 1 & & & & \\
(a+b)^1: & & & & 1 & & 1 & & & \\
(a+b)^2: & & & 1 & & 2 & & 1 & & \\
(a+b)^3: & & 1 & & 3 & & 3 & & 1 & \\
(a+b)^4: & 1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & & \vdots & & & &
\end{array}$$

A special case that is sometimes useful, but not likely worth memorizing, is that

$$\binom{2n}{n} = 2^n \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}.$$

Examples for Chapter 3

The first Example is number 7 from the 1998 AHSME. Its solution requires care in the manipulations.

EXAMPLE 1 Suppose that $N > 1$. What is the value of $\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}}$?

(A) $N^{1/27}$ **(B)** $N^{1/9}$ **(C)** $N^{1/3}$ **(D)** $N^{13/27}$ **(E)** N

Answer (D) We first rewrite the expression using exponential notation. Then

$$\begin{aligned}\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} &= \left(N\left(N \cdot N^{1/3}\right)^{1/3}\right)^{1/3} \\ &= N^{1/3}N^{(1/3)(1/3)}N^{(1/3)(1/3)(1/3)} \\ &= N^{1/3} \cdot N^{1/9} \cdot N^{1/27} = N^{1/3+1/9+1/27}.\end{aligned}$$

Since

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}=\frac{13}{27} \quad \text{we have} \quad \sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} = N^{13/27}. \qquad \square$$

The second Example is number 3 from the 1974 AHSME.

EXAMPLE 2 What is the coefficient of x^7 in the polynomial expansion of

$$(1+2x-x^2)^4?$$

(A) -12 **(B)** -8 **(C)** 6 **(D)** 8 **(E)** 12

Answer (B) First write

$$(1+2x-x^2)^4 = (1+2x-x^2)\cdot(1+2x-x^2)\cdot(1+2x-x^2)\cdot(1+2x-x^2).$$

The highest power of x is x^8, which occurs by choosing $-x^2$ from each of the four factors. To obtain the term x^7, we need to choose $-x^2$ from three of the factors and $2x$ for the fourth. There are four distinct ways to make this choice, one for each factor, so the x^7 term in the expansion is

$$4 \cdot (2x) \cdot (-x^2)^3 = -8x^7.$$

OR

Although this is not a direct application of the Binomial Theorem, the theorem gives

$$\begin{aligned}(1+2x-x^2)^4 &= \Big((1+2x)+(-x^2)\Big)^4 \\ &= (1+2x)^4+4(1+2x)^3(-x^2)+6(1+2x)^2(-x^2)^2 \\ &\quad +4(1+2x)(-x^2)^3+(-x^2)^4.\end{aligned}$$

Only $4(1+2x)(-x^2)^3 = 4x^6 - 8x^7$ has a term involving x^7, so the answer is -8.

OR

By the Zero-Coefficient Relationship for General Polynomials, the coefficient of x^7 in this polynomial of degree 8 is the negative of the sum of the zeros of $P(x) = (1+2x-x^2)^4 = (x^2-2x-1)^4$. Applying this same result to the polynomial x^2-2x-1 implies that the sum of the zeros of this polynomial is the negative of the linear term, so the sum of the zeros of x^2-2x-1 is 2. Hence the sum of the zeros of $P(x)$ is

$$-4(2) = -8 \quad \text{and the coefficient of } x^7 \text{ is } -8. \qquad \square$$

The final Example is number 15 on the 1985 AHSME. It has been chosen to show how the exponential rules can be used in a relatively high-numbered problem. It also is interesting because the answer that appears least likely, at least to me, is in fact correct!

EXAMPLE 3 Suppose that a and b are positive numbers satisfying $a^b = b^a$ and that $b = 9a$. What is a?

(A) 9 **(B)** $\dfrac{1}{9}$ **(C)** $\sqrt[9]{9}$ **(D)** $\sqrt[3]{9}$ **(E)** $\sqrt[4]{3}$

Answer (E) Since we need to determine a, we first rewrite the expression $a^b = b^a$ as $a = b^{a/b}$. Now substitute $b = 9a$ to give

$$a = b^{a/b} = (9a)^{a/(9a)} = (9a)^{1/9}, \quad \text{so} \quad a^9 = 9a.$$

Hence

$$a^8 = 9 \quad \text{and} \quad a = 9^{1/8} = \left(3^2\right)^{1/8} = 3^{2/8} = 3^{1/4} = \sqrt[4]{3}. \qquad \square$$

Exercises for Chapter 3

Exercise 1 The expression $4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9$ simplifies to which of the following?

(A) 13^{13} **(B)** 13^{36} **(C)** 36^{13} **(D)** 36^{36} **(E)** 1296^{26}

Exercise 2 The expression $\dfrac{15^{30}}{45^{15}}$ simplifies to which of the following?

(A) $\left(\dfrac{1}{3}\right)^{15}$ **(B)** $\left(\dfrac{1}{3}\right)^{2}$ **(C)** 1 **(D)** 3^{15} **(E)** 5^{15}

Exercise 3 What is the value of k if $2^{2007}-2^{2006}-2^{2005}+2^{2004} = k\cdot 2^{2004}$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Exercise 4 Suppose that $x > y > 0$. Which of the following is the same as $\dfrac{x^y y^x}{y^y x^x}$?

(A) $(x-y)^{y/x}$ **(B)** $\left(\dfrac{x}{y}\right)^{x-y}$ **(C)** 1 **(D)** $\left(\dfrac{x}{y}\right)^{y-x}$

(E) $(x-y)^{x/y}$

Exercise 5 What is the value of $\sqrt{\dfrac{8^{10}+4^{10}}{8^4+4^{11}}}$?

(A) $\sqrt{2}$ **(B)** 16 **(C)** 32 **(D)** $12^{2/3}$ **(E)** 512.5

Exercise 6 Let $f(x) = x^{(x+1)}(x+2)^{(x+3)}$. What is the value of

$$f(0) + f(-1) + f(-2) + f(-3)?$$

(A) $-\dfrac{8}{9}$ **(B)** 0 **(C)** $\dfrac{8}{9}$ **(D)** 1 **(E)** $\dfrac{10}{9}$

Exercise 7 Which of the following values of x satisfies the expression

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}?$$

(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 9

Exercise 8 Suppose that $a > 0$ and $b > 0$. Define r to be the number that results when both the base and the exponent of a^b are tripled. Suppose now that we write $r = a^b \cdot x^b$. Which of the following expressions represents x?

(A) 3 **(B)** $3a^2$ **(C)** $27a^2$ **(D)** $2a^{3b}$ **(E)** $3a^{2b}$

Exercise 9 What is the sum of all the real numbers x that satisfy

$$(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3?$$

(A) $\frac{3}{2}$ **(B)** 2 **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** $\frac{7}{2}$

Exercise 10 Suppose that $60^a = 3$ and $60^b = 5$. What is the value of $12^{(1-a-b)/(2-2b)}$?

(A) $\sqrt{3}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** $\sqrt{12}$

4
Defined Functions and Operations

4.1 Introduction

Problems on the AMC that use the material in this chapter are primarily manipulative. Often their solution requires only a careful application of a definition that may not be familiar, but is not difficult to comprehend.

4.2 Binary Operations

A **binary operation** on a set of numbers is simply a way to take two of the numbers and produce a third. Quite often a binary operation in an exam problem will be expressed using some unusual symbol, such as §. The result of the operation after it is applied to the numbers a and b would likely be written as $\S(a, b)$, or as $a \S b$.

For example, the normal rules of addition, subtraction, multiplication, and division are binary operations. These might be expressed in binary operation form, respectively, as

$$\clubsuit(a, b) = a + b,\ \diamondsuit(a, b) = a - b,\ \heartsuit(a, b) = a \cdot b,\ \text{and}\ \spadesuit(a, b) = \frac{a}{b}.$$

All of these binary operations are defined for each pair of real numbers and each produces another real number, except in the case of involving division, $\spadesuit(a, b) = a/b$. Then we cannot have $b = 0$, since division by 0 is undefined.

There are a number of definitions and properties that are associated with certain binary operations.

If § is a binary operation that is defined for each pair of numbers in the set S, then we say that:

DEFINITION 1 The binary operation § is **closed** on S if whenever a and b belong to S, then $\S(a, b)$ also belongs to S.

For example, addition is closed on S when S is the set of integers, positive integers, rational numbers, or real numbers. Multiplication is also closed on these sets.

Subtraction is closed on the sets of integers, rational numbers, and real numbers. But subtraction is not closed on the set of positive integers since the subtraction of two positive integers does not always result in a positive integer. Similarly, division is not closed on the set of integers, but is closed on the set of nonzero rational and nonzero real numbers.

DEFINITION 2 The binary operation § is **commutative** on S if for each pair a, b in S we have

$$\S(a, b) = \S(b, a).$$

For example, addition and multiplication are both commutative since for each pair of real numbers we have $a+b = b+a$ and we also have $a \cdot b = b \cdot a$.

On the other hand, subtraction and division are not commutative since in general $a - b \neq b - a$ and $a/b \neq b/a$.

DEFINITION 3 The binary operation § is **associative** on S if for each triple a, b, and c in S we have

$$\S(\S(a, b), c) = \S(a, \S(b, c)).$$

Again, addition and multiplication are associative, since

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c),$$

but subtraction is not generally associative because

$$(a - b) - c = a - b - c, \quad \text{whereas} \quad a - (b - c) = a - b + c.$$

Division also generally fails to be associative because

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc}, \quad \text{whereas} \quad \frac{a}{\frac{b}{c}} = \frac{ac}{b}.$$

DEFINITION 4 The binary operation § has an **identity** element on S if some element e exists in S such that

$$\S(a, e) = a \quad \text{and} \quad \S(e, a) = a \quad \text{for every element } a \text{ in } S.$$

It can be shown that if an identity element exists, then it is unique.

The real number 0 is the identity element for addition, and the real number 1 is the identity element for multiplication.

For subtraction there is no identity element. For if such an identity element did exist we would have $a - 0 = a$ for all real numbers a. However, we would also need to have $0 - a = a$, but this is true only when $a = 0$. A similar contradiction occurs for division, as you might expect.

DEFINITION 5 When § has an identity element e on the set S, we say that an element $\hat{a}$ in S is an **inverse** of a in S if both

$$\S(a, \hat{a}) = e \quad \text{and} \quad \S(\hat{a}, a) = e.$$

The binary operation of addition has an inverse for each element in the set of integers, or rational numbers, or real numbers. However, if S is the set of nonnegative integers, the only element in S with an inverse under addition is the identity element 0.

For multiplication, every element in the set of nonzero rational numbers has an inverse, but in the set of integers, only the numbers 1 and -1 have inverses.

In the earlier years of the exams it was not unusual for a problem to describe a specific binary operation and ask which of these properties the operation satisfied. An example of such a problem is given in the Examples. Although emphasis on definitions is not common on more recent exams, it is common to see a problem that essentially asks a similar question, but using less formal language.

On the problems that appear early on the exam, binary operation problems might simply involve repeatedly applying the operation, sometimes in a reverse manner.

4.3 Functions

The general discussion of functions and their graphs is postponed until Chapter 16, but this topic is so closely related to that of defined operations that we will give a brief introduction here.

Definition 1 A **function** f is a means of associating all the elements of one set, called the **domain** of f, with those of a second set in such a way that each element in the domain is associated with precisely one element in the second set. The **range** of f consists of those elements in the second set which are associated with some element in the domain of f.

Most often the domain and range of the function are subsets of the same set. For example, the function f that squares numbers, often expressed as

$$f(x) = x^2$$

has, unless specified otherwise, the set of all real numbers as its domain. Its range is then the set of nonnegative real numbers. The only aspect of functions that we will consider in this chapter is the substitution process associated with functional notation. In Chapter 16 we will consider functional concepts that are commonly seen in precalculus.

Consider the function f that cubes numbers, that is, $f(x) = x^3$. Suppose that we are asked to determine the number of values of x for which

$$f(x^2 + 1) = (f(x))^2 + 1.$$

This object of the problem is simply to see if functional notation can be correctly applied. Since f is the function that cubes, we have

$$f(x^2+1) = (x^2+1)^3 = (x^2)^3 + 3(x^2)^2 + 3x^2 + 1 = x^6 + 3x^4 + 3x^2 + 1,$$

and

$$(f(x))^2 + 1 = (x^3)^2 + 1 = x^6 + 1.$$

So we need to determine how many real numbers satisfy

$$x^6 + 3x^4 + 3x^2 + 1 = x^6 + 1, \quad \text{that is} \quad 0 = 3x^4 + 3x^2 = 3x^2(x^2 + 1),$$

There is only one real solution to this final equation, the number $x = 0$.

Examples for Chapter 4

The first Example is number 6 from the 1971 AHSME.

Example 1 Let $*$ be a symbol denoting the binary operation on the S of all nonzero real numbers as follows: For any two numbers a and b, define $*(a,b) = a * b = 2ab$.

Which one of the following statements is **not** true?

(A) $*$ is commutative on S **(B)** $*$ is associative on S

(C) $\frac{1}{2}$ is an identity element for $*$ in S

(D) Every element in S has an inverse for $*$

(E) $\frac{1}{2a}$ is an inverse for $*$ of the element a of S

Answer (E) Statement (A) is easily seen to be true since the commutativity of multiplication implies that

$$a * b = 2ab = 2ba = b * a \quad \text{for all real numbers } a \text{ and } b.$$

Statement (C) is also true since for all real numbers a we have

$$a * \frac{1}{2} = 2a \cdot \frac{1}{2} = a.$$

Statements (B) and (D) appear to be more complicated to verify that statement (E), so we will consider statement (E) first. If (E) is correct, then the product of $*$ of $a \neq 0$ and $\frac{1}{2a}$ would need to be the identity element, which has been found to be $1/2$. However,

$$a * \frac{1}{2a} = 2a \cdot \frac{1}{2a} = 1.$$

Since $1 \neq 1/2$, statement (E) is incorrect.

To verify that the other statements are, in fact, true, note that

for (B): $(a * b) * c = (2ab) * c = 2(2ab)c = 4abc = 2a(2bc) = a * (2bc) = a * (b * c)$,

for (D): For any element a in S we have $a * (1/4a) = 1/2$ and $1/2$ is the identity, so $1/(4a)$ is the inverse for a. □

The second Example is number 14 from the 1996 AHSME.

Example 2 The function $E(n)$ is defined for each positive integer n to be the sum of the even digits of n. For example, $E(5681) = 6 + 8 = 14$. What is the value of $E(1) + E(2) + \cdots + E(100)$?

(A) 200 **(B)** 300 **(C)** 400 **(D)** 900 **(E)** 2250

Answer (C) Rather than determine the values of $E(n)$ individually, consider the fact that the numbers 00, 01, 02, ..., 99 contain an equal number of each of the digits $0, 1, \ldots, 9$. There are a total of $2 \cdot 100 = 200$ of these digits, so there are 20 of each digit. Hence the total of all the digits in the sum

$$E(0) + E(1) + E(2) + \cdots + E(99) = 20(0 + 2 + 4 + 6 + 8) = 400.$$

Since $E(0) = 0$ and $E(100) = 0$, the required sum is also 400. □

The final Example is number 17 from the 1981 AHSME.

EXAMPLE 3 The function f has the property that, for all nonzero real numbers,

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x.$$

How many nonzero real number solutions are there to the equation $f(x) = f(-x)$?

(A) None **(B)** 1 **(C)** 2 **(D)** All nonzero real numbers

(E) Infinitely many, but not all, nonzero real numbers

Answer (C) We need to determine some way to eliminate the term $f(1/x)$ from the equation that describes $f(x)$. Then we can compare $f(x)$ and $f(-x)$. The first thing to try is to substitute $1/x$ for x in the defining equation and hope that this leads to a simplification. Doing this gives

$$f\left(\frac{1}{x}\right) + 2f(x) = 3 \cdot \frac{1}{x}, \quad \text{so} \quad f\left(\frac{1}{x}\right) = 3 \cdot \frac{1}{x} - 2f(x).$$

Now substituting $f(1/x)$ into the original equation produces

$$f(x) + 2\left(3 \cdot \frac{1}{x} - 2f(x)\right) = 3x, \quad \text{so} \quad -3f(x) + \frac{6}{x} = 3x$$

and

$$f(x) = -\frac{1}{3}\left(3x - \frac{6}{x}\right) = -x + \frac{2}{x} = \frac{2 - x^2}{x}.$$

If $f(x) = f(-x)$ and $x \neq 0$, then

$$\frac{2 - x^2}{x} = \frac{2 - (-x)^2}{-x}, \quad \text{so} \quad 2x - x^3 = -2x + x^3.$$

Hence

$$0 = 2x^3 - 4x = 2x(x^2 - 2).$$

The only nonzero solutions to this equation are $x = \pm\sqrt{2}$. □

Exercises for Chapter 4

Exercise 1 Define the operation "$\circ$" by $x \circ y = 4x - 3y + xy$, for all real numbers x and y. For how many real numbers y does $12 = 3 \circ y$?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** more than 4

Exercise 2 Define $[a, b, c]$ to mean $\dfrac{a+b}{c}$, when $c \neq 0$. What is the value of

$$\big[[60, 30, 90], [2, 1, 3], [10, 5, 15]\big]?$$

(A) 0 **(B)** 0.5 **(C)** 1 **(D)** 1.5 **(E)** 2

Exercise 3 The operation $x * y$ is defined by $x * y = (x + 1)(y + 1) - 1$. Which one of the following is **false**?

(A) $x * y = y * x$ for all real x and y.

(B) $x * (y + z) = (x * y) + (x * z)$ for all real x, y, and z.

(C) $(x - 1) * (x + 1) = (x * x) - 1$ for all real x.

(D) $x * 0 = x$ for all real x.

(E) $x * (y * z) = (x * y) * z$ for all real x, y, and z.

Exercise 4 Define $x \heartsuit y$ to be $|x - y|$ for all real numbers x and y. Which of the following statements is **not** true?

(A) $x \heartsuit y = y \heartsuit x$ for all x and y **(B)** $2(x \heartsuit y) = 2x \heartsuit 2y$ for all x and y

(C) $x \heartsuit 0 = x$ for all x **(D)** $x \heartsuit x = 0$ for all x

(E) $x \heartsuit y > 0$ if $x \neq y$

Exercise 5 Let $P(n)$ and $S(n)$ denote the product and sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose that N is a two-digit integer such that $N = P(N) + S(N)$. What is the units digit of N?

(A) 2 **(B)** 3 **(C)** 6 **(D)** 8 **(E)** 9

Exercise 6 Let $\clubsuit(n)$ denote the sum of the digits of the positive integer n. For example, $\clubsuit(8) = 8$ and $\clubsuit(123) = 1 + 2 + 3 = 6$. For how many two-digit values of n is $\clubsuit(\clubsuit(n)) = 3$?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 10

Exercise 7 Let f be a function satisfying $f(xy) = f(x)/y$ for all positive real numbers x and y, and $f(500) = 3$. What is $f(600)$?

(A) 1 **(B)** 2 **(C)** $\dfrac{5}{2}$ **(D)** 3 **(E)** $\dfrac{18}{5}$

Exercise 8 For any real number a and positive integer k, define

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-(k-1))}{k(k-1)(k-2)\cdots(2)(1)}.$$

What is the value of

$$\binom{-\frac{1}{2}}{100} \Big/ \binom{\frac{1}{2}}{100}?$$

(A) -199 **(B)** -197 **(C)** -1 **(D)** 197 **(E)** 199

Exercise 9 A real-valued function f which is not identically zero has the property that for each pair of real numbers a and b,

$$f(a+b) + f(a-b) = 2f(a) + 2f(b).$$

Which of the following statements must be true?

(A) $f(0) = 1$ **(B)** $f(-x) = -f(x)$ **(C)** $f(-x) = f(x)$

(D) $f(x+y) = f(x) + f(y)$

(E) A number $T > 0$ exists with $f(x+T) = f(x)$.

Exercise 10 Suppose that $f(x) > 0$ for all real numbers x, and that $f(x+y) = f(x)f(y)$ for all x and y. Which of the following statements must be true?

I. $f(0) = 1$

II. $f(-x) = \frac{1}{f(x)}$ for all values of x

III. $f(x) = \sqrt[3]{f(3x)}$ for all values of x

IV. $f(x) \geq 1$ for all values of x

(A) III and IV only **(B)** I, III and IV only **(C)** I, II, and IV only

(D) I, II and III only **(E)** All are true

5
Triangle Geometry

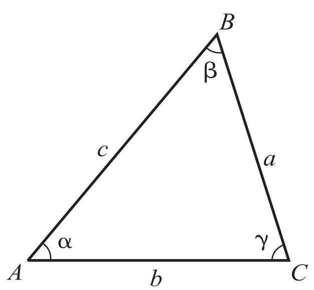
5.1 Introduction

Every AMC has included problems on triangle geometry, generally in the medium to relatively difficult range. The higher-range problems often require some geometric construction to obtain the solution. Although the subject matter necessary to solve these problems is generally included in a standard high school geometry course, the emphasis in the course may not be sufficient for the solution of all of these problems.

5.2 Definitions

We will assume the basic concepts of geometry, such as the definition of a point, a line, an angle, and so on, are known. Our starting point will be the triangle, a geometric figure in the plane having three angles and three line segments as sides.

There are a number of concepts associated with the geometry of triangles that we list first so that they can be easily found. Any reference to



parts of a general triangle will use notation similar to that in the following triangle.

Most of the problems on the AMC use degree measurement for angles, but occasionally it is necessary to translate into radian measure. This is easily done by using the fact that 180 degrees is the same as π radians, so

$$1 \text{ degree} = \pi/180 \text{ radians} \quad \text{and} \quad 1 \text{ radian} = 180/\pi \text{ degrees}.$$

The most basic result concerning the angles of a triangle is that

$$\alpha + \beta + \gamma = 180^\circ.$$

The basic fact for the sides of a triangle is the **Triangle Inequality**,

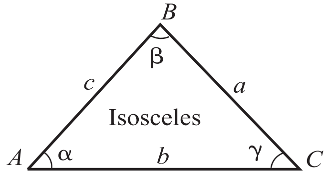
$$a < b + c, \quad b < a + c, \quad \text{and} \quad c < a + b,$$

which is alternatively expressed using the notation

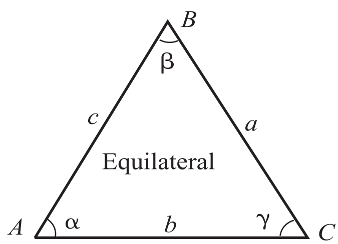
$$BC < AC + AB, \quad AC < BC + AB, \quad \text{and} \quad AB < BC + AC.$$

Some definitions that will be used throughout this chapter and in later chapters are given next.

DEFINITION 1 A triangle that has two sides of equal length is an **isosceles** triangle. The base angles of an isosceles triangle are equal.



$a = c$ and $\alpha = \gamma$

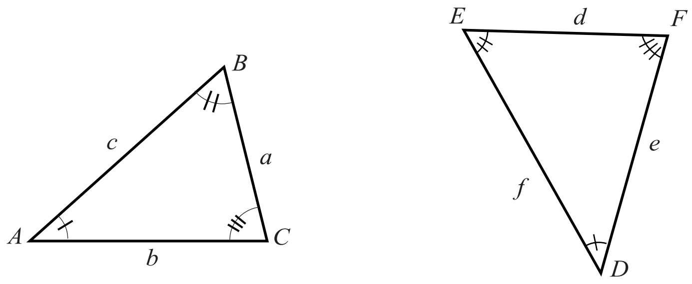


$a = b = c$ and $\alpha = \beta = \gamma$

DEFINITION 2 A triangle that has all three sides of equal length is an **equilateral** triangle. A triangle is equilateral if and only if it is *equiangular*, that is, its angles are all equal.

DEFINITION 3 Two triangles are **similar** if there is a correspondence between their vertices such that the corresponding angles are equal.

In the following figure $\triangle ABC$ is similar to $\triangle DEF$ since $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.



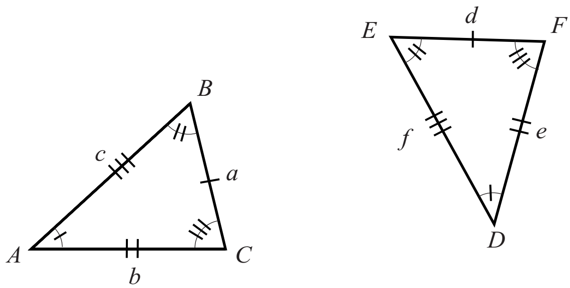
Some particularly useful results concerning similar triangles are:

- Two triangles are similar if two angles of one triangle are equal to two angles of the other triangle.
- Similar triangles have proportional sides, that is, in triangles ABC and DEF

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}, \quad \text{or alternatively,} \quad \frac{BC}{EF} = \frac{AC}{DF} = \frac{AB}{DE}.$$

DEFINITION 4 Two triangles are **congruent** if there is a correspondence between their vertices such that corresponding angles are equal and the corresponding sides are equal.

In the following figure $\triangle ABC$ is similar to $\triangle DEF$ and is also congruent because $a = d$, $b = e$, and $c = f$.



Results about congruent triangles are some of the most basic in all of geometry.

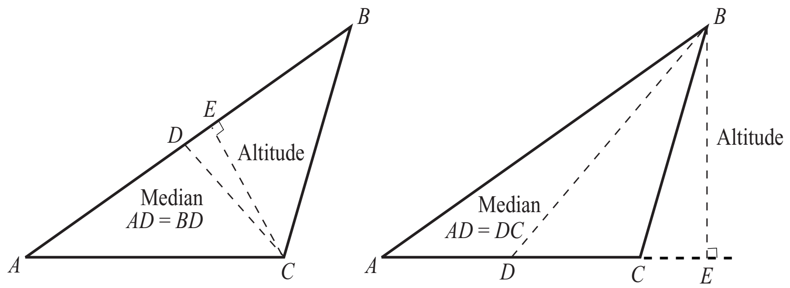
- Two triangles are congruent if the three sides of one are equal to the three sides of the other. This is called congruence by Side-Side-Side. (Often abbreviated as SSS.)

- Two triangles are congruent if two sides and the included angle of one are equal to two sides and the included angle of the other. This is called congruence by Side-Angle-Side. (SAS)

- Two triangles are congruent if two angles and an included side of one are equal to two angles and the included side of the other. This is called congruence by Angle-Side-Angle. (ASA)

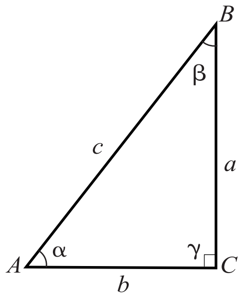
DEFINITION 5 A **median** of a triangle is a line segment from a vertex of the triangle to the midpoint of the opposite side.

DEFINITION 6 An **altitude** of a triangle is a line segment from a vertex of the triangle to the line containing the opposite side and that is perpendicular to that line.



5.3 Basic Right Triangle Results

A **right triangle** has one of its angles 90°. Many of the AMC triangle problems involve right triangles directly or use right triangles in the construction of the solution. We begin with some very basic results and then look at a few that may not be as familiar. For the material in this section we assume that right $\triangle ABC$ has the form and notation shown in the figure. In this figure side c is called the **hypotenuse** of $\triangle ABC$ and sides a and b are called the **legs**.



The most basic result concerning right triangles is the **Pythagorean Theorem**:

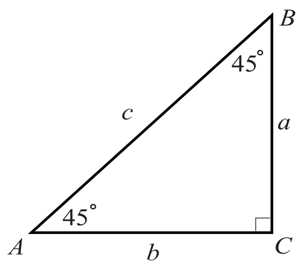
$$a^2 + b^2 = c^2.$$

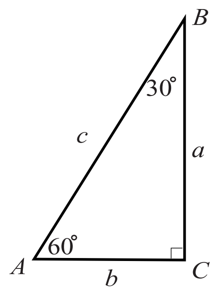
Next are some results about some special triangles.

Special Right Triangles:

RESULT 1 If $\angle\alpha = 45°$ and $\angle\beta = 45°$, then $\triangle ABC$ is called a 45–45–90° triangle. In this case

$$\frac{a}{c} = \frac{b}{c} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{a}{b} = 1.$$





RESULT 2 If $\angle\alpha = 60°$ and $\angle\beta = 30°$, then $\triangle ABC$ is called a 30–60–90° triangle. In this case

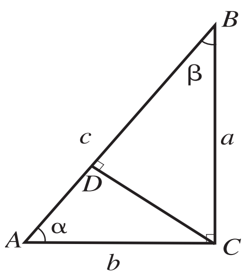
$$\frac{a}{c} = \frac{\sqrt{3}}{2}, \quad \frac{b}{c} = \frac{1}{2}, \quad \text{and} \quad \frac{a}{b} = \sqrt{3}.$$

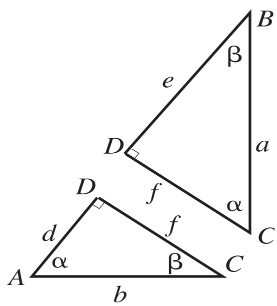
Result 3 The Right Triangle Altitude Theorem: In the right $\triangle ABC$, draw the altitude from the right angle at C to the side AB and label the intersection as D. This gives three similar right triangles: $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$. With the sides of the triangles labeled as shown in the figure, we have

$$\frac{a}{b} = \frac{e}{f} = \frac{f}{d}, \quad \frac{a}{c} = \frac{e}{a} = \frac{f}{b}, \quad \text{and} \quad \frac{b}{c} = \frac{f}{a} = \frac{d}{b},$$

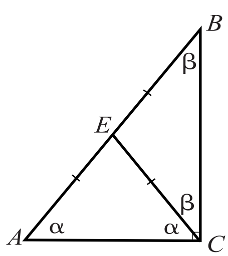
or alternatively,

$$\frac{BC}{AC} = \frac{BD}{CD} = \frac{CD}{AD}, \quad \frac{BC}{AB} = \frac{BD}{BC} = \frac{CD}{AC}, \quad \text{and} \quad \frac{AC}{AB} = \frac{CD}{BC} = \frac{AD}{AC}.$$





Result 4 The Right Triangle Median Theorem: In the right $\triangle ABC$, construct E on the line segment $\overline{AB}$ so that $\angle BCE = \beta$. Then $\angle ECA = \alpha$. So $\triangle AEC$ and $\triangle BEC$ are both isosceles, which implies that $AE = BE = CE$.

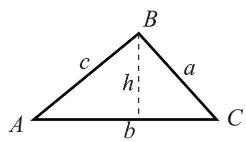


5.4 Areas of Triangles

There are a number of ways to determine the area of a triangle. Problems sometimes can be solved by using alternative methods for expressing the area of a triangle and using the common value to find unknown quantities. The most common method of determining the area of a triangle involves a base and an altitude.

Result 1 If b is the length of a base of $\triangle ABC$ and h is the length of the corresponding altitude, then the area of the triangle is

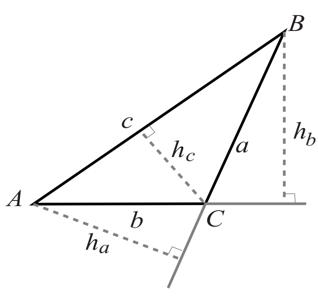
$$\text{Area}(\triangle ABC) = \frac{1}{2}bh.$$



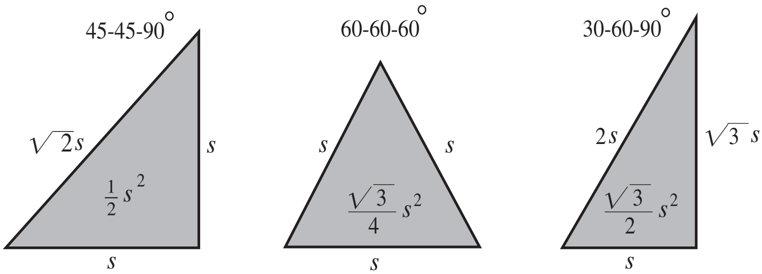
Any specific side can be the base of a triangle, and the corresponding altitude is the length of the perpendicular line segment from the base to the opposite vertex. So in the following figure we have

$$\text{Area}(\triangle ABC) = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

Note that this implies that $ah_a = bh_b = ch_c$.



Since 45–45–90°, 60–60–60°, and 30–60–90° triangles occur so often, it is worthwhile to remember the areas of these triangles, which are given in the figure.



The following formula determines the area of a triangle using only the lengths of the sides.

Result 2 Heron's Formula: The semi-perimeter, s, of $\triangle ABC$ is half its perimeter,

$$s = \frac{1}{2}(a + b + c).$$

The area of the triangle in terms of the semi-perimeter and the length of the sides is

$$\text{Area}(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

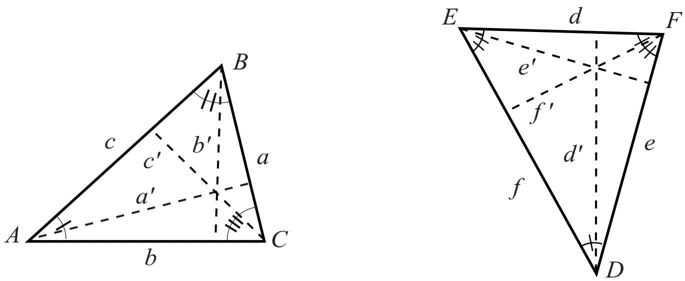
Result 3 There is a convenient method for determining the area of a triangle whose vertices in the plane are given by $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. It is most easily remembered if you are familiar with matrix determinant notation:

$$\begin{aligned}\text{Area}(\triangle ABC) &= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| \\ &= \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|.\end{aligned}$$

The final area result concerns the respective areas of similar triangles. Since altitudes of similar triangles have the same ratios as their corresponding sides, the ratios of the areas of a pair of similar triangles is as the square of the ratio of the corresponding sides (or the ratio of their corresponding altitudes).

Result 4 If $\triangle ABC$ is similar to $\triangle DEF$, then

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle DEF)} = \left(\frac{a}{d}\right)^2 = \left(\frac{b}{e}\right)^2 = \left(\frac{c}{f}\right)^2 = \left(\frac{a'}{d'}\right)^2 = \left(\frac{b'}{e'}\right)^2 = \left(\frac{c'}{f'}\right)^2.$$

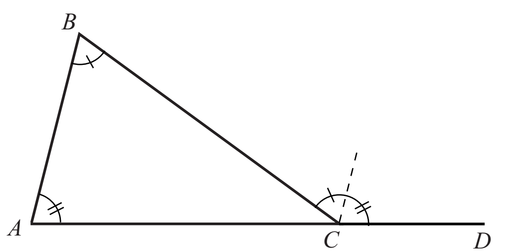


A generalization of this result concerning polygons will be presented in Chapter 7.

5.5 Geometric Results about Triangles

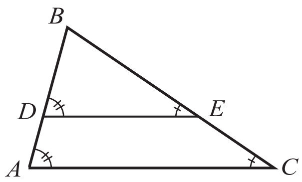
The following results will be needed frequently in solutions to the triangle geometry problems.

Result 1 The Exterior Angle Theorem: Suppose that in $\triangle ABC$ side $\overline{AC}$ is extended from C to point D. Then $\angle DCB = \angle ABC + \angle CAB$.



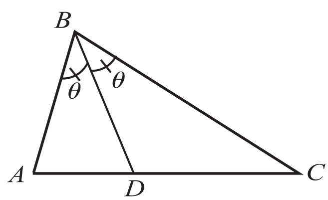
Result 2 The Side-Splitter Theorem: Suppose that in $\triangle ABC$, we have D on side $\overline{AB}$ and E on side $\overline{BC}$. Then DE is parallel to AC if and only if

$$\frac{BE}{BD} = \frac{BC}{BA}.$$



Result 3 The Angle-Bisector Theorem: If D is on side $\overline{AC}$ of $\triangle ABC$, then BD bisects $\angle ABC$ if and only if

$$\frac{AD}{CD} = \frac{AB}{BC}.$$

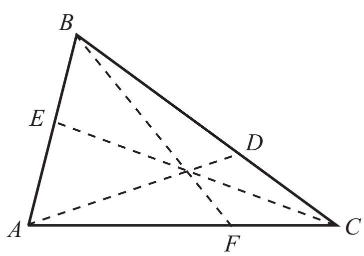


The next result appears to be infrequently taught in high school geometry courses but it has occasional application on the AMC. Some of these will be seen in this chapter and more in Chapter 6 on Circle Geometry and Chapter 7 on Polygons.

Result 4 Ceva's Theorem: Suppose that in $\triangle ABC$ we have D on side $\overline{BC}$, E on side $\overline{AB}$, and F on side $\overline{AC}$. The lines AD, CE, and BF intersect at a common point if and only if

$$AF \cdot CD \cdot BE = FC \cdot DB \cdot EA.$$

Three or more lines that intersect at a common point are called **concurrent**.

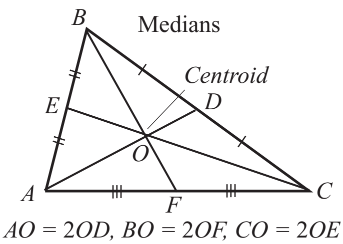


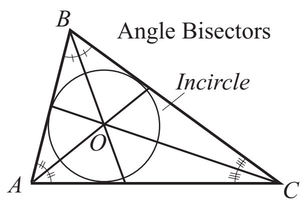
Some particularly useful consequences of Ceva's Theorem are that AD, CE, and BF are concurrent if any of the following are true:

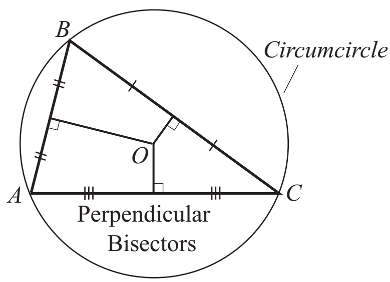
- AD, CE, and BF are the medians of $\triangle ABC$.
- AD, CE, and BF are the angle bisectors of $\triangle ABC$.
- AD, CE, and BF are the altitudes of $\triangle ABC$.

Ceva's Theorem also implies that

- Perpendicular bisectors of the sides of $\triangle ABC$ are concurrent.







Result 5 Some particularly important concurrent results for a $\triangle ABC$, illustrated in the figure above, are:

- The medians intersect at the **centroid**, or center of mass, of $\triangle ABC$, and the centroid trisects each of the medians. This implies that in the triangle in the left figure we have

$$AO = 2OD = \frac{2}{3}AD, \quad BO = 2OF = \frac{2}{3}BF,$$

and

$$CO = 2OE = \frac{2}{3}CE.$$

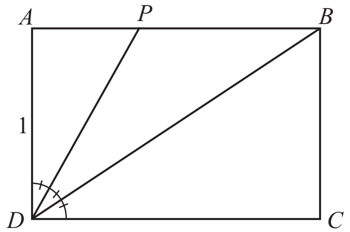
- The angle bisectors meet at the center of the **inscribed circle** of $\triangle ABC$, the only circle tangent to all three sides of $\triangle ABC$.
- The perpendicular bisectors meet at the center of the **circumscribed circle** of $\triangle ABC$, the only circle that passes through all three vertices of $\triangle ABC$.

Examples for Chapter 5

The first Example is number 7 from the 2000 AMC 10.

EXAMPLE 1 In rectangle $ABCD$ we have $AD = 1$, P is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of $\triangle BDP$?

(A) $3 + \frac{\sqrt{3}}{3}$ **(B)** $2 + \frac{4\sqrt{3}}{3}$ **(C)** $2 + 2\sqrt{2}$ **(D)** $\frac{3 + 3\sqrt{5}}{2}$
(E) $2 + \frac{5\sqrt{3}}{3}$



Answer (B) Since $\overline{BD}$ and $\overline{PD}$ trisect the right $\angle ADC$, we have

$$\angle CDB = \angle BDP = \angle PDA = 30^\circ.$$

So $\triangle DAP$ and $\triangle DAB$ are both 30–60–90° triangles. Since $AD = 1$, we have

$$AP = \frac{\sqrt{3}}{3}, \quad DP = \frac{2\sqrt{3}}{3}, \quad AB = \sqrt{3}, \quad DB = 2,$$

and $DC = AB = \sqrt{3}$. Hence

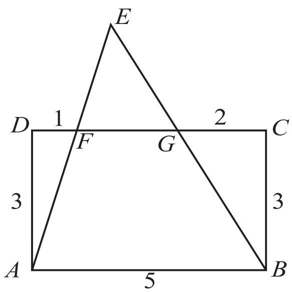
$$BD + DP + PB = BD + DP + (AB - AP)$$
$$= \frac{2\sqrt{3}}{3} + 2 + \left(\sqrt{3} - \frac{\sqrt{3}}{3}\right),$$

so the perimeter of $\triangle BDP$ is $2 + 4\sqrt{3}/3$.

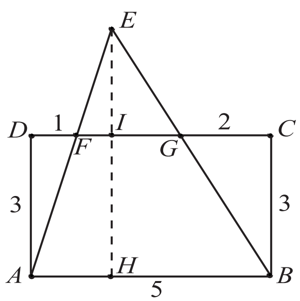
The second Example is number 20 from the 2003 AMC 10B and number 14 from the AMC 12B.

EXAMPLE 2 In rectangle $ABCD$, we have $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ with $DF = 1$ and $GC = 2$, and lines AF and BG intersect at E. What is the area of $\triangle AEB$?

(A) 10 **(B)** $\dfrac{21}{2}$ **(C)** 12 **(D)** $\dfrac{25}{2}$ **(E)** 15



Answer (D) Let H be on line AB with $EH \perp AB$, and label as I the intersection of EH and DC.



Since $DC = AB = 5$, we have $FG = 5 - DF - GC = 2$, and the Side-Spitter Theorem implies that

$$\frac{2}{5} = \frac{FG}{AB} = \frac{EI}{EH} = \frac{EI}{EI + IH} = \frac{EI}{EI + 3}.$$

The left and right ends of this equality imply that

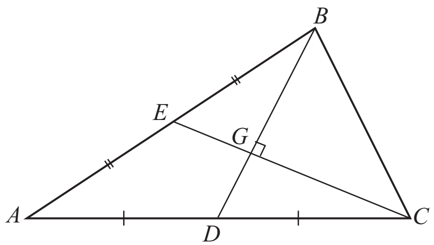
$$2EI + 6 = 5EI, \quad \text{so} \quad EI = 2.$$

Thus $EH = EI + IH = 2 + 3 = 5$ and the area of $\triangle AEB$ is

$$\frac{1}{2} AB \cdot EH = \frac{5 \cdot 5}{2} = \frac{25}{2}.$$ □

The final Example is number 15 from the 1997 AHSME.

EXAMPLE 3 Medians BD and CE of $\triangle ABC$ are perpendicular, $BD = 8$, and $CE = 12$. What is the area of $\triangle ABC$?



(A) 24 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 96

Answer (D) The solution to this problem relies on knowing Result 5 in Section 5.5 concerning the intersection properties of medians of triangles. Without knowing this fact, it would be difficult to solve.

Let G represent the point of intersection of the lines BD and CE. Since BD and CE are medians, we have

$$BG = \frac{2}{3} BD = \frac{16}{3}, \quad GD = \frac{1}{3} BD = \frac{8}{3}, \quad CG = \frac{2}{3} CE = 8,$$

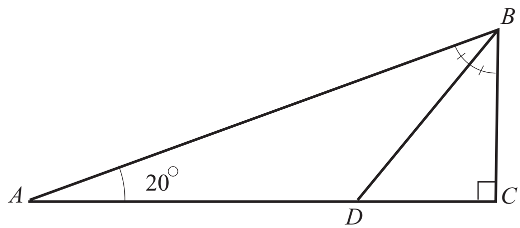
and $GE = \frac{1}{3} CE = 4$.

Note that the base AC of $\triangle ABC$ is twice the base DC of $\triangle DBC$, and that the triangles have the same altitude with respect to these bases. So the area of $\triangle ABC$ is twice that of $\triangle DBC$. Since $\angle CGB$ is a right angle, CG is an altitude of $\triangle DBC$, which implies that

$$\text{Area}(\triangle ABC) = 2\text{Area}(\triangle DBC) = 2\left(\frac{1}{2}\cdot BD\cdot CG\right) = 8\cdot 8 = 64. \quad \square$$

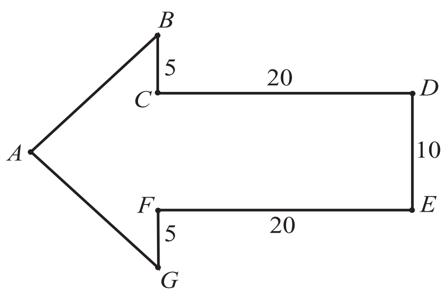
Exercises for Chapter 5

Exercise 1 In the figure, $\triangle ABC$ has a right angle at C, $\angle A = 20°$, and BD is the bisector of $\angle ABC$. What is $\angle BDC$?



(A) 40° **(B)** 45° **(C)** 50° **(D)** 55° **(E)** 60°

Exercise 2 In the arrow-shaped polygon shown, there are right angles at vertices A, C, D, E, and F, $BC = FG = 5$, $CD = FE = 20$, $DE = 10$, and $AB = AG$. Which of the following is closest to the area of the polygon?

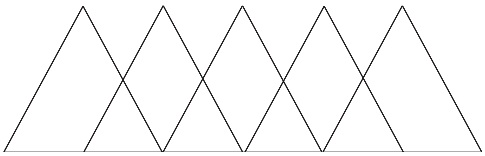


(A) 288 **(B)** 291 **(C)** 294 **(D)** 297 **(E)** 300

Exercise 3 The sides of a triangle have lengths of 15, 20, and 25. What is the length of the shortest altitude?

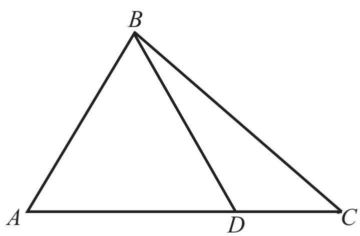
(A) 6 **(B)** 12 **(C)** 12.5 **(D)** 13 **(E)** 15

Exercise 4 Five equilateral triangles, each with side length $2\sqrt{3}$, are arranged so they are all on the same side of a line containing one side of each. Along this line, the midpoint of the base of one triangle is the vertex of the next, as shown. What is the area of the region of the plane that is covered by the union of the five triangular regions?



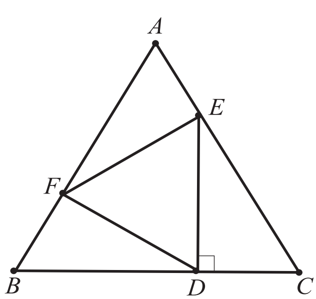
(A) 10 **(B)** 12 **(C)** 15 **(D)** $10\sqrt{3}$ **(E)** $12\sqrt{3}$

Exercise 5 In $\triangle ABC$ we have $AB = 5$, $BC = 7$, and $AC = 9$. Also, D is on $\overline{AC}$ with $BD = 5$. What is AD/DC?



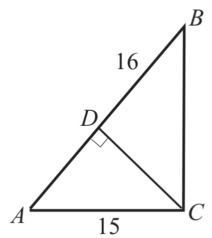
(A) $\frac{4}{3}$ **(B)** $\frac{7}{5}$ **(C)** $\frac{11}{6}$ **(D)** $\frac{13}{5}$ **(E)** $\frac{19}{8}$

Exercise 6 Equilateral $\triangle DEF$ is inscribed in equilateral $\triangle ABC$ with $DE \perp BC$, as shown. What is the ratio of the area of $\triangle DEF$ to the area of $\triangle ABC$?



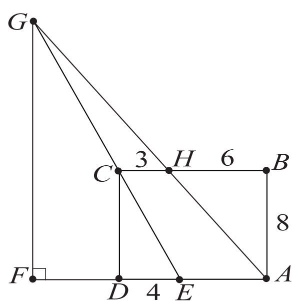
(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

Exercise 7 Right $\triangle ABC$ with hypotenuse $\overline{AB}$ has $AC = 15$. Altitude $\overline{CD}$ is constructed and $DB = 16$, as shown. What is the area of $\triangle ABC$?



(A) 120 **(B)** 144 **(C)** 150 **(D)** 216 **(E)** $144\sqrt{5}$

Exercise 8 In rectangle $ABCD$, we have $AB = 8$, $BC = 9$, H is on $\overline{BC}$ with $BH = 6$, E is on $\overline{AD}$ with $DE = 4$, line EC intersects line AH at G, and F is on line AD with $GF \perp AF$. What is the length GF?



(A) 16 **(B)** 20 **(C)** 24 **(D)** 28 **(E)** 30

Exercise 9 Right $\triangle ABC$ has its right angle at C. Let M and N be the midpoints of AC and BC, respectively, with $AN = 19$ and $BM = 22$. What is AB?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

Exercise 10 Point D is on side BC of $\triangle ABC$ with $AB = 3$, $AC = 6$, and $\angle CAD = \angle DAB = 60°$. What is the length AD?

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5 **(E)** 4

6

Circle Geometry

6.1 Introduction

This chapter continues the subject of geometry in the plane. There are many types of problems that use circles in their solution, some involving triangles as well as circles. Many of the problems that involve circles are most easily solved using equations to represent that circle, but these will be postponed to a later chapter. Here we consider only those problems that strictly involve plane geometry.

There are numerous definitions and results in this material, and it is important to have complete familiarity with the notation.

6.2 Definitions

We begin with the basic definitions and include here all the terminology that will be used for the problems that involve circles. The most basic and frequently used are those involving the area and circumference of a circle.

Definition 1 Circles:

- A **circle** is a set of all points that are a fixed distance from a given point.
- The **center** of the circle is the given point.
- Any line segment from the center that has the fixed distance as its length is a **radius** of the circle.

The term *radius* is also used to describe the fixed distance from the center to the points on the circle. This could cause confusion, but context will make the distinction clear.

Definition 2 Lines and Circles:

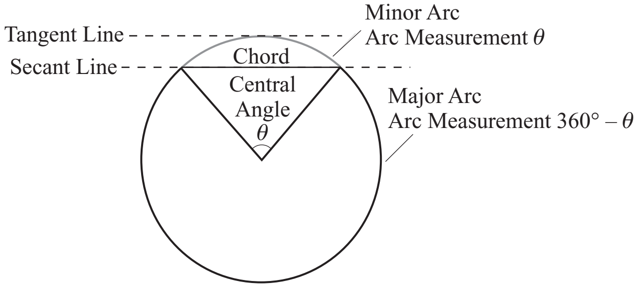
- A line that has exactly one point in common with a circle is called a **tangent** line to the circle.
- A line that intersects two points of a circle is called a **secant** line of the circle. The line segment of the secant line that joins the two points on the circle is called a **chord**.
- A **diameter** of a circle is a chord that passes through the center of the circle.

The term diameter is also used to describe the length of a diameter, which is twice the length of the radius.

A basic way to relate circle geometry to triangle geometry is to use an angle that has its vertex at the center of the circle.

Definition 3 A **central angle** of a circle is an angle whose vertex is at the center of the circle. A central angle partitions the circle into two portions. The larger and smaller portions are the **major arc** and **minor arc**, respectively. The minor arc that joins points A and B on a circle is denoted $\overset{\frown}{AB}$. This is also used to describe the measure of the arc, which is the same as the measure of the central angle.

Since a chord of a circle determines a central angle, and conversely, it is also common to use the terms major and minor arcs to describe the portions of the circle partitioned by a chord. When the chord is a diameter, the partitioned portions have the same length and are called **semicircles**.

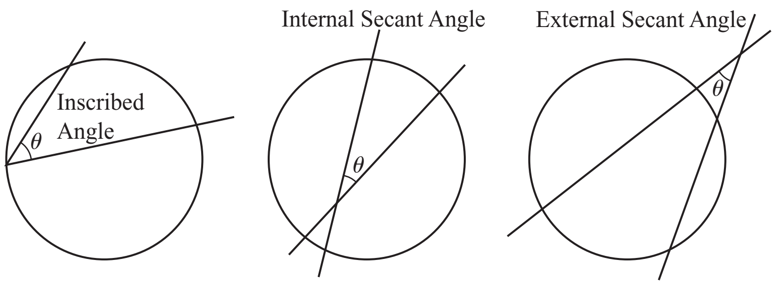


There are a few other angles associated with circles, the most common of these is the *inscribed angle*.

DEFINITION 4 An **inscribed angle** of a circle is an angle whose vertex is at a point on the circle and each of whose sides intersect the circle in a chord.

DEFINITION 5 A **secant angle** to a circle is an angle whose sides are secants of the circle.

Central and inscribed angles are also secant angles, in addition to those angles formed by lines that intersect inside the circle and lines that intersect outside the circle. All the angles formed in the figure below are secant angles.



6.3 Basic Results of Circle Geometry

The important number π is associated with both the area and the circumference of a circle. This number is irrational, in fact not even algebraic (it is not a zero of any polynomial with integer coefficients), but we can approximate it as $\pi \approx 3.14159265\ldots$.

RESULT 1 The area of a circle with radius r is πr^2.

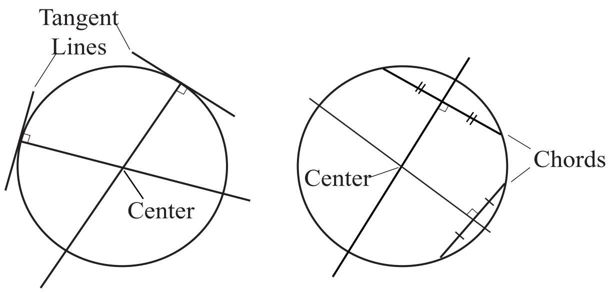
RESULT 2 The circumference of a circle with radius r is $2\pi r$.

Note that Results 1 and 2 imply that the ratio of the area of a circle to its circumference is $r/2$.

It is often important to be able to determine the center of a given circle. The next two results provide ways to determine this point.

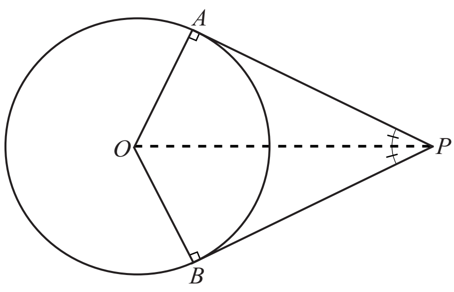
RESULT 3 A secant line passes through the center of a circle if and only if the line is perpendicular to the tangent line at each point of intersection with the circle.

RESULT 4 The secant line that is perpendicular to a chord of a circle passes through the center if and only if the line bisects the chord.



RESULT 5 Suppose that P is a point external to a circle with center O.

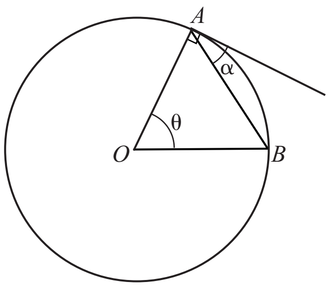
- There are precisely two lines that pass through P that are tangent to the circle.
- The line segments determined by P and these tangent points are of equal length.
- The angle formed by the line segments is bisected by the line OP.



6.4 Results Involving the Central Angle

The relationship between the central angle of a circle and the tangent and secant lines provides many of the results needed to solve problems in circle geometry. First, let us consider the case of tangent lines.

Result 1 Tangent-Chord Theorem: The central angle determined by a chord to a circle that is not a diameter is twice the angle formed by the chord and a tangent line with one endpoint in common with the chord. In the figure this implies that $\theta = 2\alpha$.

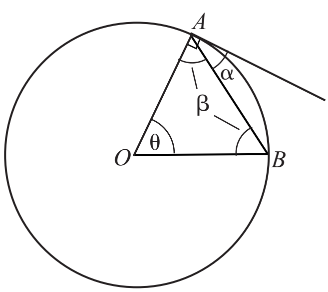


It is interesting to see why this last result is true because we will soon see similar results concerning the various secant angles.

Let the central angle be θ and the angle formed by the chord and tangent be α, as shown. The radii OA and OB have the same length, so $\triangle AOB$ is isosceles with base angles denoted β. Since the tangent line is perpendicular to OA we have both

$$\theta + 2\beta = 180^\circ \quad \text{and} \quad \beta + \alpha = 90^\circ.$$

This implies that $\theta = 2\alpha$.

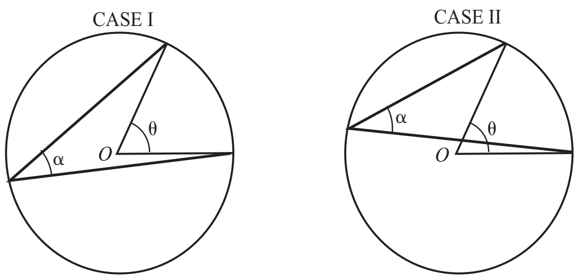


The primary result in this section relates the central angle of a circle to an inscribed circle that has the same arc. It is the basis for a number of other important results.

Result 2 The Central Angle Theorem: Any inscribed angle has measure half of the central angle with the same arc.

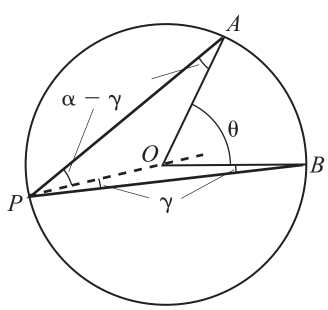
Since the Central Angle theorem is the core of a number of results, we will give an indication of its proof. There are two situations to consider, as shown below.

- Case I: Neither of the line segments of the inscribed angle intersects a side of the central angle.
- Case II: One of the line segments of the inscribed angle intersects a side of the central angle.



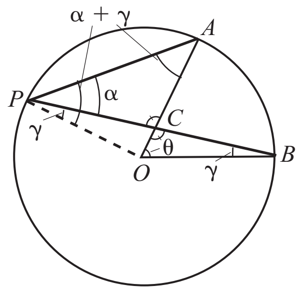
For Case I, we construct the line OP and consider the isosceles triangles $\triangle POA$ and $\triangle POB$. If the angle at B is γ, then the angle at A is $\alpha - \gamma$. Also, θ is the sum of the exterior angles of $\triangle POA$ and $\triangle POB$, so applying the Exterior Angle Theorem to both of these triangles gives

$$\theta = 2(\alpha - \gamma) + 2\gamma = 2\alpha.$$



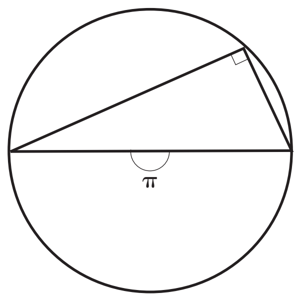
To show Case II, again construct OP and label as C the intersection of the side of the central angle and the inscribed angle, as shown. Since the angle at C has the same measure in $\triangle PCA$ and $\triangle OCB$, the sum of the remaining angles in these triangles must also agree. Hence

$$\alpha + (\alpha + \gamma) = \theta + \gamma, \quad \text{so} \quad \theta = 2\alpha.$$



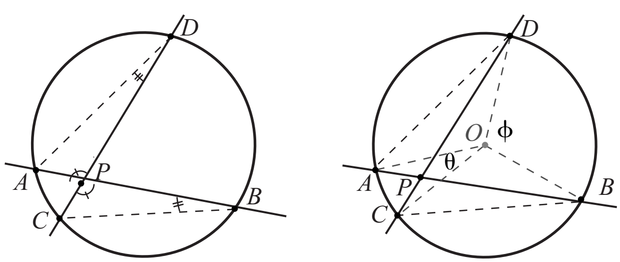
The Central Angle Theorem is often used for problem solving, but, additionally, it can be used to produce some other interesting results. The most important of these concerns inscribed right angles, which are listed first.

Result 3 An inscribed angle is a right angle if and only if the associated central angle is formed by a diameter.



Result 4 The Internal Secant Theorem: Suppose that two secant lines intersect inside a circle at point P, as shown. Then $AP \cdot PB = CP \cdot PD$.

Internal Secant Theorem



Moreover, the sum of the arcs formed by the lines is twice the measure of the angle of intersection, that is,

$$\angle APC = \angle BPD = \frac{1}{2}(\phi + \theta) = \frac{1}{2}(\overset{\frown}{BD} + \overset{\frown}{AC}).$$

The length result in the Internal Secant Theorem follows from the fact that $\angle DAB = \angle DCB$ because they are inscribed angles cutting the same arc $\overset{\frown}{BD}$. Since $\angle DPA = \angle BPC$, we have $\triangle APD$ similar to $\triangle CPB$. Equating the ratios of sides AP to CP and PB to PD gives $AP \cdot PB = CP \cdot PD$.

The angle result follows from the fact that $\angle BPD$ is an exterior angle to $\triangle APD$. Since $\angle APC = \angle BPD$, it follows that

$$\angle APC = \angle PAD + \angle ADP = \angle BAD + \angle ADC.$$

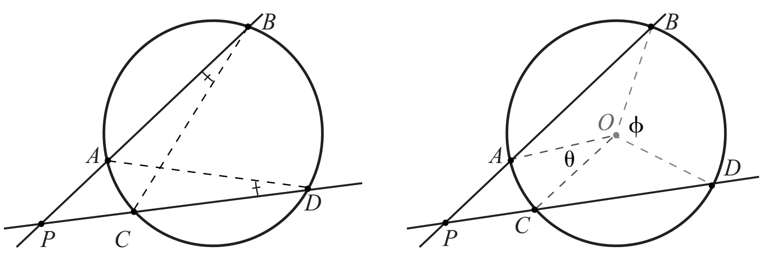
However, $\angle BAD$ and $\angle ADC$ are inscribed angles corresponding, respectively, to the central angles ϕ and θ, which implies that

$$\angle APC = \angle BAD + \angle ADC = \frac{1}{2}\phi + \frac{1}{2}\theta = \frac{1}{2}(\phi + \theta).$$

Result 5 The External Secant Theorem: Suppose that two secant lines intersect outside a circle at point P, as shown. Then $AP \cdot PB = CP \cdot PD$. Moreover, the absolute value of the difference of the arcs formed by the lines is twice the measure of the angle of intersection, that is,

$$\angle DPB = \frac{1}{2}(\phi - \theta) = \frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{AC}).$$

External Secant Theorem



The length result in the External Secant Theorem follows from the fact that $\angle DAB = \angle DCB$ because they are inscribed angles cutting the same arc $\widehat{BD}$. Since $\angle DPB$ is in both $\triangle APC$ and $\triangle BPD$, these triangles are similar. Equating the ratios of sides AP to CP and PB to PD gives $AP \cdot PB = CP \cdot PD$.

The angle result follows from the fact that $\angle DCB$ is an exterior angle to $\triangle PCB$, so

$$\angle DCB = \angle PBC + \angle CPB = \angle ABC + \angle DPB$$

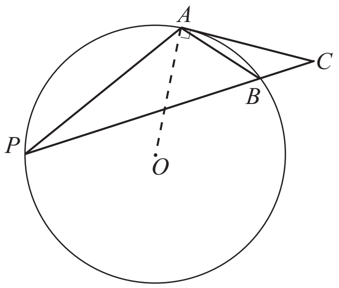
and $\angle DPB = \angle DCB - \angle ABC$.

Since $\angle DCB$ and $\angle ABC$ are inscribed angles corresponding, respectively, to the central angles ϕ and θ, we have

$$\angle DPB = \angle DCB - \angle ABC = \frac{1}{2}\phi - \frac{1}{2}\theta = \frac{1}{2}(\phi - \theta).$$

Result 6 The Power of a Point: Suppose that $\angle APB$ is an inscribed angle intersecting a circle at chord $\overline{AB}$, and that C is the point of intersection of the tangent line to the circle at A and the extension of $\overline{PB}$, as shown below. Then $\triangle ABC$ is similar to $\triangle PAC$ and $AC^2 = BC \cdot PC$.

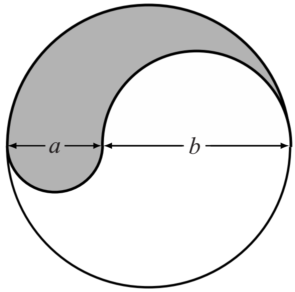
This Power of a Point follows from the Central Angle Theorem and a similar result concerning the angle formed by a tangent line and a chord. These results state that in the figure shown below we have $\angle CAB = \angle BPA = \frac{1}{2}\angle BOA$.



Examples for Chapter 6

The first Example is number 16 from the 1998 AHSME.

EXAMPLE 1 The figure shown is the union of a circle and semicircles of diameters a and b, all of whose centers are collinear. What is the ratio of the area of the shaded region to that of the unshaded region?



(A) $\sqrt{\dfrac{a}{b}}$ **(B)** $\dfrac{a}{b}$ **(C)** $\dfrac{a^2}{b^2}$ **(D)** $\dfrac{a+b}{2b}$ **(E)** $\dfrac{a^2+2ab}{b^2+2ab}$

Answer (B) The shaded region consists of a semicircle of diameter $a+b$ with the addition of a semicircle of diameter a and the deletion of a semicircle of diameter b. So

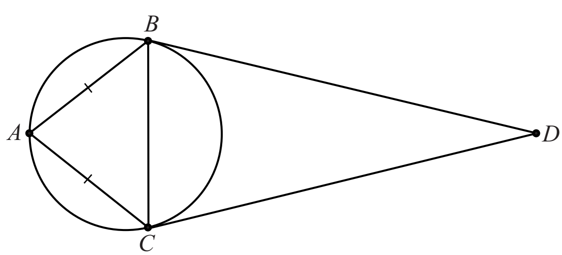
$$\text{Shaded Area} = \frac{1}{2}\left(\pi\left(\frac{a+b}{2}\right)^2 + \pi\left(\frac{a}{2}\right)^2 - \pi\left(\frac{b}{2}\right)^2\right)$$
$$= \frac{\pi}{8}\left(a^2+2ab+b^2+a^2-b^2\right) = \frac{a}{4}(a+b)\pi.$$

By symmetry, the area of the unshaded region is $b(a+b)\pi/4$, so the ratio is

$$\frac{\text{Shaded Area}}{\text{Unshaded Area}} = \frac{a(a+b)\pi/4}{b(a+b)\pi/4} = \frac{a}{b}. \qquad \square$$

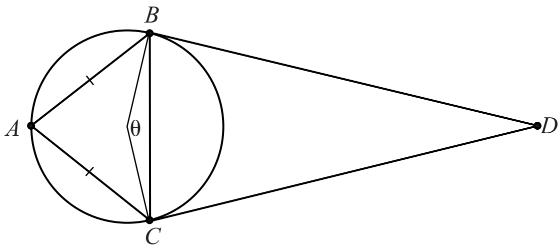
The next Example is number 14 from the 1990 AHSME.

Example 2 An acute isosceles $\triangle BAC$ is inscribed in a circle. Tangents to the circle are drawn at B and C, meeting at point D with $\angle ABC = \angle ACB = 2\angle CDB$. What is the radian measure of $\angle BAC$?



(A) $\dfrac{3}{7}\pi$ **(B)** $\dfrac{4}{9}\pi$ **(C)** $\dfrac{5}{11}\pi$ **(D)** $\dfrac{6}{13}\pi$ **(E)** $\dfrac{7}{15}\pi$

Answer (A) Let θ be the central angle generated by the chord BC.



By the Tangent Chord Theorem we have $\angle DBC = \theta/2$, and by the Inscribed Angle Theorem we have $\angle BAC = \theta/2$. As a consequence,

$$\angle DBC = \angle BAC.$$

It is given that $\triangle BAC$ is isosceles, and $\triangle BDC$ is also isosceles because BC and CD are tangent lines from the common point D. Since $\angle ABC = 2\angle BDC$, we have

$$\begin{aligned}\pi &= \angle BAC + \angle ABC + \angle ACB \\ &= \angle BAC + 2\angle ABC = \angle BAC + 4\angle BDC\end{aligned}$$

and

$$\begin{aligned}\pi &= \angle BDC + \angle DBC + \angle DCB \\ &= \angle BDC + 2\angle DBC = \angle BDC + 2\angle BAC.\end{aligned}$$

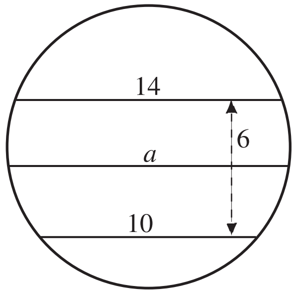
Hence $\angle BDC = \pi - 2\angle BAC$ and

$$\begin{aligned}\pi &= \angle BAC + 4\angle BDC \\ &= \angle BAC + 4(\pi - 2\angle BAC) = 4\pi - 7\angle BAC.\end{aligned}$$

This implies that $\angle BAC = 3\pi/7$. □

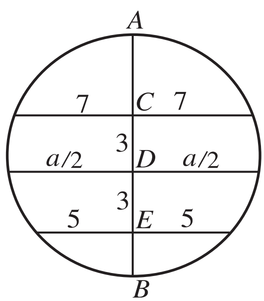
The final Example is number 28 from the 1995 AHSME.

EXAMPLE 3 Two parallel chords in a circle have lengths 10 and 14, and the distance between them is 6. What is a, the length of the parallel chord that is midway between them?



(A) 12 **(B)** $2\sqrt{39}$ **(C)** $2\sqrt{42}$ **(D)** $4\sqrt{11}$ **(E)** $2\sqrt{46}$

Answer (E) Construct the diameter $\overline{AB}$ that bisects the given line segments, as shown.



The given line segments whose centers are at C, D, and E are all secant lines. Applying the Interior Secant Theorem consecutive to these secant lines gives

$$AC \cdot CB = 7^2 = 49, \quad AD \cdot DB = \left(\frac{a}{2}\right)^2 = \frac{a^2}{4},$$

and $AE \cdot EB = 5^2 = 25$. But

$$AD = AC + 3, \quad AE = AC + 6, \quad DB = CB - 3,$$

and $EB = BC - 6$. So

$$\begin{aligned} 25 = AE \cdot EB &= (AC + 6)(CB - 6) \\ &= AC \cdot CB + 6(CB - AC) - 36 \\ &= 13 + 6(CB - AC), \end{aligned}$$

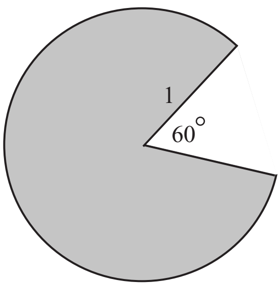
which implies that $CB - AC = 2$. As a consequence,

$$\begin{aligned}\frac{a^2}{4} &= (AC+3)(CB-3)\\ &= AC \cdot CB + 3(CB - AC) - 9 = 40 + 3 \cdot 2 = 46,\end{aligned}$$

and $a = \sqrt{4 \cdot 46} = 2\sqrt{46}$. □

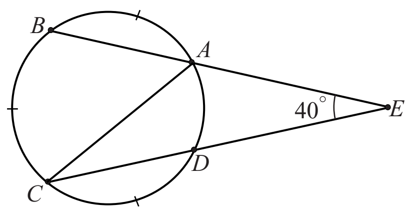
Exercises for Chapter 6

Exercise 1 In an arcade game, the "monster" is the sector of a circle of radius 1 cm, as shown in the figure. The missing piece (the mouth) has central angle 60°. What is the perimeter of the monster in cm?



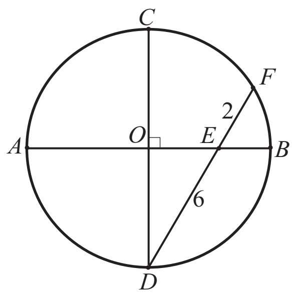
(A) $\pi + 2$ **(B)** 2π **(C)** $\frac{5}{3}\pi$ **(D)** $\frac{5}{6}\pi + 2$ **(E)** $\frac{5}{3}\pi + 2$

Exercise 2 In the figure, $\angle E = 40°$ and $\overparen{AB}$, $\overparen{BC}$, and $\overparen{CD}$ have the same length. What is $\angle ACD$?



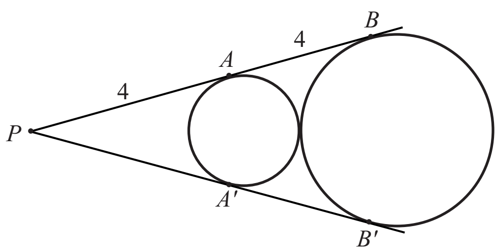
(A) 10° **(B)** 15° **(C)** 20° **(D)** 22.5° **(E)** 30°

Exercise 3 In the figure, $\overline{AB}$ and $\overline{CD}$ are diameters of the circle with center O and AB is perpendicular to CD. Chord $\overline{DF}$ intersects $\overline{AB}$ at E with $DE = 6$ and $EF = 2$. What is the area of the circle?



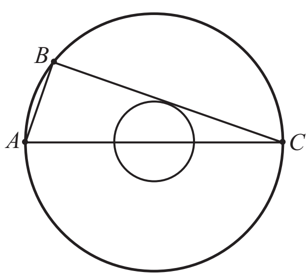
(A) 23π **(B)** $\frac{47}{2}\pi$ **(C)** 24π **(D)** $\frac{49}{2}\pi$ **(E)** 25π

Exercise 4 Two circles are externally tangent. Lines PAB and $PA'B'$ are common tangents with A and A' on the smaller circle and B and B' on the larger circle. In addition, $PA = AB = 4$. What is the area of the smaller circle?



(A) 1.44π **(B)** 2π **(C)** 2.56π **(D)** $\sqrt{8}\pi$ **(E)** 4π

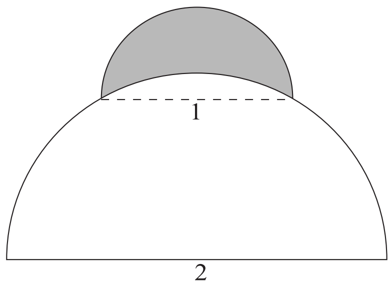
Exercise 5 In the figure, the ratio of the radii of two concentric circles is 1:3, $\overline{AC}$ is a diameter of the larger circle, $\overline{BC}$ is a chord of the larger circle



that is tangent to the smaller circle, and $AB = 12$. What is the radius of the larger circle?

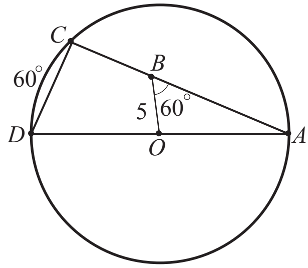
(A) 13 **(B)** 18 **(C)** 21 **(D)** 24 **(E)** 26

Exercise 6 A semicircle of diameter 1 sits on top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. What is the area of this lune?



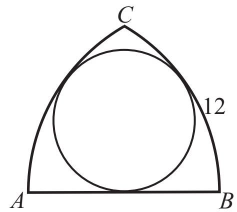
(A) $\frac{1}{12}(2\pi - 3\sqrt{3})$ **(B)** $\frac{1}{12}(3\sqrt{3} - \pi)$ **(C)** $\frac{1}{24}(6\sqrt{3} - \pi)$
(D) $\frac{1}{24}(6\sqrt{3} + \pi)$ **(E)** $\frac{1}{12}(3\sqrt{3} + \pi)$

Exercise 7 In a circle with center O, $\overline{AD}$ is a diameter, ABC is a chord, $BO = 5$ and $\angle ABO = \overset{\frown}{CD} = 60°$. What is BC?



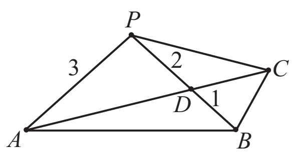
(A) 3 **(B)** $5 - \sqrt{3}$ **(C)** $3 + \sqrt{3}$ **(D)** $5 - \frac{\sqrt{3}}{2}$ **(E)** 5

Exercise 8 Circular arcs $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$ have centers at B and A, respectively, and the circle in the figure is tangent to $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$. The length of $\overset{\frown}{BC}$ is 12. What is the circumference of the circle?



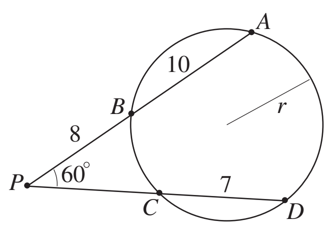
(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

Exercise 9 Point P is equidistant from A and B, $\angle APB = 2\angle ACB$, and $\overline{AC}$ intersects $\overline{BP}$ at point D with $PB = 3$ and $DB = 1$, as shown. What is $AD \cdot CD$?



(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Exercise 10 A circle of radius r has chords $\overline{AB}$ of length 10 and $\overline{CD}$ of length 7 which are extended through B and C, respectively, to intersect outside the circle at P. In addition, $\angle APD = 60°$ and $BP = 8$. What is r^2?



(A) 70 **(B)** 71 **(C)** 72 **(D)** 73 **(E)** 74

7
Polygons

7.1 Introduction

In Chapter 5 we considered the geometric properties of triangles, whose sides are composed of three straight line segments. In this chapter we expand the topic to more general geometric figures whose sides are straight line segments. These are called *polygons*.

7.2 Definitions

Definition 1 A **polygon** is a geometric figure in the plane whose sides consist of straight line segments, and no two consecutive sides lie on the same straight line.

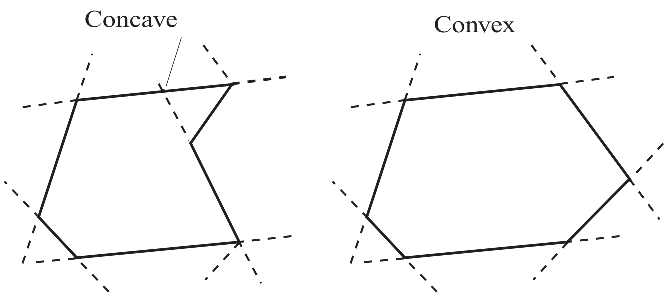
Definition 2 An n-sided polygon is called an ***n*-gon**. The most common n-gons have special names:

- A 3-gon is a triangle;
- A 4-gon is a quadrilateral;
- A 5-gon is a pentagon;
- A 6-gon is a hexagon;
- A 7-gon is a heptagon;
- An 8-gon is an octagon;

and so on.

When $n > 3$ we classify n-gons as either concave or convex. We will be primarily interested in *convex* polygons.

Definition 3 A polygon is **convex** if every line segment between two points in the interior of the polygon is contained entirely within the polygon. A polygon that is not convex is said to be **concave**. When a polygon is concave the extension of some side of the polygon intersects some other side.



The most commonly seen n-gons are the convex polygons whose sides all have the same lengths and meet their adjacent sides at the same angle.

DEFINITION 4 A polygon that is equilateral and equiangular is said to be a **regular** polygon.

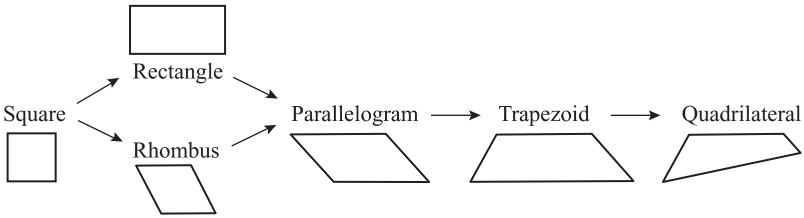
7.3 Results about Quadrilaterals

Some of the results we consider in this section will be special cases of the results in the general polygons section, but quadrilaterals are seen so frequently that it is good to have them for ready access.

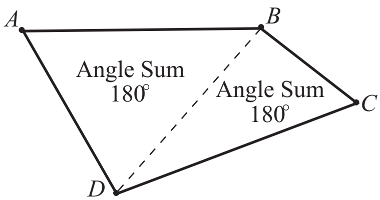
DEFINITION 1 Quadrilaterals have some special definitions:

- A regular quadrilateral is a **square**;
- An equiangular quadrilateral is a **rectangle**;
- An equilateral quadrilateral is a **rhombus**;
- A quadrilateral with both pairs of opposite sides parallel is a **parallelogram**.
- A quadrilateral with two sides parallel is a **trapezoid**.
  - If the two nonparallel sides of a trapezoid are have equal length it is **isosceles**.

From the definitions, we have the hierarchy of quadrilaterals shown as follows. Every statement about a specific quadrilateral is also true about those quadrilaterals that point to it.



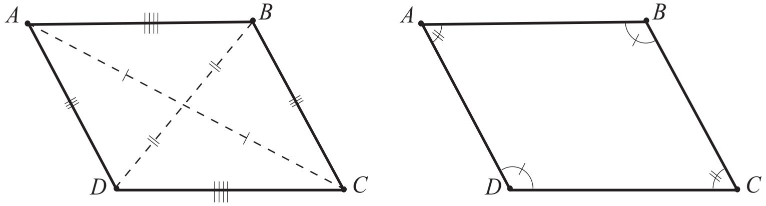
RESULT 1 The sum of the interior angles of any quadrilateral is 360°.



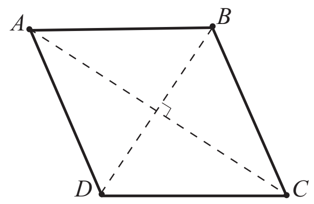
This result holds for both concave and convex quadrilaterals since in either case we can subdivide the quadrilateral into two triangles. It implies, in particular, that the interior angles of a rectangle are all 90°.

RESULT 2 A quadrilateral is a parallelogram if and only if any of the following is true:

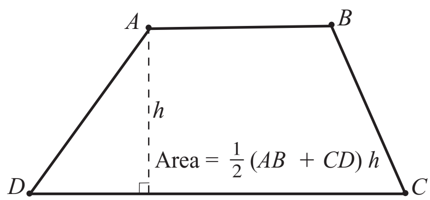
- The diagonals bisect each other.
- The lengths of the opposite sides are equal.
- The opposite angles are equal.
- The consecutive angles are supplementary, that is, they sum to 180°.
- The sum of the squares of the diagonals is twice the sum of the square of a pair of two adjacent sides, that is, $AC^2 + BD^2 = 2(AB^2 + BC^2)$.



Result 3 A parallelogram is a rhombus if and only if its diagonals are perpendicular.



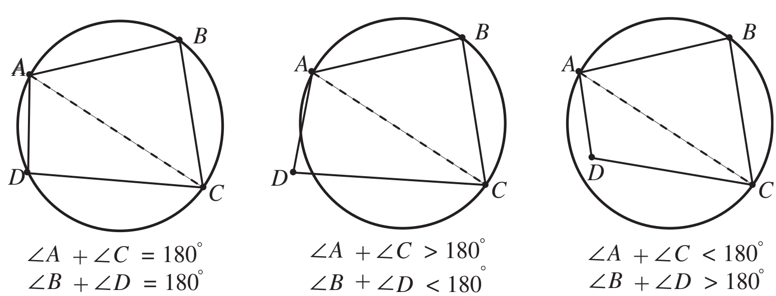
Result 4 The area of a trapezoid is the product of the average of the lengths of the parallel sides and the length of a perpendicular line segment that intersects the parallel sides.



As a consequence,

- The area of a rhombus is the product of one of its sides and the length of a perpendicular line segment to the opposite side.
- The area of a rectangle is the product of a pair of consecutive sides.

It is often important to know whether there is a circle that passes through the vertices of a particular quadrilateral $ABCD$. This will be the case if



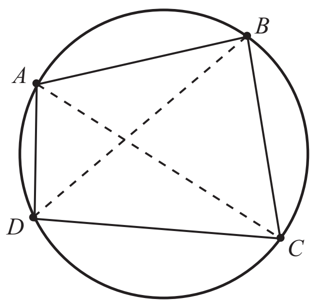
and only if the unique circle that passes through three of the vertices, A, B, and C also passes through D. Considering the figure, we see that this occurs if and only if the arc ADC and the arc ABC give the entire circle that passes through A, B, and C. The Inscribed Circle Theorem applied to this conclusion gives the following result.

Result 5 A (unique) circle passes through the vertices of a quadrilateral if and only if its opposite angles are supplementary. Such a quadrilateral is called **cyclic**.

The following useful result holds for cyclic quadrilaterals.

Result 6 Ptolemy's Theorem: The sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals, that is

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$



The final result concerning quadrilaterals involves their diagonals. The application of this result is more often seem on the AIME problems, but it also has occasional application in the latter AMC problems.

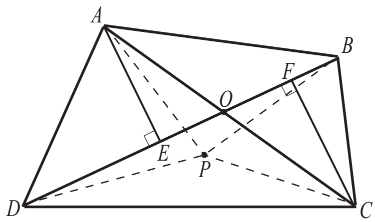
Result 7 If P is a point in the interior of the convex quadrilateral $ABCD$, then

$$\text{Area}(ABCD) \leq \frac{1}{2}(AP + PC)(BP + PD).$$

Moreover, equality occurs if and only if the diagonals of $ABCD$ intersect at right angles at the point P.

This result is not difficult to verify. Consider the figure shown, where E and F represent the altitudes from BD to $\triangle DAB$ and $\triangle BDC$, respectively, and O is the intersection of the diagonals. Then

$$\text{Area}(ABCD) = \text{Area}(\triangle DAB) + \text{Area}(\triangle BDC) = \frac{1}{2}(AE + FC)BD.$$



Since we have

$$DP + PB \geq DB \quad \text{and} \quad AP + PC \geq AO + OC \geq AE + FC,$$

we see that

$$\frac{1}{2}(AP + PC)(DP + PB) \geq \frac{1}{2}(AE + FC)BD = \text{Area}(ABCD).$$

Equality will occur only if all the inequalities are equalities. But this implies that P lies on BD and that E, O, F, and P occur at the same point, that is, when the diagonals of $ABCD$ intersect at right angles at the point P.

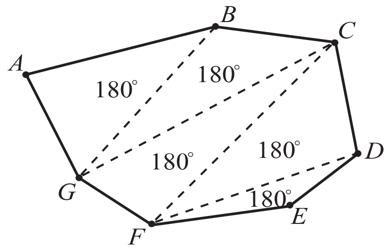
7.4 Results about General Polygons

Many of the results about general, and even regular, polygons need to be postponed until the trigonometry material has been discussed. In this chapter we will consider only a few of the basic results that find frequent application on the AMC problems.

Result 1 The sum of the interior angles of an n-gon is $(n-2)180°$.

This result follows from the fact that every n-gon can be partitioned into $n-2$ triangles, each of which has an angle sum of $180°$. From this follows the result that is most frequently needed about n-gons.

n sides
$n-2$ triangles



RESULT 2 For a regular n-gon we have

- Each interior angle has measure $\dfrac{n-2}{n}180°$.
- Each exterior angle has measure $360° - \dfrac{n-2}{n}180° = \dfrac{n+2}{n}180°$.

DEFINITION 1 Two polygons with the same number of sides are **similar** if the corresponding sides are proportional and if the angles of one are equal to the corresponding angles of the other.

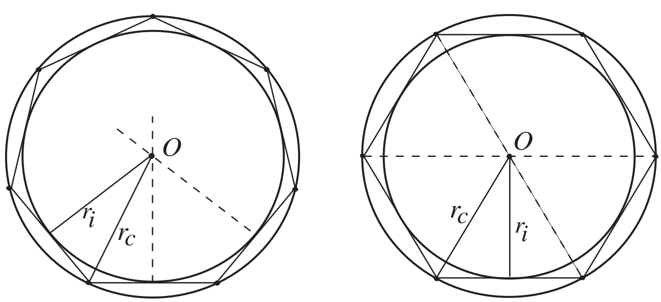
For similar polygons we have an area result that corresponds to the area result for similar triangles.

RESULT 3 If two polygons are similar, then the ratio of their areas is as the square of the ratio of a pair of corresponding sides.

There is a close connection between regular polygons and circles.

RESULT 4 Regular polygons have inscribed and circumscribed circles with a common center.

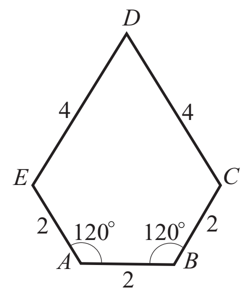
- The radius of the inscribed circle is the distance from the center along a perpendicular bisector of a side.
- The radius of the circumscribed circle is the distance from the center to a vertex.



Examples for Chapter 7

The first Example is number 14 from the 1993 AHSME.

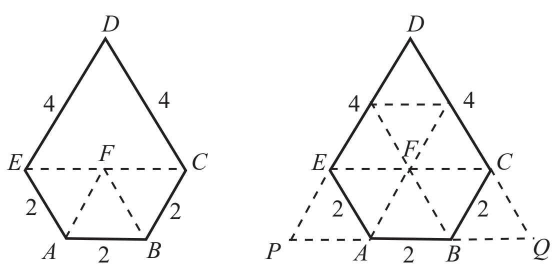
Example 1 The convex polygon $ABCDE$ has $\angle A = \angle B = 120°$, $EA = AB = BC = 2$, and $CD = DE = 4$. What is the area of $ABCDE$?



(A) 10 **(B)** $7\sqrt{3}$ **(C)** 15 **(D)** $9\sqrt{3}$ **(E)** $12\sqrt{5}$

Answer (B) Draw the line segment $\overline{EC}$ and label the midpoint of this segment F, as shown. In the figure on the left we see that the polygon is partitioned into one equilateral triangle with side length 4 and three equilateral triangles with side length 2. So the area of the polygon is

$$\frac{\sqrt{3}}{4}(4)^2 + 3\left(\frac{\sqrt{3}}{4}(2)^2\right) = \frac{\sqrt{3}}{4}(16+12) = 7\sqrt{3}.$$



OR

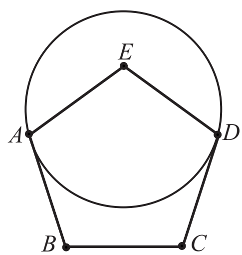
Consider the figure on the right, where the line segments $\overline{DE}$, $\overline{DC}$, and $\overline{AB}$ have been extended until they intersect at P and Q, and the equilateral

$\triangle DPQ$ with side length 6 has been partitioned into equilateral triangles with side length 2. Since 7 of the 9 smaller equilateral triangles make up the polygon, the polygon has area

$$\left(\frac{7}{9}\right)\left(\frac{\sqrt{3}}{4}\right)(6)^2 = \frac{7}{9}\cdot\frac{36}{4}\cdot\sqrt{3} = 7\sqrt{3}. \qquad \square$$

The next Example is number 17 from the 1995 AHSME.

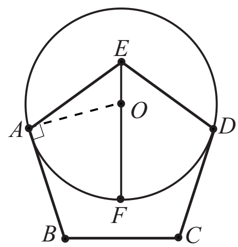
EXAMPLE 2 Given a regular pentagon $ABCDE$, a circle can be drawn that is tangent to $\overline{DC}$ at D and to $\overline{AB}$ at A. What is the degree measure of the minor arc $\widehat{AD}$?



(A) 72 **(B)** 108 **(C)** 120 **(D)** 135 **(E)** 144

Answer (E) Construct the line segment from E through the center of the circle, label the center as O and the intersection of the line segment with the circle as F. Since the measure of each of the interior angles of the pentagon is $((5-2)/5)180^\circ = 108^\circ$, we have

$$\angle AEO = \frac{1}{2}\angle AED = 54^\circ \quad \text{and} \quad \angle EAO = \angle EAB - 90^\circ = 18^\circ.$$



The Exterior Angle Theorem implies that

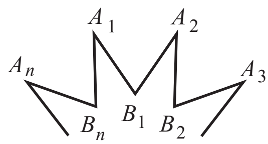
$$\angle AOF = \angle AEO + \angle EAO = 54 + 18 = 72^\circ.$$

Since $\angle AOF$ is a central angle, arc

$$\overset{\frown}{AD} = 2\,\overset{\frown}{AF} = 2\angle AOF = 144^\circ. \qquad \square$$

The final Example is number 20 from the 1992 AHSME.

EXAMPLE 3 A typical "n-pointed regular star" is shown. It is a polygon all of whose $2n$ edges have the same length, the acute angles $A_1, A_2, \ldots, A_n$ are all congruent, and the acute angles $B_1, B_2, \ldots, B_n$ are all congruent. Suppose that $\angle B_1 - \angle A_1 = 10^\circ$. What is the value of n?



(A) 12 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** 60

Answer (D) The sum of the interior angles of an n-sided polygon is $(n-2)180^\circ$, so this $2n$-sided polygon has interior angle sum given by

$$(2n-2)180^\circ = \angle A_1 + \cdots + \angle A_n + (360^\circ - \angle B_1) + \cdots + (360^\circ - \angle B_n).$$

But

$$\angle A_1 = \angle A_2 = \cdots = \angle A_n$$

and

$$\angle A_1 + 10^\circ = \angle B_1 = \angle B_2 = \cdots = \angle B_n,$$

so

$$(n-1)360^\circ = n\angle A_1 + 360^\circ n - n\angle A_1 - 10^\circ n.$$

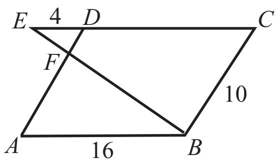
Hence $360n - 360 = 350n$ which implies that $n = \frac{360}{10} = 36$. $\square$

Exercises for Chapter 7

Exercise 1 A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. What is the distance, in feet, between the stripes?

(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 25

Exercise 2 A parallelogram $ABCD$ has $\angle ABC = 120°$, $AB = 16$, and $BC = 10$. Extend $\overline{CD}$ through D to E so that $DE = 4$, and label as F the intersection of AD and BE. What is the value of DF?

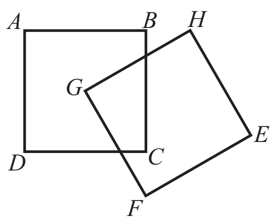


(A) 1 **(B)** 1.5 **(C)** 2 **(D)** 2.5 **(E)** 3

Exercise 3 Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

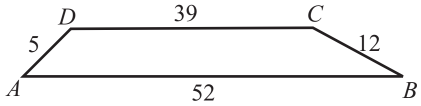
(A) $\frac{2}{3}\pi$ **(B)** 2π **(C)** $\frac{5}{2}\pi$ **(D)** $\frac{8}{3}\pi$ **(E)** 3π

Exercise 4 Squares $ABCD$ and $EFGH$ are congruent with side length 10, and G is the center of $ABCD$. What is the area of the region covered by the union of these squares?



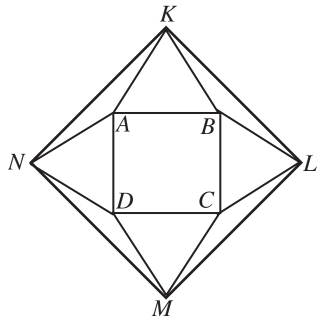
(A) 75 **(B)** 100 **(C)** 125 **(D)** 150 **(E)** 175

Exercise 5 In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. What is the area of $ABCD$?



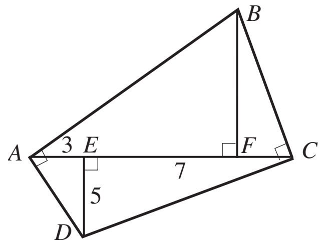
(A) 182 **(B)** 195 **(C)** 210 **(D)** 234 **(E)** 260

Exercise 6 Square $ABCD$ has area 16. Points K, L, M, and N lie outside $ABCD$ so that $\triangle AKB$, $\triangle BLC$, $\triangle CMD$, and $\triangle DNA$ are all equilateral. What is the area of $KLMN$?



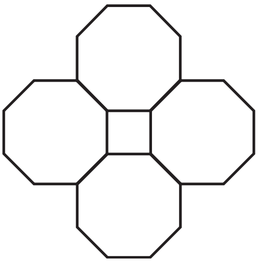
(A) 32 **(B)** $16 + 16\sqrt{3}$ **(C)** 48 **(D)** $32 + 16\sqrt{3}$ **(E)** 64

Exercise 7 The quadrilateral $ABCD$ has right angles at A and at C. Points E and F are on $\overline{AC}$ with DE and BF perpendicular to AC. In addition, $AE = 3$, $DE = 5$, and $CE = 7$. What is BF?



(A) 3.6 **(B)** 4 **(C)** 4.2 **(D)** 4.5 **(E)** 5

Exercise 8 A regular polygon of m sides is exactly enclosed by m regular polygons of n sides each. (The illustration shows the situation when $m = 4$ and $n = 8$.) What is the value of n when $m = 10$?

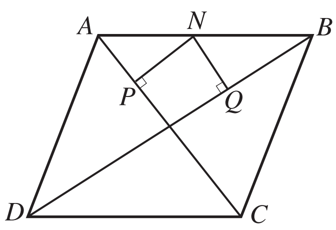


(A) 5 **(B)** 6 **(C)** 14 **(D)** 20 **(E)** 26

Exercise 9 A point P in the interior of the convex quadrilateral $ABCD$ with area 2002 has $PA = 24$, $PB = 32$, $PC = 28$, and $PD = 45$. Find the perimeter of $ABCD$.

(A) $4\sqrt{2002}$ **(B)** $2\sqrt{8465}$ **(C)** $2(48 + \sqrt{2002})$ **(D)** $2\sqrt{8633}$
(E) $4(36 + \sqrt{113})$

Exercise 10 Let $ABCD$ be a rhombus with $AC = 16$ and $BD = 30$. Let P and Q be the feet of the perpendiculars from N, which is on $\overline{AB}$, to $\overline{AC}$ and $\overline{BD}$, respectively. Which of the following is closest to the minimum possible value of PQ?



(A) 6.5 **(B)** 6.75 **(C)** 7 **(D)** 7.25 **(E)** 7.5

8
Counting

8.1 Introduction

This chapter considers problems that involve permutations, combinations, partitioning, and other counting-oriented problems. Some AMC problems involve nothing more than the application of these ideas, others use these counting techniques as a first step when solving a more complicated problem.

8.2 Permutations

Definition 1 A **permutation** of a collection of distinguishable objects is an arrangement of the objects in some specific order.

For example, *acbd* and *dabc* are both permutations of the letters a, b, c, and d. What generally interests us is the number of different permutations that are possible from a given collection. In this case there are 24 different permutations of these 4 letters. This is because any one of the 4 letters could be first. Then there are 3 possible choices remaining for the second letter, 2 possible choices remaining for the third, and, of course, only one choice remaining for the last. This is a special case of the following general result.

Result 1 The number distinct permutations of N distinguishable objects is

$$N \cdot (N-1) \cdot (N-2) \cdots 2 \cdot 1 = N!$$

When some of the objects are not distinguishable, the number of distinct permutations is reduced. Suppose that we want the number of distinct permutations of the five letters a, a, b, c, and d. If the letters were distinguishable the number would be 5!. However, we cannot distinguish between the two a's so we must reduce the number by a factor of 2, since the first time an a appeared it could be any one of the 2 possibilities. In a similar manner, if the set of letters was a, a, a, b, and c. We would need to reduce the number by a larger factor 3! since the first time an a appeared in a permutation, it could be any one of the three possibilities, the second a appearing could be any of the 2 remaining a's, and there would be only one possibility remaining for the third appearing a. This logic leads to a general permutation result for collections of objects not all of which can be distinguished.

RESULT 2 Suppose that we have N objects in m distinguishable classes, where there are n_1 indistinguishable objects in class 1, n_2 indistinguishable objects in class 2, $\ldots$, n_m indistinguishable objects in class m, with $N = n_1 + n_2 + \cdots + n_m$. Then the number of distinct permutations of the N objects is

$$\frac{N!}{n_1! \cdot n_2! \cdots n_m!}.$$

8.3 Combinations

DEFINITION 1 A **combination** is a way to choose a certain number of objects from a given collection, where the order in which the objects are chosen is not important.

As an example, suppose we are again given the letters a, b, c, and d. If only 1 object is chosen there are 4 distinct combinations, which are $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$. If 2 objects are chosen there are 6 distinct combinations, which are $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$. There are 4 distinct combinations if 3 objects are chosen, the same as if only 1 object was chosen, since choosing 3 of the 4 is the same as not choosing 1 of the 4. Finally, there is only 1 possible combination if all 4 (or 0) are chosen. Notice that the pattern of distinct combinations when 0, 1, 2, 3, or 4 objects are chosen from a set of 4 objects is 1, 4, 6, 4, 1, which is the same as the coefficients in the binomial expansion of the term

$$(a+b)^4 = 1 \cdot a^4 + 4 \cdot a^3 b + 6 \cdot a^2 b^2 + 4 \cdot ab^3 + 1 \cdot b^4.$$

This connection is true in general.

Result 1 The number of different combinations of k objects chosen from n objects is the kth binomial coefficient,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

The important thing to notice concerning permutations and combinations is:

- For a permutation, all the objects are selected and the distinguishing feature is the order of selection.
- For a combination, not all the objects need be selected and the order of selection does not matter.

8.4 Counting Factors

Suppose that we want to know the number of even multiples of a positive integer m that are less than or equal to another positive integer n. If n itself happens to be a multiple of m, say $n = k \cdot m$, then the answer is obviously $k = n/m$, since the multiples are $m, 2m, \ldots, km = n$. On the other hand, if n is not a multiple of m, then some integer k must exist with $km < n < (k+1)m$, and k is the largest integer that is less than n/m. To facilitate discussion, we make the following definition.

Definition 1 For any real number x the **floor** function of x, denoted $\lfloor x \rfloor$, is the largest integer that is less than or equal to x.

For positive real numbers x the floor function simply rounds the number down to its integer portion. For example, $\lfloor 1.7 \rfloor = 1$, $\lfloor \sqrt{5} \rfloor = 2$, and $\lfloor 357 \rfloor = 357$.

Result 1 The number of multiples of the positive integer m that are less than or equal to the positive integer n is $\left\lfloor \frac{n}{m} \right\rfloor$.

Be careful when using this result. For example, the number of multiples of 3 that are between 5 and 10, inclusive, is

$$\left\lfloor \frac{10}{3} \right\rfloor - \left\lfloor \frac{5}{3} \right\rfloor = 3 - 1 = 2, \quad \text{it is } \textit{not} \quad \left\lfloor \frac{10-5}{3} \right\rfloor = \left\lfloor \frac{5}{3} \right\rfloor = 1.$$

A frequent application of the floor function is to determine how many numbers in a certain set are divisible by some numbers but not by others. Consider the following problem.

PROBLEM 1 Determine the number of integers not exceeding 50 that are divisible by 2 or by 3.

First we note that there are

$$\left\lfloor \frac{50}{2} \right\rfloor = 25 \quad \text{numbers that are divisible by 2,}$$

and there are

$$\left\lfloor \frac{50}{3} \right\rfloor = 16 \quad \text{numbers that are divisible by 3,}$$

so we might conclude that there are $25 + 16 = 41$ that satisfy this property. However, this is not correct because we have counted twice all the multiples of $6 = 2 \cdot 3$. Hence we need to subtract this overcount, the

$$\left\lfloor \frac{50}{2 \cdot 3} \right\rfloor = \left\lfloor \frac{50}{6} \right\rfloor = 8 \quad \text{numbers that are divisible by 6.}$$

This gives the true result $25 + 16 - 8 = 33$.

Suppose now that we expand the problem as follows.

PROBLEM 2 Determine the number of integers not exceeding 50 that are divisible by 2 or by 3 or by 5.

The numbers that are divisible by 2, 3, and 5, respectively, are

$$\left\lfloor \frac{50}{2} \right\rfloor = 25, \quad \left\lfloor \frac{50}{3} \right\rfloor = 16, \quad \text{and} \quad \left\lfloor \frac{50}{5} \right\rfloor = 10.$$

and $25 + 16 + 10 = 51$ which is more than the number in the set. However, this total has counted twice all those that are multiples of both 2 and 3, those that are multiples of both 2 and 5, and those that are multiples of both 3 and 5, and there are

$$\left\lfloor \frac{50}{2 \cdot 3} \right\rfloor = 8, \quad \left\lfloor \frac{50}{2 \cdot 5} \right\rfloor = 5 \quad \text{and} \quad \left\lfloor \frac{50}{3 \cdot 5} \right\rfloor = 3$$

of these, respectively.

However, subtracting these to give

$$25 + 16 + 10 - 8 - 5 - 3 = 35$$

results in subtracting twice those numbers that are multiples of all of 2 and 3 and 5, that is, the multiples of 30. Since there is only $\lfloor 50/30 \rfloor = 1$ of these, the true total is

$$25 + 16 + 10 - 8 - 5 - 3 + 1 = 36.$$

In this case it is easy to see that this is the correct total, since the only numbers *not* divisible by any of 2, 3, or 5 are the 14 numbers

$$1,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ \text{and } 49.$$

The demonstration problems given here are special cases of a general result known as the Inclusion-Exclusion Theorem. You can find a more complete discussion of this and other counting results in the excellent book *Mathematics of Choice: How to Count Without Counting* by Ivan Niven.

Result 2 The Inclusion-Exclusion Principle: Suppose that N objects are given. Let

- $N(\alpha)$ be the number of these objects having property α,
- $N(\beta)$ be the number of these objects having property β,
- $N(\gamma)$ be the number of these objects having property γ, etc.

Let

- $N(\alpha, \beta)$ be the number of these objects having properties α and β,
- $N(\alpha, \gamma)$ be the number of these objects having properties α and γ,
- $N(\beta, \gamma)$ be the number of these objects having properties β and γ, etc

Let $N(\alpha, \beta, \gamma)$ be the number of object having properties, α, β, and γ, etc. Then the number of the N objects having all of properties $\alpha, \beta, \gamma, \ldots$ is

$$\begin{aligned}(N(\alpha) + N(\beta) + N(\gamma) + \cdots)& \\ - (N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) + \cdots)& \\ + (N(\alpha, \beta, \gamma) + \cdots) - \cdots .&\end{aligned}$$

Examples for Chapter 8

The first Example is number 16 from the 2003 AMC 10B.

EXAMPLE 1 A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night for a year?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (E) Suppose that the restaurant offers M main courses. Since the choosing of appetizer, main course, and desert are independent events, there are

$$(\text{Appetizers})(\text{Main courses})(\text{Deserts}) = (2M)(M)(3) = 6M^2$$

distinct ways to choose a meal. To cover all possible years, we need to find the smallest integer value of M such that

$$6M^2 \geq 366, \quad \text{that is,} \quad M^2 \geq 61, \quad \text{so} \quad M \geq 8. \qquad \square$$

The next Example is number 21 from the 2003 AMC 10A.

EXAMPLE 2 Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

(A) 22 **(B)** 25 **(C)** 27 **(D)** 28 **(E)** 729

Answer (D) The straightforward way to work this problem is to consider all the possible ways the cookies can be chosen. To simplify the counting process, suppose that the ordered triple (c, o, p) denotes the number of chocolate chip, oatmeal and peanut butter cookies chosen. Then if only one type of cookie is selected, the number of possible triples is

$$\binom{3}{1} = 3: \quad (6, 0, 0),\ (0, 6, 0),\ (0, 0, 6).$$

If two types of cookie are selected with 5 of one and 1 of the other, the number of possible triples is

$$\binom{3}{1} \cdot \binom{2}{1} = 3 \cdot 2 = 6.$$

These are (5, 1, 0), (5, 0, 1), (1, 5, 0), (0, 5, 1), (1, 0, 5), (0, 1, 5).

Similarly if two types of cookie are selected with 4 of one and 2 of the other, the number of possible triples is

$$\binom{3}{1} \cdot \binom{2}{1} = 3 \cdot 2 = 6.$$

These are (4, 2, 0), (4, 0, 2), (2, 4, 0), (0, 4, 2), (2, 0, 4), (0, 2, 4).

If two types of cookie are selected with 3 of one and 3 of the other, the number of possible triples is

$$\binom{3}{1} = 3 : \quad (3, 3, 0),\ (3, 0, 3),\ (0, 3, 3).$$

If all three types of cookies are chosen, the distribution could be 4 of one and 1 of each of the others, so the number of possible triples is

$$\binom{3}{1} = 3 : \quad (4, 1, 1),\ (1, 4, 1),\ (1, 1, 4);$$

or 3 of one, 2 of a second, and 1 of the third, and the number of possible triples is

$$\binom{3}{1} \cdot \binom{2}{1} = 3 \cdot 2 = 6,$$

which are (3, 2, 1), (3, 1, 2), (2, 3, 1), (2, 1, 3), (1, 2, 3); or for two of each, the distribution (2, 2, 2).

So the total possibilities are

$$3 \cdot \binom{3}{1} + 3\binom{3}{1} \cdot \binom{2}{1} + 1 = 3 \cdot 3 + 3 \cdot 6 + 1 = 9 + 18 + 1 = 28.$$

OR

Although the first solution is not difficult, it is long, time-consuming, and, unless all the possibilities are listed, as we did, it would be easy to miscount. In addition, the answer is not a multiple of 3, as we might expect given that we start with 3 types of cookies and are choosing 6. To see an alternate approach that makes the answer more intuitive, suppose that we extend the problems by adding two positions for the divider strips to the cookies, as shown below.

$$\underline{C_1}\quad \underline{C_2}\quad \underline{C_3}\quad \underline{C_4}\quad \underline{C_5}\quad \underline{C_6}\quad \underline{D_1}\quad \underline{D_2}$$

The number of C's to the left of the divider D_1 gives the number of Chocolate Chip cookies chosen, the number of C's between the dividers D_1 and D_2 gives the number of Oatmeal cookies chosen, and the number of C's to the right of the divider D_2 gives the number of Peanut Butter cookies chosen. Some sample situations are shown in the following figure.

1 Chocolate Chip, 3 Oatmeal, 2 Peanut Butter:

$$\underline{C_1}\quad \underline{D_1}\quad \underline{C_2}\quad \underline{C_3}\quad \underline{C_4}\quad \underline{D_2}\quad \underline{C_5}\quad \underline{C_6}$$

0 Chocolate Chip, 1 Oatmeal, 5 Peanut Butter:

$$\underline{D_1}\quad \underline{C_1}\quad \underline{D_2}\quad \underline{C_2}\quad \underline{C_3}\quad \underline{C_4}\quad \underline{C_5}\quad \underline{C_6}$$

4 Chocolate Chip, 2 Oatmeal, 0 Peanut Butter:

$$\underline{C_1}\quad \underline{C_2}\quad \underline{C_3}\quad \underline{C_4}\quad \underline{D_1}\quad \underline{C_5}\quad \underline{C_6}\quad \underline{D_2}$$

By considering all possible positions that the dividers can occupy, we can find the number of possible ways the types of cookies can be chosen. Since there are 8 possible slots for the 2 dividers, the dividers can occupy the slots in

$$\binom{8}{2} = \frac{8 \cdot 7}{2} = 28 \text{ ways.}$$

So the types of cookies can also be chosen in this number of ways. □

The final Example is number 25 from the 2001 AMC 10 and number 12 from the AMC 12.

EXAMPLE 3 A spider has one sock and one shoe for each of its eight legs. On each leg the sock must be put on before the shoe. In how many different orders can the spider put on its socks and shoes?

(A) $8!$ **(B)** $2^8 \cdot 8!$ **(C)** $(8!)^2$ **(D)** $\dfrac{16!}{2^8}$ **(E)** $16!$

Answer (D) Without the condition that on each of the legs the sock must go on before the shoe, the solution to the problem would be easy. The shoes and socks total 16 objects, so there would be 16! orders. Since the requirement of putting the respective socks on before the shoes reduces this number, answer choice (E) is clearly wrong. The problem is to determine the reduction.

Suppose that we let S_i be the shoe that is placed on the ith leg, and s_i be the sock that is placed on the ith leg, where $1 \le i \le 8$. If we write the order that the spider placed objects on its legs, it must start with one of the socks, say s_i. Then it can place any of its remaining socks but only the shoe S_i. After the second object is placed, what immediately can follow depends on whether the second object chosen was the shoe S_i or one of the remaining socks.

It seems clear that this problem is too difficult to solve by trying to find a pattern that will enumerate all the possibilities. Even if the spider had only 2 legs, there would be 6 possibilities

$$s_1S_1s_2S_2,\ s_1s_2S_1S_2,\ s_1s_2S_2S_1,\ s_2S_2s_1S_1,\ s_2s_1S_1S_2,\ \text{or}\ s_2s_1S_2S_1.$$

However, this simplified enumeration gives the clue to the pattern. For the two-legged spider, the total number of orders, $4! = 24$, has been reduced by multiplying by $1/4 = (1/2)^2$ because, for each leg, half of the time the sock will be first, which is a success, and half of the time the shoe will be first, which is a failure.

Since there must be a success on each of the 8 legs of our original spider, the total numbers of ways that the spider can be successful is the total number of ways the 16 objects can be chosen, 16!, multiplied by the likelihood the spider will be successful on each leg $(1/2)^8$. So the total number of successful orders is $16!/2^8$. □

Exercises for Chapter 8

Exercise 1 At a party, each man danced with exactly three women and each woman danced with exactly two men. Twelve men attended the party. How many women attended the party?

(A) 8 **(B)** 12 **(C)** 16 **(D)** 18 **(E)** 24

Exercise 2 Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

(A) 24 **(B)** 256 **(C)** 768 **(D)** 40,320 **(E)** 120,960

Exercise 3 A bag initially contains red marbles and blue marbles only, with more blue than red. Red marbles are added to the bag until only $1/3$ of the marbles in the bag are blue. Then yellow marbles are added to the bag until only $1/5$ of the marbles in the bag are blue. Finally, the number of blue marbles in the bag is doubled. What fraction of the marbles now in the bag are blue?

(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

Exercise 4 Nebraska, the home of the AMC, changed its license plate scheme. Each old license plate consisted of a letter followed by four digits. Each new license plate consists of three letters followed by three digits. By how many times is the number of possible license plates increased?

(A) $\frac{26}{10}$ **(B)** $\frac{26^2}{10^2}$ **(C)** $\frac{26^2}{10}$ **(D)** $\frac{26^3}{10^3}$ **(E)** $\frac{26^3}{10^2}$

Exercise 5 Using the letters A, M, O, S, and U, we can form $5! = 120$ five-letter "words". If these "words" are arranged in alphabetical order, then what position does the "word" *USAMO* occupy?

(A) 112 **(B)** 113 **(C)** 114 **(D)** 115 **(E)** 116

Exercise 6 Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

Exercise 7 Let n be a five-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

(A) 8180 **(B)** 8181 **(C)** 8182 **(D)** 9000 **(E)** 9090

Exercise 8 A 7-digit telephone number $d_1d_2d_3d_4d_5d_6d_7$ is called *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ (possibly both). Assume that each d_i can be any of the ten decimal digits $0, 1, 2, \ldots, 9$. What is the number of distinct memorable telephone numbers?

(A) 19,810 **(B)** 19,910 **(C)** 19,990 **(D)** 20,000
(E) 20,100

Exercise 9 Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. The three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can professors Alpha, Beta, and Gamma choose their chairs?

(A) 12 **(B)** 36 **(C)** 60 **(D)** 84 **(E)** 630

Exercise 10 How many 15-letter arrangements of 5 A's, 5 B's, and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

(A) $\sum_{i=0}^{5} \binom{5}{i}^3$ **(B)** $3^5 \cdot 2^5$ **(C)** 2^{15} **(D)** $\frac{15!}{(5!)^3}$ **(E)** 3^{15}

9
Probability

9.1 Introduction

Nearly every AMC examination involves problems on probability. These problems often incorporate the notions of permutations, combinations, and other counting techniques. Probability problems are sometimes difficult to interpret and a careful reading of the problem is most important.

9.2 Definitions and Basic Notions

The basic notions of probability that appear in problems on the AMC examinations involve situations where it is necessary to count the number of possible successes of some specified outcome as well as the number of all possible outcomes. When the events are equally likely to occur, the probability of success is the quotient of these two numbers.

Definition 1 The **probability** that an event occurs is defined as

$$\frac{\text{The number of distinct ways that the event can occur}}{\text{The number of all possible outcomes}}.$$

For example, suppose we have the following problem.

Problem 1 What is the probability of randomly drawing a face card from a deck of 52 ordinary playing cards?

For each of the four suits in the deck, there are three face cards, a Jack, a Queen, and a King, for a total of 12 face cards in all. Since there are 52 total cards in the deck, the probability that a random draw will yield a face card is 12/52, which reduces to 3/13 (quite a reasonable result, since each suit has 13 cards, of which 3 are face cards.)

This is a rather trivial example, because there are no complicating conditions. It would be highly unlikely that an AMC problem would have such an easy solution. More likely, the situation would be complicated by asking something like the following.

PROBLEM 2 What is the probability that when two cards are randomly drawn, without replacing the first card, that at least one of the cards is a face card?

Now we need to analyze the situation more carefully. First note that we could be successful in three distinct ways.

- Case I: the first card could be a face card, and the second card a non-face card;
- Case II: the first card could be a non-face card, and the first card a face card; or
- Case III: both cards could be face cards.

For Case I we know that there are 12 choices of face card when we draw the first card. If this first card drawn is a face card, then there are 40 choices of non-face cards for the second draw. So the number of ways of choosing a face card followed by a non-face card is $12 \cdot 40$. Similarly for Case II, the number of ways of choosing first a non-face card, and then a face card is $40 \cdot 12$. Finally for Case III, if the first card drawn is a face card, there remain only 11 face cards left for the second draw, so the number of ways that we can draw two face cards is $12 \cdot 11$. Since we are successful if any of these three distinct outcomes occurs, we can be successful in

$$12 \cdot 40 + 40 \cdot 12 + 12 \cdot 11 = 12(40 + 40 + 11) = 12 \cdot 91 = 1092$$

distinct ways. There are 52 cards that we could draw the first time, and 51 the second time, so the probability of being successful is

$$P = \frac{12 \cdot 91}{52 \cdot 51} = \frac{7}{17} \approx 0.412, \text{ or about } 41\% \text{ of the time.}$$

Alternatively, we could have counted the number of distinct successes in a slightly different way. We are successful if the first card is a face card regardless of which of the remaining 51 cards is chosen second. We are also successful if the first card is a non-face card but the second is a face card. So the number of distinct ways that a success can occur can be expressed as

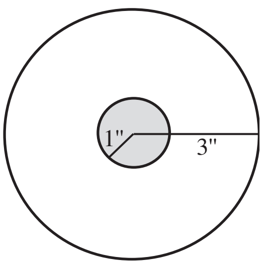
$$12 \cdot 51 + 40 \cdot 12 = 12(51 + 40) = 12 \cdot 91 = 1092.$$

Finally, we can take a negative approach to the problem. We are successful unless we have a non-face card on both choices. A non-face card can occur in $40 \cdot 39$ distinct ways, so a face card will occur on at least one of the cards in

$$52 \cdot 51 - 40 \cdot 39 = 4 \cdot 13(51 - 10 \cdot 3) = 52 \cdot 21 = 1092$$

distinct ways.

The notion of probability also occurs in situations where the actual counting of events is impossible. Consider, for example, determining the probability that a dart will hit the bulls eye with radius 1 inch in the center of a target with radius 3 inches, if it is assumed that the dart lands at a random point on the target.



We cannot determine either the number of ways the event can occur or the number of possible outcomes. Instead we use the areas of the respective regions to determine the probability.

$$P = \frac{\text{Area(Bulls eye)}}{\text{Area(Target)}} = \frac{\pi \cdot 1^2}{\pi \cdot 3^2} = \frac{1}{9}.$$

There are often numerous ways to solve a probability problem. The way to best proceed is generally the method that requires the least amount of computation. This will be illustrated in the Examples and in the Exercises.

9.3 Basic Results

Some basic facts concerning probabilities follow directly from the definition. Since the least number of occurrences of an event is at least zero, and at most the number of all possible outcomes, we have the following.

Result 1 Suppose that $P(A)$ represents the probability that an event A occurs. Then

- $0 \leq P(A) \leq 1$.
- $P(A) = 0$ only when the event cannot occur.
- $P(A) = 1$ when the event must occur.

In addition, since an event occurring and not occurring are distinct outcomes and one or the other must occur, we have the following result.

Result 2 If $P(A)$ represents the probability that A occurs, then the probability that A does not occur is $1 - P(A)$.

Many of the probability problems you will see involve the probability that two or more events will occur. To describe probabilities in this situation we use the following definition.

Definition 1 Suppose A and B are two events.

- The probability that both A and B occur is denoted $P(A \cap B)$.
- The probability that at least one of A or B occurs is denoted $P(A \cup B)$.

The following result is used repeatedly in probability problems.

Result 3 **The Inclusion-Exclusion Principle:** If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This implies that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ and that $P(A) + P(B) = P(A \cup B) + P(A \cap B)$.

This result is useful, for example, when determining the probability of choosing one card from a deck of 52 ordinary playing cards and having that card be either a Heart or a King. Choosing a Heart and choosing a King are linked events, since there is one King that is also a Heart. Hence

$$\begin{aligned} P(\text{Heart or King}) &= P(\text{Heart}) + P(\text{King}) - P(\text{Heart and King}) \\ &= \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{13+4-1}{52} = \frac{16}{52} = \frac{4}{13}. \end{aligned}$$

We could also solve the problem, of course, by noting that there are precisely 13 Hearts and 3 Kings that are not Hearts, so there are $13+3 = 16$ cards that could be chosen to be successful, of the 52 possible choices. This also gives the probability of success as $16/52 = 4/13$.

One other subject that falls within the realm of probability is the notion of *odds*. This is a form for expressing probability that is used in gambling, most commonly in horse racing. In earlier days of the AMC examinations, the use of this term was more frequent, and assumed to be understood. When it has been used on more recent examinations the definition has been given in the problem. It is a term that also occurs in various lotteries, however, so by knowing the definition and doing a few simple calculations you might become a hero in your neighborhood amongst those misguided citizens who expect to get rich quick while in truth enriching the treasury of the state.

DEFINITION 2 The **odds** that an event A occurs are r to s, also written $r : s$, when the probability that A occurs is

$$P(A) = \frac{r}{r+s}.$$

For example, if the odds are $5 : 9$ that a certain horse will win a race, then the probability of a win for the horse is expected to be

$$\frac{5}{5+9} = \frac{5}{14} \approx 0.35 \text{ or about } 35\%.$$

If the probability that a team will win a game is $2/3 = 2/(2+1)$, then the odds of that team winning are $2 : 1$.

Examples for Chapter 9

The first Example is number 10 from the 2004 AMC 10A.

Example 1 Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same?

(A) $\dfrac{19}{128}$ **(B)** $\dfrac{23}{128}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{35}{128}$ **(E)** $\dfrac{1}{2}$

Answer (D) To be successful, the number of heads on both coins could be either 0, 1, 2, or 3. In the case of both having 0 heads we have the probabilities

$$\text{for coin A: } \left(\frac{1}{2}\right)^3, \quad \text{for coin B: } \left(\frac{1}{2}\right)^4.$$

Since both must occur for a success with 0 heads this probability is

$$\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^7.$$

In the case of both coins having exactly 1 head, we have three possible places that the head could occur on the first coin and four possible tosses that the head could occur on the second coin. So we have the probabilities in this case of

$$\text{for coin A: } 3\left(\frac{1}{2}\right)^3, \quad \text{for coin B: } 4\left(\frac{1}{2}\right)^4,$$

and the probability of a success with each having exactly 1 heads is

$$3\left(\frac{1}{2}\right)^3 \cdot 4\left(\frac{1}{2}\right)^4 = 12\left(\frac{1}{2}\right)^7.$$

Similarly, for a success with exactly 2 heads on each coin, there are $\binom{3}{2} = 3$ ways of this occurring on the first coin and $\binom{4}{2} = 6$ ways on the second, so the probability in this case is

$$3\left(\frac{1}{2}\right)^3 \cdot 6\left(\frac{1}{2}\right)^4 = 18\left(\frac{1}{2}\right)^7.$$

Finally, there is only one way to have 3 heads on the first coin, but 4 ways to have 3 heads on the second, so the probability in this case is

$$\left(\frac{1}{2}\right)^3 \cdot 4\left(\frac{1}{2}\right)^4 = 4\left(\frac{1}{2}\right)^7.$$

Since we have a success when any of these cases occurs, we sum the individual probabilities to find that the probability of a success is

$$(1+12+18+4)\left(\frac{1}{2}\right)^7 = \frac{35}{128}. \qquad \square$$

The next Example is number 21 from the 2003 AMC 10B.

Example 2 A bag contains two Red beads and two Green beads. Reach into the bag and pull out a bead, replacing it with a Red bead regardless of the color. What is the probability that all the beads are Red after three such replacements?

(A) $\frac{1}{8}$ **(B)** $\frac{5}{32}$ **(C)** $\frac{9}{32}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{7}{16}$

Answer (C) For all the beads to be Red after three replacements, two of the three beads chosen must have been Green. So the selections for a successful conclusion must have been any one of the following:

- Case I: Red, Green, Green
- Case II: Green, Red, Green,
- Case III: Green, Green, Red (since they are all now red).

In Case I, the probability of first choosing a Red is $1/2$. Since no change in the colors have been made, the probability of next choosing a Green is also $1/2$. But for the final choice, there are 3 Red marbles in the bag and only 1 Green, so the probability of then choosing a Green is $1/4$. Hence the probability in Case I is

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{16}.$$

In Case II, the probability of first choosing a Green is $1/2$. But this Green is replaced by a Red, so the probability of next choosing a Red is $3/4$. Selecting a Green on the final choice is then $1/4$. Hence the probability in Case II is

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{3}{32}.$$

Finally, in Case III, the probability of first choosing a Green is $1/2$. This Green is replaced by a Red, so the probability of next choosing a Green is

1/4. Now all the marbles in the bag are Red. So the probability in Case III is

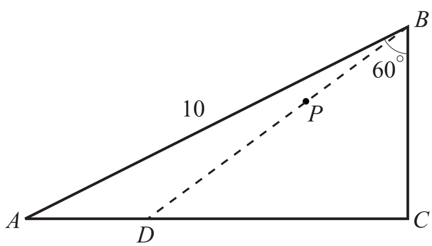
$$\frac{1}{2} \cdot \frac{1}{4} \cdot 1 = \frac{1}{8}.$$

Since these cases will each result in a success, the probability of having only Red marbles at the end of three replacements is

$$\frac{1}{16} + \frac{3}{32} + \frac{1}{8} = \frac{2+3+4}{32} = \frac{9}{32}. \qquad \square$$

The final Example is number 22 from the 2002 AMC 12A.

EXAMPLE 3 Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $\angle ABC = 60°$, and $AB = 10$. Point P is randomly chosen inside $\triangle ABC$, and $\overline{BP}$ is extended to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?



(A) $\frac{2-\sqrt{2}}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{3}}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5-\sqrt{5}}{5}$

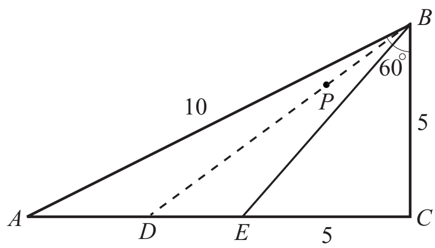
Answer (C) Since $AB = 10$ and $\angle ABC = 60°$, we have

$$BC = 5 \quad \text{and} \quad AC = \sqrt{10^2 - 5^2} = 5\sqrt{3}.$$

It is common in inequality problems to first consider the boundary case. Suppose that E is the point on $\overline{AC}$ with $BE = 5\sqrt{2}$. Then

$$CE = \sqrt{(5\sqrt{2})^2 - 5^2} = 5,$$

and $\triangle ECB$ is an isosceles right triangle.



So $BD < 5\sqrt{2}$ when D is between E and C, that is, when P is inside $\triangle BEC$. The probability that this will occur is

$$\frac{\text{Area}(\triangle BEC)}{\text{Area}(\triangle BAC)} = \frac{\frac{1}{2}5^2}{\frac{1}{2}5^2\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Hence the probability that $BD > 5\sqrt{2}$ is

$$1 - \frac{\sqrt{3}}{3} = \frac{3-\sqrt{3}}{3}. \qquad \square$$

Exercises for Chapter 9

Exercise 1 What is the probability that a randomly drawn positive factor of 60 is less than 7?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Exercise 2 Two eight-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the product of the two top numbers is greater than their sum?

(A) $\frac{1}{2}$ **(B)** $\frac{47}{64}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{55}{64}$ **(E)** $\frac{7}{8}$

Exercise 3 A point (x, y) is picked randomly from inside the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$, and $(0, 1)$. What is the probability that $x < y$?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

Exercise 4 Each face of a cube is painted either Red or Blue, each with probability $1/2$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

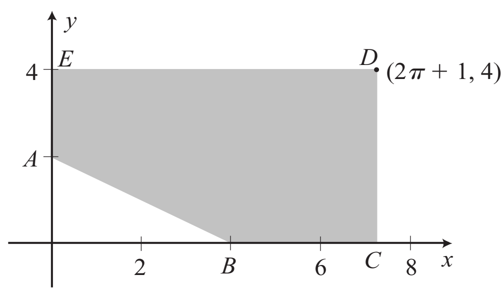
Exercise 5 A box contains exactly five chips, three Red and two White. Chips are randomly removed one at a time without replacement until all the Red chips are drawn or all the White chips are drawn. What is the probability that the last chip drawn is White?

(A) $\frac{3}{10}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{7}{10}$

Exercise 6 Juan rolls a fair regular eight-sided die. Then Amal rolls a fair regular six-sided die. What is the probability that the product of the two rolls is a multiple of 3?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{2}{3}$

Exercise 7 A point P is selected at random from the interior of the pentagon with vertices $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 4)$, and $E = (0, 4)$. What is the probability that $\angle APB$ is obtuse?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{5}{16}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{1}{2}$

Exercise 8 Let S be the set of permutations of the numbers 1, 2, 3, 4, 5 for which the first term of the permutation is not 1. A permutation is chosen randomly from S. The probability that the second term of the chosen permutation is 2, in lowest terms, is a/b. What is $a + b$?

(A) 5 **(B)** 6 **(C)** 11 **(D)** 16 **(E)** 19

Exercise 9 On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

(A) $\frac{5}{11}$ **(B)** $\frac{10}{21}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{11}{21}$ **(E)** $\frac{6}{11}$

Exercise 10 A point P is chosen at random in the interior of equilateral $\triangle ABC$. What is the probability that the area of $\triangle ABP$ is greater than both the area of $\triangle ACP$ and the area of $\triangle BCP$?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

10
Prime Decomposition

10.1 Introduction

This first chapter on number theory topics considers problems that use the Fundamental Theorem of Arithmetic, which states that every positive integer has a unique prime factorization. Many AMC problems use this decomposition as a first step in finding the number of ways that a product can be factored or that certain things can occur.

10.2 The Fundamental Theorem of Arithmetic

The positive integers, or *natural numbers*, are the fundamental building blocks of arithmetic, and the prime numbers form the basis for the natural numbers.

DEFINITION 1 A natural number greater than 1 is said to be **prime** if its only natural number divisors are 1 and itself. Natural numbers greater than 1 that are not prime are **composite**.

The Fundamental Theorem of Arithmetic is truly fundamental because it justifies the statement that the prime numbers form the basis for the natural numbers.

Result 1 The Fundamental Theorem of Arithmetic: Every natural number, other than 1, can be factored into a product of primes in only one way, apart from the order of the factors.

The proof of the Fundamental Theorem of Arithmetic is not difficult, but is somewhat long and not particularly surprising. If you are interested in seeing it, I can recommend no better treatment than that given by Ivan Niven in his excellent book *Numbers: Rational and Irrational*. This book should be in every young mathematician's library, not only for the topics he presents, but for the clarity with which they are presented. Too often it is assumed that excellent mathematicians cannot describe mathematical topics at an elementary level. This and other of Niven's writings provide an excellent counterexample.

The Fundamental Theorem of Arithmetic is often used in the AMC problems to determine the factors of an unknown integer.

PROBLEM 1 Find positive integers x and y that satisfy both

$$xy = 40 \quad \text{and} \quad 31 = 2x + 3y.$$

The problem could be solved by using the first equation to write $y = 40/x$ and then substituting into the second to produce the quadratic

$$31 = 2x + 3\left(\frac{40}{x}\right) = \frac{2x^2 + 120}{x}, \quad \text{so} \quad 0 = 2x^2 - 31x + 120.$$

This quadratic can be factored, though perhaps not obviously, as

$$0 = 2x^2 - 31x + 120 = (2x - 15)(x - 8).$$

Since x must be an integer, we have $x = 8$ and $y = 40/8 = 5$.

Alternatively, we can solve the problem by noting that since 40 has the unique prime factorization $40 = 2^3 \cdot 5$, there are only 8 possibilities for the pair (x, y). These are $(1, 40)$, $(2, 20)$, $(4, 10)$, $(8, 5)$, $(5, 8)$, $(10, 4)$, $(20, 2)$, and $(40, 1)$.

It is easy to verify that only $(x, y) = (8, 5)$ additionally satisfies $31 = 2x + 3y$.

Notice in the second solution that the prime decomposition $40 = 2^3 \cdot 5$ gives $(3 + 1)(1 + 1) = 4 \cdot 2 = 8$ ways in which 40 can be factored. This is because for x we have 4 choices for powers of 2, which are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$, as well as 2 choices for powers of 5, which are $5^0 = 1$ and $5^1 = 5$. Once the factor x is chosen, the factor y is also determined. This prime decomposition-factoring equivalence can be stated as follows.

RESULT 2 Suppose the natural number n has the prime decomposition $n = p_1^{n_1} \cdot p_2^{n_2} \dots p_k^{n_k}$, where $p_1, p_2, \dots, p_k$ is a collection of distinct primes.

Then the number of distinct divisors of n is

$$\tau(n) = (n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

This result also gives an important fact concerning the number of ways a positive integer can be factored. This follows from the fact that two factors are required to form a product that will give n.

RESULT 3 Suppose the natural number n has the prime decomposition $n = p_1^{n_1} \cdot p_2^{n_2} \dots p_k^{n_k}$, where $p_1, p_2, \dots, p_k$ is a collection of distinct primes. Then the number of distinct ways that n can be factored is

$$\frac{1}{2}\tau(n) = \frac{1}{2}(n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

RESULT 4 If n is composite, then it has a prime factor p with $p \le \sqrt{n}$.

This follows from the observation that if $p \cdot m = n = \sqrt{n} \cdot \sqrt{n}$, where p is prime and n is composite, then either $p \le \sqrt{n}$ or $m \le \sqrt{n}$. If $p \le \sqrt{n}$, the result holds since p is prime. On the other hand, if $m \le \sqrt{n}$, then m has a prime decomposition into primes that do not exceed m, each of these is also a prime factor of n, so the result again holds.

Examples for Chapter 10

The first Example is number 6 from the 1998 AHSME.

EXAMPLE 1 Suppose that 1998 is written as a product of two positive integers whose difference is as small as possible. What is this difference?

(A) 8 **(B)** 15 **(C)** 17 **(D)** 47 **(E)** 93

Answer (C) The prime decomposition of 1998 can be found by repeatedly dividing by known factors until a prime remains. Since 1998 is even, we divide by 2 until an odd factor remains. This comes quickly, $1998 = 2^1 \cdot 999$. If it wasn't originally clear that $9 = 3^2$ is a factor, it should now be, so $1998 = 2^1 \cdot 3^2 \cdot 111$. The number 111 is divisible by 3, and since 37 is prime, the prime decomposition is

$$1998 = 2^1 \cdot 3^3 \cdot 37^1.$$

There are consequently $(1+1)(3+1)(1+1)/2 = 8$ factorizations of 1998, assume that the order of the factors is not important. These factorizations are

$$1 \cdot 1998,\ 2 \cdot 999,\ 3 \cdot 666,\ 6 \cdot 333,\ 9 \cdot 222,\ 18 \cdot 111, \text{ and } 37 \cdot 54.$$

The smallest difference occurs when the factors are most nearly equal, which gives $54 - 37 = 17$.

We could have simplified the solution by noting that the smallest difference in the factors occurs when one of the factors is as close as possible to $\sqrt{1998} \approx 45$. This eliminates the need to determine all possible factorizations. □

The second Example is number 15 from the 2005 AMC 10A.

EXAMPLE 2 How many positive cubes divide $3! \cdot 5! \cdot 7!$?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (E) Written as a product of primes, we have

$$3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

A cube that is a factor has a prime factorization of the form $2^p \cdot 3^q \cdot 5^r \cdot 7^s$, where p, q, r, and s are all multiples of 3. There are 3 possible values for p, which are 0, 3, and 6. There are 2 possible values for q, which are 0 and 3. The only value for r and for s is 0. Hence there are $6 = 3 \cdot 2 \cdot 1 \cdot 1$ distinct cubes that divide $3! \cdot 5! \cdot 7!$. They are $1 = 2^0 3^0 5^0 7^0$, $8 = 2^3 3^0 5^0 7^0$, $27 = 2^0 3^3 5^0 7^0$, $64 = 2^6 3^0 5^0 7^0$, $216 = 2^3 3^3 5^0 7^0$, and $1728 = 2^6 3^3 5^0 7^0$. □

The final Example is number 21 from the 2001 AMC 12.

EXAMPLE 3 The product of four positive integers a, b, c, and d is $8!$, and they satisfy the equations

$$\begin{aligned} ab + a + b &= 524, \\ bc + b + c &= 146, \quad \text{and} \\ cd + c + d &= 104. \end{aligned}$$

What is $a - d$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Answer (D) A critical observation in the solution to this problem is that the three equations can be rewritten as

$$525 = ab + a + b + 1 = (a+1)(b+1),$$
$$147 = bc + b + c + 1 = (b+1)(c+1), \quad \text{and}$$
$$105 = cd + c + d + 1 = (c+1)(d+1).$$

Now factor the constant terms in each equation to obtain facts about the products.

$$(a+1)(b+1) = 525 = 3 \cdot 5^2 \cdot 7,$$
$$(b+1)(c+1) = 147 = 3 \cdot 7^2, \quad \text{and}$$
$$(c+1)(d+1) = 105 = 3 \cdot 5 \cdot 7.$$

Since $(a+1)(b+1)$ has a factor of $5^2 = 25$, but $(b+1)(c+1)$ has no factor of 5, $(a+1)$ must be divisible by 25. In a similar manner, $(d+1)$ must be divisible by 5.

Because $(b+1)(c+1) = 3 \cdot 7^2$, the possibilities for $(b+1)$ and $(c+1)$ are either $(b+1) = 7$ and $(c+1) = 3 \cdot 7$ or are $(b+1) = 3 \cdot 7$ and $(c+1) = 7$. However, if $(b+1) = 7$, then $(a+1) = 3 \cdot 25 = 75$ and $a = 74$. But we are given that $a \cdot b \cdot c \cdot d = 8!$, and 74 does not divide $8!$. So we cannot have $(b+1) = 7$. Hence we must have $(b+1) = 3 \cdot 7$ and $(c+1) = 7$. This implies that $(a+1) = 25$ and $(d+1) = 3 \cdot 5 = 15$. Thus $a = 24$, $b = 20$, $c = 6$, and

$$d = \frac{8!}{(24 \cdot 20 \cdot 6)} = \frac{8!}{3 \cdot 8 \cdot 4 \cdot 5 \cdot 6} = 2 \cdot 7 = 14.$$

In conclusion, $a - d = 10$. □

Exercises for Chapter 10

Exercise 1 A standard six-sided die is rolled and P is the product of the five numbers that are visible. What is the largest number that is certain to divide P?

(A) 6 **(B)** 12 **(C)** 24 **(D)** 144 **(E)** 720

Exercise 2 What is the sum of the digits of the decimal form of the product $2^{2004} \cdot 5^{2006}$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 7 **(E)** 10

Exercise 3 The number $25^{64} \cdot 64^{25}$ is the square of a positive integer N. What is the sum of the decimal digits of N?

(A) 7 **(B)** 14 **(C)** 21 **(D)** 28 **(E)** 35

Exercise 4 Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. What is the number of possible values of k?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 6

Exercise 5 Let $N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1$. How many positive integers are factors of N?

(A) 3 **(B)** 5 **(C)** 69 **(D)** 125 **(E)** 216

Exercise 6 How many perfect squares are divisors of the product $1! \cdot 2! \cdot 3! \cdots 9!$?

(A) 504 **(B)** 672 **(C)** 864 **(D)** 936 **(E)** 1008

Exercise 7 How many positive integers less than 50 have an odd number of positive integer divisors?

(A) 3 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** 11

Exercise 8 For how many values of n will an n-sided regular polygon have interior angles with integer degree measures?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

Exercise 9 Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal $0.\overline{ab}$ is expressed as a fraction in lowest terms. How many different denominators are possible?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 8 **(E)** 9

Exercise 10 Suppose that n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors. How many positive divisors does $6n$ have?

(A) 32 **(B)** 34 **(C)** 35 **(D)** 36 **(E)** 38

11
Number Theory

11.1 Introduction

Number Theory is a subject used to describe a multitude of different types of problems, with the major commonality being that the solutions to these problems are integers. The study began in the previous chapter on Prime Decomposition, but problems that have integer solutions are sufficiently common and varied to require a chapter of their own.

11.2 Number Bases and Modular Arithmetic

Some of the problems considered in this section involve the expression of numbers in bases other than 10. These should not cause any difficulty if we keep in mind how our common base-10 representation is defined. A common base-10 number written, for example, in the form *abcd* is simply

$$a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d,$$

where a, b, c, and d are integers between 0 and 9, with $a \neq 0$. In a similar manner, a base-7 number is written in the form $abcd_7$ (we assume that the base is 10 unless a subscript is specified) with

$$a \cdot 7^3 + b \cdot 7^2 + c \cdot 7 + d,$$

where a, b, c, and d are integers between 0 and 6, with $a \neq 0$. Other base representations are defined in a similar manner. There are notational difficulties when the base is greater than 10, since we have only 10 numerals to use for the representation, but we will not consider these.

Although there are other ways to change between number base systems, it is easiest to simply go through the familiar base-10 representation.

Suppose, as an example, that we want to rewrite the base-6 number 3425_6 in base 8. First change this into base-10 representation as

$$\begin{aligned} 3425_6 &= 3 \cdot 6^3 + 4 \cdot 6^2 + 2 \cdot 6 + 5 \\ &= 3 \cdot 216 + 4 \cdot 36 + 2 \cdot 6 + 5 \\ &= 648 + 144 + 12 + 5 = 809. \end{aligned}$$

Since $8^2 = 64$ and $8^3 = 512$, this implies that

$$3425_6 = 809 = 512+297 = 8^3+4\cdot 64+43 = 8^3+4\cdot 8^2+5\cdot 8+3 = 1453_8.$$

Closely associated with base representation of numbers is the concept of modular arithmetic.

DEFINITION 1 Given a positive integer n, we say that the integer a is equal to the integer b **modulo** n, written $a \bmod n \equiv b$, if n divides $a - b$.

A common use of modulo notation occurs when a is a positive integer greater than n and the integer b satisfies $0 \le b < n$. Then $a \bmod n \equiv b$ precisely when b is the remainder that results when a is divided by n.

For example,

$$37 \bmod 5 \equiv 2, \quad \text{and} \quad 9 \bmod 4 \equiv 1.$$

Note also that

$$-11 \bmod 3 \equiv 4, \quad \text{since } -11 - 4 = -15 \text{ is divisible by } 3.$$

The following result is useful for simplifying calculations involving integer division.

RESULT 1 Suppose that n is a positive integer, and that $a \bmod n \equiv b$, $c \bmod n \equiv d$. Then

- $(a + c) \bmod n \equiv b + d$,
- $(a \cdot c) \bmod n \equiv b \cdot d$,
- $a^k \bmod n = b^k$, for any positive integer k.

To see the value of this result, suppose that we want to know the remainder when the number 3^{2006} is divided by 8. Since $3^2 = 9$, we have

$3^2 \bmod 8 = 1$. This implies that

$$3^{2006} \bmod 8 \equiv \left(3^2\right)^{1003} \bmod 8 \equiv \left(3^2 \bmod 8\right)^{1003} \equiv 1^{1003} = 1.$$

So the remainder is 1.

For a more substantive example, consider finding the remainder when 3^{2006} is divided by 11. First note that

$$3^5 = 243 = 2 \cdot 121 + 1 = 2 \cdot 11^2 + 1.$$

Thus $3^5 \bmod 11 = 1$, and

$$\begin{aligned} 3^{2006} \bmod 11 &\equiv 3^{2005} \cdot 3 \bmod 11 \\ &\equiv \left(3^5 \bmod 11\right)^{401} \cdot 3 \bmod 11 \equiv 1^{401} \cdot 3 = 3. \end{aligned}$$

Hence 3^{2006} divided by 11 leaves a remainder of 3.

11.3 Integer Division Results

The introduction of modular arithmetic permits us to illustrate some useful techniques for determining certain integer factors of numbers.

Result 1 When finding the prime decomposition of an integer n:

- n is divisible by 2 if and only if the units digit of n is even.
- n is divisible by 3 if and only if the sum of its digits is divisible by 3.
- n is divisible by 5 if and only if the units digit of n is a 0 or a 5.
- n is divisible by 7 if and only if 7 divides the integer that results from first truncating n by removing its units digit, and then subtracting twice the value of this digit from the truncated integer.
- n is divisible by 9 if and only if the sum of its digits is divisible by 9.
- n is divisible by 11 if and only if the alternating (by positive and negative signs) sum of its digits is divisible by 11.

The division results by 2 and 5 should be clear. Let us first consider the situation of division by 3. Suppose that the positive integer n is written in its decimal expansion as

$$n = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0.$$

Since $10 \bmod 3 \equiv 1$, for any positive integer j we have $10^j \bmod 3 \equiv (10 \bmod 3)^j \equiv 1^j \bmod 3 \equiv 1$. Hence

$$\begin{aligned}
n \bmod 3 &\equiv \left(a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0\right) \bmod 3 \\
&\equiv (a_k \bmod 3) \cdot (10^k \bmod 3) + (a_{k-1} \bmod 3) \cdot (10^{k-1} \bmod 3) \\
&\quad + \cdots + (a_2 \bmod 3) \cdot (10^2 \bmod 3) + (a_1 \bmod 3) \cdot (10 \bmod 3) \\
&\quad + (a_0 \bmod 3) \\
&\equiv (a_k \bmod 3) \cdot 1 + (a_{k-1} \bmod 3) \cdot 1 \\
&\quad + \cdots + (a_2 \bmod 3) \cdot 1 + (a_1 \bmod 3) \cdot 1 + (a_0 \bmod 3) \\
&\equiv (a_k + a_{k+1} + \cdots + a_2 + a_1 + a_0) \bmod 3.
\end{aligned}$$

So the remainder when n is divisible by 3 is the same as the remainder when the sum of the digits is divisible by 3. This remainder is 0 if and only if 3 divides n.

The proof of the result for divisibility by 3 is a direct result of the fact that $10 \bmod 3 \equiv 1$. This in turn implies that for any positive integer j we have $10^j \bmod 3 \equiv 1$. Since we also have $10 \bmod 9 \equiv 1$, and consequently that $10^j \bmod 9 \equiv 1$ for any positive integer j, the proof for divisibility by 9 follows this same line.

The divisibility by 11 result follows from the observation that

$$10 \bmod 11 \equiv -1,$$

so for any positive integer j we have $10^j \bmod 11 \equiv (-1)^j$.

The result concerning divisibility by 7 is unusual, and in most instances probably not worth the bother. However, showing that it is true is a nice application of the power of modular arithmetic. First, we can show that the result is always true provided that it is true for all integers with at most 5 digits, that is, when

$$n = a_5 \cdot 10^5 + a_4 \cdot 10^4 + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0.$$

Then we note that since $10 \bmod 7 \equiv 3$, we have

$$10^2 \bmod 7 \equiv 2, \quad 10^3 \bmod 7 \equiv 6, \quad 10^4 \bmod 7 \equiv 4,$$

and $10^5 \bmod 7 \equiv 5$.

Suppose that $n \bmod 7 = b$, where $0 \le b \le 6$. Then

$$b \equiv n \bmod 7 \equiv 5a_5 + 4a_4 + 6a_3 + 2a_2 + 3a_1 + a_0 \bmod 7,$$

so that

$$a_0 \bmod 7 \equiv b - (5a_5 + 4a_4 + 6a_3 + 2a_2 + 3a_1) \bmod 7.$$

Now consider the reduced number, which has the form

$$n_1 = a_5 \cdot 10^4 + a_4 \cdot 10^3 + a_3 \cdot 10^2 + a_2 \cdot 10 + a_1 - 2a_0.$$

To see that 7 also divides n_1 if and only if $b = 0$, note that

$$\begin{aligned} n_1 \bmod 7 &\equiv (a_5 \cdot 10^4 + a_4 \cdot 10^3 + a_3 \cdot 10^2 + a_2 \cdot 10 + a_1 - 2a_0) \bmod 7 \\ &\equiv (4a_5 + 6a_4 + 2a_3 + 3a_2 + a_1) \bmod 7 \\ &\quad - 2b + 2(5a_5 + 4a_4 + 6a_3 + 2a_2 + 3a_1) \bmod 7 \\ &\equiv 14a_5 + 14a_4 + 14a_3 + 7a_2 + 7a_1 - 2b \bmod 7 \\ &\equiv -2b \bmod 7. \end{aligned}$$

Now, 7 divides n if and only if $b = 0$. But for $1 \le b \le 6$ the number 7 does not divide $-2b$. So $b = 0$ if and only if $-2b = 0$, which implies that 7 divides n if and only if 7 divides n_1.

To illustrate the division results, consider finding the prime decomposition of the number 390225. The sum of the digits of this number is $3 + 9 + 0 + 2 + 2 + 5 = 21$. Since 21 is divisible by 3 but not by 9, we know that 390225 is also divisible by 3 but not by 9. In addition, $3 - 9 + 0 - 2 + 2 - 5 = -11$ is divisible by 11, so 11 is also a factor of 390225. Since the unit digit of 390225 is a 5, we also know that 5 is a factor.

Dividing consecutively by these factors gives

$$390225 = 3 \cdot 130075 = 3 \cdot 5 \cdot 26015 = 3 \cdot 5^2 \cdot 5203 = 3 \cdot 5^2 \cdot 11 \cdot 473.$$

Note now that $4 - 7 + 3 = 0$ is also divisible by 11, which, since 43 is prime, gives us the complete prime decomposition

$$390225 = 3 \cdot 130075 = 3 \cdot 5 \cdot 26015 = 3 \cdot 5^2 \cdot 5203 = 3 \cdot 5^2 \cdot 11^2 \cdot 43.$$

For the result concerning divisibility by 7, first look at $n = 2233$. The reduced integer is

$$n_1 = 223 - 2 \cdot 3 = 217,$$

and the reduced integer for 217 is

$$n_2 = 21 - 2 \cdot 7 = 7.$$

Since 7 divides 7, 7 also divides 217, which implies that 7 also divides 2233.

Consider, on the other hand, the number $n = 59210$. Then

$$n_1 = 5921 - 2(0) = 5921, \quad n_2 = 590, \quad \text{and} \quad n_3 = 59.$$

Since 7 does not divide 59, 7 also does not divide 59210.

11.4 The Pigeon Hole Principle

The Pigeon Hole principle follows from the simple observation that if there are n boxes, the Pigeon Holes, into which more than n objects, the Pigeons, have been placed, then at least one of the boxes must have received more than one of the objects. This principle is frequently applied in problems where we need to determine a minimal number of objects to ensure that some integral number property is satisfied.

PROBLEM 1 Suppose that we draw cards consecutively and without replacement from a 52 card deck. How many do we need to draw to ensure that there will be a pair?

We can consider as boxes (the Pigeon Holes) the possible values of the cards (which represent the Pigeons). Then there are 13 boxes, 10 of which represent the cards numbered 1 (or Ace), through 10, and three additional boxes representing Jacks, Queens, and Kings. It is, of course, possible that the first two cards drawn could go into the same box, and hence represent a pair. But even in the most widely distributed situation, after drawing 13 cards and placing them in distinct boxes, the 14th card drawn must go into a box that already contains a card. So by the 14th draw we must have a pair.

The Pigeon Hole Principle is a very simple concept, but determining the correct specification for the Holes and the Pigeons is not always obvious.

There is an extension of the Pigeon Hole Principle that is quite easy to see.

Result 1 Extended Pigeon Hole Principle: If $(m - 1) \cdot n + 1$ objects have been placed in n boxes, then at least one of the boxes must contain m of the objects.

Examples for Chapter 11

The first Example is number 16 from the 1981 AHSME.

Example 1 The base-3 representation of x is

$$12112211122211112222_3.$$

What is the first digit (on the left) of the base-9 representation of x?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (E) Since $9 = 3^2$, we will group the base-3 digits in pairs, starting from the right. Then

$$\begin{aligned}
x &= (12)(11)(22)(11)(12)(22)(11)(11)(22)(22)_3 \\
&= (1 \cdot 3 + 2) \cdot 3^{18} + (1 \cdot 3 + 1) \cdot 3^{16} + (2 \cdot 3 + 2) \cdot 3^{14} \\
&\quad + (1 \cdot 3 + 1) \cdot 3^{12} + (1 \cdot 3 + 2) \cdot 3^{10} + (2 \cdot 3 + 2) \cdot 3^{8} \\
&\quad + (1 \cdot 3 + 1) \cdot 3^{6} + (1 \cdot 3 + 1) \cdot 3^{4} + (2 \cdot 3 + 2) \cdot 3^{2} + (2 \cdot 3 + 2) \\
&= 5 \cdot 9^9 + 4 \cdot 9^8 + 8 \cdot 9^7 + 4 \cdot 9^6 + 5 \cdot 9^5 + 8 \cdot 9^4 + 4 \cdot 9^3 \\
&\quad + 4 \cdot 9^2 + 8 \cdot 9^1 + 8 \\
&= 5484584488_9.
\end{aligned}$$

Notice that we don't actually need the base-9 value, we only needed to know the base-9 leading digit, which comes from the highest power of 9. □

The second Example is number 25 from the 1999 AHSME.

Example 2 There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that

$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

where $0 \le a_i < i$ for $i = 2, 3, \ldots, 7$. What is $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer (B) First rewrite the equation to eliminate the denominators by multiplying by 7!. This produces

$$5 \cdot 6! = a_2(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) + a_3(7 \cdot 6 \cdot 5 \cdot 4) + a_4(7 \cdot 6 \cdot 5) + a_5(7 \cdot 6) + a_6 \cdot 7 + a_7.$$

Since $0 \le a_i < i$ for each i, and all the terms on the right except the last are multiples of 7, consider this equation modulo 7. Thus

$$\begin{aligned} a_7 &\equiv 5 \cdot 6! \bmod 7 \\ &\equiv (5 \bmod 7)(720 \bmod 7) \\ &\equiv (5 \bmod 7)(6 \bmod 7) \equiv (30 \bmod 7) \equiv 2. \end{aligned}$$

Since $0 \le a_7 < 7$, this implies that $a_7 = 2$ and

$$5 \cdot 6! - 2 = a_2(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) + a_3(7 \cdot 6 \cdot 5 \cdot 4) + a_4(7 \cdot 6 \cdot 5) + a_5(7 \cdot 6) + a_6 \cdot 7.$$

In a similar manner, we consider this equation modulo 6. Since all the terms on the right side of the equation, with the exception of $a_6 \cdot 7$, have a factor of 6, we have

$$-2 \bmod 6 \equiv (a_6 \bmod 6)(7 \bmod 6) \equiv (a_6 \bmod 6)(1 \bmod 6),$$

so $a_6 \equiv -2 \bmod 6 \equiv 4$. Hence

$$5 \cdot 6! - 2 - 4 \cdot 7 = a_2(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) + a_3(7 \cdot 6 \cdot 5 \cdot 4) + a_4(7 \cdot 6 \cdot 5) + a_5(7 \cdot 6).$$

To find a_5 we consider this equation modulo 5, which results in $a_5 = 0$, since 5 divides both $5 \cdot 6!$ and $-2 - 4 \cdot 7 = -30$. As a consequence,

$$5 \cdot 6! - 2 - 4 \cdot 7 = a_2(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) + a_3(7 \cdot 6 \cdot 5 \cdot 4) + a_4(7 \cdot 6 \cdot 5),$$

which when divided by $7 \cdot 6 \cdot 5$ reduces to

$$17 = 12a_2 + 4a_3 + a_4.$$

We could continue to reduce modulo 4 and 3, but it is clear from this equation that we must have $a_4 = a_3 = a_2 = 1$. In conclusion, then

$$a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 1 + 1 + 1 + 0 + 4 + 2 = 9. \qquad \square$$

The final Example is number 29 from the 1990 AHSME.

EXAMPLE 3 A subset of the integers $1, 2, \ldots, 100$ has the property that none of its members is 3 times another. What is the largest number of members such a subset can have?

(A) 50 **(B)** 66 **(C)** 67 **(D)** 76 **(E)** 78

Answer (D) Consider first the subset that contains all the integers that are not a multiple of 3, the set

$$S = \{1, 2, 4, 5, 7, 8, \ldots, 98, 100\}.$$

The number of elements of S is

$$100 - \left\lfloor \frac{100}{3} \right\rfloor = 100 - 33 = 67.$$

In addition to these, we can add some of the multiples of 3. We cannot add 3 and 6, since 1 and 2 are in S. However, since 3 and 6 are not in S we can add 9 and 18. Adding 9 and 18 excludes the addition of 27 and 54, which permits the addition of 81 (note that $3 \cdot 54$ is too large to be considered).

In addition, since we have not added 12 and 15 we can add 36 and 45. We can also add 63 and 72, since we have not added 21 and 24. Finally, we can add 90 and 99 since we have not added 30 or 33. Thus we can add to S the 9 additional elements $\{9, 18, 36, 45, 63, 72, 81, 90, 99\}$. This gives a total of 76 elements in S.

An easier strategy is to add to the original 67 elements those positive integers less than 100 whose prime decomposition has an even number of 3's, that is, the nine numbers,

$$9, 18 = 2 \cdot 9, 36 = 4 \cdot 9, 45 = 5 \cdot 9, 63 = 7 \cdot 9, 72 = 8 \cdot 9,$$

$$81 = 9^2, 90 = 10 \cdot 9, \text{ and } 99 = 11 \cdot 9.$$

Each of these is three times a number whose prime decomposition has an odd number of 3's, all of which were excluded from the original set. □

Exercises for Chapter 11

Exercise 1 The number 2^{1000} is divided by 13. What is the remainder?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 7 **(E)** 11

Exercise 2 A three-digit base-10 number n is selected at random. What is the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals?

(A) $\frac{76}{225}$ **(B)** $\frac{38}{125}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{152}{225}$ **(E)** $\frac{76}{125}$

Exercise 3 The two-digit integers from 19 to 92 are written consecutively to form the large integer

$$N = 19202122 \cdots 909192.$$

Suppose that 3^k is the highest power of 3 that is a factor N. What is k ?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Exercise 4 A cryptographer devises the following method for encoding positive integers. First, the integer is expressed in base 5. Second, a 1-to-1 correspondence is established between the digits that appear in the base-5 expression and the elements of the set $\{V, W, X, Y, Z\}$. Using this correspondence, the cyptographer finds that three consecutive integers in increasing order are coded as VYZ, VYX, and VVW. What is the base-10 expression for the integer coded as XYZ?

(A) 48 **(B)** 71 **(C)** 82 **(D)** 108 **(E)** 113

Exercise 5 Suppose that the base-8 representation of a perfect square is $ab3c$, where $a \neq 0$. What is c?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** 7

Exercise 6 In year N, the 300th day of the year is a Tuesday. In year $N+1$, the 200th day is also a Tuesday. On what day of the week did the 100th day of year $N - 1$ occur?

(A) Thursday **(B)** Friday **(C)** Saturday **(D)** Sunday
(E) Monday

Exercise 7 A circular table has 60 chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone. What is the smallest possible value for N?

(A) 15 **(B)** 20 **(C)** 30 **(D)** 40 **(E)** 58

Exercise 8 A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks, and 40 black socks. Socks are randomly selected one at a time from the drawer, without replacement. What is the smallest number of socks that must be selected to ensure that the selection contains at least 10 pairs?

(A) 21 **(B)** 23 **(C)** 24 **(D)** 30 **(E)** 50

Exercise 9 Label one disk "1", two disks "2", three disks "3", ..., and fifty disks "50". Put these $1 + 2 + 3 + \cdots + 50 = 1275$ labeled disks in a box. Disks are then drawn from the box at random without replacement. What is the minimum number of disks that must be drawn to ensure drawing at least ten disks with the same label?

(A) 10 **(B)** 51 **(C)** 415 **(D)** 451 **(E)** 501

Exercise 10 Let S be a subset of $\{1, 2, 3, \ldots, 50\}$ such that no pair of distinct elements in S has a sum divisible by 7. What is the maximum number of elements in S?

(A) 6 **(B)** 7 **(C)** 14 **(D)** 21 **(E)** 23

12
Sequences and Series

12.1 Introduction

This chapter considers the common arithmetic and geometric sequences and series, as well as sequences and series that are defined inductively and recursively. Pattern recognition often plays a major role in the solution of these problems.

12.2 Definitions

By a sequence of numbers we simply mean an ordered way of writing the numbers. To be more precise, we use the notion of a function.

DEFINITION 1 A **sequence** of numbers is a function that assigns to each positive integer a distinct number. The number assigned by the sequence to the integer n is commonly denoted using a subscript, such as a_n. These numbers are called the **terms** of the sequence.

DEFINITION 2 A **series** of numbers is a sequence that is formed by adding the terms of another sequence.

As an example of this definition, consider the sequence of numbers 1, 4, 7, ..., where the ... (called ellipsis) is used to indicate that this sequence continues in the manner described by the first few terms of the sequence. If we denote the first term by a_1, that is, $a_1 = 1$, then $a_2 = 4$, $a_3 = 7$, and if the pattern continues, it is not difficult to believe that $a_n = 3n - 2$.

The series associated with the sequence $\{a_n\}$ has its first terms

$$S_1 = a_1 = 1,$$

$$S_2 = a_1 + a_2 = 1 + 4 = 5,$$

and

$$S_3 = a_1 + a_2 + a_3 = 5 + 7 = 12.$$

If our pattern for a_n is correct, we will show later in this section that the series associated with this sequence has

$$S_n = \frac{3n^2}{2} - \frac{n}{2}.$$

There are two special types of sequences that we will frequently see on the AMC. The first is the arithmetic sequence.

Definition 3 An **arithmetic sequence** (also called an *arithmetic progression*) $\{a_n\}$ is defined by $a_n = a + (n-1)d$ for some constants a and $d \neq 0$ and each positive integer n. The number d is called the *common difference*.

The sequence that we considered prior to this definition is an arithmetic sequence with first term $a = 1$ and common difference $d = 3$. The sum of the first n terms of an arithmetic sequence are found using the fact, from Chapter 1, that

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

For an arithmetic sequence with $a_n = a + (n-1)d$ we have

$$\begin{aligned} & a_1 + a_2 + a_3 + \cdots + a_n \\ &\quad = a + (a+d) + (a+2d) + \cdots + (a + (n-1)d) \\ &\quad = n \cdot a + d(1 + 2 + \cdots + (n-1)) \\ &\quad = na + \frac{d}{2}(n-1)n. \end{aligned}$$

This gives the following result.

Result 1 If $\{a_n\}$ is an arithmetic sequence with first term a and common difference d, then

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{n}{2}\left(2a + (n-1)d\right).$$

Since $2a + (n-1)d = a + (a + (n-1)d) = a_1 + a_n$, this result also implies that the sum of the first n terms of an arithmetic sequence is n times the average of the first and last terms.

Result 2 If $\{a_n\}$ is an arithmetic sequence with first term a and common difference d, then

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{n}{2}(a_1 + a_n).$$

A special case of this result occurs when the common difference is 1. In this case we have $a_1 = k$ for some integer k, $a_n = k + n - 1$, and

$$a_1 + a_2 + \cdots + a_n = k + (k+1) + \cdots + (k + n - 1) = \frac{n}{2}(2k + n - 1).$$

The other special type of sequence that frequently occurs is the geometric sequence.

Definition 4 A **geometric sequence** $\{a_n\}$ is defined for each positive integer n by $a_n = a \cdot r^{n-1}$, where a is a constant and r is a constant that is neither 0 nor 1. The number r is called the *common ratio*.

The sum of the first n terms of a geometric sequence has a nice form but we need to work a bit harder to show it than in the case of arithmetic sequences. Let S_n denote the sum of the first n terms of a geometric sequence $\{a_n\}$ that has first term a and common ratio r, that is,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

If we multiply each side of this equation by r and subtract the resulting equation from the preceding equation we obtain

$$\begin{aligned} S_n - rS_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - r(a + ar + \cdots + ar^{n-1}) \\ &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n) \\ &= a + (ar - ar) + (ar^2 - ar^2) + \cdots + \left(ar^{n-1} - ar^{n-1}\right) + ar^n \\ &= a - ar^n. \end{aligned}$$

So

$$(1-r)S_n = S_n - rS_n = a - ar^n = a(1-r^n).$$

This gives us a sum formula for a geometric series.

RESULT 3 If $\{a_n\}$ is a geometric sequence with first term a and common ratio r, then

$$a_1 + a_2 + \cdots + a_n = a\frac{1-r^n}{1-r}.$$

This formula has a particularly important application when we have $|r| < 1$. In this case, r^n approaches 0 as n becomes increasingly large, and

$$a + ar + ar^2 + ar^3 \cdots = a\frac{1-0}{a-r} = \frac{a}{1-r}.$$

For example, we have

$$1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots = \frac{1}{1-1/3} = \frac{1}{2/3} = \frac{3}{2}.$$

The other type of series that commonly occurs on the AMC has terms that are recursively defined or repeat themselves after a finite number of terms. These generally require determining a formula for an arbitrary term of the sequence. To verify that the formula actually holds in general requires mathematical induction, a topic that you should research if you are truly interested in mathematical problem solving.

Examples for Chapter 12

The first Example is number 13 from the 2002 AMC 12B.

EXAMPLE 1 The sum of 18 consecutive positive integers is a perfect square. What is the smallest possible value of this sum?

(A) 169 **(B)** 225 **(C)** 289 **(D)** 361 **(E)** 441

Answer (B) If the first term of the sequence is denoted a, then this arithmetic sequence with common difference $d = 1$ has the sum

$$\begin{aligned} a + (a+1) + \cdots + (a+17) &= 18a + (1 + 2 + \cdots + 17) \\ &= 18a + \frac{1}{2}(17 \cdot 18) = 9(2a + 17). \end{aligned}$$

For the sum to be a perfect square, the term $2a+17$ must be a perfect square. By inspection, we can see that this first occurs when $a = 4$. This gives the minimal sum, which is $9 \cdot 25 = 225$. □

The next Example is number 14 from the 1981 AHSME.

EXAMPLE 2 In a geometric sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 91. What is the sum of the first four terms?

(A) 28 **(B)** 32 **(C)** 35 **(D)** 49 **(E)** 84

Answer (A) Let a denote the first term of the geometric sequence and r denote the common ratio. The stated conditions imply that

$$7 = a + ar$$

and that

$$\begin{aligned} 91 &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 \\ &= (a + ar)(1 + r^2 + r^4) = 7(1 + r^2 + r^4). \end{aligned}$$

As a consequence,

$$13 = 1 + r^2 + r^4$$

and

$$0 = r^4 + r^2 - 12 = (r^2 + 4)(r^2 - 3).$$

Hence $r^2 = 3$, and the sum of the first four terms of the sequence is

$$a + ar + ar^2 + ar^3 = (a + ar)(1 + r^2) = 7(1 + 3) = 28. \qquad \square$$

The final Example is number 24 from the 2004 AMC 10A.

EXAMPLE 3 Let $a_1, a_2, \ldots,$ be a sequence with the following properties.

(i) $a_1 = 1$, and

(ii) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$?

(A) 1 **(B)** 2^{99} **(C)** 2^{100} **(D)** 2^{4950} **(E)** 2^{9999}

Answer (D) Computing a few of the initial terms gives

$$a_2 = a_{2\cdot 1} = 1 \cdot a_1 = 1 = 2^0,$$

$$a_{2^2} = a_4 = a_{2\cdot 2} = 2 \cdot a_2 = 2 = 2^1,$$

$$a_{2^3} = a_8 = a_{2\cdot 4} = 4 \cdot a_4 = 8 = 2^{1+2},$$

and

$$a_{2^4} = a_{16} = a_{2\cdot 8} = 8 \cdot a_8 = 64 = 2^{1+2+3}.$$

It appears from this sample that for each positive integer n we have

$$a_{2^n} = 2^{1+2+\cdots+(n-1)} = 2^{(n-1)n/2}.$$

An induction argument confirms that this formula holds for all positive integers. To see this, first note that the pattern holds when n is one of the first few positive integers. Suppose that it is true for a given positive integer n, then

$$a_{2^{n+1}} = a_{2\cdot 2^n} = 2^n \cdot a_{2^n} = 2^n \cdot 2^{(n-1)n/2} = 2^{n+(n-1)n/2} = 2^{n(n+1)/2},$$

so it is also true for the next integer. As a consequence, the formula must be true for all positive integers. This is the essence of a proof by mathematical induction. Hence

$$a_{2^{100}} = 2^{(99\cdot 100)/2} = 2^{4950}. \qquad \square$$

Exercises for Chapter 12

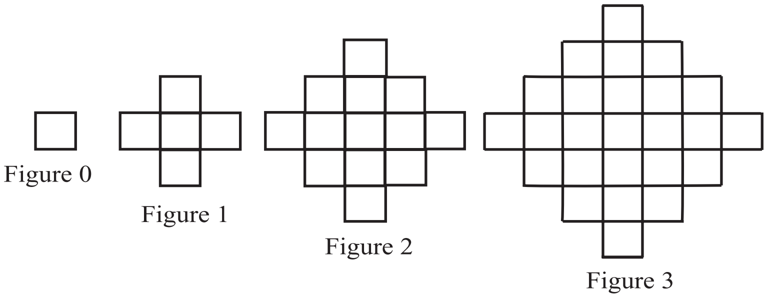
Exercise 1 A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows are there?

(A) 5 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Exercise 2 The second and fourth terms of a geometric sequence are 2 and 6. Which of the following is a possible first term?

(A) $-\sqrt{3}$ **(B)** $-\dfrac{2\sqrt{3}}{3}$ **(C)** $-\dfrac{\sqrt{3}}{3}$ **(D)** $\sqrt{3}$ **(E)** 3

Exercise 3 Figures 0, 1, 2, and 3 consists of 1, 5, 13, and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in Figure 100?



(A) 10,401 **(B)** 19,801 **(C)** 20,201 **(D)** 39,801
(E) 40,801

Exercise 4 Let 1, 4, ... and 9, 16, ... be two arithmetic sequences. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

(A) 3722 **(B)** 3732 **(C)** 3914 **(D)** 3924 **(E)** 4007

Exercise 5 Let $a_1, a_2, \ldots, a_k$ be a finite arithmetic sequence with

$$a_4 + a_7 + a_{10} = 17,$$

$$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} = 77,$$

and $a_k = 13$. What is k?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

Exercise 6 A sequence of three real numbers forms an arithmetic sequence whose first term is 9. If the first term is unchanged, 2 is added to the second term, and 20 is added to the third term, then the three resulting numbers form a geometric sequence. What is the smallest possible value for the third term of the geometric progression?

(A) 1 **(B)** 4 **(C)** 36 **(D)** 49 **(E)** 81

Exercise 7 Consider the sequence of numbers: 4, 7, 1, 8, 9, 7, 6, ... For $n > 2$, the nth term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. What is the smallest value of n for which $S_n > 10{,}000$?

(A) 1992 **(B)** 1999 **(C)** 2001 **(D)** 2002 **(E)** 2004

Exercise 8 Alice, Bob, and Carol repeatedly take turns tossing a fair regular six-sided die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first to toss a six.

(A) $\frac{1}{3}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{5}{18}$ **(D)** $\frac{25}{91}$ **(E)** $\frac{36}{91}$

Exercise 9 Suppose that the sequence $\{a_n\}$ is defined by

$$a_1 = 2, \quad \text{and} \quad a_{n+1} = a_n + 2n, \quad \text{when } n \geq 1.$$

What is a_{100}?

(A) 9900 **(B)** 9902 **(C)** 9904 **(D)** 10100 **(E)** 10102

Exercise 10 The increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that $a_{n+2} = a_n + a_{n+1}$, for all $n \geq 1$. Suppose that $a_7 = 120$. What is a_8?

(A) 128 **(B)** 168 **(C)** 193 **(D)** 194 **(E)** 210

13
Statistics

13.1 Introduction

Most of the AMC statistics problems involve the concepts of mean, median, and mode. These problems are not generally difficult, but the concepts might not be familiar to all students. Students taking the AMC 10 will not likely see problems involving statistics, unless the definitions of the statistical concepts are given in the problem.

Statistics problems can also involve graph interpretation, as well as counting methods such as permutations and combinations.

13.2 Definitions

Suppose that we are given a collection of numbers, $\{a_1, a_2, \ldots, a_n\}$. There are various ways to describe the way in which these numbers are distributed.

DEFINITION 1 The **Arithmetic Mean** of a collection of n numbers a_1, $a_2, \ldots, a_n$ is the average of these numbers

$$\text{Arithmetic Mean} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Often this is simply called the *mean* or *average* of the numbers.

Although the arithmetic mean tells the average of the numbers, it may not sufficiently describe how the numbers are distributed. For example, the mean of the numbers 0, 0, 0, and 100 is 25, but none of the numbers is close to the mean. There are many other ways to describe how the numbers are distributed, but only the following three are commonly used on the AMC examinations.

DEFINITION 2 Consider the collection of numbers $\{a_1, a_2, \ldots, a_n\}$, with $a_1 \le a_2 \le \cdots \le a_n$.

- The **median** of the collection is the middle term of the collection when n is odd, and is the average of the two middle terms when n is even.
- The **mode** of the collection is the term(s) that occur most frequently.

In addition, we have a definition that is occasionally used in statistic-oriented problems.

DEFINITION 3 The **range** of a collection of integers $\{a_1, a_2, \ldots, a_n\}$ is defined to be the length of the smallest closed interval that contains all the integers in the collection. Hence the range is the difference between the largest and smallest integers.

Looking again at our collection $a_1 = 0, a_2 = 0, a_3 = 0$, and $a_4 = 100$, we see that the median is 0, the average of a_2 and a_3. The mode is also 0, and the range is $100 - 0 = 100$.

For a more illustrative example, consider the set of integers {2, 3, 3, 7, 8, 11}. This set has

$$\text{Mean} = \frac{2+3+3+7+8+11}{6} = \frac{34}{6} = \frac{17}{3}, \quad \text{Median} = \frac{3+7}{2} = 5,$$
$$\text{Mode} = 3, \quad \text{and} \quad \text{Range} = 11 - 2 = 9.$$

13.3 Results

One of the most frequently used results concerns the connection between the mean and the sum of a collection of numbers.

RESULT 1 For the collection of the n numbers $a_1, a_2, \ldots, a_n$ we have

$$a_1 + a_2 + \cdots + a_n = n \cdot \text{Mean}.$$

In the previous chapter we saw some interesting results about arithmetic and geometric sequences. We can use these results to determine the values of the various statistical concepts in the case when the collection of numbers happens to come from one of these types of sequences.

First consider the arithmetic sequences.

RESULT 2 Consider the collection of n numbers, the arithmetic sequence with first term a, and common difference d.

- The mean and median are both $a + \frac{1}{2}(n-1)d$.
- The mode consists of all the terms in the collection.

To show the first of these results, recall that

$$\begin{aligned} a + (a+d) + \cdots + (a+(n-1)d) &= na + d(1+2+\cdots+(n-1)) \\ &= na + \frac{d}{2}(n-1)n. \end{aligned}$$

Dividing by n gives the mean, $a + \frac{1}{2}(n-1)d$. When n is odd this will be the middle term of the sequence. When n is even, it will be the average of the two middle terms. In either case it is also the mode.

The case for geometric sequences is not as concise, nor as frequently used.

RESULT 3 Consider the n numbers in the geometric sequence with first term a and common ratio r, where $|r| \neq 1$.

- The mean is $\frac{a}{n} \cdot \frac{1-r^n}{1-r}$.
- The median is $ar^{(n-1)/2}$ if n is odd, and is $\frac{a(1+r)}{2} r^{(n-2)/2}$ if n is even.
- The mode consists of all the terms in the collection.

To show the first result, recall that in the previous chapter we found that the sum of the terms in the sequence is

$$a + ar + \cdots + ar^{(n-1)} = a \cdot \frac{1-r^n}{1-r}, \quad \text{so} \quad \text{Mean} = \frac{a}{n} \cdot \frac{1-r^n}{1-r}.$$

The values for the median and mode depend on the sign of the ratio r as well as its magnitude. The cases are too specialized to be of general use.

Examples for Chapter 13

The first Example is number 16 from the 1991 AHSME.

Example 1 One hundred students at Century High School participated in the AMC 12 last year, and their mean score was 100. The number of non-seniors taking the AMC 12 was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of non-seniors. What was the mean score of the seniors?

(A) 100 **(B)** 112.5 **(C)** 120 **(D)** 125 **(E)** 150

Answer (D) Let S denote the number of seniors. Then the number of non-seniors is $1.5S$ and

$$S + 1.5S = 100, \quad \text{so} \quad S = \frac{100}{2.5} = 40.$$

So there are 40 seniors and 60 non-seniors. Now let M be the mean of the seniors. Then the mean of the non-seniors is $\frac{2}{3}M$. The sum of all the scores is the overall mean times the number of students, as well as the sum of the product of the number of each type of student times the mean of each type. So

$$10{,}000 = (100) \cdot (100) = 40 \cdot M + 60 \cdot \frac{2}{3}M = 80M,$$

and $M = 10{,}000/80 = 125$. □

The next Example is number 10 from the 2004 AMC 12A.

Example 2 The sum of 49 consecutive integers is 7^5. What is their median?

(A) 7 **(B)** 7^2 **(C)** 7^3 **(D)** 7^4 **(E)** 7^5

Answer (C) Let a denote the smallest integer. Since the integers are consecutive, they form an arithmetic sequence with common difference $d = 1$. Hence the mean and median are the same with value

$$\text{Median} = \text{Mean} = \frac{7^5}{49} = \frac{7^5}{7^2} = 7^3.$$

The final Example is number 9 from the 2000 AMC 12.

EXAMPLE 3 Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

(A) 71 **(B)** 76 **(C)** 80 **(D)** 82 **(E)** 91

Answer (C) Since the average of the first two scores entered is an integer, these scores must both be even or both be odd.

If they were the two odd scores, then they would add to $71+91 = 162$. Since 162 is divisible by 3, the third score added would also have to be divisible by 3. Otherwise the average of the first three scores would not be an integer. But none of the remaining scores, 76, 80, and 82 is divisible by 3. As a consequence, the first two scores could not have been the odd scores.

Hence the first two scores added must both be even. Since $76 + 80 = 156$, and $80+82 = 162$ are both divisible by 3, but none of the 71, 91 and 82 is divisible by 3, the first pair chosen could not have been either 76 and 80 or 80 and 82. So the first two integers chosen must have been 76 and 82, and $76 + 82 = 158$ has a remainder of 2 when divided by 3. Consequently, the third integer chosen must have a remainder of 1 when divided by 3, and of those remaining, only 91 has this property. So the first scores selected must have been 76, 82, and 91. Since this sum is odd, the fourth number chosen must be the remaining odd number 71. Note that $76 + 82 + 91 + 71 = 320$ is divisible by 4. Hence the last score chosen must have been 80. □

Exercises for Chapter 13

Exercise 1 Six numbers from a list of nine integers are 7, 8, 3, 5, 9, and 5. What is the largest possible value of the median of all nine numbers in this list?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Exercise 2 Consider the sequence

$$1, -2, 3, -4, 5, -6, \ldots,$$

whose nth term is $(-1)^{n+1} \cdot n$. What is the average of the first 200 terms of the sequence?

(A) -1 **(B)** -0.5 **(C)** 0 **(D)** 0.5 **(E)** 1

Exercise 3 A speaker talked for sixty minutes to a full auditorium. Twenty percent of the audience heard the entire talk and ten percent slept through the entire talk. Half of the remainder heard one third of the talk and the other half heard two thirds of the talk. What was the average number of minutes of the talk heard by members of the audience?

(A) 24 **(B)** 27 **(C)** 30 **(D)** 33 **(E)** 36

Exercise 4 The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Exercise 5 All the students in an algebra class took a 100-point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Exercise 6 In the sixth, seventh, eighth, and ninth basketball games of the season, a player scored 23, 14, 11, and 20 points, respectively. Her points-per-game average was higher after nine games than it was after the first five games. Her average after ten games was greater than 18. What is the least number of points she could have scored in the tenth game?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

Exercise 7 The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

(A) 5 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 36

Exercise 8 The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. What is the largest integer that can be an element of this collection?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Exercise 9 The sequence $a_1, a_2, a_3, \ldots,$ satisfies $a_1 = 19$ and $a_9 = 99$. Also, a_n is the arithmetic mean of the first $n - 1$ terms whenever $n \geq 3$. What is a_2?

(A) 29 **(B)** 59 **(C)** 79 **(D)** 99 **(E)** 179

Exercise 10 A list of integers has mode 32 and mean 22. The smallest number on the list is 10. The median m of the list is a member of the list. If the list member m were replaced by $m + 10$, the mean and median of the new list would be 24 and $m + 10$, respectively. If m were instead replaced by $m - 8$, the median of the new list would be $m - 4$. What is m?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

14
Trigonometry

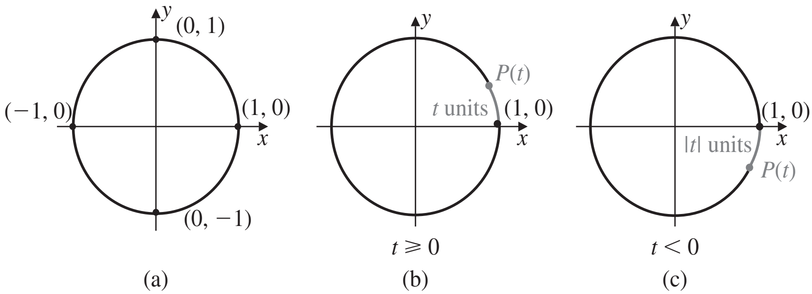
14.1 Introduction

This chapter begins the consideration of mathematical topics expected to be known for the AMC 12 exam but not for the AMC 10. There have not been many problems involving trigonometry on the more recent exams because the use of calculators makes many of the traditional problems trivial. However, the topic is important and the subject matter dealing with this subject is quite general. Students taking the AMC 10 examinations will not see problems involving trigonometry.

14.2 Definitions and Results

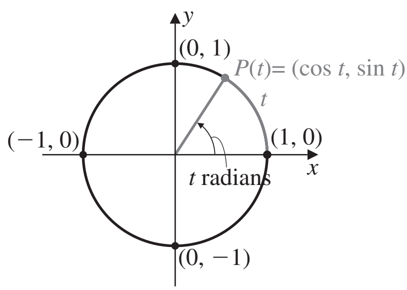
The two very basic definitions in trigonometry are the sine and the cosine of a given number or given angle. There are two standard and equivalent ways to define these concepts; one uses right triangles, and the other uses the unit circle. Defining the sine and cosine for angles using right triangles is generally the first definition that is presented, but the unit circle approach is more appropriate when the sine and cosine are needed for functional representation. We will give the unit circle definition, since it is more general, and may not be as familiar.

Place a unit circle in the xy-plane. For each positive real number t, define $P(t)$ as the point on this unit circle that is a distance t along the circle, measured counterclockwise, from the point $(1, 0)$. For each negative number t, define $P(t)$ as the point on this unit circle that is a distance $|t|$ along the circle, measured clockwise, from the point $(1, 0)$. Finally, define $P(0) = (1, 0)$. In this way we have, for each real number t, a unique pair $(x(t), y(t))$ of xy-coordinates on the unit circle to describe the point $P(t)$. These coordinates provide the two basic trigonometric functions.



Definition 1 The Sine and Cosine: Suppose that the coordinates of a point $P(t)$ on the unit circle are $(x(t), y(t))$. Then the **sine of *t***, written $\sin t$, and the **cosine of *t***, written $\cos t$, are defined by

$$\sin t = y(t) \quad \text{and} \quad \cos t = x(t).$$



These definitions are also used for the sine and cosine of an angle with radian measure t. So the trigonometric functions serve two purposes, directly as functions with domain the set of real numbers and indirectly as functions with domain the set of all possible angles, where the angles are given in radian measure. Since an angle of π radians is the same as an angle of 180 degrees, we also can determine the sine and cosine of angles measured in degrees.

There are some results that follow quickly from these definitions of the sine and cosine. The first, and most important, identity in trigonometry follows from the fact that for any number t the point $P(t) = (\cos t, \sin t)$ lies on the circle with equation $x^2 + y^2 = 1$.

Result 1 The Pythagorean Identity: For all real numbers t we have

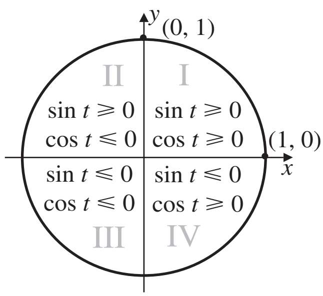
$$(\sin t)^2 + (\cos t)^2 = 1.$$

In addition, since the points (x, y) on the circle $x^2 + y^2 = 1$ satisfy $-1 \le x \le 1$ and $-1 \le y \le 1$, the same bounds hold for the sine and cosine functions.

Result 2 Bounds on the Sine and Cosine: For all real numbers t,

$$-1 \le \sin t \le 1 \quad \text{and} \quad -1 \le \cos t \le 1.$$

The signs of the sine and cosine functions are also easily determined once it is known in which quadrant of the plane $P(t)$ lies. For example, if $P(t)$ lies in quadrant II, the x-coordinate is negative and the y-coordinate is positive, so $\cos t \le 0$ and $\sin t \ge 0$.

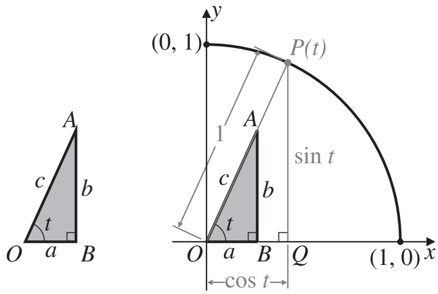


The points $P(t) = (\cos t, \sin t)$ and $P(-t) = (\cos(-t), \sin(-t))$ are obtained in the same manner, except that in the first instance the rotation is counterclockwise from $(1, 0)$ and in the second the rotation is clockwise from $(1, 0)$.

Result 3 For all values of t, we have

- $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$;
- $\cos(t + 2n\pi) = \cos t$ and $\sin(t + 2n\pi) = \sin t$, for each integer n.

To complete this section, let us show that the unit circle definition is consistent with the definition of the sine and cosine that is given in terms of right triangles. The following figure shows a right triangle OAB with right angle at B using the common side labeling $a = OB$, $b = AB$, and $c = OA$ superimposed on a unit circle with B on the positive x-axis. The radian measure of $\angle BOA$ is denoted t.



The point where the vertical line through the point $P(t)$ intersects the x-axis is labeled Q. The horizontal and vertical sides of $\triangle OQP$ are, respectively, $\cos t$ and $\sin t$, and the hypotenuse of this triangle has length 1. Since $\triangle OQP$ is similar to $\triangle OBA$, we have

$$\frac{\cos t}{1} = \frac{a}{c}, \quad \frac{\sin t}{1} = \frac{b}{c}, \quad \text{so} \quad \cos t = \frac{a}{c} \quad \text{and} \quad \sin t = \frac{b}{c}.$$

This gives the right triangle definition of the sine and cosine of the angle t.

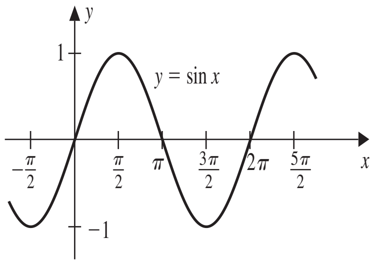
14.3 Important Sine and Cosine Facts

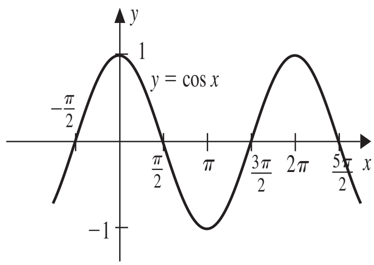
There are certain values of t for which the sine and the cosine are relatively easy to determine. Using these and the symmetry of the unit circle provides us with much of the specific information we need.

Result 1 The following values of the sine and cosine functions should be considered essential knowledge.

- $\sin 0 = 0$ and $\cos 0 = 1$;
- $\sin \dfrac{\pi}{6} = \dfrac{1}{2}$ and $\cos \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$;
- $\sin \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ and $\cos \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$;
- $\sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and $\cos \dfrac{\pi}{3} = \dfrac{1}{2}$;
- $\sin \dfrac{\pi}{2} = 1$ and $\cos \dfrac{\pi}{2} = 0$.

The graphs of the sine and cosine functions are shown in the next figure.





There are two basic identities that are frequently needed, and from which other important identities quite easily follow.

Result 2 For each pair of real numbers t_1 and t_2 we have

- $\sin(t_1 \pm t_2) = \sin t_1 \cos t_2 \pm \cos t_1 \sin t_2$;
- $\cos(t_1 \pm t_2) = \cos t_1 \cos t_2 \mp \sin t_1 \sin t_2$.

Special cases of these results follow when we have $t_1 = t_2$. These are summarized in the following **double-angle** and **half-angle** formulas.

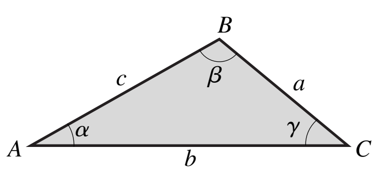
Result 3 For every real number t we have

- $\sin(2t) = 2 \sin t \cos t$,
- $\cos(2t) = (\cos t)^2 - (\sin t)^2 = 2(\cos t)^2 - 1 = 1 - 2(\sin t)^2$,
- $\left(\sin\left(\frac{t}{2}\right)\right)^2 = \frac{1 - \cos t}{2}$ and $\left(\cos\left(\frac{t}{2}\right)\right)^2 = \frac{1 + \cos t}{2}$.

The final results we will consider are used to determine the lengths of missing sides or the measure of missing angles in a triangle that is not a right triangle. These are some of the results most frequently needed to solve AMC problems that involve trigonometry.

Result 4 Let the angles and sides in $\triangle ABC$ be denoted as follows:

$\alpha = \angle CAB,\ \beta = \angle ABC,\ \gamma = \angle BCA, a = BC,\ b = AC,$ and $c = AB$.



- **The Law of Cosines:**

$$a^2 = b^2 + c^2 - 2bc\cos\alpha.$$

We also, of course, have

$$b^2 = a^2 + c^2 - 2ac\cos\beta, \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

- **The Law of Sines:**

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

14.4 The Other Trigonometric Functions

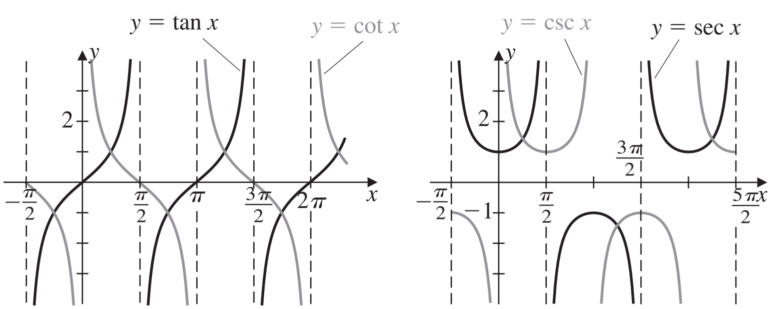
In addition to the basic sine and cosine functions, there are trigonometric functions defined as the reciprocals and quotients of these functions. If you know the basic definitions of these additional functions and definitions and the important facts about the sine and cosine functions, you should be able to handle all the AMC problems that involve trigonometry.

Definition 1 For each real number t for which the quotient is defined, we have

$$\textbf{Tangent of } t\textbf{:} \quad \tan t = \frac{\sin t}{\cos t}, \qquad \textbf{Cotangent of } t\textbf{:} \quad \cot t = \frac{1}{\tan t},$$

$$\textbf{Secant of } t\textbf{:} \quad \sec t = \frac{1}{\cos t}, \qquad \textbf{Cosecant of } t\textbf{:} \quad \csc t = \frac{1}{\sin t}.$$

The graphs of these trigonometric functions are shown in the next figure.

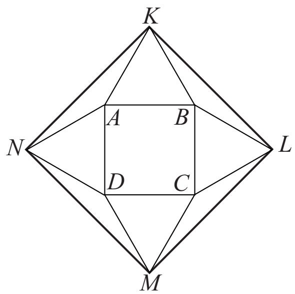


There are numerous facts and identities involving these trigonometric functions, but all can be derived from the knowledge of the corresponding facts concerning the sine and cosine functions.

Examples for Chapter 14

The first Example is number 14 from the 2003 AMC 12A. It was given as Exercise 6 in Chapter 7 where the solution used only geometry. Here we will use the Law of Cosines to solve the problem.

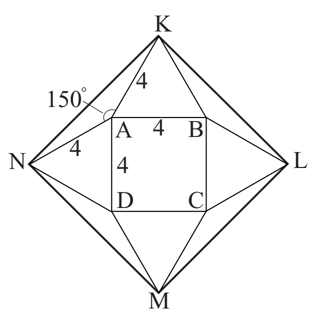
EXAMPLE 1 Points K, L, M, and N lie exterior to the square $ABCD$ so that $\triangle AKB$, $\triangle BLC$, $\triangle CMD$, and $\triangle DNA$ are equilateral triangles, as shown. The area of $ABCD$ is 16. What is the area of $KLMN$?



(A) 32 **(B)** $16 + 16\sqrt{3}$ **(C)** 48 **(D)** $32 + 16\sqrt{3}$ **(E)** 64

Answer (D) Since $\triangle ABK$ and $\triangle ADN$ are both equilateral triangles we have

$$4 = AB = AK = AN$$



and

$$\angle NAK = 360^\circ - \angle BAK - \angle BAD - \angle DAN = 150^\circ.$$

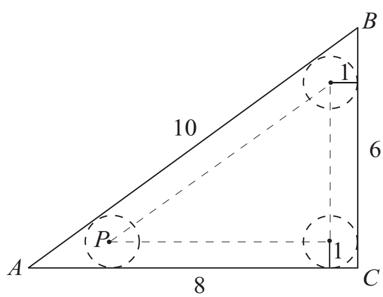
The Laws of Cosines applied to $\triangle AKN$ implies that the area of the square $KLMN$ is

$$\begin{aligned} KN^2 &= NA^2 + KA^2 - 2 \cdot NA \cdot KA \cos 150^\circ \\ &= 16 + 16 - 2(4)(4)\left(-\frac{\sqrt{3}}{2}\right) = 32 + 16\sqrt{3}. \end{aligned}$$ □

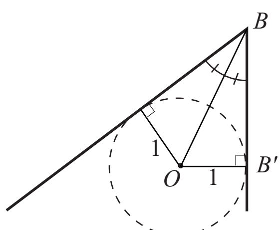
The next Example is number 27 from the 1993 AHSME.

Example 2 The sides of $\triangle ABC$ have lengths 6, 8, and 10. A circle with center P and radius 1 rolls around the inside of $\triangle ABC$, always remaining tangent to at least one side of the triangle. How far has P traveled when it first returns to its original position?

(A) 10 **(B)** 12 **(C)** 14 **(D)** 15 **(E)** 17



Answer (B) The triangle traced by the point P is similar to $\triangle ABC$, so its perimeter is the same fraction of the perimeter of $\triangle ABC$ as the fraction of one of its sides is to the corresponding side of $\triangle ABC$. Let O be the vertex



closest to B of the triangle formed by the point P, and let B' be the point on $\overline{BC}$ that is distance 1 from O. The figure shows a reproduction of $\angle ABC$ from which we see that $2\angle OBB' = \angle ABC$.

So

$$\begin{aligned} BB' = \cot\angle OBB' = \cot\frac{1}{2}\angle ABC &= \frac{\cos\frac{1}{2}\angle ABC}{\sin\frac{1}{2}\angle ABC} \\ &= \frac{\sqrt{(1+\cos\angle ABC)/2}}{\sqrt{(1-\cos\angle ABC)/2}} \\ &= \sqrt{\frac{1+\frac{6}{10}}{1-\frac{6}{10}}} \\ &= \sqrt{\frac{16/10}{4/10}} = \sqrt{4} = 2. \end{aligned}$$

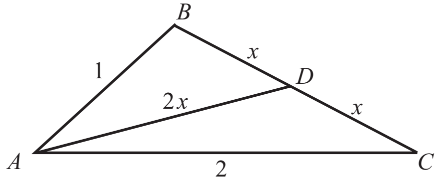
Hence the vertical side of the triangle traced by P has length $6-2-1=3$, which is half the length of the side BC of $\triangle ABC$. As a consequence, the triangle traced by P has half the perimeter of $\triangle ABC$, or $(6+8+10)/2 = 12$. □

The final Example is number 23 from the 2002 AMC 12B.

Example 3 In $\triangle ABC$ we have $AB = 1$ and $AC = 2$. Side $\overline{BC}$ and the median from A to $\overline{BC}$ have the same length. What is BC?

(A) $\dfrac{1+\sqrt{2}}{2}$ **(B)** $\dfrac{1+\sqrt{3}}{2}$ **(C)** $\sqrt{2}$ **(D)** $\dfrac{3}{2}$ **(E)** $\sqrt{3}$

Answer (C) Let D be the midpoint of the line segment $\overline{BC}$, and let $x = BD = DC$.



Then $AD = 2x$, and applying the Law of Cosines to $\triangle ADC$ gives

$$4x^2 = x^2 + 4 - 4x\cos\angle C,$$

and to $\triangle ABC$ gives

$$1 = 4x^2 + 4 - 8x \cos \angle C.$$

Multiplying the first equation by 2 and subtracting the second equation to eliminate the term involving $\cos \angle C$ produces

$$8x^2 - 1 = 2x^2 + 8 - 4x^2 - 4, \quad \text{which simplifies to} \quad x^2 = \frac{1}{2}.$$

Hence $x = \sqrt{2}/2$ and $BC = 2x = \sqrt{2}$. □

Exercises for Chapter 14

Exercise 1 Which of the following is the same as the expression

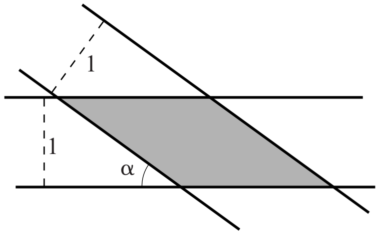
$$\sin(x - y) \cos y + \cos(x - y) \sin y?$$

(A) 1 **(B)** $\sin x$ **(C)** $\cos x$ **(D)** $\sin x \cos 2y$ **(E)** $\cos x \cos 2y$

Exercise 2 Two strips of width 1 overlap at an angle of α, as shown in the shaded region. What is the area of the overlap?

(A) $\sin \alpha$ **(B)** $\dfrac{1}{\sin \alpha}$ **(C)** $\dfrac{1}{1 - \cos \alpha}$ **(D)** $\dfrac{1}{(\sin \alpha)^2}$

(E) $\dfrac{1}{(1 - \cos \alpha)^2}$



Exercise 3 We are given that $\sin x = 3 \cos x$. What is $\sin x \cos x$?

(A) $\dfrac{1}{6}$ **(B)** $\dfrac{1}{5}$ **(C)** $\dfrac{2}{9}$ **(D)** $\dfrac{1}{4}$ **(E)** $\dfrac{3}{10}$

Exercise 4 Two rays (a *ray* is a half line) with common endpoint O form a $30°$ angle. Point A lies on one ray, point B on the other ray, and $AB = 1$. What is the maximum possible value of OB?

(A) 1 **(B)** $\dfrac{1+\sqrt{3}}{\sqrt{2}}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\dfrac{4}{\sqrt{3}}$

Exercise 5 Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, for some x with $0 < x < \pi/2$. What is the length of the hypotenuse of the triangle?

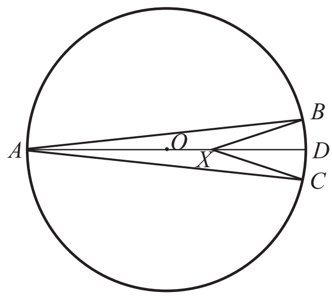
(A) $\dfrac{4}{3}$ **(B)** $\dfrac{3}{2}$ **(C)** $\dfrac{2\sqrt{5}}{5}$ **(D)** $\dfrac{3\sqrt{5}}{5}$ **(E)** $\dfrac{2\sqrt{5}}{3}$

Exercise 6 Points A, B, C, and D are on a circle of diameter 1, and X is on diameter $\overline{AD}$. In addition, $BX = CX$ and $3\angle BAC = \angle BXC = 36°$. What is the length AX?

(A) $\cos 6° \cos 12° \sec 18°$ **(B)** $\cos 6° \sin 12° \csc 18°$

(C) $\cos 6° \sin 12° \sec 18°$ **(D)** $\sin 6° \sin 12° \csc 18°$

(E) $\sin 6° \sin 12° \sec 18°$



Exercise 7 In $\triangle ABC$, we have $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$. What is the degree measure of $\angle C$?

(A) 30 **(B)** 60 **(C)** 90 **(D)** 120 **(E)** 150

Exercise 8 An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

(A) $\dfrac{1}{6}$ **(B)** $\dfrac{1}{5}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{1}{3}$ **(E)** $\dfrac{1}{2}$

Exercise 9 Suppose that $\sum_{n=0}^{\infty}(\cos\theta)^{2n} = 5$. What is $\cos 2\theta$?

(A) $\frac{1}{5}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{\sqrt{5}}{5}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{4}{5}$

Exercise 10 In $\triangle ABC$, side $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ meet in point D, and BD bisects $\angle ABC$. In addition, $AD = 9$ and $DC = 7$. What is the area of $\triangle ABD$?

(A) 14 **(B)** 21 **(C)** 28 **(D)** $14\sqrt{5}$ **(E)** $28\sqrt{5}$

15

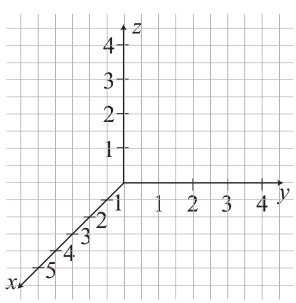
Three-Dimensional Geometry

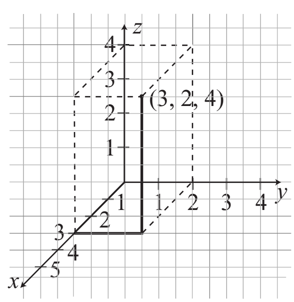
15.1 Introduction

Many AMC exams have some three-dimensional problems among the latter offerings. Often these problems are not difficult, but involve topics that are unfamiliar to many students. In this chapter the properties of standard three-dimensional objects are considered, and it is shown how these concepts are derived from more familiar planar objects. Students taking the AMC 10 are less likely to see problems involving three-dimensional geometry.

15.2 Definitions and Results

Many of the problems involving 3 dimensions require little more than the ability to visualize in three-dimensional space. To do this more easily, it is common practice to use a perspective view of objects in space. In the figure on the left, a three-dimensional coordinate system is shown with the y- and z- axes in the plane of the paper and the x-axis projecting perpendicular to the front. To show this we draw the x-axis by bisecting the obtuse angle





formed by the other two axes. In addition, we make the scale on the x-axis shorter than that on the other axes, by a factor of $\sqrt{2}$. To avoid clutter, the negative portions of the coordinate axes are generally omitted, unless they are needed to give a true representation.

The point with coordinates $x = 3$, $y = 2$, and $z = 4$ is shown on the graph at right. A rectangular box has been added to better illustrate the three-dimensionality of this representation.

Most concepts involving objects in three-dimensional space are straightforward generalizations of planar concepts.

DEFINITION 1 The **distance** between points (a, b, c) and (x, y, z) in space is

$$D((a, b, c), (x, y, z)) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

In Chapter 7 we saw that a polygon in the plane is a geometric object whose boundaries are straight line segments. The polygon is convex if every line segment between points in the interior of the polygon is contained entirely within the polygon. A convex polygon is regular if its edges are congruent, that is, they all have the same length. In space we have similar concepts, but here planes take the place of lines.

DEFINITION 2 A **polyhedron** is a solid bounded by parts of intersecting planes. The intersection of the polyhedron with one of its bounding planes is called a **face** of the polyhedron and a line segment formed by the intersection of two faces is called an **edge**. The points where the edges of a polyhedron meet are called **vertices**.

Leonhard Euler found the following interesting relationship among the faces, edges, and vertices of a polyhedron.

Result 1 Euler's Formula: If a polyhedron with a solid interior has F faces, E edges, and V vertices, then

$$F + V - E = 2.$$

The following definition for polyhedra is similar to this same concept that we saw in Chapter 7 as applied to polygons.

DEFINITION 3 The polyhedron is **convex** if every line segment between points in the interior of the polyhedron is contained entirely within the polyhedron. A convex polyhedron is **regular** if its faces are congruent to each other.

Let us now consider some common polyhedra and other solids that are commonly seen. We have listed the volume formulas for these objects. The surface area is also needed at times, but this is just the sum of the areas of the faces of the solid.

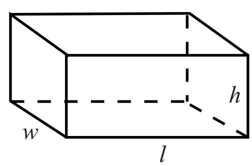
DEFINITION 4 A **rectangular solid** is a polyhedra with six rectangular faces meeting at right angles. Its volume is the product of its length, width and height.

DEFINITION 5 Suppose that M and N are parallel planes and R is a polygonal region in M. Let l be a line intersecting both M and N. The set of all line segments parallel to line l that join a point of region R to a point of N is a **prism**. The volume of a prism is the product of its altitude and the area of its base.

DEFINITION 6 Suppose that R is a polygonal region in plane M and P is a point not on the plane. The set of all segments that join P to a point of R is a **pyramid**. The volume of a pyramid is one-third the product of its altitude and the area of its base.

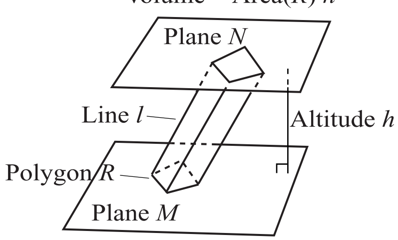
Rectangular Solid

Volume = $l{\cdot}w{\cdot}h$



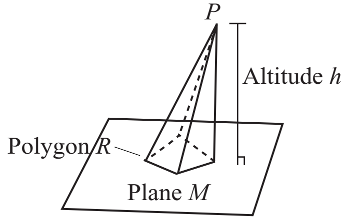
Prism

Volume = Area(R)$\cdot h$



Pyramid

Volume = $\frac{1}{3}$ Area(R)$\cdot h$



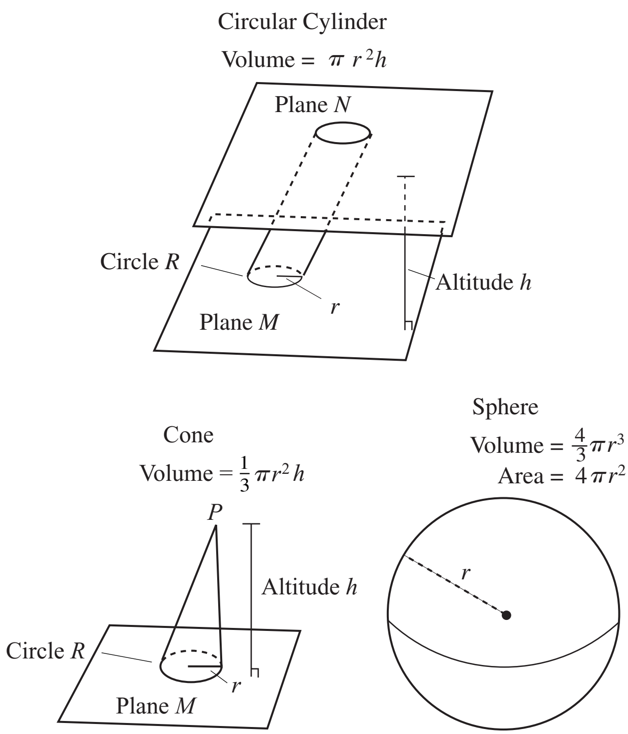
The solids given next are not polyhedra since they involve circular faces or cross sections.

Definition 7 Suppose M and N are parallel planes and R is a circular region in M with radius r. Let l be a line intersecting both M and N. Then the set of all segments parallel to line l that join a point of region R to a point of N form a solid called a **circular cylinder**. The volume of a circular cylinder is πr^2 times its altitude.

The special case when the line l is perpendicular to the circular region R produces a **right-circular cylinder**. The altitude of a right-circular cylinder is the same as its height.

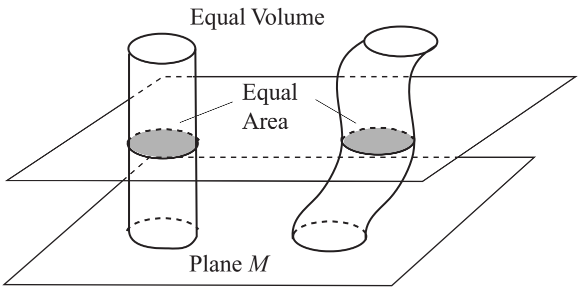
Definition 8 Suppose that R is a circular region with radius r in a plane M and P is a point not on the plane. The set of all segments that join P to a point of R is a **cone**. The volume of a cone is $\pi r^2/3$ times its altitude.

Definition 9 A **sphere** is the set of all points in space that are a given distance, r, from a fixed point. The volume of a sphere is $4\pi r^3/3$, and its surface area is $4\pi r^2$.



The final result, which can at times be employed to reduce the complication when determining the volume of a solid, is credited to the early 17th century Italian mathematician Beneventura Cavalieri. However, it is included more for its interest rather than for its utility for solving AMC problems.

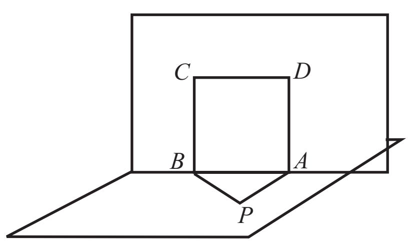
Result 2 Cavalieri's Principle: Consider two geometric solids S_1 and S_2 and a plane M. If every plane parallel to M that intersects S_1 also intersects S_2 and the intersected cross-sections always have the same area, then the solids S_1 and S_2 have the same volume.



Examples for Chapter 15

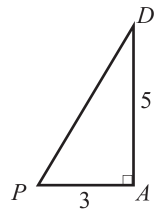
The first Example is number 9 from the 1996 AHSME.

EXAMPLE 1 Triangle PAB and square $ABCD$ lie in perpendicular planes. Suppose that $PA = 3$, $PB = 4$, and $AB = 5$. What is PD?



(A) 5 **(B)** $\sqrt{34}$ **(C)** $\sqrt{41}$ **(D)** $2\sqrt{13}$ **(E)** 8

Answer (B) Since the planes are perpendicular, $\triangle PAD$ has a right angle at A. In addition, $AD = AB = 5$.



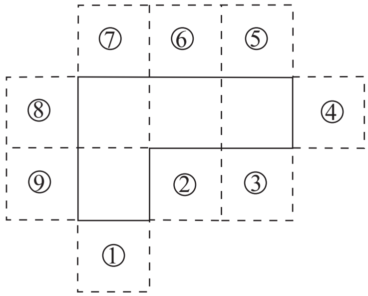
By the Pythagorean Theorem we have

$$PD = \sqrt{AD^2 + AP^2} = \sqrt{5^2 + 3^2} = \sqrt{34}. \qquad \square$$

Notice that we did not need the fact that $PB = 4$. It implies that $\triangle BAP$ has a right angle at P, but this is not needed in the problem, since any point P on the horizontal plane whose distance from A is 3 has $PD = \sqrt{34}$. Information that is given to attempt to lead a problem solver down the wrong path is commonly known as a "red herring".

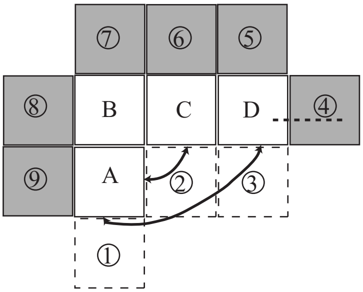
The next Example is number 13 from the 2003 AMC 12A.

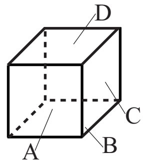
EXAMPLE 2 The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (E) When the sides are folded, the square A must share an edge of the cube with squares C and D, as shown at left.

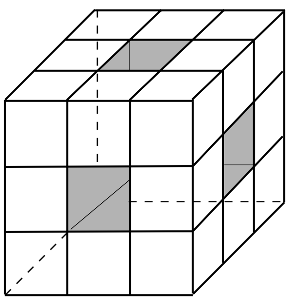




This eliminates the possibility of using squares labeled 1, 2, or 3 as a face of the open cube. When folded into a cube with face A on the front and side C at the right, as shown on the right, we have open spaces at the top and at the left of the cube. So any of the 6 squares labeled 4 through 9 can be used to complete the open cube. □

The final Example is number 27 from the 1998 AHSME.

EXAMPLE 3 A $9 \times 9 \times 9$ cube is composed of twenty-seven $3 \times 3 \times 3$ cubes.



The big cube is "tunneled" as follows: First, the $3 \times 3 \times 3$ cubes which make up the center of each face are removed as well as the interior center $3 \times 3 \times 3$ cube, as shown. Second, each of the twenty remaining $3 \times 3 \times 3$ cubes is diminished in the same way. That is, the unit cubes in the center of each face as well as each interior center cube are removed. What is the surface area of the final figure?

(A) 384 **(B)** 729 **(C)** 864 **(D)** 1024 **(E)** 1056

Answer (E) First note that the six faces of the original cube have a combined a surface area of $9 \times 9 \times 6 = 486$ square units. The first step of the operation removes from each face one square from the surface, but adds four squares in the interior. So after the first removal the surface area is

$$486 + 6 \cdot (-1 + 4)(3 \times 3) = 486 + 162 = 648 \text{ square units.}$$

The second step must consider two types of cubes.

Case I: There are eight cubes on the corners, which have three faces exposed. From each face we delete a unit square and add a total of $6 \cdot 4 = 24$ internal unit squares. So each cube contributes a total of

$$-3 + 24 = 21 \text{ square units.}$$

Case II: There are 12 cubes on edges with two external and two internal faces exposed. From each face we again delete a unit square and add a total of $6 \cdot 4 = 24$ internal unit squares. So each cube contributes a total of

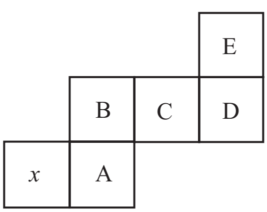
$$-4 + 24 = 20 \text{ square units.}$$

The surface area of the final solid is consequently

$$648 + 8 \cdot 21 + 12 \cdot 20 = 648 + 168 + 240 = 1056 \text{ square units.} \qquad \square$$

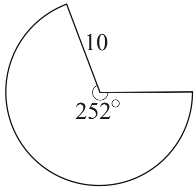
Exercises for Chapter 15

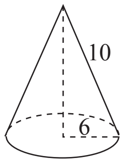
Exercise 1 The figure shown can be folded into the shape of a cube. In the resulting cube, which of the lettered faces is opposite the face marked x?

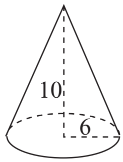


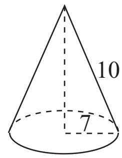
(A) A **(B)** B **(C)** C **(D)** D **(E)** E

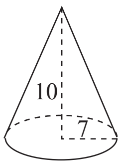
Exercise 2 Which of the cones below can be formed from a 252° sector of a circle of radius 10 by aligning the two straight sides?

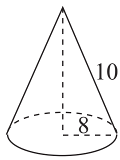


(A) 

(B) 

(C) 

(D) 

(E) 

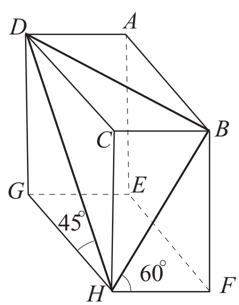
Exercise 3 Three cubes having volumes 1, 8, and 27 are glued together at their faces. What is the smallest possible surface area that the resulting polyhedron can have?

(A) 36 **(B)** 56 **(C)** 70 **(D)** 72 **(E)** 74

Exercise 4 An $11 \times 11 \times 11$ wooded cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that have at least one face that can be seen from a single point?

(A) 328 **(B)** 329 **(C)** 330 **(D)** 331 **(E)** 332

Exercise 5 In the rectangular solid shown, we have $\angle DHG = 45°$ and $\angle FHB = 60°$. What is the cosine of $\angle BHD$?

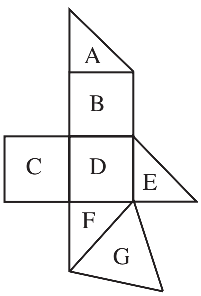


(A) $\dfrac{\sqrt{3}}{6}$ **(B)** $\dfrac{\sqrt{2}}{6}$ **(C)** $\dfrac{\sqrt{6}}{3}$ **(D)** $\dfrac{\sqrt{6}}{4}$ **(E)** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

Exercise 6 A wooden cube with edge length $n > 2$ units is painted black on all sides. The cube is cut into n^3 smaller cubes each of unit edge length by making slices parallel to its faces. The number of smaller cubes with just one face painted black is equal to the number of smaller cubes completely free of paint. What is n?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Exercise 7 In the figure, A, E, and F are isosceles right triangles; B, C, and D are squares with sides of length 1, and G is an equilateral triangle. The figure can be folded along the edges of these polygons to form a polyhedron. What is the volume of the polyhedron?

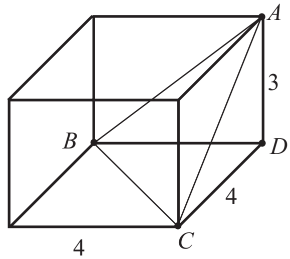


(A) $\dfrac{1}{2}$ **(B)** $\dfrac{2}{3}$ **(C)** $\dfrac{3}{4}$ **(D)** $\dfrac{5}{6}$ **(E)** $\dfrac{4}{3}$

Exercise 8 Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look identical to the other.)

(A) 210 **(B)** 560 **(C)** 840 **(D)** 1260 **(E)** 1680

Exercise 9 On a $4 \times 4 \times 3$ rectangular parallelepiped, vertices A, B, and C are adjacent to vertex D. Consider the plane containing the points A, B, and C. What is the perpendicular distance from D to this plane?



(A) $\frac{6\sqrt{34}}{17}$ **(B)** $\sqrt{5}$ **(C)** $\sqrt{6}$ **(D)** $2\sqrt{2}$ **(E)** $2\sqrt{3}$

Exercise 10 Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

(A) $3 + \frac{\sqrt{30}}{2}$ **(B)** $3 + \frac{\sqrt{69}}{3}$ **(C)** $3 + \frac{\sqrt{123}}{4}$ **(D)** $\frac{52}{9}$
(E) $3 + 2\sqrt{2}$

16
Functions

16.1 Introduction

The properties of functions are now well known to juniors and seniors in high school, and the recent AMCs have included increasing numbers of problems associated with functions and functional notation. Many of these problems require only careful manipulation of the functional definitions.

Students taking the AMC 10 will not likely see problems involving the function concepts considered in this chapter.

16.2 Definitions

A function is a specific way to associate the elements of one set with the elements of another set.

DEFINITION 1 A **function** from set A to set B is a means of associating every element of the set A with a specific element in the set B. The set A is called the **domain** of the function, and the collection of elements in B that the function associates with some element in A is called the **range** of the function.

Most of the problems on the AMC will concern functions whose domain and range are both subsets of real numbers. The domain of the function will often not be specified, but assumed to be the largest set of real numbers for which the function values are defined.

For example, it is common to describe a function, which we will call f, that associates each real number with the principle square root of the number simply by writing

$$f(x) = \sqrt{x}.$$

The variable used to describe the function is generally x, but this is just a place marker. It would probably be better to write the description of the function as

$$f(\_\_) = \sqrt{\_\_}$$

to better indicate that whatever fills in the blank on the left also fills in the blank on the right.

Unless otherwise specified, the domain of the function is assumed to be the largest set of real numbers for which the function is defined and produces a real number. In the case of $f(x) = \sqrt{x}$, the domain is the interval $[0, \infty)$ since this is the largest set of real numbers on which the square root is defined and produces another real number. The range is also the interval $[0, \infty)$, since it is precisely these numbers that can result from taking the square root of the numbers in the domain.

A somewhat more complicated example is given by

$$f(x) = 1 + \sqrt{\frac{x}{(x+2)(2-x)}}.$$

To determine the domain of this function, we need to find when

$$\frac{x}{(x+2)(2-x)} \geq 0.$$

There are algebraic ways involving sets of inequalities to do this, but it is easier to consider the problem geometrically. First note that the function values can change from positive to negative only when passing through points where the function is zero or where it is undefined. (This is a consequence of the Intermediate Value Theorem for continuous functions.) Since $x/((x+2)(2-x))$ is zero when $x = 0$ and is undefined when $x = -2$ and when $x = 2$, this implies that the sign of the values of

$$\frac{x}{(x+2)(x-2)}$$

is constant on each of the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$.

Trying a single point in each of these intervals gives us the conclusion that $x/((x+2)(2-x)) > 0$ precisely when x is in $(0, 2)$ or in $(-\infty, -2)$.

In addition, $x/((x+2)(2-x)) = 0$ precisely when $x = 0$, so the domain of the function, f, given by

$$f(x) = 1 + \sqrt{\frac{x}{(x+2)(2-x)}} \quad \text{is } (-\infty, -2) \cup [0, 2).$$

It is generally more difficult to determine the range of a function, but in this case we can find it by doing a little analysis on the ends of the domain. First note that $f(0) = 1 + 0 = 1$, and since the principle square root is never negative, this is the smallest possible value in the range. In addition, the closer x gets to 2, but remaining less than 2, the smaller the denominator of the fraction under the square root. This will force the fraction to become large, and its square root will also become large. It is reasonable to surmise, therefore, that every value in the interval $[1, \infty)$ will be in the range of f.

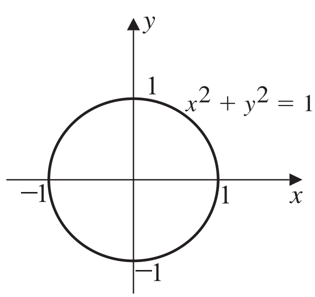
You might occasionally see functions on the AMC whose domains and ranges include some complex numbers that are not also real numbers. We will postpone discussion of this type of function until Chapter 18.

16.3 Graphs of Functions

The **graph of an equation** in the variables x and y consists of the set of points (x, y) in the xy-plane whose coordinates satisfy the equation. For example, the graph of the equation

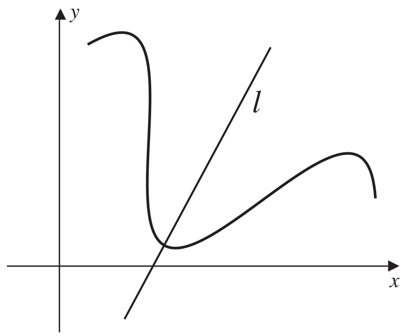
$$x^2 + y^2 = 1$$

is the circle centered at the origin and having radius 1, since the points on this circle are precisely those that satisfy this equation.



There are certain features of graphing that occasionally appear on the AMC 12. Most have to do with symmetry.

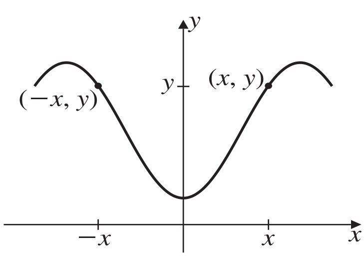
DEFINITION 1 A graph is **symmetric to a line** when the portion of the graph on one side of the line is the mirror image of the portion on the other side. The graph shown in the following figure is symmetric to the line l.



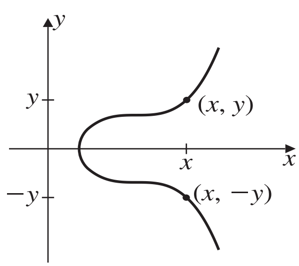
Symmetry of a graph to a line is easiest to determine when the line is one of the coordinate axes. Axis symmetry is also called *symmetry with respect to the axis.*

Definition 2 Axis Symmetry

- The graph of an equation has ***y*-axis symmetry** if $(-x, y)$ is on the graph whenever (x, y) is on the graph.
- The graph of an equation has ***x*-axis symmetry** if $(x, -y)$ is on the graph whenever (x, y) is on the graph.



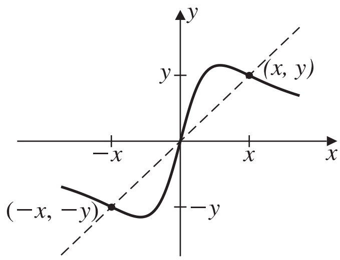
y-axis symmetry



x-axis symmetry

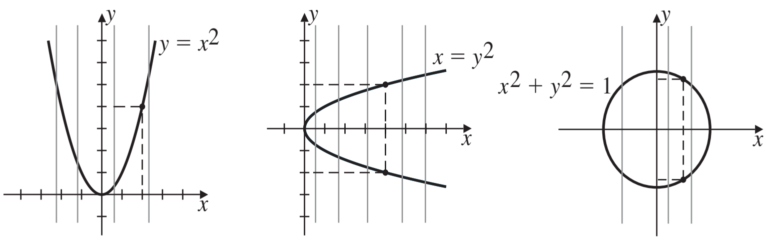
Symmetry is also defined with *respect to a point* in the plane. In this case, the graph has a mirror reflection property with respect to the point. This feature might be difficult to detect for arbitrary points in the plane, but it is easy when the point is the origin.

DEFINITION 3 The graph of an equation has **origin symmetry** if $(-x, -y)$ is on the graph whenever (x, y) is on the graph.



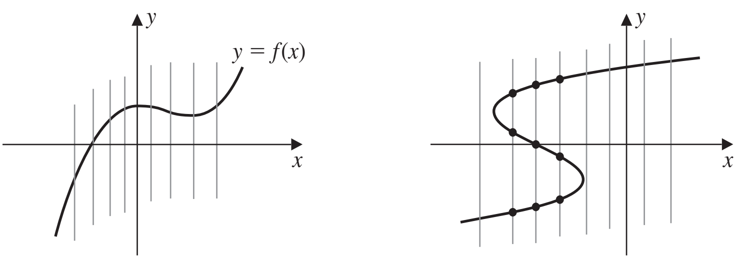
Origin symmetry

In this chapter we are primarily concerned with functions, that is, graphs of equations of the form $y = f(x)$, where f is a function. These graphs have special properties.



A function (a) Not a function (b) Not a function (c)

DEFINITION 4 An equation describes y as a function of x if and only if every vertical line intersects the graph of the equation at most once.



A function Not a function

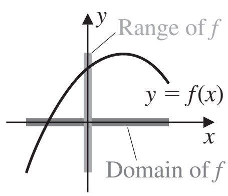
Note that the definition of a function does not permit its graph to have x-axis symmetry, unless it is the graph of the function $f(x) = 0$.

However, the graph of a function can have y-axis symmetry, in which case $f(-x) = f(x)$, for each value of x in its domain. It could also have have origin symmetry, in which case $f(-x) = -f(x)$, for each value of x in its domain.

It is easy to determine the domain and the range of a function f if the graph of $y = f(x)$ is known.

Definition 5 The Domain and Range of a Function

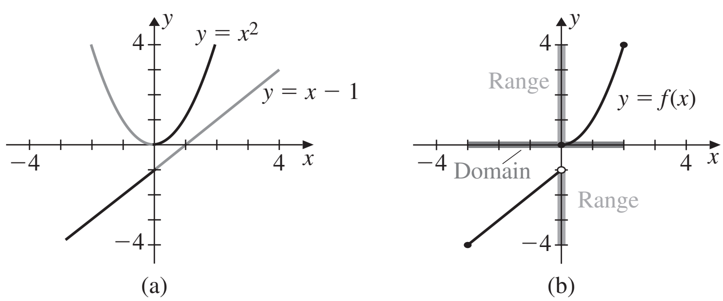
- The **domain** of a function f is described by those values on the horizontal axis through which a vertical line intersects the graph of $y = f(x)$.
- The **range** of a function f is described by those values on the vertical axis through which a horizontal line intersects the graph of $y = f(x)$.



For example, consider the graph of the function defined by

$$f(x) = \begin{cases} x - 1, & \text{if } -3 \le x < 0, \\ x^2, & \text{if } 0 \le x \le 2. \end{cases}$$

The graph on the left shows the two equations $y = x - 1$ and $y = x^2$ that make up the graph $y = f(x)$, which is shown on the right.



Since the vertical lines that intersect the graph are those in the interval $[-3, 2]$, this is the domain of the function. Horizontal lines intersect the graph when $-4 \le y < -1$ or when $0 \le y \le 4$, so the range of f is $[-4, -1) \cup [0, 4]$.

16.4 Composition of Functions

The composition of a pair, or more, of functions is simply the result of following the operation of one function with the operation of another. Suppose, for example, that the functions f and g are defined by

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \frac{1}{x-2}.$$

The compositions of f and g are defined by

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x-2}\right) = \sqrt{\frac{1}{x-2}}$$

and

$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \frac{1}{\sqrt{x}-2}.$$

Notice that the values of these compositions are not the same. In fact, they are not even defined on the same set. The domain of $f \circ g$ is $(2, \infty)$ whereas the domain of $g \circ f$ is $[0, 4) \cup (4, \infty)$.

A particularly important composition results when f and g have the property that

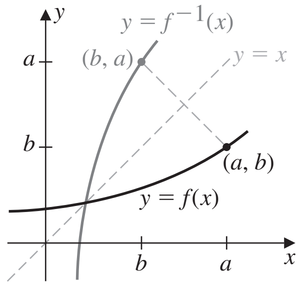
$$(f \circ g)(x) = x, \text{ for each } x \text{ in the domain of } g$$

and, in addition,

$$(g \circ f)(x) = x, \text{ for each } x \text{ in the domain of } f.$$

In this case the functions f and g are said to be **inverse functions** of one another. In this case the function g is denoted by $g = f^{-1}$, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line through the origin that has slope 1.

Definition 1 The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$.



The graphs of a function and its inverse both increase or decrease together, but the steepness of the graph of the inverse function is inversely related to the steepness of the graph of the function. You will prove that this is true in calculus.

Examples for Chapter 16

The first Example is number 14 from the 1995 competition.

EXAMPLE 1 Suppose that $f(x) = ax^4 - bx^2 + x + 5$ and that $f(-3) = 2$. What is $f(3)$?

(A) -5 **(B)** -2 **(C)** 1 **(D)** 3 **(E)** 8

Answer (E) We first need to relate the value of $f(-x)$ to the value of $f(x)$, and then apply this relationship when $x = 3$. Note that $(-x)^n = x^n$ when n is an even integer, and $(-x)^n = -x^n$ when n is an odd integer. Applying this information implies that

$$\begin{aligned} f(-x) &= a(-x)^4 - b(-x)^2 + (-x) + 5 \\ &= ax^4 - bx^2 - x + 5 \\ &= (ax^4 - bx^2 + x + 5) - 2x = f(x) - 2x. \end{aligned}$$

So

$$f(3) = f(-(3)) - 2(-3) = 2 + 6 = 8. \qquad \square$$

The next Example is number 21 from the 1991 competition.

Example 2 For all real numbers x, except $x = 0$ and $x = 1$, the function f is defined by

$$f\left(\frac{x}{x-1}\right) = \frac{1}{x}.$$

Suppose $0 < \theta < \pi/2$. What is $f\left((\sec\theta)^2\right)$?

(A) $(\sin\theta)^2$ **(B)** $(\cos\theta)^2$ **(C)** $(\tan\theta)^2$ **(D)** $(\cot\theta)^2$
(E) $(\csc\theta)^2$

Answer (A) The description of the function is complicated in this problem, so we will introduce a new variable to simplify the functional relationship. First let $y = x/(x-1)$ and solve for x in terms of y. When $x \neq 0$ and $x \neq 1$ we have

$$y(x-1) = x \quad \text{which implies} \quad yx - y = x, \quad \text{and} \quad yx - x = y.$$

Solving for x in terms of y gives

$$x(y-1) = y \quad \text{and} \quad x = \frac{y}{y-1}.$$

Note that this expression is not defined when $y = 1$. In addition, if $y = 0$ then $x = 0$, which was not defined in the original expression. For $y \neq 0$ and $y \neq 1$ we have

$$f(y) = f\left(\frac{x}{x-1}\right) = \frac{1}{x} = \frac{1}{y/(y-1)} = \frac{y-1}{y} = 1 - \frac{1}{y}.$$

When $y = (\sec\theta)^2$, for $0 < \theta < \frac{\pi}{2}$, this gives

$$f\left((\sec\theta)^2\right) = 1 - \frac{1}{(\sec\theta)^2} = 1 - (\cos\theta)^2 = (\sin\theta)^2. \qquad \square$$

The final Example is number 26 from the 1993 competition.

Example 3 The real-valued function f is defined by

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}.$$

What is the maximum value of $f(x)$?

(A) $\sqrt{7} - 1$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $\sqrt{55} - \sqrt{5}$

Answer (C) The function consists of two quadratic terms under square roots and, unless we can simplify the expression, it is going to be difficult to solve the problem. One of the first things to consider when there is a quadratic in a problem is completing the square on the quadratic terms. In this case the quadratic terms are

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2$$

and

$$\begin{aligned} 14x - x^2 - 48 &= -(x^2 - 14x + 48) \\ &= -(x^2 - 14x + 49) + 1 = 1 - (x - 7)^2. \end{aligned}$$

For x to be in the domain of f we must have both

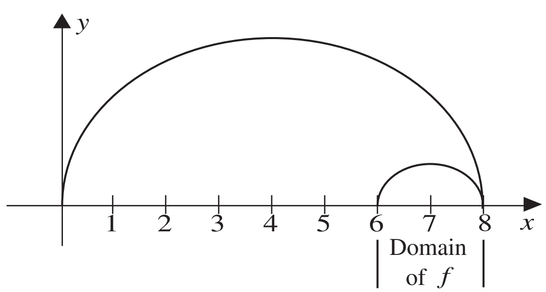
$$0 \le 16 - (x - 4)^2, \text{ which implies } |x - 4| \le 4, \text{ so } \quad 0 \le x \le 8,$$

and

$$0 \le 1 - (x - 7)^2, \text{ which implies } |x - 7| \le 1, \text{ so } \quad 6 \le x \le 8,$$

So we need only consider the values of x in the interval [6, 8], and for these values we have

$$f(x) = \sqrt{16 - (x - 4)^2} - \sqrt{1 - (x - 7)^2}.$$



As shown in the figure, the graph of $y = \sqrt{16 - (x - 4)^2}$ is the semicircle in the first quadrant with center at (4, 0) and radius 4. The graph of $y = \sqrt{1 - (x - 7)^2}$ is the semicircle in the first quadrant with center at (7, 0) and radius 1. Seen in this form, it is clear that the value in [6, 8] that maximizes $f(x)$ is $x = 6$ since this value maximizes $\sqrt{16-(x-4)^2}$ and also minimizes $\sqrt{1-(x-7)^2}$. Hence, the maximum value of $f(x)$ is

$$f(6) = \sqrt{16-(6-4)^2} - \sqrt{1-(6-7)^2} = \sqrt{12} - \sqrt{0} = 2\sqrt{3}.$$

Students who have studied calculus might be tempted to solve this problem by taking the derivative of $f(x)$, setting this to zero and solving for x. Doing this will create quite an algebraic equation, which will take longer to resolve than the given algebra solution. Remember, no problem on the AMC has a calculus solution that is easier that some non-calculus solution. □

Exercises for Chapter 16

Exercise 1 Suppose that for all $x > 0$ we have $f(2x) = \dfrac{2}{2+x}$. What is $2f(x)$?

(A) $\dfrac{2}{1+x}$ **(B)** $\dfrac{2}{2+x}$ **(C)** $\dfrac{4}{1+x}$ **(D)** $\dfrac{4}{2+x}$ **(E)** $\dfrac{8}{4+x}$

Exercise 2 The function f is defined for positive integers n by:

$$f(n) = \begin{cases} n+3, & \text{if } n \text{ is odd,} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Suppose k is an odd integer and that $f(f(f(k))) = 27$. What is the sum of the digits of k?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 15

Exercise 3 Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b, and c are constants. Suppose that $f(-7) = 7$. What is $f(7)$?

(A) -17 **(B)** -7 **(C)** 14 **(D)** 17 **(E)** 21

Exercise 4 The function f satisfies $f(2+x) = f(2-x)$ for all real numbers x. Moreover, $f(x) = 0$ has exactly four distinct real roots. What is the sum of these roots?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Exercise 5 Suppose that the function f, for $x \neq -3/2$, is defined by

$$f(x) = \frac{cx}{2x+3},$$

and that $f(f(x)) = x$ for all real numbers in its domain. What is the value of c?

(A) -3 **(B)** $-\frac{3}{2}$ **(C)** $\frac{3}{2}$ **(D)** 3 **(E)** 5

Exercise 6 Let $f\left(x^2+1\right) = x^4 + 5x^2 + 3$. What is $f\left(x^2-1\right)$?

(A) $x^4 + 5x^2 + 1$ **(B)** $x^4 + x^2 - 3$ **(C)** $x^4 - 3x^2 + 1$
(D) $x^4 - 5x^2 + 1$ **(E)** $x^4 + x^2 + 3$

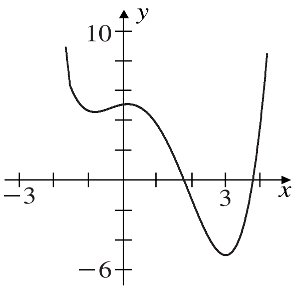
Exercise 7 Suppose that $x^2 + y^2 = 14x + 6y + 6$. What is the maximum value of $3x + 4y$?

(A) 72 **(B)** 73 **(C)** 74 **(D)** 75 **(E)** 76

Exercise 8 What is the number of real solutions of the equation $\frac{x}{100} = \sin x$?

(A) 61 **(B)** 62 **(C)** 63 **(D)** 64 **(E)** 65

Exercise 9 The graph shows a portion of the curve defined by a quartic polynomial of the form $P(x) = x^4 + ax^3 + bx^2 + cx + d$. Which of the following is the smallest?



(A) $P(-1)$ **(B)** The product of the zeros of P.
(C) The product of the non-real zeros of P.
(D) The sum of the coefficients of P.
(E) The sum of the real zeros of P.

Exercise 10 Let $f(x) = x^2 + 6x + 1$, and let R denote the set of points (x, y) in the coordinate plane such that

$$f(x) + f(y) \le 0 \quad \text{and} \quad f(x) - f(y) \le 0.$$

Which of the numbers is closest to the area of R?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

17
Logarithms

17.1 Introduction

Logarithm properties often cause students difficulty, but the number of concepts to master in order to do these problems is rather small. As a consequence, even though most problems involving logarithms come later in the exams, they are often not difficult. This is another topic that would be more heavily emphasized if calculators were excluded from the exam. Students taking the AMC 10 examinations will not see problems involving logarithms.

17.2 Definitions and Results

DEFINITION 1 If $a \neq 1$ and x are positive real numbers, then we say that ***y* is the logarithm of *x* to the base *a***, written $y = \log_a x$, when x is the value of a raised to the power y. That is,

$$y = \log_a x \iff x = a^y.$$

Because of the relationship of the logarithm function to an exponential function, we have some immediate results.

RESULT 1 For any positive number $a \neq 1$, we have

$$\log_a 1 = 0, \quad \log_a a = 1, \quad \text{and} \quad \log_a \frac{1}{a} = -1.$$

RESULT 2 If $a \neq 1$ and x and y are positive real numbers with $\log_a x = \log_a y$, then $x = y$.

Result 3 If $a \neq 1$, $b \neq 1$, x and y are positive real numbers and r is any real number, then

- $a^{\log_a x} = x$
- $\log_a a^r = r$
- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a x^r = r \log_a x$
- $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y.$

Suppose that $a \neq 1$. The first property in Result 3 states that

$$x = a^{\log_a x}.$$

If, in addition, $b \neq 1$, then we can take the base b logarithm of this expression to produce

$$\log_b x = \log_b a^{\log_a x} = \log_a x \cdot \log_b a.$$

Dividing both sides by $\log_b a$ gives the following.

Result 4 For any positive $a \neq 1$ and $b \neq 1$, and any positive number x we have

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

This result means that all logarithm functions have the same form; any one of them is simply a multiple of any other. This relationship between logarithm functions gives the following particularly useful formula when $x = b$, since in this case $\log_b x = \log_b b = 1$. This is the most frequently used result for solving AMC logarithm problems.

Result 5 For any positive $a \neq 1$ and and $b \neq 1$, we have

$$\log_a b = \frac{1}{\log_b a}.$$

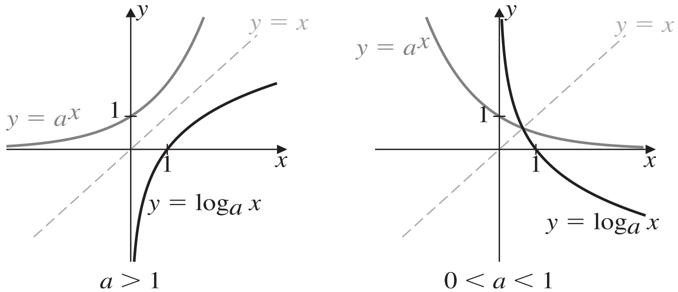
Another result concerning logarithms that is occasionally needed concerns the occurrences when the logarithm to a certain base results in a rational number.

Result 6 Suppose that $a \neq 1$ is a positive number. Then $\log_a b$ results in an integer (respectively, a rational number), if and only if $b = a^r$, where r is an integer (respectively, where r is a rational number).

Another important feature of logarithms in mathematics in general is that they are each inverse functions for a specific exponential function.

DEFINITION 2 A logarithmic function has the form $f(x) = \log_a x$ where $a \neq 1$ is a positive real number.

Graphed below is the function $f(x) = \log_a x$ and the function $g(x) = a^x$ for the situations when $a > 1$ and when $0 < a < 1$. Since these are inverse to one another, the graph $y = f(x)$ is the graph $y = g(x)$ reflected about the line $y = x$.



Examples for Chapter 17

The first Example is number 8 from the 1996 AHSME.

EXAMPLE 1 Suppose that $3 = k \cdot 2^r$ and that $15 = k \cdot 4^r$. What is r?

(A) $-\log_2 5$ **(B)** $\log_5 2$ **(C)** $\log_{10} 5$ **(D)** $\log_2 5$ **(E)** $\frac{5}{2}$

Answer (D) If we solve for k in the equation $3 = k \cdot 2^r$ and substitute this value into the equation $15 = k \cdot 4^r$, we have

$$k = \frac{3}{2^r}, \quad \text{so} \quad 15 = \frac{3}{2^r} \cdot 4^r = 3 \cdot 2^r.$$

Therefore $2^r = 15/3 = 5$, and $r = \log_2 5$. □

The next Example is number 17 from the 2003 AMC 12B.

EXAMPLE 2 Suppose that $\log xy^3 = 1$ and $\log x^2y = 1$. What is $\log xy$?

(A) $-\frac{1}{2}$ **(B)** 0 **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** 1

Answer (D) The basic logarithm properties give

$$1 = \log xy^3 = \log x + 3\log y \quad \text{and} \quad 1 = \log x^2y = 2\log x + \log y.$$

Multiply the first equation by 2 and subtract the corresponding terms of the second equation to produce $1 = 5\log y$. Then

$$\log y = \frac{1}{5}, \quad \log x = 1 - 3\log y = \frac{2}{5},$$

and $\log xy = \log x + \log y = \dfrac{3}{5}$.

OR

Since $1 = \log x^2y$, we have

$$2 = 2\log x^2y = \log x^4y^2.$$

In addition, $1 = \log xy^3$, so

$$\begin{aligned} 3 = 1 + 2 &= \log xy^3 + \log x^4y^2 \\ &= \log(xy^3)(x^4y^2) = \log(xy)^5 = 5\log xy. \end{aligned}$$

Hence $\log xy = \dfrac{3}{5}$. □

The final Example is number 22 from the 2002 AMC 12B.

EXAMPLE 3 For all integers n greater than 1, define

$$a_n = \frac{1}{\log_n 2002}.$$

Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. What is $b - c$?

(A) -2 **(B)** -1 **(C)** $\dfrac{1}{2002}$ **(D)** $\dfrac{1}{1001}$ **(E)** $\dfrac{1}{2}$

Answer (B) First recall that from the basic logarithm property given in Result 5 of Section 17.2, we can re-express a_n as

$$a_n = \frac{1}{\log_n 2002} = \log_{2002} n.$$

So

$$\begin{aligned} b - c &= a_2 + a_3 + a_4 + a_5 - a_{10} - a_{11} - a_{12} - a_{13} - a_{14} \\ &= \log_{2002} 2 + \log_{2002} 3 + \log_{2002} 4 + \log_{2002} 5 - \log_{2002} 10 \\ &\quad - \log_{2002} 11 - \log_{2002} 12 - \log_{2002} 13 - \log_{2002} 14 \\ &= \log_{2002} \frac{2 \cdot 3 \cdot 4 \cdot 5}{10 \cdot 11 \cdot 12 \cdot 13 \cdot 14}. \end{aligned}$$

So

$$b - c = \log_{2002} \frac{1}{11 \cdot 13 \cdot 14} = \log_{2002} \frac{1}{2002} = -\log_{2002} 2002,$$

and $b - c = -1$. □

Exercises for Chapter 17

Exercise 1 Suppose that

$$\log_2(\log_3(\log_5(\log_7 N))) = 11.$$

How many different prime numbers are factors of N?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 7

Exercise 2 Suppose that

$$\log_2(\log_2(\log_2 x)) = 2.$$

How many digits are in the base-10 representation for x?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

Exercise 3 How many positive integers b have the property that $\log_b 729$ is also a positive integer?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Exercise 4 Let $f(n) = \log_{2002} n^2$ for all positive integers n. Define

$$N = f(11) + f(13) + f(14).$$

Which of the following relations is true?

(A) $N > 1$ **(B)** $N = 1$ **(C)** $1 < N < 2$ **(D)** $N = 2$
(E) $N > 2$

Exercise 5 For some real numbers a and b, the equation

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots, and the sum of the base-2 logarithms of the roots is 5. What is the value of a?

(A) -256 **(B)** -64 **(C)** -8 **(D)** 64 **(E)** 256

Exercise 6 For any positive integer n, define

$$f(n) = \begin{cases} \log_8 n, & \text{if } \log_8 n \text{ is rational,} \\ 0, & \text{otherwise.} \end{cases}$$

What is $\displaystyle\sum_{n=1}^{1997} f(n)$?

(A) $\log_8 2047$ **(B)** 6 **(C)** $\dfrac{55}{3}$ **(D)** $\dfrac{58}{3}$ **(E)** 585

Exercise 7 What is the value of the expression

$$N = \frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}\ ?$$

(A) 0.01 **(B)** 0.1 **(C)** 1 **(D)** 2 **(E)** 10

Exercise 8 The graph, G, of $y = \log_{10} x$ is rotated $90°$ counter-clockwise about the origin to obtain a new graph, G'. Which of the following is an equation for G'?

(A) $y = \log_{10}\left(\dfrac{x+90}{9}\right)$ **(B)** $y = \log_x 10$ **(C)** $y = \dfrac{1}{x+1}$
(D) $y = 10^{-x}$ **(E)** $y = 10^x$

Exercise 9 What is the value of the sum

$S = \log_{10}(\tan 1°) + \log_{10}(\tan 2°) + \cdots + \log_{10}(\tan 88°) + \log_{10}(\tan 89°)$?

(A) 0 **(B)** $\frac{1}{2}\log_{10}\left(\frac{1}{2}\sqrt{3}\right)$ **(C)** $\frac{1}{2}\log_{10} 2$ **(D)** $\frac{1}{2}\log_{10} 3$
(E) 1

Exercise 10 Let $a \geq b > 1$. What is the largest possible value of

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} \text{ ?}$$

(A) -2 **(B)** 0 **(C)** 2 **(D)** 3 **(E)** 4

18
Complex Numbers

18.1 Introduction

Most of the AMC 12 examinations include a problem that deals with complex numbers. These problems generally come near the end of the examination, since many contestants are not familiar with this subject and the manipulations can be intricate. However, there are just a few concepts needed to solve these problems, so they can sometimes be relatively easy. Students taking the AMC 10 examinations will not see problems involving complex numbers.

18.2 Definitions

DEFINITION 1 The set of complex numbers, denoted $\mathcal{C}$, consists of all expressions of the form $z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. That is,

$$\mathcal{C} = \{z = a + bi \mid a \text{ and } b \text{ are real and } i^2 = -1.\}$$

The number a is called the **real part** of z and the number b is called the **imaginary part** of z. Note that a complex number whose imaginary part is zero is a real number.

Complex numbers can be written in a variety of ways, but the standard form of a complex number is always assumed to be

$$(\text{real part}) + (\text{imaginary part})i.$$

DEFINITION 2 Two complex numbers are *equal* if and only if they have the same real and imaginary parts, that is,

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d.$$

The *sum* and *product* of two complex numbers $z = a + bi$ and $w = c + di$ are defined by

$$z + w = (a + c) + (b + d)i \quad \text{and} \quad z \cdot w = (ac - bd) + (ad + bc)i$$

These arithmetic operations are the natural consequence of the fact that the real and imaginary parts of a complex number must be kept separate and that $i^2 = -1$. In particular, the multiplication property comes from

$$\begin{aligned} z \cdot w = (a + bi) \cdot (c + di) &= a \cdot c + (a \cdot d)i + i(b \cdot c) + i(b \cdot d)i \\ &= ac + ad \cdot i + bc \cdot i + bd \cdot i^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

DEFINITION 3 The **complex conjugate**, $\overline{z}$, of the complex number $z = a + bi$ is the complex number $\overline{z} = a - bi$.

Notice that z is a real number if and only if $z = \overline{z}$. Conjugation of complex numbers satisfies various important algebraic properties. For example,

$$\overline{\overline{z}} = z, \quad \overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}, \quad \text{and} \quad \overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}}.$$

DEFINITION 4 The *absolute value*, or **magnitude** of the complex number $z = a + bi$ is defined by

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \overline{z}}.$$

Notice that it is not the case that $|z|^2 = z^2$, unless the imaginary part of z is 0. However, for any complex number $z = a + bi$, we have

$$|\overline{z}| = |a - bi| = \sqrt{a^2 + b^2} = |z|.$$

As in the case of real numbers, for any pair of complex numbers z and w we have the *triangle inequality*

$$|z + w| \le |z| + |w|.$$

Recall that in Section 3.4 we re-expressed a quotient involving radicals by multiplying by the conjugate of the denominator. In the example of that section we saw that

$$\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{3+\sqrt{6}}{1} = 3+\sqrt{6}.$$

We can use the complex conjugate of a complex number in a similar manner to re-express the reciprocal of a complex number in standard form:

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{\overline{z}}{|z|^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i.$$

Note that this also implies that

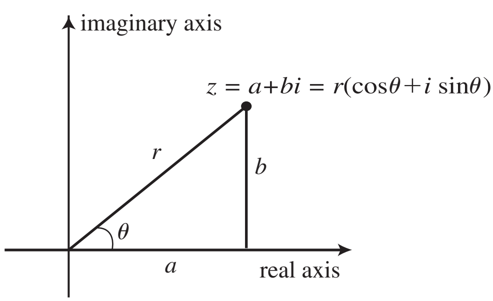
$$\frac{1}{z} = \frac{a-bi}{a+bi} = \frac{\overline{z}}{|z|^2}.$$

DEFINITION 5 The complex plane is the same as the usual xy-plane except the x-axis is now the **real axis** and the y-axis is now the **imaginary axis**. The distance from the point $z = a + bi$ to the origin $0 + 0i$ is $|z|$. As a consequence, the distance between two complex numbers z and w is $|z-w|$.

DEFINITION 6 Every complex number $z = a + bi$ can also be written in a **polar form** as

$$z = r \cdot (\cos\theta + i\sin\theta),$$

where $r = |z|$ and θ is the angle (in radians) that the ray from the origin through the point z makes with the positive real axis. The angle θ is the **argument** of z.



The complex conjugate of $z = a + ib$, has the polar form

$$\overline{z} = r \cdot (\cos(-\theta) + i \sin(-\theta)) = r \cdot (\cos\theta - i \sin\theta),$$

since the argument of $\overline{z}$ is the negative of the argument of z and the magnitudes of z and $\overline{z}$ are the same. As a consequence, the reciprocal relationship for complex numbers can also be expressed as

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{r(\cos\theta - i \sin\theta)}{r^2} = \frac{1}{r}(\cos\theta - i \sin\theta).$$

18.3 Important Complex Number Properties

Since the standard form of a complex number is

$$\text{(real part)} + \text{(imaginary part)}i,$$

it is natural to try to write complex functions in this form as well. A most important result is due to Leonhard Euler, one of the most prolific mathematicians of all time. It involves the transcendental number $e \approx 2.71$, and gives an important connection between the exponential function with e as its base and the sine and cosine functions.

Result 1 Euler's Formula: For all real numbers θ, we have

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Euler's Formula implies that for any value of θ we have

$$\left|e^{i\theta}\right| = |\cos\theta + i \sin\theta| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1.$$

Euler's formula permits us to write the polar form of a complex number as

$$z = r(\cos\theta + i \sin\theta) = r \cdot e^{i\theta},$$

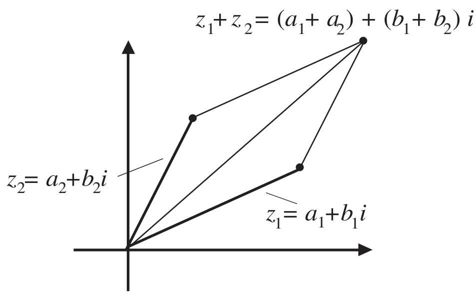
where $r = |z|$ is the magnitude of z and θ is its argument.

Euler's Formula also implies that an expression in the form e^{x+iy}, where x and y are real number variables, can be expressed as

$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y).$$

So the magnitude, that is, the distance from the origin, of e^{x+iy} depends only on the real variable x. The imaginary variable y tells where the number lies on the circle of radius e^x centered at the origin.

The sum of two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ is illustrated by the vector addition shown in the figure below. The numbers a_1 and a_2 are represented in the horizontal-coordinate direction and b_1 and b_2 in the vertical-coordinate direction.



For multiplication, the standard definition does not provide a convenient vector representation. First write the complex numbers in polar form and apply Euler's formula, then

$$z_1 = a_1 + b_1 i = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

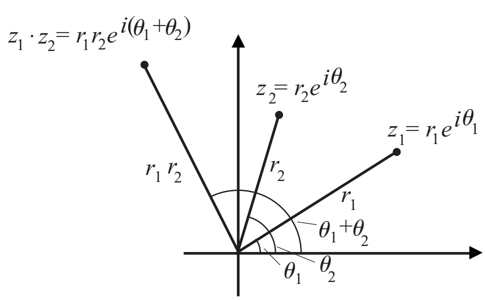
and

$$z_2 = a_2 + b_2 i = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}.$$

Using the properties of exponentials and Euler's formula again gives

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

This implies that the product of z_1 and z_2 is found by multiplying the magnitudes and adding the arguments, as shown below.



Result 2 Reciprocals of Complex Numbers: Euler's Formula gives yet another way to find the reciprocal of a complex number. The rules of exponents imply that if $z = re^{i\theta}$, then

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

Result 3 De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This result follows easily from Euler's formula and the rules of exponents, since

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

De Moivre's formula gives us a convenient way to find nth roots of real and complex numbers.

Result 4 Roots of Complex Numbers: Suppose that n is a positive integer and z is a given complex number. There are n distinct nth roots of z, which are defined by

$$w_i = r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right),$$

for each $k = 0, 1, \ldots, n-1$.

To show this result, we first write w and z in a polar form, that is, as

$$w = s(\cos\phi + i\sin\phi) \quad \text{and} \quad z = r(\cos\theta + i\sin\theta).$$

where $r \ge 0$ and $s \ge 0$. In order for

$$z = r(\cos\theta + i\sin\theta) = w^n = s^n(\cos\phi + i\sin\phi)^n = s^n(\cos n\phi + i\sin n\phi),$$

we must have

$$s = r^{1/n} \quad \text{and} \quad n\phi = \theta + 2k\pi, \text{ for some integer } k.$$

This gives n distinct nth roots of z, as specified in Result 4.

Examples for Chapter 18

The first Example is number 12 from the 1994 AHSME.

EXAMPLE 1 Define i such that $i^2 = -1$. What is $(i - i^{-1})^{-1}$?

(A) 0 **(B)** $-2i$ **(C)** $2i$ **(D)** $-\dfrac{i}{2}$ **(E)** $\dfrac{i}{2}$

Answer (D) First simplify the expression i^{-1}. This gives

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i.$$

Then

$$(i - i^{-1})^{-1} = (i + i)^{-1} = (2i)^{-1} = 2^{-1}i^{-1} = \frac{1}{2}(-i) = -\frac{i}{2}. \qquad \square$$

The next Example is number 22 from the 1990 AHSME.

EXAMPLE 2 The six solutions of $z^6 = -64$ are written in the form $a + bi$, where a and b are real numbers. What is the product of those solutions with $a > 0$?

(A) -2 **(B)** 0 **(C)** $2i$ **(D)** 4 **(E)** 16

Answer (D) De Moivre's Formula implies that the six 6th roots of

$$-64 = 64(\cos \pi + i \sin \pi)$$

are

$$z_k = 64^{1/6}\left(\cos\left(\frac{\pi + 2k\pi}{6}\right) + i \sin\left(\frac{\pi + 2k\pi}{6}\right)\right),$$

for $k = 0, 1, 2, 3, 4,$ and 5.

These are evenly spaced around the circle with radius $64^{1/6} = 2$, beginning with

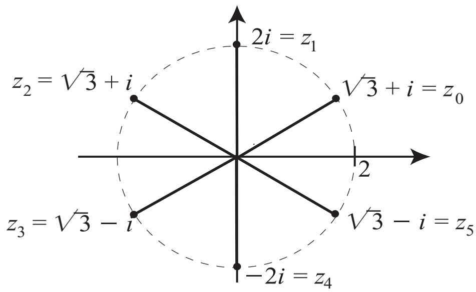
$$z_0 = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = \sqrt{3} + i.$$

As shown in the figure, the only solutions to $z^6 = -64$ with positive real part are

$$z_0 = \sqrt{3} + i \quad \text{and} \quad z_5 = \sqrt{3} - i,$$

and they are complex conjugates. So the product is

$$z_0 \cdot z_5 = z_0 \cdot \overline{z_0} = |z_0|^2 = (\sqrt{3})^2 + (1)^2 = 4. \qquad \square$$



The final Example is number 24 from the 1981 AHSME.

EXAMPLE 3 Suppose that n is a positive integer and that $z + \dfrac{1}{z} = 2\cos\theta$, where $0 < \theta < \pi$. What is the value of $z^n + \dfrac{1}{z^n}$?

(A) $2\cos\theta$ **(B)** $2^n\cos\theta$ **(C)** $2(\cos\theta)^n$ **(D)** $2\cos n\theta$
(E) $2^n(\cos\theta)^n$

Answer (D) The given equation can be expressed as a quadratic in z as

$$z^2 - (2\cos\theta)z + 1 = 0,$$

and the quadratic formula implies that

$$z = \frac{2\cos\theta}{2} \pm \frac{\sqrt{(4\cos\theta)^2 - 4}}{2} = \cos\theta \pm \sqrt{-(\sin\theta)^2} = \cos\theta \pm i\sin\theta.$$

Since the magnitude of z is 1, the reciprocal relationship of complex numbers implies that

$$\frac{1}{z} = \overline{z} = \cos(-\theta) \pm i\sin(-\theta) = \cos\theta \mp i\sin\theta.$$

Applying De Moivre's Theorem to these equations gives

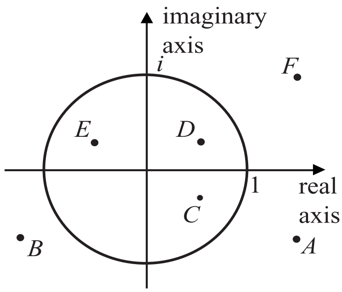
$$\begin{aligned} z^n + \frac{1}{z^n} = z^n + \left(\frac{1}{z}\right)^n &= (\cos n\theta \pm i\sin n\theta) + (\cos n\theta \mp i\sin n\theta) \\ &= 2\cos n\theta. \end{aligned} \qquad \square$$

Exercises for Chapter 18

Exercise 1 Four complex numbers lie at the vertices of a square in the complex plane. Three of the numbers are $1+2i$,$-2+i$ and $-1-2i$. What is the fourth number?

(A) $2+i$ **(B)** $2-i$ **(C)** $1-2i$ **(D)** $-1+2i$ **(E)** $-2-i$

Exercise 2 The diagram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. Which of these numbers might be the reciprocal of F?



(A) A **(B)** B **(C)** C **(D)** D **(E)** E

Exercise 3 Define a sequence of complex numbers by $z_1 = 0$, and $z_{n+1} = z_n^2 + i$ for each positive integer n. What is $|z_{2005}|$?

(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** $\sqrt{2004}$ **(E)** $\sqrt{2^{1002}}$

Exercise 4 Let S be the set of points z in the complex plane such that $(3+4i)z$ is a real number. Which describes the graph of S?

(A) A line through the origin. **(B)** A line not through the origin.
(C) A circle. **(D)** A hyperbola. **(E)** A parabola.

Exercise 5 Simplify $(i+1)^{2008} - (i-1)^{2008}$.

(A) -2^{1004} **(B)** $-2^{1004}i$ **(C)** 0 **(D)** $2^{1004}i$ **(E)** 2^{1004}

Exercise 6 What is the value of $\sum_{n=0}^{40} i^n \cos(45+90n)^\circ$?

(A) $\frac{\sqrt{2}}{2}$ **(B)** $-10i\sqrt{2}$ **(C)** $\frac{21\sqrt{2}}{2}$ **(D)** $\frac{\sqrt{2}}{2}(21-20i)$
(E) $\frac{\sqrt{2}}{2}(21+20i)$

Exercise 7 The complex number z satisfies $z + |z| = 2 + 8i$. What is $|z|^2$?

(A) 68 **(B)** 100 **(C)** 169 **(D)** 208 **(E)** 289

Exercise 8 The zeros of the polynomial

$$P(x) = x^4 + ax^3 + bx^2 + cx + d$$

are complex numbers lying on the unit circle, and a, b, c, and d are real numbers. What is the sum of the reciprocals of the roots of $P(x)$?

(A) a **(B)** b **(C)** c **(D)** $-a$ **(E)** $-b$

Exercise 9 Suppose that $x = (-1 + i\sqrt{3})/2$ and $y = (-1 - i\sqrt{3})/2$. Which of the following statements is **not** correct?

(A) $x^5 + y^5 = -1$ **(B)** $x^7 + y^7 = -1$ **(C)** $x^9 + y^9 = -1$
(D) $x^{11} + y^{11} = -1$ **(E)** $x^{13} + y^{13} = -1$

Exercise 10 What is the product of the real parts of the solutions to the equation $z^2 - z = 5 - 5i$?

(A) -25 **(B)** -6 **(C)** -5 **(D)** $5/4$ **(E)** 25

Solutions to Exercises

Solutions for Chapter 1: Arithmetic Ratios

Exercise 1 Answer (B) Suppose that Jenny starts with x jelly beans. Since she eats 20% per day, she has $0.8x$ at the end of day 1 and $0.8(0.8x) = 0.64x$ at the end of day 2. So the number of jelly beans she started with satisfies

$$32 = 0.64x \quad \text{and} \quad x = \frac{32}{0.64} = \frac{1}{0.02} = 50.$$

OR

We could also solve this problem in a classic manner by arbitrarily guessing the solution and then adjusting the guess. Suppose we assume that Jenny originally had 125 jelly beans (or any number for which consecutive products by 80% are integers). Then at the end of the first day she would have $(4/5)125 = 100$, and at the end of the second day she would have $(4/5)100 = 80$. Since she instead had only 32 at the end of the second day, she must have started with

$$\frac{32}{80} \cdot 125 = \frac{2}{5} \cdot 125 = 50. \qquad \square$$

Exercise 2 Answer (B) Wanda and Darren work every 4 and 3 days, respectively, so they will work together again in 12 days. Beatrice will also work with them that day since she works every 6 days. In fact, Wanda, Darren, and Beatrice will work together every 12 days from now. But Chi will not work together with them again for

$$\text{lcm}\{3, 4, 6, 7\} = 7 \cdot 3 \cdot 4 = 84 \text{ days.} \qquad \square$$

Exercise 3 Answer (D) This problem requires only the careful balancing of equations, being certain that we realize what we need to determine. Since

$$b \text{ hops} = c \text{ skips}, \quad d \text{ jumps} = e \text{ hops}, \quad \text{and} \quad f \text{ jumps} = g \text{ meters},$$

we have

$$1 \text{ meter} = \frac{f}{g} \text{ jumps}, \quad 1 \text{ jump} = \frac{e}{d} \text{ hops}, \quad \text{and} \quad 1 \text{ hop} = \frac{c}{b} \text{ skips}.$$

Thus

$$1 \text{ meter} = \frac{f}{g} \cdot \frac{e}{d} \cdot \frac{c}{b} \text{ skips} = \frac{cef}{bdg} \text{ skips.} \qquad \square$$

Exercise 4 Answer (C) Let

- r and h be the radius and height of the original can,
- R and H be the radius and height of the new can.

Since the diameter is increased by 25%, the radius is increased by this same percent. That is, $R = 1.25r = \frac{5}{4}r$. The volume of a cylindrical can is the product of the area of its base and its height so

$$\pi r^2 h = \pi R^2 H = \pi \left(\frac{5}{4} r\right)^2 H,$$

and

$$H = \frac{\pi r^2}{\pi \left(\frac{5}{4} r\right)^2} h = \frac{1}{25/16} h = \frac{16}{25} h.$$

The height has been decreased by $(1 - 16/25) = 9/25 = 36/100 = 36\%$. $\square$

We could also solve this problem by assuming convenient values for the initial diameter and height. Then using the fact that the volume remains constant and the diameter increases by 25%, we can find that the new diameter must be 16/25 of the original diameter. It is even easier to observe that the volume depends quadratically (as the square) on the diameter, but only linearly on the height. So changing the diameter to $5/4$ of the original value while keeping the volume constant requires a change of $1/(5/4)^2 = 16/25$ in the height.

Exercise 5 Answer (B) Let D, R, and T be, respectively, the distance, rate, and time that would give the Bird his exact arrival time. Since 3 minutes is $1/20$ hours and the distance traveled in each case is the same, we have

$$D = 40\left(T + \frac{1}{20}\right) \quad \text{and} \quad D = 60\left(T - \frac{1}{20}\right).$$

Equating the values of D gives

$$40T + 2 = 60T - 3, \quad \text{so} \quad 20T = 5, \quad \text{and} \quad T = 1/4 \text{ hours.}$$

With this value of T we have

$$D = 40\left(\frac{1}{4} + \frac{1}{20}\right) = 12 \text{ miles}, \quad \text{so} \quad R = \frac{12 \text{ miles}}{1/4 \text{ hours}} = 48\,\frac{\text{miles}}{\text{hour}}. \quad \square$$

Note the similarity, both in the problem and the answer, to that given in Problem 2 at the beginning of the chapter.

Exercise 6 Answer (B) A sphere with radius r has volume $V_s = (4/3)\pi r^3$, and a cone with base radius r and height h has volume $V_c = (1/3)\pi r^2 h$.

In this problem the radius of the sphere is the same as the base radius of the cone, and the volume of the melted sphere is $3/4$ of the original sphere and is the same as the volume of the cone. So

$$\frac{3}{4}V_s = \frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^2 h = V_c$$

and

$$h = \frac{\pi r^3}{(1/3)\pi r^2} = 3r.$$

Hence the ratio of the height, h, to the radius, r, of the cone is $3 : 1$. $\square$

When this problem was given in 2003, the formula for the volume of a sphere and a cone were given with the AMC 10 problem but not with the AMC 12 problem. The committee felt that by grade 11 these formulas should be familiar, but might not be known by a student in grade 9.

Exercise 7 Answer (C) We first define two time variables. Let

- T_c denote the amount of time after noon as indicated on Cassandra's watch.
- T_t denote the true amount of time after noon.

When the true time is 1:00 we have

$$T_c = 57 + \frac{36}{60} = 57 + \frac{3}{5} \quad \text{and} \quad T_t = 60.$$

So

$$60T_c = \left(57 + \frac{3}{5}\right) T_t = \frac{288}{5} T_t, \quad \text{and} \quad T_t = \frac{60 \cdot 5}{288} T_c = \frac{25}{24} T_c.$$

When Cassandra's watch reads 10:00 PM, that is, $T_c = 600$ minutes past noon, the true number of minutes past noon is

$$T_t = \frac{25}{24}(600) = 600 + 25.$$

Hence the true time is 10:25 PM. □

Exercise 8 Answer (B) Since Jack runs uphill at 15 km/hr, it takes him $5/15 = 1/3$ hours to reach the top of the hill. He has a head start of 10 minutes, or 1/6 hours, so it takes him only $1/6$ hours to reach the top of the hill after Jill starts to run. In that $1/6$ hours, Jill has run $16(1/6) = 8/3$ km up the hill, and has $5 - 8/3 = 7/3$ km remaining to reach the top.

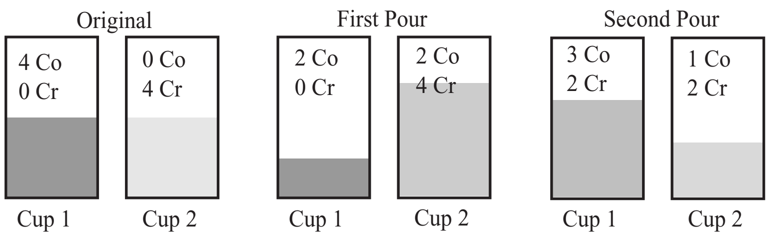
The situation then is as follows: Jack is running down the hill at 20 km/hr, and Jill is $7/3$ km below the top running up at 16 km/hr. They meet at some distance D from the top of the hill. Since the time T for both is the same when they meet,

$$\frac{D}{20} = T \quad \text{and} \quad T = \frac{7/3 - D}{16} = \frac{7}{48} - \frac{D}{16}.$$

So

$$\frac{7}{48} = \frac{D}{20} + \frac{D}{16} = \frac{9}{80} D \quad \text{and} \quad D = \frac{7}{48} \cdot \frac{80}{9} = \frac{35}{27} \text{ km}. \qquad \square$$

Exercise 9 Answer (D) After the first pouring, half of the coffee is in the second cup together with all the cream, a total of 6 ounces of liquid. When half the well-stirred liquid in the second cup is poured into the first cup, the first cup will increase by three ounces, one of coffee and two of cream. So the first cup will now have 5 ounces of liquid, 3 of coffee and 2 of cream. The figure shows the situation.



The fraction of cream in the first cup after two pours is therefore $2/5$. □

Exercise 10 Answer (C) Let

- R_s be the speed of Sam.
- R_w be the speed of Walt.
- T_1 be the time it takes Sam to finish the original race.
- T_2 be the time it takes Sam to finish the second race.

The conditions concerning the original race state that

$$h = R_s \cdot T_1 \quad \text{and} \quad h - d = R_w \cdot T_1 \quad \text{so} \quad R_s = \frac{h}{h-d} R_w.$$

The conditions for the second race state that

$$h + d = R_s T_2 = \frac{h}{h-d} R_w T_2,$$

so

$$R_w T_2 = (h+d)\frac{h-d}{h} = \frac{h^2 - d^2}{h} = h - \frac{d^2}{h}.$$

At time T_2, Sam's distance from the original starting line is h and Walt's distance is $R_w T_2 = h - d^2/h$. So Sam is ahead by the amount d^2/h at the end of the second race. □

The original problem stated that the winner of both races was the same, but this information is not needed to solve the problem.

Solutions for Chapter 2: Polynomials

Exercise 1 Answer (D) The graph of $y = P(x)$ is a line, and the slope of the line is constant. So

$$\frac{P(12) - P(2)}{12 - 2} = \frac{P(6) - P(2)}{6 - 2} = \frac{12}{4} = 3,$$

and

$$P(12) - P(2) = 10 \cdot 3 = 30. \qquad \square$$

Exercise 2 Answer (B) Since

$$3x_2^2 - hx_2 = b = 3x_1^2 - hx_1$$

we have

$$\begin{aligned} 0 &= 3x_2^2 - 3x_1^2 - hx_2 + hx_1 \\ &= 3(x_2 + x_1)(x_2 - x_1) - h(x_2 - x_1) = (3(x_2 + x_1) - h)(x_2 - x_1). \end{aligned}$$

We are given that $x_1 \neq x_2$, so $3(x_1 + x_2) - h = 0$, which implies that $x_1 + x_2 = h/3$.

OR

Since x_1 and x_2 are the two solutions to the quadratic equation

$$0 = 3x^2 - hx - b, \quad \text{they also satisfy} \quad 0 = x^2 - \frac{h}{3}x - \frac{b}{3}.$$

The sum of the solutions of the latter equation is the negative of the linear term. This implies that $x_1 + x_2 = h/3$. $\square$

Exercise 3 Answer (D) The Factor Theorem implies that the remainder is a constant, C, when $x + 1$ divides $x^{51} + 51$. Hence for some polynomial $Q(x)$ (of degree 50) we have,

$$x^{51} + 51 = (x + 1)Q(x) + C \quad \text{so} \quad C = x^{51} + 51 - (x + 1)Q(x).$$

When $x = -1$, this implies that

$$C = (-1)^{51} + 51 - (-1+1)Q(-1) = -1 + 51 - 0 = 50.$$

OR

The Linear Factor Theorem, Result 1 of Section 2.4, implies that the remainder when $P(x) = x^{51} + 51$ is divided by $x + 1$ is $P(-1)$. Now we can apply Result 5 of Section 2.4 to deduce that this remainder is

$$P(-1) = 51 + (-1)^{51} = 51 - 1 = 50. \qquad \square$$

Exercise 4 Answer (C) The intersections occur when $P(x) = Q(x)$, or equivalently, when $P(x) - Q(x) = 0$. Since $P(x)$ and $Q(x)$ are both polynomials of degree 4 with leading coefficient 1, the polynomial $P(x) - Q(x)$ is of degree at most 3. So $P(x) - Q(x)$ can have at most three real zeros. These zeros give the intersection points of the graphs. $\square$

It can be seen that three intersections are possible by considering, for example, the polynomials $P(x) = x^4$ and $Q(x) = x^4 + x^3 - x$. The three intersections of $y = P(x)$ and $y = Q(x)$ occur when x is 0, 1, and -1.

Exercise 5 Answer (E) The original quadratic has the form

$$y(x) = a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k,$$

and the reflected quadratic has the form

$$y_r(x) = -a(x-h)^2 + k = -ax^2 + 2ahx - ah^2 + k,$$

By Result 5 of Section 2.4 we have

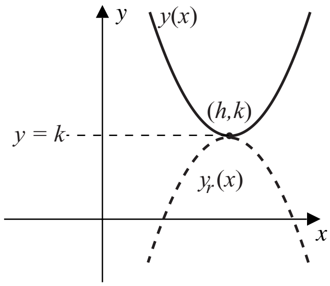
$$y(1) = a + b + c = a - 2ah + ah^2 + k$$

and

$$y_r(1) = d + e + f = -a + 2ah - ah^2 + k,$$

so

$$a + b + c + d + e + f = y(1) + y_r(1) = 2k.$$



Exercise 6 Answer (C) Since $(x-19)(x-99)$ is a quadratic polynomial, the remainder when this is divided into $P(x)$ will be linear, that is,

$$P(x) = (x-19)(x-99)Q(x) + ax + b, \quad \text{for some constants } a \text{ and } b.$$

The Linear Factor Theorem implies that

$$99 = P(19) = 19a + b \quad \text{and} \quad 19 = P(99) = 99a + b.$$

Subtracting these equations and substituting gives

$$80a = -80 \quad \text{so} \quad a = -1,$$

and $b = 99 - (-1)19 = 118$. The remainder is therefore $-x + 118$. □

Exercise 7 Answer (A) First note that the y-intercept occurs when $x = 0$, so $c = P(0) = 2$. We will now use Result 4 of Section 2.4, the Zeros-Coefficient Relationship for General Polynomials. It implies that the product of the zeros of $P(x)$ is $-c = -2$, and we are told that this is the same as the average of the zeros. The sum of the zeros, which is $-a$, is 3 times the average, so

$$a = -3(-c) = -3(-2) = 6.$$

To find b we note that the sum of the coefficients of $P(x)$ is the same as the product of the zeros, which we found to be $-c = -2$. Hence

$$1 + a + b + c = -c \quad \text{and} \quad b = -2c - 1 - a = -4 - 1 - 6 = -11. \ \square$$

Exercise 8 Answer (B) First note that

$$P\left(\frac{x}{3}\right) = x^2 + x + 1 = 9\left(\frac{x}{3}\right)^2 + 3\left(\frac{x}{3}\right) + 1,$$

so

$$P(x) = 9x^2 + 3x + 1 \quad \text{and} \quad P(3x) = 81x^2 + 9x + 1.$$

If $7 = P(3x) = 81x^2 + 9x + 1$, then

$$0 = 81x^2 + 9x - 6 = 81\left(x^2 + \frac{1}{9}\,x - \frac{2}{27}\right).$$

The Zero-Coefficient Relationship for Quadratic Polynomials implies that the sum of the zeros of the quadratic is the negative of the linear term that is within the parentheses. Hence the sum of all the values of x for which $P(3x) = 7$ is $-1/9$. □

Exercise 9 Answer (B) If the equations have a common solution, then

$$x^2 + ax + 1 = x^2 - x - a \quad \text{so} \quad (a+1)(x+1) = 0.$$

As a consequence, we must have either $a = -1$ or $x = -1$. If $a = -1$, the equation becomes

$$x^2 - x + 1 = 0$$

which has no real solution since its discriminant is $(-1)^2 - 4(1)(1) < 0$. On the other hand, if $x = -1$, then

$$a = x^2 - x = (-1)^2 - (-1) = 2.$$

So there is exactly one value of a for which the equations have a common real solution. For the value $a = 2$ the equations are, respectively,

$$0 = x^2 + 2x + 1 = (x+1)^2 \quad \text{and} \quad 0 = x^2 - x - 2 = (x+1)(x-2),$$

which have the common solution $x = -1$. □

Exercise 10 Answer (B) If the zeros of $x^2 + mx + n$ are denoted by a and b, then the zeros of $x^2 + px + q$ are a^3 and b^3. The relationships of the zeros to the coefficients of these quadratics implies that

$$a \cdot b = n \quad \text{and} \quad a^3 \cdot b^3 = q,$$

as well as

$$a + b = -m \quad \text{and} \quad a^3 + b^3 = -p.$$

Expanding the product $(a+b)^3$ gives

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = (a^3+b^3) + 3ab(a+b).$$

Hence we have

$$(-m)^3 = (-p) + 3n(-m), \quad \text{which implies that} \quad p = m^3 - 3mn. \quad \square$$

Solutions for Chapter 3: Exponentials and Radicals

Exercise 1 Answer (C) We can use the laws of exponents to regroup the expression either as

$$4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9 = (4\cdot 9)^4\,(4\cdot 9)^9 = 36^4 \cdot 36^9 = 36^{(4+9)} = 36^{13}$$

or as

$$4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9 = \left(4^4 \cdot 4^9\right)\cdot\left(9^4 \cdot 9^9\right) = 4^{13}\cdot 9^{13} = (4\cdot 9)^{13} = 36^{13}. \quad \square$$

Exercise 2 Answer (E) First express the numerator and denominator using the common base 15. Then

$$\frac{15^{30}}{45^{15}} = \frac{15^{30}}{15^{15}\cdot 3^{15}} = \frac{15^{30-15}}{3^{15}} = \frac{15^{15}}{3^{15}} = \left(\frac{15}{3}\right)^{15} = 5^{15}. \quad \square$$

Exercise 3 Answer (C) Since the exponent of the side with the unknown term k is 2004, we will express all the terms using this exponent. This gives

$$\begin{aligned} k\cdot 2^{2004} &= 2^{2004}\cdot 2^3 - 2^{2004}\cdot 2^2 - 2^{2004}\cdot 2 + 2^{2004} \\ &= (8-4-2+1)2^{2004} = 3\cdot 2^{2004}, \end{aligned}$$

so $k = 3$. $\square$

Exercise 4 Answer (D) Applying the rules of exponents simplifies the expression to

$$\frac{x^y y^x}{y^y x^x} = \frac{x^y x^{-x}}{y^y y^{-x}} = \frac{x^{y-x}}{y^{y-x}} = \left(\frac{x}{y}\right)^{y-x}. \quad \square$$

Exercise 5 Answer (B) To simplify this expression write the terms in a common base. We use the base 2, since $4 = 2^2$ and $8 = 2^3$. So

$$\sqrt{\frac{8^{10}+4^{10}}{8^4+4^{11}}} = \sqrt{\frac{\left(2^3\right)^{10}+\left(2^2\right)^{10}}{\left(2^3\right)^4+\left(2^2\right)^{11}}} = \sqrt{\frac{2^{30}+2^{20}}{2^{12}+2^{22}}} = \sqrt{\frac{2^{20}\left(2^{10}+1\right)}{2^{12}\left(1+2^{10}\right)}}$$
$$= \sqrt{2^8} = 2^4 = 16. \qquad \square$$

Exercise 6 Answer (E) First recall that $0^x = 0$ when $x \neq 0$. Then note that for any value of x the expression for $f(x)$ contains only terms whose base and exponents are different. This implies that $f(0) = 0$ and $f(-2) = 0$. Hence the expression reduces to

$$f(-1) + f(-3) = (-1)^0(1)^2 + (-3)^{-2}(-1)^0 = 1 + \frac{1}{9} = \frac{10}{9}. \qquad \square$$

Exercise 7 Answer (B) First simplify the expression by letting $y = 1/x$. Then write all the expressions in the common base 5. This simplifies the equation to

$$5^{-4} = \frac{5^{48y}}{5^{26y} \cdot 5^{34y}} = \frac{5^{48y}}{5^{60y}} = \frac{1}{5^{12y}} = 5^{-12y}.$$

So

$$-4 = -12y, \quad y = \frac{1}{3}, \quad \text{and} \quad x = \frac{1}{y} = 3. \qquad \square$$

Exercise 8 Answer (C) We have

$$a^b \cdot x^b = r = (3a)^{(3b)} = 3^{(3b)}a^{(3b)} = \left(3^3\right)^b \left(a^3\right)^b$$
$$= 27^b \cdot a^{(3b)} = a^b \cdot 27^b a^{(2b)}.$$

Dividing the left and right sides of the equation by a^b gives

$$x^b = 27^b a^{2b} = \left(27a^2\right)^b, \quad \text{so} \quad x = 27a^2. \qquad \square$$

Exercise 9 Answer (E) To simplify the notation, we first let

$$a = 2^x - 4 \quad \text{and} \quad b = 4^x - 2.$$

Making these substitutions in the given equation produces

$$a^3 + b^3 = (4^x + 2^x - 6)^3 = \left((2^x - 4) + (4^x - 2)\right)^3 = (a+b)^3.$$

Now apply the Binomial Theorem to the term $(a+b)^3$ to deduce that

$$a^3+b^3=(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

and

$$0=3a^2b+3ab^2=3ab(a+b).$$

Hence there are three possibilities:

- If $a=0$, then $2^x=4$ and $x=2$.
- If $b=0$, then $4^x=2$ and $x=1/2$.
- If $a+b=0$, then $2^x+4^x=6$, and $x=1$.

The values of x satisfying the equation consequently sum to $2+\frac{1}{2}+1=\frac{7}{2}$.
□

There are numerous ways to show that the equation $2^x+4^x=6$ is satisfied only when $x=1$. One way is to note that the functions defined by 2^x and by 4^x are always increasing. So for $x<1$, we have $2^x+4^x<6$ and for $x>1$, we have $2^x+4^x>6$. Another is to consider the equation

$$\begin{aligned}0&=2^x+4^x-6=4^x+2^x-6=\left(2^x\right)^2+2^x-6\\&=\left(2^x-2\right)\left(2^x+3\right).\end{aligned}$$

The second factor is nonzero for all real numbers x, so the only real solution occurs when $2^x=2$, that is, when $x=1$.

Exercise 10 Answer (B) As we have seen in the previous exercises, the key to solving many of the exponential problems is to first write all the terms in a common base. The problem tells us that

$$60^a=3 \quad \text{and} \quad 60^b=5,$$

so we first try to write the unknown quantities using base 60. Since

$$12=\frac{60}{5} \quad \text{and} \quad 5=60^b, \quad \text{we have} \quad 12=\frac{60}{60^b}=60^{1-b}.$$

This gives

$$\begin{aligned} 12^{(1-a-b)/(2-2b)} &= \left(60^{(1-a-b)/(2-2b)}\right)^{1-b} \\ &= 60^{(1-a-b)/2} \\ &= \left(60^{(1-a-b)}\right)^{1/2}, \end{aligned}$$

which simplifies to

$$12^{(1-a-b)/(2-2b)} = \left(\frac{60}{60^a \cdot 60^b}\right)^{1/2} = \sqrt{\frac{60}{3 \cdot 5}} = \sqrt{4} = 2. \qquad \square$$

Solutions for Chapter 4: Defined Functions and Operations

Exercise 1 Answer (E) For any value of y we have

$$12 = 3 \circ y = 4 \cdot 3 - 3y + 3 \cdot y = 12.$$

So the equation $3 \circ y = 12$ is true for all values of y, that is, it is an *identity*.

$\square$

Exercise 2 Answer (E) First we need

$$[60, 30, 90] = \frac{30 + 60}{90} = 1, \quad [2, 1, 3] = \frac{2 + 1}{3} = 1,$$

and

$$[10, 5, 15] = \frac{10 + 5}{15} = 1.$$

Then

$$[[60, 30, 90], [2, 1, 3], [10, 5, 15]] = [1, 1, 1] = \frac{1 + 1}{1} = 2. \qquad \square$$

Exercise 3 Answer (B) We need to examine each of the statements until we find one that is false for this binary operation.

Statement (A) describes commutativity, and this is true since

$$x * y = (x + 1)(y + 1) - 1 = (y + 1)(x + 1) - 1 = y * x.$$

However (B) is false unless $x = 0$ since

$$(x * (y + z)) = (x + 1)(y + z + 1) - 1,$$

whereas,

$$\begin{aligned} x * y + x * z &= (x+1)(y+1) - 1 + (x+1)(z+1) - 1 \\ &= (x+1)(y+z+2) - 2. \end{aligned}$$

We could stop now, since the problem implies that only one of the statements is false. However, let us verify that the others are true.

For (C), we have

$$\begin{aligned} (x-1) * (x+1) &= (x-1+1)(x+1+1) - 1 \\ &= x^2 + 2x - 1 = (x^2 + 2x + 1) - 2 \\ &= (x+1)(x+1) - 2 = (x * x) - 1. \end{aligned}$$

For (D), we have

$$x * 0 = (x+1)(0+1) - 1 = x + 1 - 1 = x,$$

and for (E), which describes associativity of $*$, we have

$$\begin{aligned} x * (y * z) &= (x+1)((y * z) + 1) - 1 \\ &= (x+1)(((y+1)(z+1) - 1) + 1) - 1 \\ &= (x+1)(y+1)(z+1) - 1, \end{aligned}$$

and

$$\begin{aligned} (x * y) * z &= ((x * y + 1)(z+1) - 1 \\ &= (((x+1)(y+1) - 1) + 1)(z+1) - 1 \\ &= (x+1)(y+1)(z+1) - 1. \end{aligned}$$ □

Note the amount of work that needs to be done if you do not happen to find the false statement early. It is certainly worthwhile to give a hard look at the statements before doing the algebra, and try to choose the one that seems to be the least likely to be true.

Exercise 4 Answer (C) Before beginning the verification of the various statements, note that $x \heartsuit y$ never produces a negative number. So if $x < 0$, statement (C) is clearly false.

The others are, in fact, true since:
For (A),

$$x\heartsuit y = |x - y| = |y - x| = y\heartsuit x,$$

For (B),

$$2(x\heartsuit y) = 2|x - y| = |2x - 2y| = 2x\heartsuit 2y.$$

For (D),

$$x\heartsuit x = |x - x| = |0| = 0.$$

For (E),

$$x\heartsuit y = |x - y| \geq 0 \quad \text{and is 0 if and only if } x = y. \qquad \square$$

Since $x\heartsuit y = |x - y|$ describes the distance from x to y, the answer choices can be rephrased as:

(A) The distance from x to y is the same as the distance from y to x.

(B) Twice the distance from x to y is the same as the distance from $2x$ to $2y$.

(C) The distance from x to 0 is x. (Clearly false if $x < 0$, since distance is not negative.)

(D) The distance from x to x is 0.

(E) The distance from x to y is positive unless $x = y$.

Exercise 5 Answer (E) Probably the only reasonable approach to solving a problem of this type is to first write the integer N in the form $a \cdot 10 + b$, where a one of the numbers $1, 2, \ldots, 9$, and b is one of the numbers $0, 1, \ldots, 9$. Then use the given equation to determine b. Because

$$a \cdot 10 + b = N = P(N) + S(N) = a \cdot b + (a + b), \quad \text{we have} \quad 9a = a \cdot b.$$

Since $a \neq 0$, we have $b = 9$. $\square$

Exercise 6 Answer (E) First write n as $n = a \cdot 10 + b$, where a and b are single-digit non-negative integers with $a \neq 0$. Note that the maximum possible value for $\clubsuit(n)$ when n is a two-digit integer is $9 + 9 = 18$. Let $m = \clubsuit(n)$. In order to have $\clubsuit(m) = 3$, we must have m as 3, 12, or 21. However $m = 21 > 18$ does not give valid solutions for n. Let us consider the other possibilities. If

$$3 = m = \clubsuit(n), \quad \text{then } n = 12, 21, \text{ or } 30,$$

and if

$$12 = m = \clubsuit(n), \quad \text{then } n = 39, 93, 48, 84, 57, 75, \text{ or } 66.$$

This gives 10 total possibilities for n. $\square$

Exercise 7 Answer (C) We know the value of $f(500)$ and need the value of $f(600)$. Since we are given a product relationship for the function, we will write 600 in terms of 500 in a way that involves a product. Since $600 = 500 \cdot (6/5)$, we have

$$f(600) = f\left(500 \cdot \frac{6}{5}\right) = \frac{f(500)}{\frac{6}{5}} = \frac{3}{\frac{6}{5}} = 3 \cdot \frac{5}{6} = \frac{5}{2}.$$

OR

First write $f(500)$ in terms of $f(1)$. This gives

$$3 = f(500) = \frac{f(1)}{500}, \quad \text{so} \quad f(1) = 1500.$$

Then

$$f(600) = \frac{f(1)}{600} = \frac{1500}{600} = \frac{5}{2}.$$ $\square$

Notice that this solution illustrates that a function f with the specified properties does exist. The fact that $f(1) = 1500$ implies that

$$f(x) = f(1 \cdot x) = \frac{f(1)}{x} = \frac{1500}{x}.$$

Exercise 8 Answer (A) This is an example of a problem that is easier to solve in general than in particular, since the specific numbers tend to get in the way of the simplification. We first determine the value of the quotient

$$\frac{\binom{a}{k}}{\binom{a+1}{k}} = \frac{\frac{a(a-1)(a-2)\cdots(a-(k-2))(a-(k-1))}{k(k-1)(k-2)\cdots(2)(1)}}{\frac{(a+1)(a)(a-1)\cdots((a+1)-(k-2))(a+1-(k-1))}{k(k-1)(k-2)\cdots(2)(1)}} = \frac{a+1-k}{a+1}.$$

When $a = -1/2$ and $k = 100$ this produces

$$\binom{-\frac{1}{2}}{100} \Big/ \binom{\frac{1}{2}}{100} = \frac{-\frac{1}{2} + 1 - 100}{-\frac{1}{2} + 1} = \frac{-\frac{199}{2}}{\frac{1}{2}} = -199. \qquad \square$$

Exercise 9 Answer (C) We need to find the true statement, so let's start with the easiest, the one in (A). If we let $a = 0$ and $b = 0$, then

$$f(a+b) + f(a-b) = 2f(a) + 2f(b)$$

implies that

$$f(0) + f(0) = 2f(0) + 2f(0) \quad \text{so} \quad f(0) = 0 \neq 1.$$

We can look at (B) and (C) together by letting $a = 0$ and $b = x$. Then

$$f(x) + f(-x) = 2f(0) + 2f(x) = 2f(x), \quad \text{so} \quad f(x) = f(-x),$$

and (C) is true. We are done!

To show that statement (D) cannot be true, suppose that it does hold for all values of x and y, and let x be such that $f(x) \neq 0$. Then we would have

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x) = f(x) + f(x) = 2f(x)$$

and

$$f(x) = 0.$$

This contradicts the assumption that $f(x) \neq 0$.

Statement (E) implies that f must be a periodic function. To show that this need not be the case, consider $f(x) = x^2$. This non-periodic function satisfies the given relationship since

$$\begin{aligned} f(a+b) + f(a-b) &= (a+b)^2 + (a-b)^2 \\ &= a^2 + 2ab + b^2 + a^2 - 2ab + b^2 \\ &= 2a^2 + 2b^2 \\ &= 2f(a) + 2f(b). \end{aligned} \qquad \square$$

Exercise 10 Answer (D) We show that (I) is true by observing that $f(0) = f(0+0) = f(0) \cdot f(0)$. Since we are given that $f(x) > 0$ for all x, we can divide both sides by $f(0)$ to produce $f(0) = 1$.

To show that (II) is true, consider

$$1 = f(0) = f(x + (-x)) = f(x) \cdot f(-x). \quad \text{Then} \quad f(-x) = \frac{1}{f(x)}.$$

Statement (III) is also true, since

$$\begin{aligned} f(3x) = f(x + 2x) = f(x) \cdot f(2x) &= f(x) \cdot f(x + x) \\ &= f(x) \cdot f(x) \cdot f(x) \\ &= (f(x))^3 . \end{aligned}$$

By this time you should be suspicious that these conditions might describe some common class of functions, and indeed they do. All of the conditions are satisfied by the exponential functions, those of the form $f(x) = a^x$, for some positive real number a. When $a = 1$, we have condition (IV) satisfied, but in any other case this is not true, since for a positive number $a \neq 1$ we have either

$$f(1) = a < 1, \quad \text{in the case that } 0 < a < 1$$

or

$$f(-1) = a^{-1} < 1, \quad \text{in the case that } 1 < a.$$

Hence statement IV does not necessarily hold, only statements I, II, and III are always true. □

Solutions for Chapter 5: Triangle Geometry

Exercise 1 Answer (D) Since $\angle ABC = 90 - 20 = 70°$, we have $\angle DBC = 35°$ and

$$\angle BDC = 90 - 35 = 55°.$$ □

Exercise 2 Answer (E) First note that

$$BG = BC + CF + FG = 5 + DE + 5 = 5 + 10 + 5 = 20.$$

Since $\triangle BAG$ is a 45–45–90° triangle, we have

$$AB = AG = BG \cdot \frac{\sqrt{2}}{2} = 20 \cdot \frac{\sqrt{2}}{2} = 10\sqrt{2}.$$

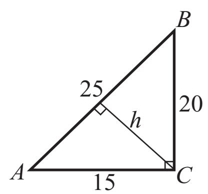
The area of the polygon is

$$\text{Area}(\triangle ABC) + \text{Area}(\square CDEF) = \frac{1}{2}\left(10\sqrt{2}\right)^2 + 10 \cdot 20 = 300. \quad \square$$

Exercise 3 Answer (B) Since $15 = 3 \cdot 5$, $20 = 4 \cdot 5$, and $25 = 5 \cdot 5$, we have

$$15^2 + 20^2 = 5^2(3^2 + 4^2) = 5^2 \cdot 5^2 = 25^2,$$

and $\triangle ABC$ is a right triangle.



The two legs $\overline{AC}$ and $\overline{BC}$ are altitudes with lengths 15 and 20, respectively. The Right Triangle Altitude Theorem implies that the third altitude h satisfies

$$\frac{h}{15} = \frac{20}{25} = \frac{4}{5}, \quad \text{so} \quad h = 12$$

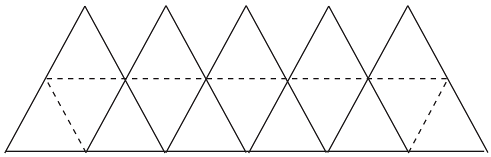
is the shortest altitude.

OR

We have

$$h = \frac{\text{Area}(\triangle ABC)}{\frac{1}{2} \cdot AB} = \frac{\frac{1}{2} \cdot 15 \cdot 20}{\frac{1}{2} \cdot 25} = 12. \qquad \square$$

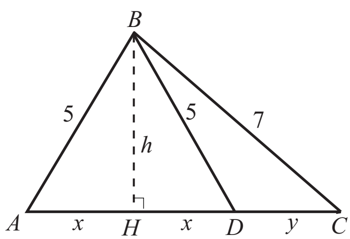
Exercise 4 Answer (E) Consider the figure with the addition of a dashed line through the intersection points of the sides of the triangles. Complete the construction by adding the dashed lines from the endpoints of this line segment, as shown.



The region is then divided into 16 non-overlapping smaller equilateral triangles. Each of these smaller triangles has side length $\sqrt{3}$ and altitude $3/2$, so the area of each small triangle is

$$\frac{1}{2} \cdot \sqrt{3} \cdot \frac{3}{2} = \frac{3\sqrt{3}}{4}, \quad \text{and the total area is} \quad 16\left(\frac{3\sqrt{3}}{4}\right) = 12\sqrt{3}. \quad \square$$

Exercise 5 Answer (E) Let H be the base of the altitude from B of $\triangle ABC$. Also, to simplify the notation let $h = BH$, $x = AH = HD$, and $y = DC$.



The Pythagorean Theorem applied to the right triangles BHD and BHC gives, respectively,

$$25 = x^2 + h^2 \quad \text{and} \quad 49 = (x + y)^2 + h^2 = x^2 + 2xy + y^2 + h^2.$$

So

$$49 = (x^2 + h^2) + 2xy + y^2 = 25 + 2xy + y^2 \quad \text{and} \quad 2xy + y^2 = 24.$$

We also know that $AC = 2x + y = 9$, which implies that

$$24 = 2xy + y^2 = (2x + y) \cdot y = 9y.$$

So $y = \frac{8}{3}$, and $x = \frac{1}{2}(9 - y) = \frac{19}{6}$. Hence

$$\frac{AD}{DC} = \frac{2x}{y} = \frac{19/3}{8/3} = \frac{19}{8}. \qquad \square$$

Exercise 6 Answer (C) We need to determine a ratio, not a specific value for either area, so we can simplify the situation by assuming that the interior equilateral triangle has side length $ED = 1$. Since $\angle ECD = 60^\circ$, we then have

$$EC = \frac{2\sqrt{3}}{3} \quad \text{and} \quad CD = \frac{1}{2}EC = \frac{\sqrt{3}}{3}.$$

Since $FE \perp AC$ and $DE = FE$, triangles CDE and AEF are congruent. This implies that

$$AE = DC = \frac{\sqrt{3}}{3}, \quad \text{so} \quad AC = AE + EC = \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} = \sqrt{3}.$$

Since $\triangle AEF$ is similar to $\triangle ABC$, the ratio of their areas is the same as the square of the ratio of their sides, which is

$$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABC)} = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}. \qquad \square$$

Exercise 7 Answer (C) The Right Triangle Altitude Theorem implies that $\triangle ADC$, $\triangle CDB$, and $\triangle ACB$ are similar right triangles, and that

$$\frac{AD}{15} = \frac{15}{AB} = \frac{15}{AD + DB} = \frac{15}{AD + 16}.$$

So $AD^2 + 16AD = 225$ and

$$0 = AD^2 + 16AD - 225 = (AD - 9)(AD + 16).$$

Hence

$$AD = 9, \quad AB = \frac{15^2}{AD} = \frac{225}{9} = 25,$$

and

$$BC = \sqrt{25^2 - 15^2} = 5\sqrt{5^2 - 3^2} = 20.$$

This implies that

$$\text{Area}(\triangle ABC) = \frac{1}{2} \cdot AC \cdot BC = \frac{1}{2} \cdot 15 \cdot 20 = 150. \qquad \square$$

Alternatively, we could use the fact that base $AB = 25$ and its corresponding altitude $DC = \sqrt{15^2 - 9^2} = 12$ to show that

$$\text{Area}(\triangle ABC) = \frac{1}{2} \cdot AB \cdot DC = \frac{1}{2} \cdot 25 \cdot 12 = 150.$$

Exercise 8 Answer (B) First note that $CD = 8$ and $EA = 9 - 4 = 5$. Since $\triangle CDE$ is similar to $\triangle GFE$, we have

$$\frac{CD}{DE} = \frac{8}{4} = \frac{GF}{FE} = \frac{GF}{FD + 4}, \quad \text{so} \quad GF = 2FD + 8.$$

Also, since $\triangle GFA$ is similar to $\triangle ABH$, we have

$$\frac{AB}{BH} = \frac{8}{6} = \frac{GF}{FA} = \frac{GF}{FD + 9}, \quad \text{so} \quad 3GF = 4FD + 36.$$

Solving the equations $GF = 2FD + 8$ and $3GF = 4FD + 36$ for GF gives

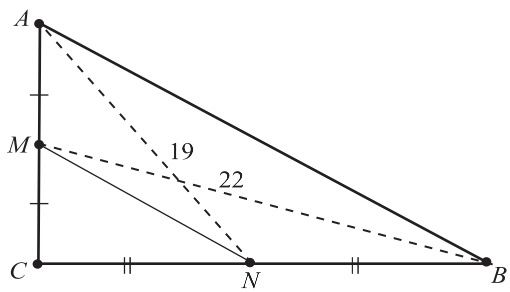
$$3GF = 4FD + 36 = 2(GF - 8) + 36 = 2GF + 20, \quad \text{so} \quad GF = 20. \; \square$$

Exercise 9 Answer (B) Since $\triangle MCB$ and $\triangle ACN$ are right triangles with $AC = 2MC$ and $BC = 2NC$, we have

$$22^2 = MC^2 + (2CN)^2 = MC^2 + 4CN^2$$

and

$$19^2 = CN^2 + (2MC)^2 = 4MC^2 + CN^2.$$



Adding the corresponding sides of these equations and using the fact that $\triangle MCN$ is also a right triangle produces

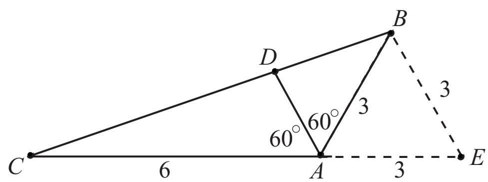
$$22^2 + 19^2 = 5(MC^2 + CN^2) = 5MN^2,$$

so

$$MN = \sqrt{\frac{22^2 + 19^2}{5}} = \sqrt{\frac{845}{5}} = \sqrt{169} = 13.$$

Applying the Side-Splitter Theorem to $\triangle ACB$ and $\triangle MCN$ gives $AB = 2MN = 26$. □

Exercise 10 Answer (A) An important part of the solution to this problem is to correctly construct a diagram similar to that shown.



In the diagram we have extended the side $\overline{CA}$ to E and formed the equilateral $\triangle ABE$. All the sides of $\triangle ABE$ have length 3, and applying the Side-Splitter Theorem gives

$$\frac{AD}{BE} = \frac{CA}{CE}, \quad \text{so} \quad AD = BE \cdot \frac{CA}{CE} = 3 \cdot \frac{6}{9} = 2.$$ □

Solutions for Chapter 6: Circle Geometry

Exercise 1 Answer (E) The circumference of the entire circle is $2\pi \cdot 1 = 2\pi$, so the circular portion has perimeter

$$\frac{360 - 60}{360} \cdot 2\pi = \frac{5}{3}\pi.$$

The straight line portions of the perimeter each have length 1, so the total perimeter is $5\pi/3 + 2$. □

Exercise 2 Answer (B) Since the arcs of a circle sum to 360°, and lengths of $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CD}$ are equal, we have

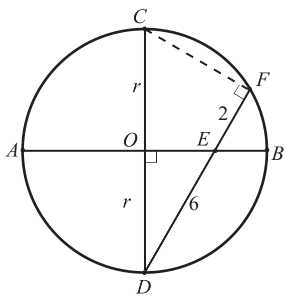
$$360° = 3\,\overset{\frown}{BC} + \overset{\frown}{AD}, \quad \text{so} \quad \overset{\frown}{BC} = \frac{1}{3}(360° - \overset{\frown}{AD}) = 120° - \frac{1}{3}\,\overset{\frown}{AD}.$$

The External Secant Theorem implies that

$$40^\circ = \angle AED = \frac{1}{2}(\widehat{BC} - \widehat{AD}) = \frac{1}{2}\left(120^\circ - \frac{1}{3}\widehat{AD} - \widehat{AD}\right)$$
$$= 60^\circ - \frac{2}{3}\widehat{AD},$$

so $\widehat{AD} = 30^\circ$. The Inscribed Angle Theorem implies that $\angle ACD = \widehat{AD}/2 = 15^\circ$. □

Exercise 3 Answer (C) First draw the chord $\overline{CF}$.

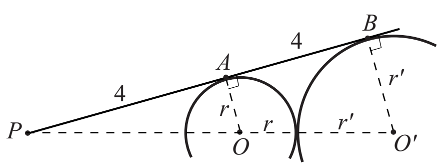


Then $\triangle CFD$ is a right triangle because $CD = 2r$ is a diameter of the circle. Since $\triangle CFD$ shares an angle with $\triangle DOE$, these triangles are similar, and we have

$$\frac{2r}{6+2} = \frac{6}{r}, \quad \text{so} \quad r^2 = 24.$$

The area of the circle is 24π. □

Exercise 4 Answer (B) With the notation in the figure, we have right $\triangle PAO$ similar to right $\triangle PBO'$, where O and O' are, respectively, the centers of the circles with radii r and r'.



So

$$\frac{PA}{r} = \frac{PB}{r'}, \quad \text{which implies that} \quad r' = \frac{PB}{PA}r = \frac{8}{4}r = 2r,$$

and

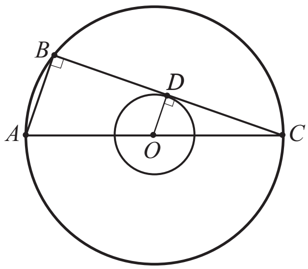
$$\frac{PO}{r} = \frac{PO'}{r'} = \frac{PO + 3r}{2r}, \quad \text{which implies that} \quad PO = 3r.$$

Applying the Pythagorean Theorem to $\triangle PAO$ gives

$$9r^2 = PO^2 = PA^2 + AO^2 = 16 + r^2, \quad \text{so} \quad r^2 = 2.$$

The area of the smaller circle is $\pi r^2 = 2\pi$. □

Exercise 5 Answer (B) First construct the radius $\overline{OD}$ from the center O of the concentric circles to the point D of tangency of the smaller circle.



Then $\angle ODC$ is a right angle, and by the conditions of the problem, $OC = 3OD$. Since the hypotenuse of $\triangle ABC$ is a diameter, $\angle ABC$ is also a right angle and $\triangle ODC$ is similar to $\triangle ABC$. Hence

$$\frac{1}{2} = \frac{OC}{AC} = \frac{OD}{AB} = \frac{OD}{12} \quad \text{and} \quad OC = 3OD = 3 \cdot 6 = 18.$$ □

Exercise 6 Answer (C) First consider the region that consists of the equilateral triangle topped with a semicircle, as shown. The area of this region is

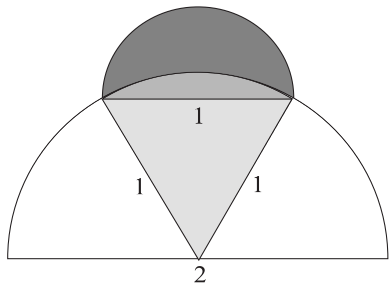
$$\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\pi \left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{8}\pi.$$

The area of the lune results from subtracting from this the area of the sector of the larger semicircle,

$$\frac{1}{6}\pi(1)^2 = \frac{1}{6}\pi.$$

Hence the area of the lune is

$$\frac{\sqrt{3}}{4} + \frac{1}{8}\pi - \frac{1}{6}\pi = \frac{\sqrt{3}}{4} - \frac{1}{24}\pi = \frac{1}{24}(6\sqrt{3} - \pi). \qquad \square$$



Exercise 7 Answer (E) Since $\overset{\frown}{CD} = \angle ABO = 60°$, the Inscribed Angle Theorem implies that $\angle BAO = 30°$, and $\triangle AOB$ is a 30–60–90° triangle with

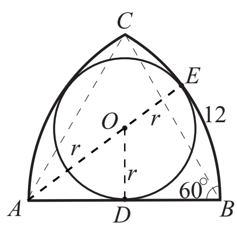
$$AB = 2 \cdot OB = 10 \quad \text{and} \quad AO = \sqrt{3}OB = 5\sqrt{3}.$$

Also, the hypotenuse of $\triangle ACD$ is a diameter of the circle, so $\angle ACD$ is a right angle and $\triangle ACD$ is similar to $\triangle AOB$. This implies that

$$\frac{AC}{AD} = \frac{AO}{AB} \quad \text{and} \quad AC = AD \cdot \frac{AO}{AB} = 2(5\sqrt{3}) \cdot \frac{5\sqrt{3}}{10} = 15.$$

Hence $BC = AC - AB = 15 - 10 = 5$. $\square$

Exercise 8 Answer (D) The construction given in the problem is a classic way to construct an equilateral $\triangle ABC$, with side length AB. The length of



$\widehat{BC}$ is one-sixth the circumference of the circle with radius AB, so

$$12 = \frac{1}{6}(2\pi \cdot AB) \quad \text{and} \quad AB = \frac{36}{\pi}.$$

Let O be the center of the circle, r be the radius, and D be the midpoint of $\overline{AB}$. The symmetry of the region implies that OD is a perpendicular bisector of $\overline{AB}$. Construct $\overline{AE}$, the line segment passing through O and intersecting the arc $\widehat{BC}$ at E. Then $AE = AB$ and $r = OE = OD$, so in the right $\triangle ADO$ we have

$$\frac{36}{\pi} = AE = AO + OE = \sqrt{AD^2 + DO^2} + OE = \sqrt{\left(\frac{18}{\pi}\right)^2 + r^2} + r.$$

Hence

$$0 = \left(\frac{36}{\pi} - r\right)^2 - \left(\left(\frac{18}{\pi}\right)^2 + r^2\right) = 3\left(\frac{18}{\pi}\right)^2 - \frac{72}{\pi}r,$$

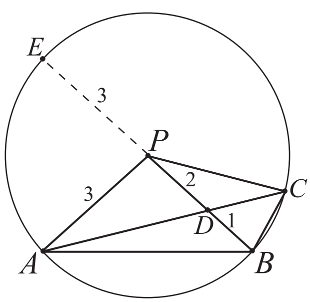
and

$$r = \frac{\pi}{72} \cdot 3\left(\frac{18}{\pi}\right)^2 = \frac{27}{2\pi}.$$

The circumference of the circle is

$$2\pi r = 2\pi\left(\frac{27}{2\pi}\right) = 27. \qquad \square$$

Exercise 9 Answer (A) Because we are being asked to find the product of portions of a line segment, we might be suspicious that one of the Secant Theorems is involved. But the Secant Theorems require a circle, and the only reasonable circle to construct has its center at P and radius $AP = BP = 3$. Let E be the intersection of the extension of $\overline{PB}$ with the circle, as shown.



Notice that the Inscribed Angle Theorem implies that this circle also passes through C since $\angle APB = 2\angle ACB$. The Internal Secant Theorem applied to the chords $\overline{AC}$ and $\overline{EB}$ gives

$$AD \cdot CD = ED \cdot DB = 5 \cdot 1 = 5. \qquad \square$$

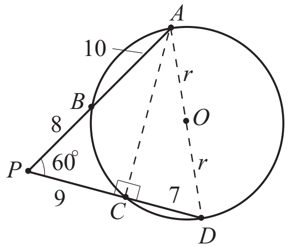
Exercise 10 Answer (D) The first step is to use the External Secant Theorem to determine the length PC of the missing portion of the secant line PD. Since

$$PA \cdot PB = PD \cdot PC = (PC + CD) \cdot PC = (PC + 7) \cdot PC$$

we have

$$PC(PC + 7) = 18 \cdot 8 \quad \text{and} \quad PC = 9.$$

Notice that in the figure the points on the circle have been moved slightly to give a truer representation of the conditions in the problem. The figure in the problem was purposely constructed slightly in error so that the following results could not be deduced simply by looking at the figure.



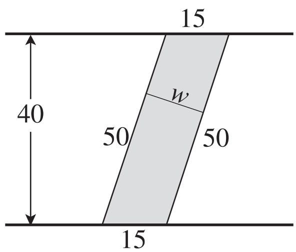
We now need to make a critical observation. Since $PA = 2PC$ and $\angle APC = 60°$, we know that $\triangle ACP$ is a 30–60–90° triangle with $\angle ACP = 90°$. As a consequence, $\triangle ACD$ is a right triangle inscribed in the circle, so $\overline{AD}$ is a diameter of the circle. To find AD, first use the 30–60–90° $\triangle ACP$ to deduce that $AC = \sqrt{3}PC = 9\sqrt{3}$. Then use the right $\triangle ACD$ to obtain

$$(2r)^2 = AD^2 = CD^2 + AC^2 = 7^2 + (9\sqrt{3})^2 = 292.$$

So $r^2 = 292/4 = 73$. $\square$

Solutions for Chapter 7: Polygons

Exercise 1 Answer (C) The crosswalk has the shape of of parallelogram with base 15 feet and altitude 40 feet, so its area is $15 \cdot 40 = 600$ square feet.



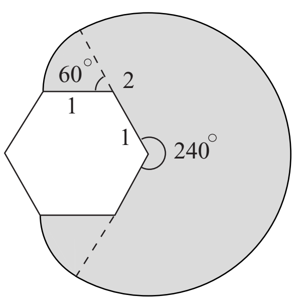
But it can also be viewed as a parallelogram with base 50 and height w, the width of the crosswalk. Hence $600 = 50 \cdot w$, and $w = 600/50 = 12$ feet. □

Exercise 2 Answer (C) Since AD is parallel to BC, we have $\angle EDF = \angle BAF$. In addition, $\angle DFE = \angle AFB$, so $\triangle DFE$ is similar to $\triangle AFB$. As a consequence,

$$\frac{16}{4} = \frac{AF}{DF} = \frac{10 - DF}{DF} \quad \text{and} \quad 4 \cdot DF = 10 - DF.$$

This implies that $DF = 2$. □

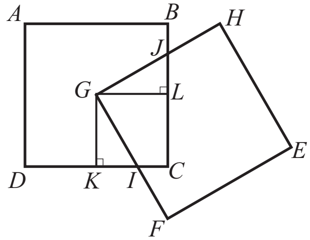
Exercise 3 Answer (E) Spot can go anywhere in a 240° sector of radius two yards, and can also cover a 60° sector of radius one yard around each of the adjoining corners.



The total area is

$$\pi(2)^2 \cdot \frac{240}{360} + 2\left(\pi(1)^2 \cdot \frac{60}{360}\right) = \frac{8}{3}\pi + \frac{1}{3}\pi = 3\pi. \qquad \square$$

Exercise 4 Answer (E) Label as I the intersection of sides $\overline{CD}$ and $\overline{GF}$, and as J the intersection of $\overline{BC}$ and $\overline{GH}$.

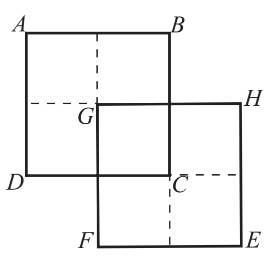


Drop perpendiculars from G, the center of square $ABCD$, to sides $\overline{CD}$, and $\overline{BC}$ and label the intersections as K and L, respectively. Then right $\triangle GKI$ is congruent to right $\triangle GLJ$. Hence the area of the overlap of the two squares is the same as the area of the square $GKCL$, which is $1/4$ of the area of the squares $ABCD$ and $EFGH$. As a consequence, the total area is

$$\text{Area}(ABCD) + \text{Area}(EFGH) - \text{Area}(GKCL) = 100 + 100 - 25 = 175.$$

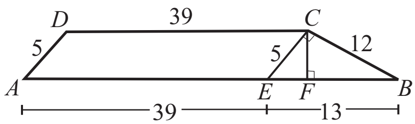
OR

Since the orientation of square $GHEF$ relative to square $ABCD$ is not relevant, we can assume that $GF \perp DC$ and, as a consequence, $GH \perp BC$.



Hence the area of the union of the squares is $7/4$ the area of square $ABCD$, that is, $(7/4) \cdot 100 = 175$. $\square$

Exercise 5 Answer (C) To determine the area of this trapezoid we need to know the distance between the parallel line segments $\overline{AB}$ and $\overline{DC}$. We could determine this distance by extending the line segments $\overline{AD}$ and $\overline{BC}$ until they intersect and then use similar triangles. Instead, let us consider an easier method.

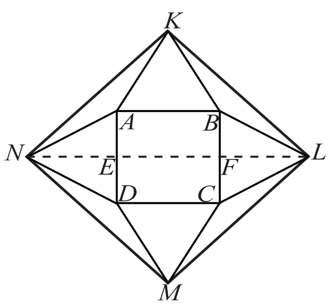


Construct the segment $\overline{EC}$ parallel to $\overline{AD}$ with E on $\overline{AB}$. Then $AECD$ is a parallelogram. Because $EC = 5$ and $CB = 12$, $\triangle ECB$ is a right triangle with hypotenuse $EB = 13$. Now construct the altitude $\overline{FC}$ of this triangle from side EB. By the Right Triangle Altitude Theorem, $\triangle EFC$ is similar to $\triangle ECB$, so $CF/CE = CB/EB$. As a consequence,

$$CF = CE \cdot \frac{CB}{EB} = 5 \cdot \frac{12}{13} \quad \text{and} \quad \text{Area}(ABCD) = \frac{1}{2}(52+39)\frac{60}{13} = 210.$$

□

Exercise 6 Answer (D) Construct line segment $\overline{NL}$ and label as E and F its intersections with AD and BC, respectively.



Since $\triangle DNA$ is equilateral with side length $AD = 4$, and E is the midpoint of AD, the symmetry of the figure gives

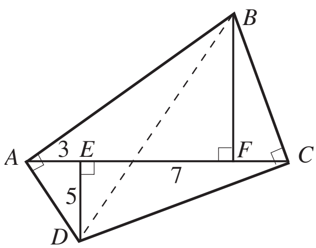
$$NL = NE + EF + FL = \frac{\sqrt{3}}{2} \cdot 4 + 4 + \frac{\sqrt{3}}{2} \cdot 4 = 4(1 + \sqrt{3}).$$

However, $\triangle NKL$ is a 45–45–90° triangle, so the area of square $KLNM$ is

$$KN^2 = \left(\frac{NL}{\sqrt{2}}\right)^2 = \left(\frac{4(1+\sqrt{3})}{\sqrt{2}}\right)^2 = 8(1+2\sqrt{3}+3) = 32+16\sqrt{3}. \ \square$$

Exercise 7 Answer (C) First construct the diagonal joining B and D. Since $\angle BAD$ is a right angle, $\angle BAE + \angle EAD = 90°$. But $\triangle AFB$ is a right triangle, so $\angle BAE + \angle ABF = 90°$ as well. Thus $\angle EAD = \angle ABF$, and right $\triangle AED$ is similar to right $\triangle DAB$. Hence

$$\frac{AF}{BF} = \frac{DE}{AE}, \quad \text{so} \quad AF = \frac{DE}{AE}BF = \frac{5}{3}BF.$$



Similarly, $\angle CBF = \angle DCE$, so right $\triangle CFB$ is similar to right $\triangle DEC$. So

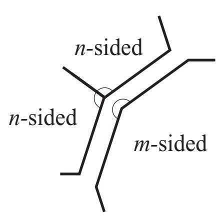
$$\frac{CF}{BF} = \frac{ED}{EC} \quad \text{and} \quad CF = \frac{ED}{EC}BF = \frac{5}{7}BF.$$

Thus

$$AE + EC = 3 + 7 = 10 = AF + FC = \frac{5}{3}BF + \frac{5}{7}BF = \frac{50}{21}BF$$

and $BF = 21/5 = 4.2$. $\square$

Exercise 8 Answer (A) The exterior angle of an m-sided regular polygon has degree measure $(m+2)180°/m$, which in our case is $\frac{12\cdot 180}{10} = 216°$.



Two of the n-sided regular polygons meet externally at each vertex of the m-sided polygon. So the interior angle measure of the n-sided regular polygons is half the exterior angle measure of the m-sided polygons.

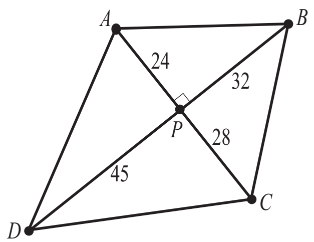
Hence

$$\frac{n-2}{n}180 = \frac{1}{2}(216) \quad \text{so} \quad \frac{n-2}{n} = \frac{3}{5} \quad \text{and} \quad n = 5. \qquad \square$$

Exercise 9 Answer (E) To solve this problem we need to hope that some special facts are true about the point P. Fortunately, we have that

$$(AP + PC)(BP + PD) = 52 \cdot 77 = 4004 = 2\text{Area}(ABCD),$$

so Result 7 of Section 7.3 implies that P is the intersection of the diagonals of $ABCD$, and that the diagonals meet at right angles.



As a consequence,

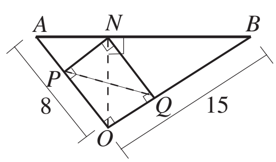
$$AB = \sqrt{24^2 + 32^2} = 40, \quad BC = \sqrt{32^2 + 28^2} = 4\sqrt{113},$$

$$CD = \sqrt{28^2 + 45^2} = \sqrt{2809} = 53, \quad \text{and} \quad DA = \sqrt{45^2 + 24^2} = 51.$$

The perimeter is consequently

$$40 + 4\sqrt{113} + 53 + 51 = 4(36 + \sqrt{113}). \qquad \square$$

Exercise 10 Answer (C) Let O represent the point of intersection of the diagonals of the rhombus $ABCD$. The diagonals of a rhombus bisect each other at right angles, so $AO = 8$, $BO = 15$, and the quadrilateral $PNQO$ is a rectangle.



The diagonals of a rectangle have equal length, so $PQ = NO$. To minimize NO we choose N as an altitude of side AB in $\triangle AOB$. Then the Right Angle Altitude Theorem applied to $\triangle AOB$ with altitude $\overline{NO}$ implies that

$$\frac{NO}{AO} = \frac{OB}{AB} = \frac{OB}{\sqrt{AO^2 + OB^2}} = \frac{15}{\sqrt{8^2 + 15^2}} = \frac{15}{17}.$$

So the minimal value of PQ is

$$PQ = NO = \frac{15}{17} \cdot 8 = \frac{120}{17} = 7 + \frac{1}{17} \approx 7.0. \qquad \square$$

Solutions for Chapter 8: Counting

Exercise 1 Answer (D) Since there are 12 men at the party, and each danced with 3 women, there was a total of $12 \cdot 3 = 36$ dancing pairs. If the number of women at the party is denoted w, then since each danced with 2 men, the number of dancing pairs is also $2w$. So $2w = 36$ and $w = 18$. $\square$

Exercise 2 Answer (C) There are only 3 distinct ways that the customer can choose the number of patties, but many ways to choose the condiments. If the customer chooses i of them, for some $0 \le i \le 8$, there are $\binom{8}{i}$ possible choices. Since the customer can choose none or any number up to 8 of the individual condiments, the number of ways that the condiments can be chosen is

$$\sum_{i=0}^{8} \binom{8}{i} = \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}$$
$$= 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256 \text{ ways.}$$

So the total number of choices is $3 \cdot 256 = 768$. $\square$

There are a number of other ways to see that there are 256 different ways to choose the condiments. We could notice that the Binomial Theorem implies that

$$256 = 2^8 = (1 + 1)^8 = \sum_{i=0}^{8} \binom{8}{i},$$

which would eliminate much of the tedious arithmetic.

Alternatively we could note that for each of the condiments, a different combination occurs when that condiment is selected or is omitted. Since there are 8 condiments that could be either selected or deleted, there are 2^8 different ways that they can be chosen.

Exercise 3 Answer (C) Suppose that the bag originally contains R red and B blue marbles, for a total of $R + B$ marbles. After adding the additional red marbles, there are $3B$ marbles in the bag, B of them are blue and $2B$ of them are red. Then yellow marbles are added until there are $5B$ marbles in the bag, of which B are blue, $2B$ are red, and $2B$ are yellow. Finally, we double the number of blues to make $2B$ of each color. So $1/3$ of the marbles in the bag are now blue.

Notice that we don't need to do all the computation described above. Just before the last blue marbles are added to the bag there are B blue marbles and $4B$ that are not blue. Doubling the blues gives $2B$ blue marbles and $4B$ that are are not blue, so $1/3$ are blue. We don't care about the color of the non-blues. □

Exercise 4 Answer (C) The old license plate scheme had

$$\text{(letter)(digit)(digit)(digit)(digit)} = 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26 \times 10^4$$

distinct plates. The new scheme has

$$\begin{aligned}\text{(letter)(letter)(letter)(digit)(digit)(digit)} &= 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \\ &= 26^3 \times 10^3\end{aligned}$$

distinct plates. So the ratio of the new scheme to the old is

$$\frac{26^3 \times 10^3}{26 \times 10^4} = \frac{26^2}{10}. \qquad \square$$

Exercise 5 Answer (D) The first 24 of the 120 words begin with A, the next 24 with M, the next 24 with O, and the next 24 with S. So there are 96 letters on the list before any words begin with U. Of those that begin with U the first 6 have the second letter A, the next 6 have M, and the next 6 have O. So there are 96 + 18 = 114 words before the word begins with *US*. The first word beginning with *US* is *USAMO*, the one we want. So it is the 115th word.

OR

Since *USAMO* is near the end of the list, it is easier to count from the end than from the front. The last words on the list are *USOMA*, *USOAM*, *USMOA*, *USMAO*, *USAOM* and, finally, *USAMO*. Since there are 120 words on the list, *USAMO* is the 115th. □

Exercise 6 Answer (D) The possibilities for (glazed, chocolate, powdered) are

$$\begin{aligned}
\text{All of one type}: &\quad (4,0,0),\ (0,4,0),\ (0,0,4);\\
\text{Three of one type, one of another}: &\quad (3,1,0),\ (3,0,1),\ (1,3,0),\\
&\quad (0,3,1),\ (1,0,3),\ (0,1,3);\\
\text{Two of each of two types}: &\quad (2,2,0),\ (2,0,2),\ (0,2,2);\\
\text{Two of one type, one of each other}: &\quad (2,1,1),\ (1,2,1),\ (1,1,2).
\end{aligned}$$

This gives a total of 15 distinct choices.

OR

Place a slot for each choice of donut and an additional two slots for dividers, as in Sample Problem 2. The distinct selection of dividers is equivalent to the distinct selection of types of cookies. Since there are

$$\binom{6}{2} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5}{2} = 15$$

ways to select the dividers, there are 15 ways to select the types of donuts. □

Listing all possibilities is not difficult in this problem, but consider the additional complication if Pat was to chose donuts from 5 types. The alternate solution for this case would require 5 dividers, so then would be $3 + 5 = 8$ slots in which to place the 5 dividers. This gives

$$\binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8 \cdot 7 \cdot 6 \cdot 3 \cdot 2}{6} = 56$$

distinct choices. I would not want to list them, would you?

Exercise 7 Answer (B) Since

$$n = 100 \cdot q + r = 99 \cdot q + (q + r),$$

n is divisible by 11 if and only if $q + r$ is divisible by 11. The five-digit numbers are those between 10,000 and 99,999 inclusive, and

$$\left\lfloor \frac{99{,}999}{11} \right\rfloor - \left\lfloor \frac{10{,}000}{11} \right\rfloor = 9090 - 909 = 8181$$

of these are divisible by 11. □

Exercise 8 Answer (C) There are $10 \cdot 10 \cdot 10 = 10^3$ different possible sequences for the first three numbers $d_1d_2d_3$. For each of these, the numbers $d_4d_5d_6$ could match and then d_7 could be any value, so there are $\left(10^3\right) \cdot 10 = 10^4$ possibilities in this form.

In like manner, there are 10^4 possibilities where $d_5d_6d_7$ match $d_1d_2d_3$ and then d_4 could be any value.

This gives a total of 20,000 possibilities, but there is overlap and the values in the overlap have been counted twice. The overlap occurs when $d_4d_5d_6$ and $d_5d_6d_7$ both match $d_1d_2d_3$. In this situation, $d_1 = d_4 = d_5$, and $d_2 = d_5 = d_6$, and $d_3 = d_6 = d_7$. That is, all the digits in the telephone number are the same, which can happen in 10 distinct ways. The Inclusion-Exclusion Principle implies that we should subtract this number, giving $20{,}000 - 10 = 19{,}990$ different memorable numbers. □

The weakness in this problem is that the person who does not answer the question will be awarded a higher score than the person who reasons logically, but neglects to count the overlap. Be very careful with counting problems to ensure that all cases have been considered.

Exercise 9 Answer (C) Consider the nine chairs

No Professor								No Professor
Chair 1	Chair 2	Chair 3	Chair 4	Chair 5	Chair 6	Chair 7	Chair 8	Chair 9

and first note that none of the professors can occupy chairs 1 or 9.

If a professor occupies chair 2 and the second professor occupies chair 4, then the third professor could occupy any of chairs 6, 7, or 8. Hence there are 3 possibilities when chairs 2 and 4 are occupied. If a professor occupies chair 2 and the next chair occupied by a professor is chair 5, then the third professor could occupy either chairs 7 or 8, so there are 2 possibilities. Finally, if chair 2 is occupied by a professor but the next chair occupied by a professor is chair 6, then there is only one possibility for the third professor, and that is chair 8. Hence there are 6 ways the professors can occupy the chairs if the first chair occupied by a professor is chair 2.

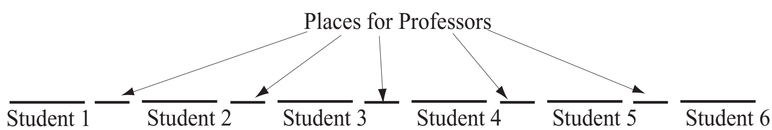
If the first chair occupied by a professor is chair 3, then the next chair occupied by a professor could be chair 5, and there are two possibilities for the third professor, chair 7 and chair 8. If the first chair occupied by a professor is chair 3 and the next occupied by a professor is chair 6 then there is only one possibility for the third professor, which is chair 8. Hence there are 3 ways the professors can occupy the chairs if the first chair occupied by a professor is chair 3.

Finally, the professors could occupy chairs 4, 6, and 8.

Summing these possibilities we see that there are $6+3+1 = 10$ ways that the chairs can be occupied by professors, but there are also $3! = 6$ ways that the individual professors can be arranged in these chairs. Since the two events are independent, there is a total of $10 \cdot 6 = 60$ ways that professors Alpha, Beta, and Gamma can choose their chairs.

OR

Alternatively, we could assume that the students arrived first and arranged themselves in a row. Then the professors arrived and placed chairs between the students.



Then there are 5 places that the professors can place their chairs, between the first and second students, the second and third students, ..., the fifth and sixth students. Of these five slots, the professors must choose 3, which can be done in $\binom{5}{3} = 10$ ways. The remainder of the solution is as above. □

Exercise 10 Answer (A) This problem is harder than it first appears. It is rather easy to see that, since the first 5 letters must be chosen from B's and C's, there are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways to satisfy the condition that there be no A's in the first 5 letters. But then when we look at the next 5 letters we have 5 A's to choose from but the number of C's to choose from depends on the number we used in the first 5 places. We need to look at the cases separately.

Suppose that we use only B's in the first 5 places. There are 5! ways to do this. Then all the C's must go in the second 5 places (because they cannot go in the last 5 places) and all the A's must go in the last 5 places. So there are $(5!)^2$ choices in this case . This gives a total of $(5!)^3$ choices when we use only B's in the first five places. (The same when we use only C's in the first 5 places.)

If we use 4 B's and 1 C in the first 5 places, we have $\binom{5}{1}$ ways they can be chosen. Then we need to use 4 C's and 1 A in the second 5 places and 1 B and 4 A's in the last 5 places. This gives a total of $\binom{5}{1}^2$ ways to choose the second and third blocks of five letters. It also gives a total of $\binom{5}{1}^3$ ways when we use 4 B's and 1 C in the first 5 places. (The same holds when we use 4 C's and 1 B in the first 5 places.)

If we use 3 B's and 2 C's we have a similar situation which gives $\binom{5}{2}^3$ total choices.

Adding this implies that the total ways to make the choices is

$$2(5!)^3 + 2\binom{5}{1}^3 + 2\binom{5}{2}^3.$$

This is not one of the answer choices, but it looks closest to the answer choice in A. And indeed, since

$$1 = \binom{5}{0} = \binom{5}{5}, \quad \binom{5}{1} = \binom{5}{4}, \quad \text{and} \quad \binom{5}{2} = \binom{5}{3},$$

we have

$$2(1)^3 + 2\binom{5}{1}^3 + 2\binom{5}{2}^3 = \binom{5}{0}^3 + \binom{5}{1}^3 + \binom{5}{2}^3 + \binom{5}{3}^3 + \binom{5}{4}^3 + \binom{5}{5}^3.$$

So the number of ways to choose A, B, and C is $\displaystyle\sum_{i=0}^{5} \binom{5}{i}^3$. □

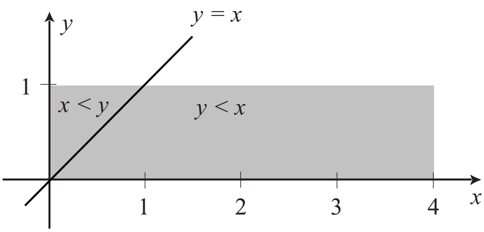
Solutions for Chapter 9: Probability

Exercise 1 Answer (E) There are 12 factors of 60, which are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Of these, there are 6 that are less than 7. So the probability that a randomly drawn positive factor is less than 7 is $6/12 = 1/2$. □

Exercise 2 Answer (C) We could write out all the possible outcomes, total them and count the number with a product greater than the sum. But there are quite a lot, $8 \cdot 8 = 64$ of them, and this would take too long. Instead, note that the sum will be greater than the product only when a 1 appears on one of the dice. In addition, the sum will be equal to the product only when a 2 appears on both dice.

The number of instances of a 1 appearing on one of the dice is $8 + 8 - 1 = 15$. (The 1 must be subtracted so that we do not count twice the appearance of a 1 on each die.) So the probability that the product of the two top numbers *is not* greater than the sum is $16/64 = 1/4$. The probability that the product of the two top numbers *is* greater than the sum is consequently $1 - 1/4 = 3/4$. □

Exercise 3 Answer (A) The region together with the line $y = x$ is shown in the figure. The area of the portion to the left of the line is $1/2$ and the total area of the rectangle is 4. So the probability that a randomly chosen point (x, y) within the rectangle will have $x < y$ is $(1/2)/4 = 1/8$. □



Exercise 4 Answer (B) Consider modifying the problem to assume that a success occurs when the painted cube can be placed on a horizontal surface so that four vertical faces are Red. This will simplify the counting process and be half the probability that the problem requests.

The event is successfully done if

- Case I: all the sides of the cube are Red, or
- Case II: all but one of the sides the cube are Red, or
- Case III: four sides of the cube are Red and two are Blue, where the Blue sides are opposite one another on the cube.

There are $2^6 = 64$ ways that the cube can be painted, one color for each side, and only one way in which all the sides can be Red, so the probability of success in Case I is $1/64$.

In Case II, exactly one of the six sides is Blue, so the probability of success in this case is $6/64 = 3/32$.

In Case III, there are three pairs of opposite sides, and exactly one pair must be Blue, so the probability in this case is $3/64$.

Hence the probability that the painted cube can be placed on a horizontal surface so that four vertical faces are Red is

$$\frac{1}{64}+\frac{3}{32}+\frac{3}{64}=\frac{5}{32}.$$

The probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are Blue is also $5/32$, so the answer to the problem is $2(5/32) = 5/16$. □

Exercise 5 Answer (D) The last chip will be White when the consecutive draws follow one of the these six cases, which occur with the probabilities shown:

- Case I: R, R, W, W Probability: $\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{10}$.
- Case II: R, W, R, W Probability: $\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{10}$.
- Case III: R, W, W Probability: $\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}=\frac{1}{10}$.
- Case IV: W, R, R, W Probability: $\frac{2}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{10}$.
- Case V: W, R, W Probability: $\frac{2}{5}\cdot\frac{3}{4}\cdot\frac{1}{3}=\frac{1}{10}$.
- Case VI: W, W Probability: $\frac{2}{5}\cdot\frac{1}{4}=\frac{1}{10}$.

The solution is: the probability that one of these situations occurs is the sum of the individual probabilities,

$$6\cdot\frac{1}{10}=\frac{3}{5}.$$

OR

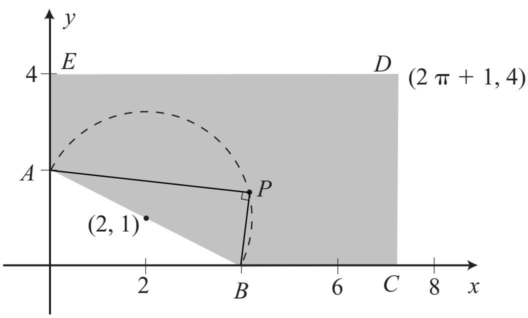
Let's look at this problem in another way. Draw all the chips even if all those of one color have been removed. The probability that the last chip drawn is a Red is the same as the probability that the first chip drawn is a Red, and this is $3/5$. So the probability that all the White chips are drawn before the Red chips are all drawn is also $3/5$. □

Exercise 6 Answer (C) For the product to be a multiple of 3, one or both of the numbers appearing on the dice must be a multiple of 3. Exactly two of the numbers that could appear on each of the dice are multiples of 3. So the Inclusion-Exclusion Principle implies that there are

$$2 \cdot 6 + 8 \cdot 2 - 2 \cdot 2 = 24$$

ways that the product is a multiple of 3. Since there are $8 \cdot 6 = 48$ possible outcomes when the two dice are rolled, the probability of the product being a multiple of 3 is $24/48 = 1/2$. □

Exercise 7 Answer (C) First recall that an obtuse angle is one with measure greater than 90°. The figure shows a point P with $\angle APB = 90°$. In this case P is a point on the semicircle with center at the midpoint (2, 1) of the line segment $\overline{AB}$ and radius $\sqrt{2^2 + 1^2} = \sqrt{5}$.



Points that lie outside this semicircle have $\angle APB < 90°$ and those inside have $\angle APB > 90°$. The area of this semicircular region is

$$\frac{1}{2}\pi(\sqrt{5})^2 = \frac{5\pi}{2},$$

and the area of the pentagon is

$$(2\pi + 1)(4) - \frac{1}{2}(2 \cdot 4) = 8\pi.$$

So the probability that the point P lies inside of the semicircle, and hence that $\angle APB$ is obtuse, is

$$\frac{5\pi/2}{8\pi} = \frac{5}{16}.$$ □

Exercise 8 Answer (E) Since the number 1 cannot be the first term, the number of ways that the acceptable permutations can be ordered is

$$4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96,$$

and the number of ways that a 2 can occur as the second term in this situation is

$$3 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 18.$$

So the probability that a 2 will occur as the second term of an acceptable permutation is $18/96 = 3/16$, and $3 + 16 = 19$.

OR

The probability that a 2 occurs as the first term of the permutation is $1/4$, and the probability of a success in this situation is 0. The probability that 2 does not occur as the first term is $3/4$; the probability that in this situation it occurs as the second term is $1/4$. Hence the probability that a 2 occurs as the second term of an acceptable permutation is

$$\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16} \quad \text{and} \quad 3 + 16 = 19. \qquad \square$$

Exercise 9 Answer (D) First note that there are

$$1 + 2 + 3 + 4 + 5 + 6 = 21$$

dots originally present on the die.

There are two cases to consider. In the first case, the face on top could originally have contained an even number, 2, 4 or 6, of dots and this face had one of its dots removed. The probability of this occurring is

$$\text{for 2: } \frac{1}{6} \cdot \frac{2}{21}, \quad \text{for 4: } \frac{1}{6} \cdot \frac{4}{21}, \quad \text{for 6: } \frac{1}{6} \cdot \frac{6}{21}.$$

In the second case, the face on top could originally have contained an odd number, 1, 3 or 5, of dots and the dot removed was from some other face. The probability of this occurring is

$$\text{for 1: } \frac{1}{6} \cdot \frac{20}{21}, \quad \text{for 3: } \frac{1}{6} \cdot \frac{18}{21}, \quad \text{for 5: } \frac{1}{6} \cdot \frac{16}{21}.$$

Hence the probability of the top face being odd after a dot has been randomly removed is

$$\frac{1}{6} \cdot \frac{1}{21}(2+4+6+20+18+16) = \frac{66}{6 \cdot 21} = \frac{11}{21}.$$

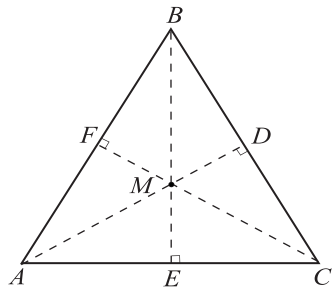
OR

Suppose that the dot has been removed from a face that originally had an odd number of dots. Then there will be 2 odd faces and 4 even faces, so the probability that the top face is odd in this case is $1/3$. On the other hand, if the dot has been removed from a face that originally had an even number of dots, there will be 4 odd faces and 2 even faces. So the probability that the top face is odd in this case is $2/3$.

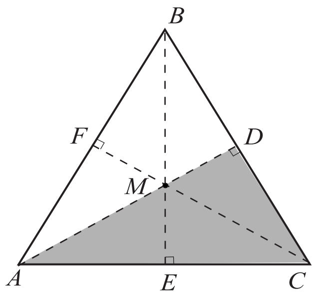
Because each dot has the probability $1/21$ of being removed, the top face is odd with probability

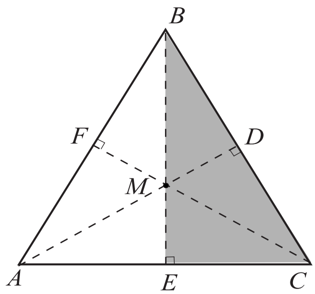
$$\left(\frac{1}{3}\right)\left(\frac{1+3+5}{21}\right) + \left(\frac{2}{3}\right)\left(\frac{2+4+6}{21}\right) = \frac{33}{63} = \frac{11}{21}. \qquad \square$$

Exercise 10 Answer (C) The figure shows the triangle to which the centroid M has been added, as well as the bisectors of the sides of the $\triangle ABC$, which are the points D, E, and F. The symmetry of the situation implies that $\triangle ABC$ is decomposed into the six congruent right triangles that have a vertex at M.



The area of $\triangle ABP$ is greater than the area of $\triangle APC$ provided that P lies below the line segment $\overline{AD}$, as shown in the figure below at the left. Similarly, the area of $\triangle ABP$ is greater than the area of $\triangle BPC$ provided that P lies to the right of the line segment $\overline{BE}$, as shown on the figure on the right.





For both conditions to hold, P must be within the union of $\triangle DMC$ and $\triangle EMC$. Since the six triangles with a vertex at M are congruent, they have the same area, and the probability that P is within the union of $\triangle DMC$ and $\triangle EMC$ is $2/6 = 1/3$. $\square$

Solutions for Chapter 10: Prime Decomposition

Exercise 1 Answer (B) The possible numbers that can be visible are 1, 2, 3, 4, 5, and 6, so the product P is a factor of the product of $6! = 2^4 \cdot 3^2 \cdot 5$. However, 4 could be the number not visible, so we need to decrease the powers of 2 in P from 4 to 2 to ensure that P divides the product. Similarly, since 3 may not be visible we need to decrease the power of 3 from 2 to 1. Since 5 may or may not be visible, we need to decrease the power of 5 from 1 to 0. The situation when one of the numbers 2 or 6 is not visible is taken care of by these reductions. So $P = 2^2 \cdot 3^1 \cdot 5^0 = 12$ is the largest number that divides the product of the visible faces in every situation. $\square$

Exercise 2 Answer (D) Problems that ask for the sum of the digits of a number are often resolved by first rearranging the calculations so that there are powers of 10. This, in turn, is done by reducing the situation, if possible, to powers of 2 and 5. Here we have

$$2^{2004} \cdot 5^{2006} = (2 \cdot 5)^{2004} \cdot 5^2 = 25 \times 10^{2004}.$$

Multiplying by 10^{2004} has no effect on the sum of the digits of the product, it only adds to the number of trailing zeros. Hence the sum of the digits is $2 + 5 = 7$. $\square$

Exercise 3 Answer (B) This is another problem that can be resolved by first rearranging the calculations so that there are powers of 10. Here we have

$$\begin{aligned} N = \left(25^{64} \cdot 64^{25}\right)^{1/2} &= \left(\left(5^2\right)^{64}\right)^{1/2} \cdot \left(\left(2^6\right)^{25}\right)^{1/2} \\ &= 5^{(2\cdot 64\cdot(1/2))} \cdot 2^{(6\cdot 25\cdot(1/2))} \\ &= 5^{64} \cdot 2^{75} = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048. \end{aligned}$$

Multiplying by 10^{64} does not effect sum of the digits of N, so the sum is $2 + 0 + 4 + 8 = 14$. □

Exercise 4 Answer (B) Suppose that the roots of the equation are p and q with $1 \le p \le q$. The sum of the roots is the negative of the coefficient of the linear term, so

$$p + q = -(-63) = 63$$

is odd. So one of the prime factors is odd and the other is even. Since 2 is the only even prime, the primes must be $p = 2$ and $q = 61$. Hence there is exactly one possible value of k, which is $k = 2 \cdot 61 = 122$. □

Exercise 5 Answer (E) This problem requires little more than recognizing that the expression can be written using the Binomial Theorem:

$$\begin{aligned} N &= 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1 \\ &= (69 + 1)^5 = 70^5 = 2^5 \cdot 5^5 \cdot 7^5. \end{aligned}$$

So there are $(5 + 1)(5 + 1)(5 + 1) = 6^3 = 216$ factors of N. □

Exercise 6 Answer (B) First write the product in its prime decomposition as

$$\begin{aligned} 1! \cdot 2! \cdots 9! &= 1^9 \cdot 2^8 \cdot 3^7 \cdots 9^1 = (2)^8(3)^7(2^2)^6(5)^5(2 \cdot 3)^4(7)^3(2^3)^2(3^2) \\ &= 2^{30} \cdot 3^{13} \cdot 5^5 \cdot 7^3. \end{aligned}$$

To have a divisor that is a square, each of the powers of each of the primes must be even. For the prime 2, there are 16 possibilities, which are 2^0, 2^2, $2^4, \ldots, 2^{30}$. Similarly there are 7 possibilities for the prime 3, which are $3^0, 3^2, 3^4, \ldots, 3^{12}$. There are three for 5, which are 5^0, 5^2, and 5^4, and two for 7, which are 7^0 and 7^2. Each possibility for each of the primes results in a unique square number, so there is a total of

$$16 \cdot 7 \cdot 3 \cdot 2 = 672 \text{ square divisors.} \qquad \square$$

Exercise 7 Answer (C) First, for $1 \le n \le 50$, we express n in terms of its prime factorization, that is, as

$$n = 2^{p_2} \cdot 3^{p_3} \cdot 5^{p_5} \cdots 47^{p_{47}},$$

where, for each i, we have $p_i \ge 0$. The number of factors of n is

$$(p_2 + 1)(p_3 + 1)(p_5 + 1) \cdots (p_{47} + 1).$$

For n to have an odd number of factors we must have all the terms in the product odd. As a consequence, each of $p_1, p_3, p_5, \ldots, p_{47}$ must be even. Hence the largest nonzero value of p_k is p_7, and the only possibilities are the 7 numbers $1, 4 = 2^2, 9 = 3^2, 16 = 2^4, 25 = 5^2, 36 = 2^2 \cdot 3^2$, and $49 = 7^2$. $\square$

Exercise 8 Answer (D) We found in Chapter 7 that the number of degrees in each interior angle of a regular n-gon is

$$\frac{n-2}{n} 180 = 180 - \frac{360}{n}.$$

So the number of degrees is an integer if and only if n divides $360 = 2^3 \cdot 3^2 \cdot 5$. Since there are $(3+1)(2+1)(1+1) = 24$ factors of 360, you might be misled into thinking that this is the correct answer. However, no polygon is formed when $n = 1$ or when $n = 2$, so there only 22 polygons with this property. $\square$

It is unlikely that the choice 24 would be included in a modern exam since it would likely result from the correct logic, but overlooking the special cases. Since the scoring policy puts a significant penalty on incorrect answers over blank answers, including the choice 24 would have the undesirable effect of giving a lower score to a student who almost has the correct answer than to one who has no notion of how the problem can be solved.

Exercise 9 Answer (C) First note that when a rational number r has the repeated decimal representation

$$r = \beta_m \beta_{m-1} \cdots \beta_1 . \overline{\alpha_1 \alpha_2 \cdots \alpha_n},$$

then

$$10^n r = \beta_m \beta_{m-1} \cdots \beta_1 \alpha_1 \alpha_2 \cdots \alpha_n . \overline{\alpha_1 \alpha_2 \cdots \alpha_n},$$

so

$$(10^n - 1) r = \beta_m \beta_{m-1} \cdots \beta_1 \alpha_1 \alpha_2 \cdots \alpha_n - \beta_m \beta_{m-1} \cdots \beta_1$$

and

$$r = \frac{\beta_m \beta_{m-1} \cdots \beta_1 \alpha_1 \alpha_2 \cdots \alpha_n - \beta_m \beta_{m-1} \cdots \beta_1}{10^n - 1}.$$

In our case we have $n = 2$, and r has the form

$$r = \frac{ab}{99}.$$

Fractions with distinct denominators occur precisely when $ab = 10a + b$ is divisible by one of the factors of $99 = 3^2 \cdot 11$. These factors are 1, 3, 9, 11, 33, and 99. However, the factor 1 is eliminated since this would give the excluded value $r = 0.\overline{99}$. Hence there are five possibilities, which are

$$3: \ 0.\overline{33}, \quad 9: \ 0.\overline{11}, \quad 11: \ 0.\overline{09}, \quad 33: \ 0.\overline{03}, \quad \text{and} \quad 99: \ 0.\overline{01}. \ \square$$

Exercise 10 Answer (C) First assume that when n is written as a product of primes it has the form

$$n = 2^{p_2} \cdot 3^{p_3} \cdot 5^{p_5} \cdot 7^{p_7} \cdots .$$

Then

$$2n = 2^{p_2+1} \cdot 3^{p_3} \cdot 5^{p_5} \cdot 7^{p_7} \cdots \quad \text{and} \quad 3n = 2^{p_2} \cdot 3^{p_3+1} \cdot 5^{p_5} \cdot 7^{p_7} \cdots .$$

So the number of factors of n, $2n$, and $3n$ are

$$n : (p_2 + 1)(p_3 + 1)(p_5 + 1)(p_7 + 1) \cdots ,$$

$$2n : (p_2 + 2)(p_3 + 1)(p_5 + 1)(p_7 + 1) \cdots = 28 = 2^2 \cdot 7,$$

and

$$3n : (p_2 + 1)(p_3 + 2)(p_5 + 1)(p_7 + 1) \cdots = 30 = 2 \cdot 3 \cdot 5.$$

Subtracting the middle expression from the last expression gives

$$\begin{aligned} 2 &= ((p_2+1)(p_3+2) - (p_2+2)(p_3+1))\,(p_5+1)(p_7+1)\cdots \\ &= (p_2 - p_3)(p_5+1)(p_7+1)\cdots . \end{aligned}$$

Since all the expressions are integers, this implies that we have two possibilities

- Case I: $1 = p_2 - p_3 \quad$ and $\quad 2 = (p_5+1)(p_7+1)\cdots$, and
- Case II: $2 = p_2 - p_3 \quad$ and $\quad 1 = (p_5+1)(p_7+1)\cdots$.

However, Case I implies that $p_2 = p_3 + 1$ and, consequently, that

$$30 = (p_2+1)(p_3+2)(p_5+1)(p_7+1)\cdots = (p_3+2)^2 \cdot 2,$$

so

$$15 = (p_3+2)^2.$$

But 15 is not the square of an integer, so Case I cannot be true.

Hence Case II must hold, which implies that $p_2 = p_3 + 2$ and, consequently, that

$$28 = 7 \cdot 4 = (p_2+2)(p_3+1) = (p_3+4)(p_3+1).$$

So $p_3 = 3$ and $p_2 = 3 + 2 = 5$. Also, since

$$1 = (p_5+1)(p_7+1)\cdots ,$$

we have $0 = p_k$ for each prime $k \geq 5$, and n has the prime factorization

$$n = 2^5 \cdot 3^3.$$

The factorization of $6n = 2 \cdot 3 \cdot n$ is therefore

$$6n = (2 \cdot 3) \cdot n = (2 \cdot 3) \cdot \left(2^5 \cdot 3^3\right) = 2^6 \cdot 3^4,$$

so $6n$ has $(6+1)(4+1) = 7 \cdot 5 = 35$ factors. □

Solutions for Chapter 11: Number Theory

Exercise 1 Answer (C) Consider 2^{1000} modulo 13 since this produces the remainder. An important observation to make to simplify the process is that since $65 = 64 + 1$, we have

$$2^6 \bmod 13 \equiv (65 - 1) \bmod 13 \equiv -1.$$

So

$$\begin{aligned} 2^{1000} \bmod 13 &\equiv 2^{996} \cdot 2^4 \bmod 13 \\ &\equiv \left(2^6\right)^{166} \bmod 13 \cdot 16 \bmod 13 \\ &\equiv \left(2^6 \bmod 13\right)^{166} \cdot 16 \bmod 13 \equiv (-1)^{166} \cdot 3 = 3. \qquad \square \end{aligned}$$

Exercise 2 Answer (D) The largest three-digit base-9 number is $9^3 - 1 = 728$ and the smallest three-digit base-11 number is $11^2 = 121$. There are 608 integers that satisfy $121 \le n \le 728$, and 900 three-digit numbers altogether, so the probability is $608/900 = 152/225$.

Exercise 3 Answer (B) Recall that N is a multiple of 3 and of 9 if and only if the sum of its digits is a multiple of 3 and of 9, respectively. Consider the sum, S, of the digits of N. Rearranging the sum and using the fact that

$$1 + 2 + \cdots + 8 = 36 \quad \text{and} \quad 1 + 2 + \cdots + 9 = 45$$

gives

$$\begin{aligned} S &= 1 + 9 + 2 + 0 + 2 + 1 + \cdots + 9 + 1 + 9 + 2 \\ &= 10 + 2 \cdot 10 + (1 + 2 + \cdots + 9) + 3 \cdot 10 + (1 + 2 + \cdots + 9) + \cdots \\ &\quad + 8 \cdot 10 + (1 + 2 + \cdots + 9) + 3 \cdot 9 + 1 + 2 \\ &= 10 \cdot (1 + 2 + \cdots + 8) + 7 \cdot (1 + 2 + \cdots + 9) + 3 \cdot 9 + 3 \\ &= 10 \cdot 36 + 7 \cdot 45 + 3 \cdot 9 + 3 = (40 + 35 + 3) \cdot 9 + 3 = 78 \cdot 9 + 3. \end{aligned}$$

So S, and as a consequence N, is divisible by 3 but not by 9, and $k = 1$.

$\square$

Exercise 4 Answer (D) Since the base-5 numbers VYZ, VYX, and VVW are consecutive integers, the base-5 values must be

$$Z = 3, \quad X = 4, \quad W = 0, \quad \text{and} \quad V = Y + 1.$$

Since the only integers that remain for Y and V are 1 and 2, we have $Y = 1$ and $V = 2$. So

$$XYZ_5 = 413_5 = 4 \cdot 5^2 + 1 \cdot 5 + 3 = 108. \qquad \square$$

Exercise 5 Answer (B) The number we seek has a two-digit base-8 representation, which we write in the form $n = s \cdot 8 + t$, where $0 \le t < 8$. Since $n^2 = ab3c_8$, we have

$$n^2 = s^2 \cdot 8^2 + 2st \cdot 8 + t^2 = a \cdot 8^3 + b \cdot 8^2 + 3 \cdot 8 + c.$$

Consider the 8's digit of n^2. Since $2st$ is even but 3 is odd, the 8's digit of t^2 must be odd. The table below shows the base-8 squares of the units digits.

t	0	1	2	3	4	5	6	7
t^2 (base 10)	0	1	4	9	16	25	36	49
t^2 (base 8)	0_8	1_8	4_8	11_8	20_8	31_8	44_8	61_8

Only 3 and 5 have odd 8's digits and in either case the units digit, which is the value of c, is 1. □

Exercise 6 Answer (A) One problem we need to address is whether either of the years $N-1$ or N is a leap year. Let us begin counting from the first day of year $N-1$. If year $N-1$ is not a leap year, then day 300 of year N is day $300 + 365 = 665$ in our counting. Then, since $665 \bmod 7 \equiv 0$, the day of the week that year $N-1$ starts is the same as the day of the week of day 300 in year N, which is Tuesday. If, in addition, year N is not a leap year, then day 200 in year $N+1$ is day $2 \cdot 365 + 200 = 930$ in our counting. But $930 \bmod 7 \equiv 6$, which implies that this day is a Monday, which is false. Hence year N must have been a leap year, so year $N-1$ is not a leap year. Since $100 \bmod 7 \equiv 2$, the day of the week of day 100 of year $N-1$ is 2 days later than Tuesday, which implies that it is Thursday. □

Exercise 7 Answer (B) Select one of the seated people and label the chair of that person as 1. Then label the remaining chairs in a consecutive clockwise manner, with 2 immediately succeeding and 60 immediately preceding the chair 1. To satisfy the condition in the minimal way, the second person should be seated in chair 4, the next in 7, and so on. That is, we can place people in the 20 chairs whose number is 1 modulo 3 without having two people next to each other. After this is done, the next person will have to sit in a chair that has a number that is either 0 or 2 modulo 3. In either case this is a chair that is next to one of the chairs that is already occupied. So the smallest possible value of N is 20. □

Exercise 8 Answer (B) If we drew 5 socks from each of the 4 colors there would be 2 pairs per color, or 8 pairs in total. The 21st sock drawn would provide a ninth pair, but if the 22nd sock drawn was the same color as the 21st sock, there would still be only 9 pairs. However, the 23rd sock would make the 10th pair, regardless of the color drawn. □

Exercise 9 Answer (C) First note that we can draw all the disks with labels 1 through 9 since we cannot have ten of any of these. This gives us

$$1+2+\cdots+9=\frac{9\cdot 10}{2}=45 \text{ disks.}$$

Now we apply an Extended Pigeon Hole Principle. We assume that there are 41 boxes, those numbered 10 through 50, into which we place the disks when they have been drawn, and assume that the disks represent the pigeons. The question is "How many disks are required so that there are at least 10 disks in one of these boxes?" The answer is that we need to draw at most

$$(10-1)\cdot 41+1=370 \text{ disks.}$$

To this we must add the 45 that we could have drawn which are labeled 1 through 9. So to ensure that at least 10 disks with the same label have been drawn we need to draw a total of $370+45=415$ disks. □

Exercise 10 Answer (E) Let E_i, for $i=0,1,2,\ldots,6$, be the set of those integers in $\{1,2,3,\ldots,50\}$ that are equivalent to i modulo 7. Then

$$E_0=\{7,14,21,28,35,42,49\}, \quad E_1=\{1,8,15,22,29,36,43,50\},$$
$$E_2=\{2,9,16,23,30,37,44\}, \quad E_3=\{3,10,17,24,31,38,45\},$$
$$E_4=\{4,11,18,25,32,39,46\}, \quad E_5=\{5,12,19,26,33,40,47\}$$

and

$$E_6=\{6,13,20,27,34,41,48\}.$$

Note that S can contain at most 1 of the elements in E_0. In addition, S can have all the elements of set E_1, but then S cannot include any element of E_6. In addition, S can contain all of E_2, but none of E_5; and all of E_3, but none of E_4. Thus the maximum number of elements in S is $1+8+7+7=23$. We could have included E_6, instead of E_1, but E_6 has one less number. □

Solutions for Chapter 12: Sequences and Series

Exercise 1 Answer (D) Let a_n denote the number of cans in the nth row counting from the top. Then

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 5, \quad \text{and, generally} \quad a_n = 1 + (n-1)\cdot 2.$$

This is an arithmetic sequence, and the sum of its first n terms is

$$\frac{n}{2}(a_1 + a_n) = \frac{n}{2}(1 + (1 + (n-1)\cdot 2) = \frac{n}{2}(2n) = n^2.$$

The sum 100 occurs, then, when $n = 10$. □

Exercise 2 Answer (B) Let a denote the first term of the geometric sequence and r denote the common ratio. The stated conditions imply that

$$2 = ar,$$

and that

$$6 = ar^3 = ar \cdot r^2 = 2 \cdot r^2, \quad \text{so} \quad r^2 = 3.$$

We have two possibilities for r, either $r_1 = \sqrt{3}$ or $r_2 = -\sqrt{3}$. These possibilities imply that either

$$a = \frac{2}{r_1} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad \text{or} \quad a = \frac{2}{r_2} = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}.$$

Only the latter is one of the answer choices. □

Exercise 3 Answer (C) There are numerous ways to recognize a pattern in the figures. The method we have chosen is to first consider the number of additional small squares that would be needed to make the figure a large square. If this completion were done, each large square would consist of $2n + 1$ small squares, for a total of $(2n + 1)^2$ small squares. However, to make the nth figure a square of this size we would need to add small squares to each of the corners. As shown, the number of small squares we would need to add to each of the four corners of the nth figure is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

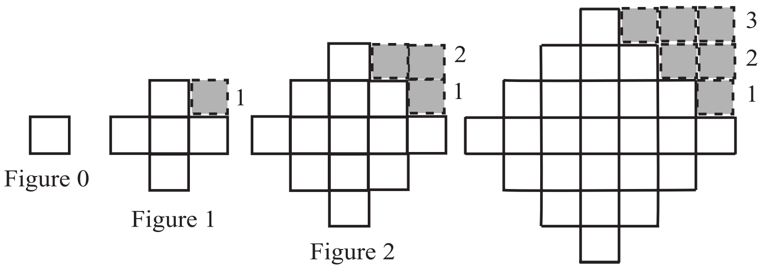


Figure 0 Figure 1 Figure 2 Figure 3

Hence the number of small squares in the nth figure must be

$$(2n+1)^2 - 4 \cdot \frac{n(n+1)}{2} = 4n^2 + 4n + 1 - 2n^2 - 2n = 2n^2 + 2n + 1.$$

When $n = 100$, this gives a figure with $2 \cdot 100^2 + 2 \cdot 100 + 1 = 20{,}201$ small squares. □

Exercise 4 Answer (A) The two sequences have, respectively, the terms

$$a_n = 1 + 3(n-1) \quad \text{and} \quad b_n = 9 + 7(n-1).$$

There are 2004 terms of a_n and 2004 terms of b_n in S, but some of these numbers are the same. Since the 2004th term of the first sequence, $a_{2004} = 1+3(2003) = 6010$, is smaller than the 2004th term of the second sequence, $b_{2004} = 9 + 7(2003) = 14{,}030$, the number 6010 is an upper bound for the numbers in S that overlap.

The first overlap is $a_6 = 1 + 3 \cdot 5 = 16 = 9 + 7 \cdot 1 = b_2$, the next is $a_{13} = 1 + 3 \cdot 12 = 37 = 9 + 7 \cdot 4 = b_5$. In general, since $\operatorname{lcm}\{3, 7\} = 21$, the terms will overlap every 21 integers after the first overlap, 16. There are

$$\left\lfloor \frac{6010-16}{21} \right\rfloor = \left\lfloor \frac{5996}{21} \right\rfloor = \left\lfloor 285 + \frac{11}{21} \right\rfloor = 285$$

such overlaps after 16, or a total of 286 cases in which the sequences have common terms. Hence the total number of elements in S is

$$2004 + 2004 - 286 = 3722.$$ □

Exercise 5 Answer (B) Let a denote the first term of the sequence and d denote the common ratio. Then the specified conditions are that

$$17 = a_4 + a_7 + a_{10} = 3a + (3 + 6 + 9)d = 3a + 18d$$

and

$$77 = 11a + (3 + 4 + \cdots + 13)d = 11a + 88d.$$

Dividing the second equation by 11 and solving for a gives

$$a = 7 - 8d \quad \text{so} \quad 17 = 3(7 - 8d) + 18d = 21 - 6d.$$

Solving first for d and then for a produces

$$6d = 4, \quad \text{so } d = \frac{2}{3} \quad \text{and} \quad a = 7 - 8d = 7 - \frac{16}{3} = \frac{5}{3}.$$

If $a_k = 13$, we must have

$$13 = a_k = \frac{5}{3} + (k-1)\frac{2}{3}, \quad \text{so} \quad k - 1 = \frac{3}{2}\left(13 - \frac{5}{3}\right) = \frac{3}{2} \cdot \frac{34}{3} = 17.$$

Thus $k = 18$. □

Exercise 6 Answer (A) Let d be the common difference of the original arithmetic sequence, so that the sequence is given by

$$a_n = 9 + (n-1)d$$

for each positive integer n. Let r be the common ratio of the geometric sequence that results when 2 is added to the second term and 20 is added to the third term. Then this sequence has terms

$$b_1 = 9, \quad 9r = b_2 = 2 + 9 + d = 11 + d,$$

and

$$9r^2 = b_3 = 20 + 9 + 2d = 29 + 2d.$$

Hence

$$d = 9r - 11 \quad \text{and} \quad 9r^2 = 29 + 2(9r - 11) = 7 + 18r.$$

This implies that

$$0 = 9r^2 - 18r - 7 = (3r + 1)(3r - 7),$$

which has solutions $r_1 = -\frac{1}{3}$ and $r_2 = \frac{7}{3}$. The possible values for the third term of the geometric sequence are

$$9r_1^2 = 9\left(-\frac{1}{3}\right)^2 = 1 \quad \text{and} \quad 9r_2^2 = 9\left(\frac{7}{3}\right)^2 = 49,$$

and the smallest of these has the value 1. □

Exercise 7 Answer (B) This is neither an arithmetic nor a geometric sequence, so there must be some other pattern that will generate higher terms. Since all the terms of the sequence are single digit integers, let's see if there is some repeating pattern. Writing out a few more terms shows that the sequence

$$4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,\ 1,\ 3,\ 4,\ 7, \ldots$$

must repeat as soon as two consecutive terms repeat in order. Hence the sequence repeats after 12 terms, and the sum of the terms in each 12-block is

$$4 + 7 + 1 + 8 + 9 + 7 + 6 + 3 + 9 + 2 + 1 + 3 = 60.$$

Since

$$\left\lfloor \frac{10{,}000}{60} \right\rfloor = 166, \quad \text{and} \quad 10{,}000 - 166(60) = 40,$$

the sum first exceeds 10,000 after 166 blocks of 12 occur and additionally, when enough additional terms are added so that the sum exceeds 40. Because

$$4 + 7 + 1 + 8 + 9 + 7 = 36 \quad \text{and} \quad 4 + 7 + 1 + 8 + 9 + 7 + 6 = 42,$$

we need seven additional terms, and the smallest value of n for which $S_n > 10{,}000$ is $12 \cdot 166 + 7 = 1999$. □

Exercise 8 Answer (D) For Carol to win on the first round, she must toss a six after each of the others has tossed a non-six. The probability of this occurring is

$$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^2.$$

For Carol to win on the second round, there must have been 5 non-six tosses followed by a six, which occurs with probability

$$\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}=\frac{1}{6}\left(\frac{5}{6}\right)^5.$$

This pattern of possible success continues. Carol wins in the third round with probability

$$\frac{1}{6}\left(\frac{5}{6}\right)^8,$$

in the fourth with probability

$$\frac{1}{6}\left(\frac{5}{6}\right)^{11},$$

and so on. Note, in fact, that it is possible, but unlikely, that no one ever tosses a six.

The probability that Carol is the first to throw a six is the sum of probabilities that she wins in each of the individual rounds, that is,

$$\frac{1}{6}\left(\frac{5}{6}\right)^2+\frac{1}{6}\left(\frac{5}{6}\right)^5+\frac{1}{6}\left(\frac{5}{6}\right)^8+\cdots=\frac{1}{6}\left(\frac{5}{6}\right)^2\left[1+\left(\frac{5}{6}\right)^3+\left(\frac{5}{6}\right)^6+\cdots\right]$$

This is an infinite geometric series with first term

$$a=\frac{1}{6}\left(\frac{5}{6}\right)^2=\frac{25}{216}\quad\text{and ratio}\quad r=\left(\frac{5}{6}\right)^3=\frac{125}{216}.$$

Since the ratio has magnitude less than 1, the series converges and the probability that Carol wins is

$$\frac{a}{1-r}=\frac{25/216}{1-125/216}=\frac{25}{91}.\qquad\square$$

Exercise 9 Answer (B) We first convert this recursively defined sequence into a sequence form that can generate the nth term directly, that is, without referring to the previous terms in the sequence. Writing the first few terms establishes a pattern for the sequence, particularly if we don't simplify the intermediate steps:

$$a_1 = 2, \quad a_2 = a_1 + 2 \cdot 1 = 2 + 2 \cdot 1,$$
$$a_3 = a_2 + 2 \cdot 2 = 2 + 2 \cdot 1 + 2 \cdot 2 = 2 + 2(1 + 2),$$

and

$$a_4 = a_3 + 2 \cdot 3 = 2 + 2(1 + 2 + 3).$$

It appears that the sequence follows the pattern

$$a_n = 2 + 2(1 + 2 + \cdots + (n - 1)) = 2 + 2 \cdot \frac{(n-1)n}{2} = 2 + (n - 1)n.$$

Induction will verify that this is true. We know that this formula holds for the integers $n = 1, 2, 3$, and 4. If it holds for any specific integer n, then

$$a_{n+1} = a_n + 2n = 2 + (n - 1)n + 2n = 2 + n^2 - n + 2n = 2 + n(n + 1).$$

Hence the formula holds for the integer $n + 1$ whenever it holds for the integer n. When $n = 100$ we have

$$a_{100} = 2 + (99 \cdot 100) = 9902. \qquad \square$$

Exercise 10 Answer (D) The terms of this sequence will all be determined if we can find the first two terms, which for simplicity we denote $a_1 = a$ and $a_2 = b$. The first eight terms can then be expressed as

$$a_1 = a, \quad a_2 = b, \quad a_3 = a + b, \quad a_4 = a + 2b, \quad a_5 = 2a + 3b,$$
$$a_6 = 3a + 5b, \quad a_7 = 5a + 8b, \quad \text{and} \quad a_8 = 8a + 13b.$$

Since a and b are positive integers and

$$120 = a_7 = 5a + 8b, \quad b \text{ is a multiple of 5, say } b = 5k.$$

Then

$$120 = 5a + 40k \quad \text{and} \quad 24 = a + 8k.$$

But this implies that a is a multiple of 8, say $a = 8j$, and

$$24 = 8j + 8k \quad \text{which reduces to} \quad 3 = j + k.$$

For a and b to be positive integers, we must have both j and k positive integers. Hence we have only two possible cases. The first case is $j = 2$ and $k = 1$. The second case is $j = 1$ and $k = 2$.

- Case I: If $j = 2$ and $k = 1$, then $a = 8j = 16$ and $b = 5k = 5$, and the sequence is not increasing. This contradicts the statement in the problem so it is not possible.
- Case II: If $j = 1$ and $k = 2$ the sequence is increasing with

$$a_1 = a = 8j = 8, \ a_2 = b = 5k = 10,$$

and

$$a_8 = 8a + 13b = 64 + 130 = 194.$$

Note that we also have $a_7 = 5a + 8b = 40 + 80 = 120$, as specified in the statement of the problem. □

Solutions for Chapter 13: Statistics

Exercise 1 Answer (D) To have the median as large as possible we need to have the missing numbers greater than any of those on the list. If we denote these numbers by a, b, and c, and arrange the list in increasing order, then we have 3, 5, 5, 7, 8, 9, a, b, and c. The median of the list in this form is 8. □

Exercise 2 Answer (B) The sum of the first two terms is $1 + (-2) = -1$, of the first four terms is $1 + (-2) + 3 + (-4) = -2$, and, in general, the sum of the first $2n$ terms is

$$(1 + (-2)) + (3 + (-4)) + \cdots + ((2n - 1) + (-2n)) = (-1)n = -n.$$

So the average of the first $2n$ terms is $-n/(2n) = -0.5$. □

Exercise 3 Answer (D) Let N denote the number of people in the auditorium. Then the number of (people-listening)(minutes) is

$$(0.2N)(60) + (0.1N)(0) + \frac{1}{2}(0.7N)(20) + \frac{1}{2}(0.7N)(40),$$

which is equivalent to $12N + 7N + 14N = 33N$. Hence the average is $33N/N = 33$ minutes. □

Exercise 4 Answer (A) Let the number of pennies, nickels, dimes, and quarters originally in her purse be denoted p, n, d, and q, respectively. Since the original average is 20 cents we have

$$20(p+n+d+q) = 1 \cdot p + 5 \cdot n + 10 \cdot d + 25 \cdot q.$$

If one quarter is added the average is 21 cents, so

$$21(p+n+d+q+1) = 1 \cdot p + 5 \cdot n + 10 \cdot d + 25 \cdot (q+1),$$

which simplifies to

$$\begin{aligned} 21(p+n+d+q) &= (1 \cdot p + 5 \cdot n + 10 \cdot d + 25 \cdot q) + 4 \\ &= 20(p+n+d+q) + 4. \end{aligned}$$

Hence $p+n+d+q=4$, and there are four coins originally in the purse.

The total value of the coins originally in the purse is $20 \cdot 4 = 80$ cents. This sum can only be made using four coins by having three quarters and one nickel, so there were no dimes. □

Exercise 5 Answer (D) Let N denote the number of students in the class. Then the total of all the scores is $76N$. Since five students scored 100, and every student scored at least 60, when we consider the totals without the five students scoring 100 we have

$$60(N-5) \leq 76N - 5(100),$$

so

$$200 = 5 \cdot 100 - 60 \cdot 5 \leq 76N - 60N = 16N.$$

Hence $N \geq 200/16 = 12.5$. But N must be an integer, so $N \geq 13$.

One way for the average of 76 to be obtained when $N = 13$ is to have the five scores be 100 and the eight remaining scores be 61. □

Exercise 6 Answer (C) The points per game average for the sixth through ninth games was

$$\frac{1}{4}(23+14+11+20) = 17,$$

so the average points she scored for the first five games must have been less than 17. Hence the total for the first five games could be at most $5(17) - 1 =$

84. The total points for the first nine games could not exceed

$$\begin{aligned}(5 \cdot 17 - 1) + (23 + 14 + 11 + 20) &= 5 \cdot 17 - 1 + 4 \cdot 17 \\ &= 9 \cdot 17 - 1 = 152.\end{aligned}$$

Since the average for the first ten games was at least 18, which gives a final total of at least 180 points, she must have scored at least $180 - 152 = 28$ points in the final game. □

Exercise 7 Answer (D) Write the numbers in increasing order as a, b, and c. Since the median is 5, we know that $b = 5$, which implies that

$$\frac{a+5+c}{3} = a + 10, \quad \text{and that} \quad \frac{a+5+c}{3} = c - 15.$$

Simplifying these two equations gives

$$-2a + c = 25 \quad \text{and} \quad a = 2c - 50.$$

If we substitute the expression for a into the first equation, we have

$$25 = -2(2c - 50) + c = -3c + 100, \quad \text{so} \quad c = 25.$$

Then $a = 2c - 50 = 2 \cdot 25 - 50 = 0$, so the sum is $0 + 5 + 25 = 30$.

OR

Assume the notation in the first solution. Since the sum of all the terms is the product of the mean and number of terms, we have both

$$\text{Sum} = a + 5 + c = 3(a + 10) = 3a + 30$$

and

$$\text{Sum} = a + 5 + c = 3(c - 15) = 3c - 45,$$

so

$$2 \cdot \text{Sum} = 3a + 30 + 3c - 45 = 3(a + 5 + c) - 30 = 3 \cdot \text{Sum} - 30$$

and Sum $= 30$. □

Exercise 8 Answer (D) Since the median and unique mode is 8, the integers listed in increasing order must have the form a, b, c, 8, 8, d, e, and f. Also, since the mode is unique, if any of the unknown integers are not unique to the set, then at least one of c and d must be equal to 8. Further, since the mean is also 8, we must have

$$a + b + c + d + e + f = 8 \cdot 6 = 48.$$

This implies that

$$(d - 8) + (e - 8) + (f - 8) = (8 - c) + (8 - b) + (8 - a),$$

so the total excess above the mean for those greater than the mean is the same as the total deficit below the mean for those that less than the mean.

Now the range of this collection of integers being 8 comes into play. Recall that the range is the length of the smallest interval that contains all the integers in the collection. If the largest integer in the collection was 15, then the smallest integer would be $15 - 8 = 7$. This is clearly impossible since it would imply that at $7 \le a$, $7 \le b$ and $7 \le c$, so the deficit would be at most 3, whereas the excess would be at least 7. If the largest integer was 14, then the smallest would be $14 - 8 = 6$. The excess is now at least $14 - 8 = 6$, and the deficit can only be 6 if $a = b = c = 6$. But since the unique mode is 8, having $a = b = c = 6$ implies that there are at least four 8's in the collection. This is possible if and only if $d = e = 8$. Hence the largest can be 14, but only for the collection 6, 6, 6, 8, 8, 8, 8, 14.

OR

If we note that the values 6, 6, 6, 8, 8, 8, 8, 14 satisfy the requirements of the problem, then the answer is at least 14. If the largest number were 15, the collection would have the ordered form 7, __, __, 8, 8, __, __, 15. But $7 + 8 + 8 + 15 = 38$, and a mean of 8 implies that the sum of all values is 64. In this case, the four missing values would sum to $64 - 38 = 26$, and their average value would be 6.5. This implies that at least one would be less than 7, which is a contradiction. Therefore, the largest integer that can be in the set is 14. □

Exercise 9 Answer (E) We first note that

$$a_3 = \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(19 + a_2)$$

and that

$$a_4 = \frac{1}{3}(a_1 + a_2 + a_3) = \frac{1}{3}\left(19 + a_2 + \frac{1}{2}(19 + a_2)\right)$$
$$= \frac{1}{3}\left(\frac{3}{2}(19 + a_2)\right) = \frac{1}{2}(19 + a_2) = a_3.$$

This leads us to suspect that there may be equality in the terms from a_3 on. To show that this is true, suppose that for some $n \geq 3$ we have $\frac{1}{2}(19+a_2) = a_3 = a_4 = \cdots = a_n$. Then

$$\begin{aligned}
a_{n+1} &= \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \\
&= \frac{1}{n}((19 + a_2) + a_3 + \cdots + a_3) \\
&= \frac{1}{n}((19 + a_2) + (n - 2)a_3) \\
&= \frac{1}{n}\left((19 + a_2) + (n - 2)\cdot\frac{1}{2}(19 + a_2)\right) \\
&= \frac{1}{n}\left(\frac{n}{2}(19 + a_2)\right) = \frac{1}{2}(19 + a_2) = a_3,
\end{aligned}$$

and our suspicions are confirmed. As a consequence,

$$99 = a_9 = \frac{1}{2}(19 + a_2) \quad \text{and} \quad a_2 = 2 \cdot 99 - 19 = 179. \qquad \square$$

Exercise 10 Answer (E) Let the numbers on the list be denoted by

$$a_1 = 10 \leq a_2 \leq \cdots \leq m \leq \cdots \leq a_n.$$

Since the mean is 22, we have

$$22n = 10 + a_2 + \cdots + m + \cdots + a_n.$$

The mean becomes 24 when m is replaced by $m + 10$ so

$$24n = 10 + a_2 + \cdots + (m + 10) + \cdots + a_n.$$

Subtracting corresponding terms in the two equations gives $2n = 10$, so $n = 5$. As a consequence, the original list is $10 \leq a_2 \leq m \leq a_4 \leq a_5$, and

$$10 + a_2 + m + a_4 + a_5 = 5 \cdot 22 = 110.$$

We are told that the mode is 32, which must be the value of at least two of the terms of the sequence. If the mode is both a_2 and m. then the sequence would have the form 10, 32, 32, a_4, a_5 with $32 \le a_4 \le a_5$. This would imply that

$$10 + a_2 + m + a_4 + a_5 \ge 10 + 4 \cdot 32 = 138,$$

which is false, since it exceeds 110.

Suppose now that the mode is both m and a_4. Then the sequence would have the form 10, a_2, 32, 32, a_5 with $10 \le a_2$ and $32 \le a_5$. This would imply that

$$10 + a_2 + m + a_4 + a_5 \le 20 + 3 \cdot 32 = 116,$$

which is also false.

Hence the mode must be $32 = a_4 = a_5$. Thus the sequence is 10, a_2, m, 32, 32 and

$$110 = 10 + a_2 + m + a_4 + a_5 = 10 + a_2 + m + 32 + 32, \quad \text{so} \quad a_2 + m = 36.$$

Replacing m by $m - 8$ changes the median to $m - 4$, which means the ordered list is now

$$10, m - 8, a_2, 32, 32$$

and that $a_2 = m - 4$. So

$$36 = a_2 + m = (m - 4) + m,$$

which implies that $m = 20$ (and that $a_2 = 16$). □

Solutions for Chapter 14: Trigonometry

Exercise 1 **Answer (B)** Using the difference formulas for the sine and cosine produces

$$\begin{aligned}
&\sin(x - y)\cos y + \cos(x - y)\sin y \\
&\quad = (\sin x \cos y - \cos x \sin y)\cos y + (\cos x \cos y + \sin x \sin y)\sin y \\
&\quad = \sin x \left((\cos y)^2 + (\sin y)^2\right) - \cos x \sin y \cos y + \cos x \cos y \sin y \\
&\quad = \sin x \left((\cos y)^2 + (\sin y)^2\right).
\end{aligned}$$

Since $(\cos y)^2 + (\sin y)^2 = 1$ we have

$$\sin(x-y)\cos y + \cos(x-y)\sin y = \sin x.$$

OR

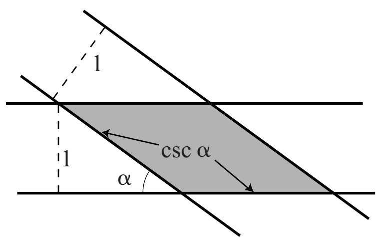
If we notice that the given expression has the form of the sum formula for the sine function, we can write

$$\sin x = \sin((x-y)+y) = \sin(x-y)\cos y + \sin y \cos(x-y).$$

This is equivalent to the given expression. □

This problem would not likely be on a current examination since it is too easy to determine the answer simply by looking at special cases. Note, for example, that when $x = y = 0$ the value is 0, which eliminates the answer choices (A), (C), and (E). In addition, the answer choice (D) is eliminated by letting $x = y = \pi/4$.

Exercise 2 Answer (B) The strips both have width 1, so the region is a rhombus with side length $\csc\alpha$.



The altitude of the rhombus is 1, so the area of the region is

$$\text{Area} = \text{Altitude} \cdot \text{Length of Base} = 1 \cdot \csc\alpha = \csc\alpha = \frac{1}{\sin\alpha}. \qquad \square$$

Exercise 3 Answer (E) Since

$$1 = (\sin x)^2 + (\cos x)^2 = (3\cos x)^2 + (\cos x)^2 = 10(\cos x)^2,$$

we have

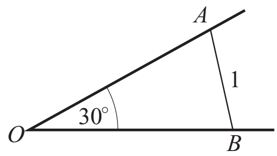
$$\cos x = \pm\frac{\sqrt{10}}{10} \quad \text{and} \quad \sin x = 3\cos x = \pm 3\frac{\sqrt{10}}{10}.$$

Since $\sin x = 3\cos x$, the sign chosen for the sine must match the sign for the cosine. Hence

$$\sin x \cos x = \left(\pm 3\frac{\sqrt{10}}{10}\right)\left(\pm\frac{\sqrt{10}}{10}\right) = \frac{3}{10}. \qquad \square$$

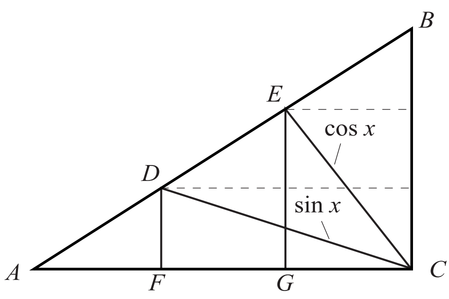
Exercise 4 **Answer (D)** We can apply the Law of Sines to $\triangle AOB$ to see that

$$\frac{OB}{\sin A} = \frac{1}{\sin 30^\circ} = \frac{1}{1/2} = 2, \quad \text{so} \quad OB = 2\sin A.$$



The maximum value for $\sin A$ is 1, which occurs when $\angle A = 90^\circ$. So the maximum value for OB is 2, which occurs when $\triangle OBA$ is a 30–60–90° triangle with right angle at A. $\square$

Exercise 5 **Answer (D)** The figure shows a right $\triangle ABC$ with right angle at C and hypotenuse AB. Let D and E be the trisection points for side $\overline{AB}$, and F and G be the points on side AC with DF and EG perpendicular to AC.



The Side-Splitter Theorem implies that in $\triangle DFC$ we have $DF = BC/3$ and $FC = 2AC/3$, and that in $\triangle EGC$ we have $EG = 2BC/3$ and

$GC = AC/3$. Applying the Pythagorean Theorem to these triangles gives the identities

$$(\sin x)^2 = \left(\frac{BC}{3}\right)^2 + \left(\frac{2AC}{3}\right)^2 \quad \text{and} \quad (\cos x)^2 = \left(\frac{2BC}{3}\right)^2 + \left(\frac{AC}{3}\right)^2.$$

Adding the corresponding terms produces

$$\begin{aligned} 1 = (\sin x)^2 + (\cos x)^2 &= \left(\frac{1}{9}BC^2 + \frac{4}{9}AC^2\right) + \left(\frac{4}{9}BC^2 + \frac{1}{9}AC^2\right) \\ &= \frac{5}{9}(BC^2 + AC^2), \end{aligned}$$

so

$$AB = \sqrt{BC^2 + AC^2} = \sqrt{\frac{9}{5}} = \frac{3\sqrt{5}}{5}.$$

Notice that we do not use the specific values of the sine and the cosine to solve this problem, only the fact that the sum of their squares is 1. Hence it is not important that our figure indicates that $\sin x < \cos x$. □

Exercise 6 Answer (B) Since $\angle BAC = 12°$, the Central Angle Theorem implies that $\angle BOC = 24°$. Also, since $\angle BXC = 36°$, the point X must lie on line segment OD. The line AD bisects $\angle BAC = 12°$, so we have $\angle BAD = 6°$. Moreover, $\triangle ABD$ is a right triangle since its hypotenuse is the diameter $\overline{AD} = 1$ of the circle. Thus $AB = \cos 6°$. Also, AD bisects $\angle BXC$, so $\angle BXD = 18°$, $\angle BXA = 180° - 18° = 162°$, and

$$\angle ABX = 180 - 6 - 162 = 12°.$$

Applying the Law of Sines to $\triangle ABX$ gives

$$\frac{AX}{\sin \angle ABX} = \frac{AB}{\sin \angle BXA},$$

so

$$AX = \frac{AB \sin 12°}{\sin 162°} = \cos 6° \sin 12° \csc 162°.$$

Since $\sin 162° = \sin 18°$, we also have $\csc 162° = \csc 18°$, which implies that

$$AX = \cos 6° \sin 12° \csc 18°.$$ □

Exercise 7 Answer (A) Because of the fundamental trigonometric relationship that the sum of the squares of the sine and cosine of any angle gives the value 1, it seems reasonable to first square and then add the corresponding terms of the given equations. Since

$$36 = 9(\sin A)^2 + 16(\cos B)^2 + 24(\sin A)(\cos B)$$

and

$$1 = 16(\sin B)^2 + 9(\cos A)^2 + 24(\sin B)(\cos A),$$

adding corresponding terms gives

$$\begin{aligned}37 &= 9\left((\sin A)^2 + (\cos A)^2\right) + 16\left((\sin B)^2 + (\cos B)^2\right)\\ &\quad + 24(\sin A \cos B + \sin B \cos A)\\ &= 25 + 24(\sin A \cos B + \sin B \cos A) = 25 + 24 \sin(A + B).\end{aligned}$$

So

$$\sin(A + B) = \frac{37 - 25}{24} = \frac{1}{2},$$

which implies that either

$$\angle(A + B) = 30^\circ \quad \text{or} \quad \angle(A + B) = 150^\circ.$$

If $\angle(A + B) = 30^\circ$, then $\angle A < 30^\circ$, which would imply that $\sin A < 1/2$ and that

$$3 \sin A + 4 \cos B < (3/2) + 4 < 6.$$

But this invalidates the first given equation. As a consequence, we must have $\angle(A + B) = 150^\circ$, which implies that $\angle C = 180^\circ - \angle(A + B) = 30^\circ$. □

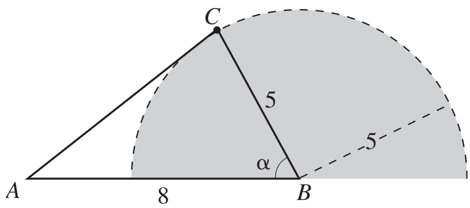
There is a unique triangle that satisfies the conditions in the problem. It has

$$\angle B = \arcsin \frac{1}{74}\left(18\sqrt{3} + 11\right) \approx 34.75^\circ$$

and

$$\angle A = 150 - \angle B \approx 115.25^\circ.$$

Exercise 8 Answer (D) The point C will lie on the semicircle of radius 5 shown in the figure.



First, we find the value of α which gives $AC = 7$. The Law of Cosines applied to $\triangle ABC$ implies that

$$7^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cos\alpha, \quad \text{so} \quad \cos\alpha = \frac{1}{80}(25 + 64 - 49) = \frac{1}{2},$$

and $\alpha = \pi/3$. Thus for $\alpha < \pi/3$ we have $AC < 7$, and when $\pi/3 \le \alpha < \pi$ we have $AC \ge 7$. So the probability that $AC < 7$ is $(\pi/3)/\pi = 1/3$. □

Exercise 9 Answer (D) The sum is an infinite geometric series

$$\sum_{n=0}^{\infty}(\cos\theta)^{2n} = 1 + (\cos\theta)^2 + (\cos\theta)^4 + \cdots$$

having first term $a = 1$ and common ratio $r = (\cos\theta)^2$. Since the series sums to 5, we have

$$5 = \sum_{n=0}^{\infty}(\cos\theta)^{2n} = \frac{a}{1-r} = \frac{1}{1-(\cos\theta)^2} = \frac{1}{(\sin\theta)^2},$$

and $(\sin\theta)^2 = 1/5$. Hence

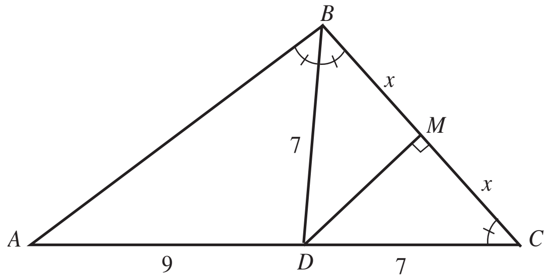
$$\cos 2\theta = 1 - 2(\sin\theta)^2 = 1 - 2 \cdot \frac{1}{5} = \frac{3}{5}.$$ □

Exercise 10 Answer (D) Let M be the midpoint of the line segment $\overline{BC}$ and $x = BM = MC$. The Angle-Bisector Theorem implies that

$$\frac{AB}{BC} = \frac{AD}{CD} = \frac{9}{7}, \quad \text{so} \quad AB = \frac{9}{7}BC = \frac{18}{7}x.$$

Because $\triangle BMD$ and $\triangle CMD$ are congruent right triangles, we have $BD = CD = 7$. In addition, since BD bisects $\angle ABC$, we have

$$\cos\angle ABD = \cos\angle DBM = \frac{BD}{7} = \frac{x}{7}.$$



Applying the Law of Cosines to $\triangle ABD$ gives another relationship involving $AB = 18x/7$. It is

$$9^2 = 7^2 + AB^2 - 2 \cdot 7 \cdot AB \cos \angle ABD = 49 + \left(\frac{18x}{7}\right)^2 - 14 \cdot \frac{18x}{7} \cdot \frac{x}{7},$$

which reduces to

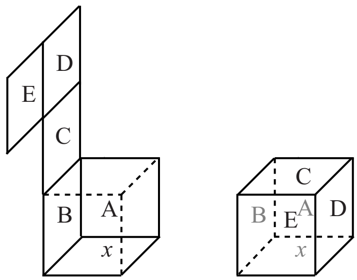
$$32 = \frac{72}{49}x^2, \quad \text{so} \quad x = \frac{14}{3}.$$

This implies that $AB = (18/7)(14/3) = 12$, which together with the fact that $AD = 9$ and $BD = 7$ permits us to use Heron's formula to find the area of $\triangle ABD$. The semi-perimeter of this triangle is $s = (12 + 9 + 7)/2 = 14$ so

$$\begin{aligned} \text{Area}(\triangle ABD) &= \sqrt{14 \cdot (14 - 12) \cdot (14 - 9) \cdot (14 - 7)} \\ &= \sqrt{14 \cdot 2 \cdot 5 \cdot 7} \\ &= 14\sqrt{5}. \end{aligned}$$ □

Solutions for Chapter 15: Three-Dimensional Geometry

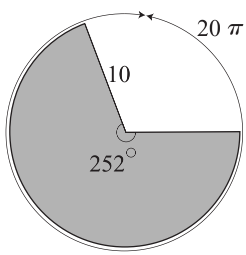
Exercise 1 **Answer (C)** Place the square marked x on the bottom and fold up side A at the back and B to the left. The situation is shown in the left figure.



Folding the remaining sides gives the figure on the right, which shows that side C must be on the top, D at the right, and E in front. So side C is opposite the face marked x. □

Exercise 2 Answer (C) The circumference of the circle from which the section is cut is $2\pi \cdot 10 = 20\pi$. The figure with the problem indicates that the portion of the circumference of the circle that makes up the side of the cone is

$$\frac{252}{360} \cdot 20\pi = 14\pi.$$

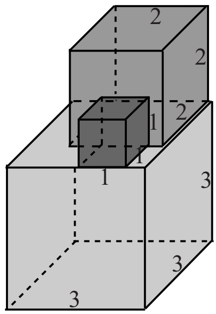


So the circle at the base of the cone has 14π as its circumference, which implies that it has radius $14\pi/(2\pi) = 7$. Since the slant height of the cone is 10, the correct figure is as shown in (C). □

Exercise 3 Answer (D) The total surface area of the three cubes before they have been placed together is

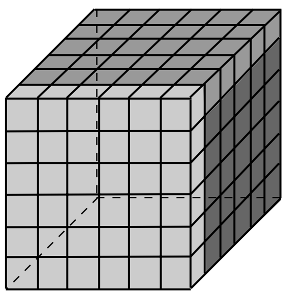
$$6 \cdot (3 \times 3) + 6 \cdot (2 \times 2) + 6 \cdot (1 \times 1) = 54 + 24 + 6 = 84 \text{ square units.}$$

Placing the cube with volume 8 on the top face of the cube with volume 27 removes $2 \times 2 = 4$ square units of the surface area from each cube. This reduces the surface area to $84 - 2 \cdot 4 = 76$ square units.



Placing the unit cube so that one of its faces adjoins the cube with volume 27 and another adjoins the cube with volume 8 reduces the surface area an additional four square units, leaving a minimal surface area of $84 - 2 \cdot 4 - 4 \cdot 1 = 72$ square units. □

Exercise 4 Answer (D) The figure shows the situation for a $6 \times 6 \times 6$ cube.



For this cube there are 6^2 cubes that can be seen on the front face, an additional $6 \cdot 5$ cubes that can be seen on the top face, and an additional 5^2 cubes that can be seen on the side face, for a total of $6^2 + 6 \cdot 5 + 5^2$ cubes. In like manner, the number that can be seen for an $11 \times 11 \times 11$ cube is

$$11^2 + 11 \cdot 10 + 10^2 = 121 + 110 + 100 = 331.$$

In general, for an $n \times n \times n$ wooden cube formed from n^3 unit cubes, the maximum number of unit cubes that have at least one face that can be seen is

$$n^2 + n(n-1) + (n-1)^2 = 3n^3 - 3n^2 + 1. \qquad \square$$

Exercise 5 Answer (D) Since no dimensions have been specified, we can assign a value to one of them and determine the others relative to that dimension. Let $GH = 1$. Since $\angle GHD = 45°$, this implies that

$$1 = GH = DG = DC = CH = BF, \quad \text{and that} \quad DH = \sqrt{2}.$$

In addition, since $\triangle HFB$ is 30–60–90° with its longest leg $BF = 1$, we have

$$BH = BD = \frac{2\sqrt{3}}{3} \quad \text{and} \quad BC = HF = \frac{1}{2}BH = \frac{\sqrt{3}}{3}.$$

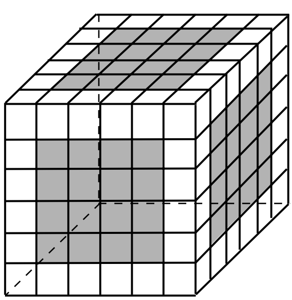
Applying the Law of Cosines to $\triangle BHD$ gives

$$BD^2 = DH^2 + BH^2 - 2 \cdot DH \cdot BH \cos \angle BHD,$$

so

$$\begin{aligned}
\cos \angle BHD &= \frac{1}{2 \cdot DH \cdot BH} \cdot \left(DH^2 + BH^2 - BD^2\right) \\
&= \frac{1}{2 \cdot \sqrt{2} \cdot \frac{2\sqrt{3}}{3}} \left((\sqrt{2})^2 + \left(\frac{2\sqrt{3}}{3}\right)^2 - \left(\frac{2\sqrt{3}}{3}\right)^2 \right) \\
&= \frac{3}{4\sqrt{6}} \cdot 2 = \frac{\sqrt{6}}{4}. \qquad \square
\end{aligned}$$

Exercise 6 Answer (E) The cubes with only one face painted are shown as shaded in the figure. Since each of the six faces has $(n-2)^2$ such unit squares, there are $6 \cdot (n-2)^2$ small cubes with exactly one face painted.

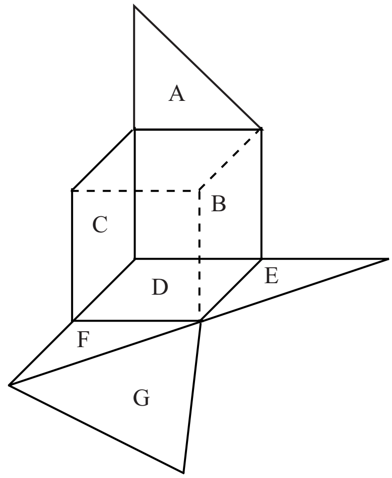


The number of interior cubes is $(n-2)^3$ so the conditions in the problem tell us that

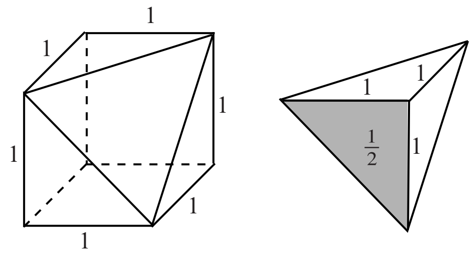
$$6 \cdot (n-2)^2 = (n-2)^3.$$

Hence $6 = n - 2$, and $n = 8$. □

Exercise 7 Answer (D) If we place square D on the bottom and fold up sides B and C we have the situation shown in the figure.

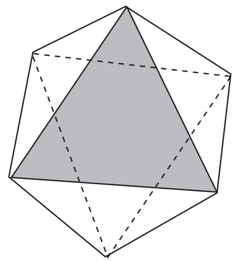


So F is the front face, E is the right face, and G is the face joining the front, right, and top faces.



The polyhedron formed is a cube from which the pyramid shown at the right has been deleted. Since the height of the pyramid is 1 and the base area is $1/2$, the pyramid has volume $(1/3) \cdot (1/2) \cdot 1 = 1/6$. So the volume of the polyhedron is $1 - 1/6 = 5/6$. □

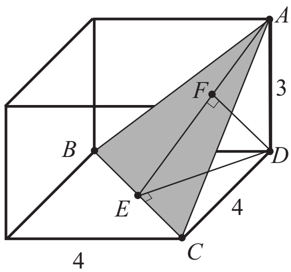
Exercise 8 Answer (E) Since there are eight faces of the octahedron there are 8! ways to color the individual faces. However, some of these are indistinguishable. Consider the shaded face in the figure.



Any of the eight faces could be in this position, and once a particular face is chosen for this position, it could be rotated in any one of three ways. Hence the number of indistinguishable ways to color the octahedron is

$$\frac{8!}{8 \cdot 3} = \frac{7!}{3} = 1 \cdot 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 1680. \qquad \square$$

Exercise 9 Answer (A) Consider the bisector E of the line segment $\overline{BC}$. Then $CE = BC/2 = 2\sqrt{2}$.



Since the base area is a square with E at its center, the diagonals of the base bisect each other and we also have $DE = CE = 2\sqrt{2}$.

Applying the Pythagorean Theorem to the right $\triangle ADE$, gives

$$AE = \sqrt{AD^2 + DE^2} = \sqrt{9+8} = \sqrt{17}.$$

Now consider the point F on AE such that DF is an altitude of right $\triangle ADE$. The Right Triangle Altitude Theorem implies that

$$\frac{DF}{AD} = \frac{DE}{AE}, \quad \text{so} \quad DF = AD \cdot \frac{DE}{AE} = 3 \cdot \frac{2\sqrt{2}}{\sqrt{17}} = \frac{6\sqrt{34}}{17}.$$

OR

We can also find DF by using the fact that it is the altitude of the tetrahedron $ABCD$ if we use the base ABC. Hence the volume of the tetrahedron is

$$\begin{aligned} V &= \frac{1}{3} DF \cdot \text{Area}(\triangle ABC) = \frac{1}{3} DF \cdot \left(\frac{1}{2} BC \cdot EA\right) \\ &= \frac{1}{6} \cdot 4\sqrt{2} \cdot \sqrt{17} DF = \frac{2\sqrt{34}}{3} DF. \end{aligned}$$

But this volume can also be computed by using the altitude AD and the base triangle BCD. Hence

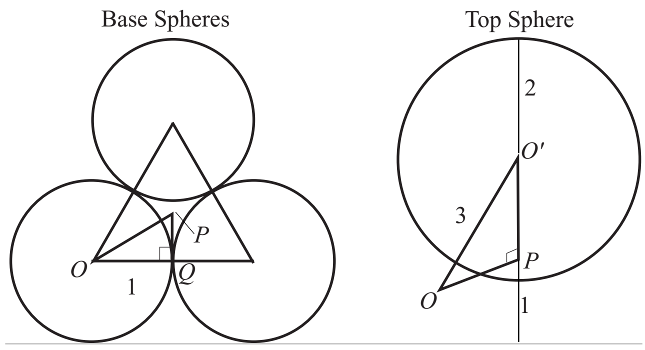
$$V = \frac{1}{3} \cdot 3 \cdot \text{Area}(\triangle BCD) = \frac{1}{2}\left(4^2\right) = 8.$$

As a consequence,

$$8 = \frac{2\sqrt{34}}{3} DF \quad \text{and} \quad DF = \frac{12}{\sqrt{34}} = \frac{6\sqrt{34}}{17}.$$

To see that DF is indeed the shortest distance to the plane, note that if a sphere were drawn centered at D of radius greater than DF, then it would intersect the plane as a circle whose center is at F. □

Exercise 10 Answer (B) The figure at left shows a view of the cross-sections of the three base spheres. The center of the sphere resting on this



base will be above the point P. Let O represent the center of one of the base spheres and Q be the midpoint of the line segment joining two of the centers of the base spheres, as shown. Since the longer leg of the 30–60–90° $\triangle PQO$ is 1, the hypotenuse is $OP = 2\sqrt{3}/3$.

The figure on the right shows the right triangle formed by O, the center O' of the large sphere, and the point P. Since $OO' = 3$, the sum of the radii of the spheres, and $OP = 2\sqrt{3}/3$, we have

$$PO' = \sqrt{3^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}.$$

But P is on the same level as the centers of the base spheres, which are 1 unit above the plane, and the center O' is 2 units from the top of the large sphere. So the distance from the plane to the top of the large sphere is

$$1 + \frac{\sqrt{69}}{3} + 2 = 3 + \frac{\sqrt{69}}{3}. \qquad \square$$

Solutions for Chapter 16: Functions

Exercise 1 Answer (E) We would like to have the description of the function in terms of x rather than $2x$, since the answer asks for it in this form. Let $y = 2x$. Then $x = y/2$ and

$$f(y) = \frac{2}{2 + y/2} = \frac{4}{4 + y} \quad \text{or equivalently} \quad f(x) = \frac{4}{4 + x}.$$

As a consequence, $\quad 2f(x) = 2 \cdot \dfrac{4}{4 + x} = \dfrac{8}{4 + x}.$ $\square$

Exercise 2 Answer (B) Since k is an odd integer, we have $f(k) = k + 3$, which is an even integer. So

$$f(f(k)) = f(k + 3) = \frac{k + 3}{2}.$$

But now there are two possibilities to consider, since we have no way of knowing if this last value is even or odd.

If $f(f(k))$ is even, then

$$27 = f(f(f(k))) = \frac{(k + 3)/2}{2} = \frac{k + 3}{4}, \quad \text{so} \quad k = 4 \cdot 27 - 3 = 105.$$

If $f(f(k))$ is odd, then

$$27 = f(f(f(k))) = \frac{k+3}{2} + 3, \quad \text{so} \quad k = (2 \cdot (27-3)) - 3 = 45.$$

To see which of these values is correct, we will reapply a recursive formula.

For $x = 105$: $f(f(f(105))) = f(f(108)) = f(54) = 27$, which is correct.

But for $x = 45$: $f(f(f(45))) = f(f(48)) = f(24) = 12$, which is incorrect.

So the correct value is $k = 105$, whose digits sum to 6. □

Exercise 3 Answer (A) First notice that for each real number x we have

$$\begin{aligned} f(-x) &= a(-x)^7 + b(-x)^3 + c(-x) - 5 \\ &= -(ax^7 + bx^3 + cx - 5) - 10 = -f(x) - 10. \end{aligned}$$

So

$$f(x) = -f(-x) - 10,$$

and

$$f(7) = f(-(-7)) = -f(-7) - 10 = -(7) - 10 = -17. \qquad \square$$

Exercise 4 Answer (E) Suppose that r is a root. Then since we know that $f(2+x) = f(2-x)$ for all x, we have

$$0 = f(r) = f(2 + (r-2)) = f(2 - (r-2)) = f(4-r),$$

so $4 - r$ is also a root, and the sum of this pair is $r + (4-r) = 4$. Since there are two pairs of roots of this type, the sum of the four roots is 8. □

Note that we cannot have $r = 2$ as one of the roots of $f(x) = 0$, since if this were the case, we would have one root satisfying $r = 4 - r$, but the others would be in pairs. This would mean that there would be an odd number of distinct roots, and four is most certainly even. An elementary function satisfying the property given in the statement of the exercises is $f(x) = x(x-1)(x-3)(x-4)$.

Exercise 5 Answer (A) Since $f(f(x)) = x$ we have

$$x = \frac{cf(x)}{2f(x)+3} = \frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+c}\right)+3} = \frac{c^2x}{2cx+6x+9}$$

so

$$2cx^2 + 6x^2 + 9x = c^2x, \quad \text{and} \quad 0 = (2c+6)x^2 + (9-c^2)x.$$

The coefficient of x^2 is 0 when $c = 3$ or $c = -3$. But the coefficient of x is 0 only when $c = -3$. So the only value of c that satisfies this equation for all values of x is $c = -3$. □

Since the statement $f(f(x)) = x$ must be true for all values of x, it is tempting to simply try some easy values to see if they will tell us what c must be. The easiest value to try is $x = 0$, but $f(f(0)) = 0$ regardless of the value of c. If we try $x = 1$ we obtain

$$1 = f(f(1)) = f\left(\frac{c}{5}\right) = \frac{c^2/5}{2c/5+3} = \frac{c^2}{2c+15},$$

and

$$0 = c^2 - 2c - 15 = (c-5)(c+3).$$

This equation has two solutions, $c = -3$ and $c = 5$. Unfortunately both are answer choices. We would need to try some other value of x to try to eliminate one of these possibilities. This is probably not worth the effort; it is easier to simply solve the problem in general.

Exercise 6 Answer (B) We first rewrite the function in terms of a simple variable such as y, rather than in terms of x^2+1. Let $y = x^2+1$ and solve for x in terms of y. However, this gives $x = \pm\sqrt{y-1}$, and there could be difficulty resolving which sign is appropriate. In fact, there will not be a problem since the powers of x in the description of the problem are all even.

$$\begin{aligned} f(y) = f\left(x^2+1\right) &= \left(\pm\sqrt{y-1}\right)^4 + 5\left(\pm\sqrt{y-1}\right)^2 + 3 \\ &= (y-1)^2 + 5(y-1) + 3 = y^2 + 3y - 1. \end{aligned}$$

So

$$f\left(x^2-1\right)=\left(x^2-1\right)^2+3\left(x^2-1\right)-1=x^4+x^2-3.$$

OR

As a slight modification, we could first note that

$$\left(x^2+1\right)^2=x^4+2x^2+1,$$

so

$$\begin{aligned}x^4+5x^2+3&=(x^4+2x^2+1)+3x^2+2\\&=\left(x^2+1\right)^2+3(x^2+1)-1.\end{aligned}$$

Now let $y=x^2+1$ to give

$$f(y)=f(x^2+1)=\left(x^2+1\right)^2+3(x^2+1)-1=y^2+3y-1,$$

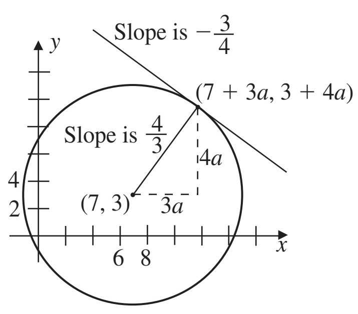
and finish the problem as done previously. □

Exercise 7 Answer (B) We have quadratics in both x and y so we first complete the square on these terms. From this we see that the set of points (x, y) that satisfy $x^2+y^2=14x+6y+6$ also satisfy the equation

$$(x^2-14x+49)+(y^2-6y+9)=6+49+9,$$

that is

$$(x-7)^2+(y-3)^2=64.$$



These are the points on the circle with radius 8 centered at $(7, 3)$. For a given constant c, the set of points that satisfy $3x + 4y = c$, or equivalently, $y = -(3/4)x + c/4$, lie on the line with slope $-3/4$ and y-intercept $c/4$. So, considered geometrically, we need to find the line with slope $-3/4$ that intersects the circle and has the largest value for its y-intercept.

This line will be tangent to the curve, as shown in the figure, and the line segment that is the radius to this tangent point has slope $4/3$. If we label the tangent point as $(7 + 3a, 3 + 4a)$, then we have

$$64 = (3a)^2 + (4a)^2 = 25a^2 \quad \text{and} \quad a = \frac{8}{5}.$$

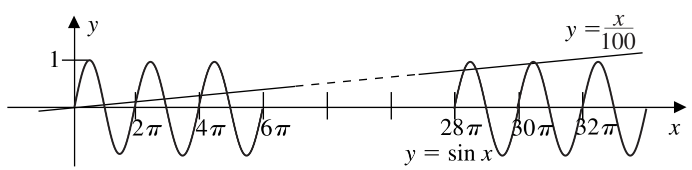
So the point that gives the maximum value of $3x + 4y$ is $(7 + 24/5, 3 + 32/5) = (59/5, 47/5)$, and this maximum value is

$$3 \cdot \frac{59}{5} + 4 \cdot \frac{44}{5} = \frac{177 + 188}{5} = \frac{365}{5} = 73. \qquad \square$$

Exercise 8 Answer (C) First note that if x is a solution to this equation, then so is $-x$, since both

$$\frac{-x}{100} = -\frac{x}{100} \quad \text{and} \quad \sin(-x) = -\sin x.$$

In addition, $x = 0$ is a solution, since both sides of the equation are zero. As a consequence, the number of solutions is odd, and we can restrict our search to the number of positive solutions.



The maximum value for the sine function is 1, so we will have no solutions when $x > 100$. The figure shows that there is one positive solution in the interval $(0, 2\pi)$, two solutions in $(2\pi, 3\pi)$, and two solutions in the first half of each succeeding interval of width 2π until we get to 100. Since

$$31\pi \approx 97.4 < 100 < 32\pi, \quad \text{and} \quad 31\pi = 15 \cdot 2\pi + \pi,$$

there are $2 \cdot 15 + 1 = 31$ positive solutions. Hence the total number of solutions is $2 \cdot 31 + 1 = 63$. $\square$

Exercise 9 Answer (C) First note that $P(-1) \approx 4.5$. From Chapter 2, we know that the product of the zeros is d, which is also the y-intercept of the graph, approximately 5. Also, the real zeros are approximately 1.7 and 3.7, whose sum is approximately 5.4. In addition, the sum of the coefficients of P is $P(1) \approx 4$. So of the choices (A), (B), (D), and (E), the smallest is (D), the sum of the coefficients of P, and this value is approximately 4.

To determine the approximate value of choice (C), note that since the product of all the zeros of P is d, which we found to be approximately 5, and the product of the real zeros is approximately $(1.7)(3.7) \approx 6$, the product of the non-real zeros must be approximately $5/6$. So (C) is the smallest of the choices. □

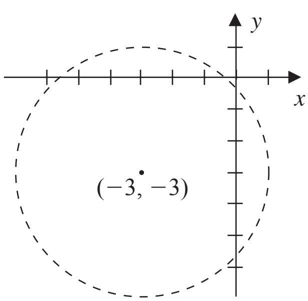
Exercise 10 Answer (E) Since a quadratic is involved, we will first complete the square. Note that

$$\begin{aligned} f(x) + f(y) &= x^2 + 6x + 1 + y^2 + 6y + 1 \\ &= (x^2 + 6x + 9) + (y^2 + 6y + 9) - 16 \\ &= (x+3)^2 + (y+3)^2 - 16. \end{aligned}$$

In addition,

$$\begin{aligned} f(x) - f(y) &= x^2 + 6x + 1 - y^2 - 6y - 1 \\ &= (x^2 - y^2) + 6(x - y) = (x - y)(x + y + 6). \end{aligned}$$

To satisfy $f(x) + f(y) \le 0$, the point (x, y) must lie inside the circle with center at $(-3, -3)$ and radius 4.



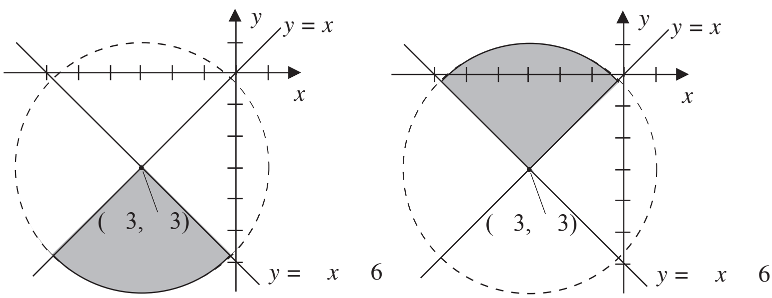
To satisfy $f(x) - f(y) \le 0$, the factors of $(x - y)(x + y + 6)$ must differ in sign. This implies that we must have either

$$\text{Case I:} \quad x - y \geq 0 \ \text{ and } \ x + y + 6 \leq 0$$

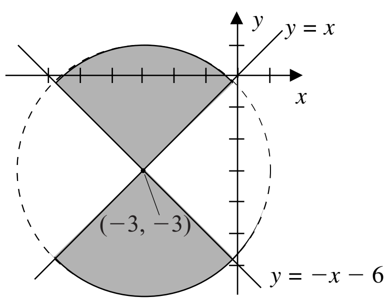
or

$$\text{Case II:} \quad x - y \leq 0 \ \text{ and } \ x + y + 6 \geq 0.$$

For Case I the condition that $x - y \geq 0$ implies that this region lies to the right of the line $y = x$, and the condition $x + y + 6 \leq 0$ implies that this region lies to the left of the line $y = -x - 6$. As a consequence, the portion of this region that lies within the circle is as shown on the left in the figure below.



For Case II the condition $x - y \leq 0$ implies that this region lies to the left of the line $y = x$, and the condition $x + y + 6 \geq 0$ implies that this region lies to the right of the line $y = -x - 6$. As a consequence, the portion of this region that lies within the circle is as shown on the right in the figure above.



Hence the total region R is within the circle with center $(-3, -3)$ and radius 4, and is bounded by the lines $y = x$ and $y = -x - 6$, as shown in the figure above. It consists of two quarter circles of radius 4, so its area is

$$\frac{1}{2}\cdot \pi 4^2 = 8\pi \approx 8(3.14) \approx 25.1. \qquad \square$$

Solutions for Chapter 17: Logarithms

Exercise 1 Answer (A) First note that

$$\log_2(\log_3(\log_5(\log_7 N))) = 11 \quad \text{implies that} \quad \log_3(\log_5(\log_7 N)) = 2^{11}.$$

Then

$$\log_3(\log_5(\log_7 N)) = 2^{11} \quad \text{implies that} \quad \log_5(\log_7 N) = 3^{2^{11}},$$

that

$$\log_5(\log_7 N) = 3^{2^{11}} \quad \text{gives} \quad \log_7 N = 5^{3^{2^{11}}},$$

and finally, that

$$\log_7 N = 5^{3^{2^{11}}} \quad \text{is equivalent to} \quad N = 7^{5^{3^{2^{11}}}}.$$

In summary then, N has only the prime factor 7. $\square$

Exercise 2 Answer (A) First unravel this logarithm expression.

$$\log_2(\log_2(\log_2 x)) = 2 \quad \text{implies that} \quad \log_2(\log_2 x) = 2^2 = 4,$$

so

$$\log_2(\log_2 x) = 4 \quad \text{implies that} \quad \log_2 x = 2^4 = 16,$$

and

$$\log_2 x = 16 \quad \text{implies that} \quad x = 2^{16}.$$

There are numerous ways to estimate the size of 2^{16}. Suppose you know that $2^{10} = 1024$. (This is denoted as K in computer terms.) Then $2^{16} = 2^{10}\cdot 2^6 = 1024\cdot 64 \approx 64{,}000$. So the number of digits in the base-10 representation of 2^{16} must be 5.

We could also note that $2^{16} = 4^8 = 16^4 = 256^2$. Since both $200^2 = 40{,}000$ and $300^2 = 90{,}000$ both have 5-digit representations, so must $256^2 = 2^{16}$. $\square$

Exercise 3 Answer (E) For $\log_b 729$ to be a positive integer, it must be the case that $729 = 3^6$ is a positive integer power of b. This is true only for $b = 3$, $b = 3^2 = 9$, $b = 3^3 = 27$, and $b = 3^6 = 729$. So there are four possibilities for b. □

Exercise 4 Answer (D) We have

$$\begin{aligned} N &= f(11) + f(13) + f(14) \\ &= \log_{2002} 11^2 + \log_{2002} 13^2 + \log_{2002} 14^2 \\ &= 2\left(\log_{2002} 11 + \log_{2002} 13 + \log_{2002} 14\right) \\ &= 2\left(\log_{2002} 11 \cdot 13 \cdot 14\right) = 2 \log_{2002} 2002. \end{aligned}$$

Since $\log_{2002} 2000 = 1$, we have $N = 2$. □

Exercise 5 Answer (A) Let the three distinct positive roots be denoted r_1, r_2, and r_3. Then by the Factor Theorem we have

$$0 = 8x^3+4ax^2+2bx+a = 8(x-r_1)(x-r_2)(x-r_3), \quad \text{so} \quad a = -8r_1r_2r_3.$$

Since we are given that

$$\log_2 r_1r_2r_3 = 5, \quad \text{we also have} \quad r_1r_2r_3 = 2^5 = 32.$$

Hence $a = -8(32) = -256$. □

Exercise 6 Answer (C) Result 6 of Section 17.2 states that $\log_8 b$ is rational if and only if $b = 8^r$ for some rational number r. Since $2 = 8^{1/3}$, this implies that integers n for which $f(n) \neq 0$ are those of the form $8^{m/3}$ for some positive integer m. The collection consists of

$$2 = 8^{1/3},\ 4 = 8^{2/3} = 2^2,\ 8 = 2^3,\ 16 = 2^4,\ 32 = 2^5,\ \ldots,\ 512 = 2^9,$$

and $1024 = 2^{10}$.

So

$$\sum_{n=1}^{1997} f(n) = \log_8 2 + \log_8 4 + \log_8 8 + \log_8 16 + \cdots + \log_8 512 + \log_8 1024,$$

$$= \log_8 8^{1/3} + \log_8 8^{2/3} + \log_8 8 + \log_8 8^{4/3} + \cdots$$
$$+ \log_8 8^{10/3} + \log_8 1024,$$
$$= \frac{1}{3}(1+2+3+\cdots+10) = \frac{1}{3}\cdot\frac{1}{2}(10\cdot 11) = \frac{55}{3}. \qquad \square$$

We could also compute this last value by noting that

$$\sum_{n=1}^{1997} f(n) = \log_8 2 + \log_8 4 + \log_8 8 + \log_8 16 + \cdots$$
$$+ \log_8 512 + \log_8 1024$$
$$= \log_8(2\cdot 4\cdot 8\cdots 1024)$$
$$= \log_8\left(8^{1/3}\cdot 8^{2/3}\cdot 8^{3/3}\cdots 8^{10/3}\right) = \log_8 8^{55/3} = \frac{55}{3}.$$

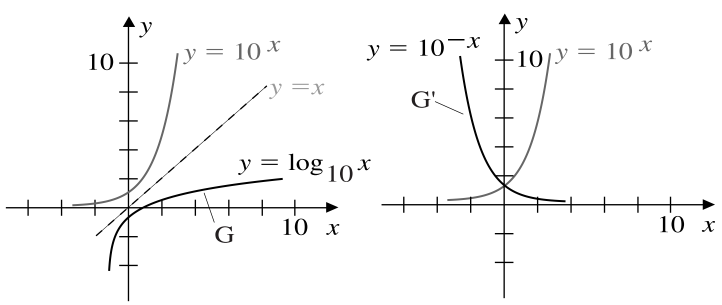
Exercise 7 Answer (C) The basic logarithm properties given in Results 5 and 3 of Section 17.2 can be used to change this expression into

$$N = \frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}$$
$$= \log_{100!} 2 + \log_{100!} 3 + \log_{100!} 4 + \cdots + \log_{100!} 100$$
$$= \log_{100!}(2\cdot 3\cdot 4\cdots 100) = \log_{100!} 100!,$$

so $N = 1$. $\square$

Exercise 8 Answer (D) In the figure shown at the left, we have the graphs of $y = 10^x$, and the reflection of that graph about the line $y = x$, which gives the graph of $y = \log_{10} x$.

The graph of $y = \log_{10} x$ contains the point $(1, 0)$, $(10, 1)$, and $(1/10, -1)$. Rotating this graph counter-clockwise by 90° gives the dark



graph on the figure at the right. It represents the function $f(x) = 10^{-x}$ and contains the points $(-1, 10)$, $(0, 1)$, and $(1, 1/10)$. So the answer is (D).

To see that the other choices cannot be true, note that on the rotated graph $y \to 0$ as $x \to \infty$, which eliminates choices (A) and (E). Also, $y = 10$ when $x = -1$, which eliminates choices (B) and (C), since neither are defined at $x = -1$. □

Exercise 9 Answer (A) First note that for any x in $(0, \pi/2)$ we can use the fact that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$ to deduce that

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin(\pi/2 - x)}{\cos(\pi/2 - x)} = \frac{\sin(\pi/2)\cos x - \sin x \cos(\pi/2)}{\cos(\pi/2)\cos x + \sin(\pi/2)\sin x}$$
$$= \frac{\cos x}{\sin x} = \frac{1}{\tan x}.$$

So for any x in $(0, \pi/2)$ we have

$$\tan x \cdot \tan(\pi/2 - x) = 1.$$

Now re-express the sum to take advantage of this fact, that is, as

$$\begin{aligned} S &= \log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ) \\ &= \log_{10}\left(\tan 1^\circ \cdot \tan 2^\circ \cdots \tan 88^\circ \cdot \tan 89^\circ\right) \\ &= \log_{10}\left((\tan 1^\circ \cdot \tan 89^\circ)(\tan 2^\circ \cdot \tan 88^\circ)\cdots(\tan 44^\circ \cdot \tan 46^\circ)\cdot \tan 45^\circ\right) \\ &= \log_{10}\left(1 \cdot 1 \cdots 1 \cdot \tan 45^\circ\right) \\ &= \log_{10} \tan 45^\circ. \end{aligned}$$

So $S = \log_{10} \tan 45^\circ = \log_{10} 1 = 0.$ □

Exercise 10 Answer (B) The basic properties of logarithms imply that

$$\begin{aligned} \log_a \frac{a}{b} + \log_b \frac{b}{a} &= \log_a a - \log_a b + \log_b b - \log_b a \\ &= 1 - \log_a b + 1 - \frac{1}{\log_a b} \\ &= \frac{2\log_a b - \left(\log_a b\right)^2 - 1}{\log_a b} \\ &= -\frac{\left(\log_a b - 1\right)^2}{\log_a b}. \end{aligned}$$

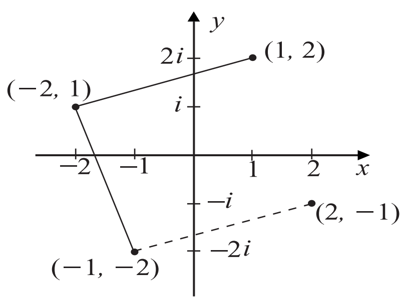
Since $a \geq b > 1$ the denominator is positive and the quantity is negative except when $\log_a b - 1 = 0$. Hence the maximum value of the quantity is 0, which occurs if and only if $\log_a b = 1$, that is, if and only if $a = b$. □

A note for those who have some knowledge of calculus. If we let $x = \log_a b$, then what we need to do is maximize the function $f(x) = 2 - x - 1/x$. If $0 = f'(x) = -1 + 1/x^2$, then $x = \pm 1$. This implies either $a = b$ or $a = 1/b$. But since both a and b are at least 1, only $a = b$ satisfies. Hence $x = \log_a b = 1$ and $f(1) = 0$.

Calculus is not required for any of the AMC problems, and the problem posers try to ensure that any calculus solution will be at least as hard as some non-calculus solution. On the other hand, if you have this powerful tool, you can certainly use it.

Solutions for Chapter 18: Complex Numbers

Exercise 1 Answer (B) We can put this in a more familiar setting by recognizing that a complex number is graphed with its real part on the horizontal (or x-) axis and its imaginary part on the vertical (or y-) axis. So the complex numbers we are given are shown on the xy-plane.



The line segment joining $(-2, 1)$ and $(1, 2)$ increases three units in the x-direction and 1 unit in the y-direction. The point at the end of the parallel line segment that begins at $(-1, -2)$ is $(-1+3, -2+1) = (2, -1)$, which represents the complex number $2 - i$.

Notice that the given pair of points $(1, 2)$ and $(-1, -2)$ is symmetric with respect to the origin, as is the given point $(-2, 1)$ and the new point $(2, -1)$. So the complex number is $2 - i$. □

Exercise 2 Answer (C) The reciprocal of a complex number $z \neq 0$ can be expressed as

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2},$$

so the reciprocal of F has the same argument as its complex conjugate. Since F is in the first quadrant, the reciprocal must be in the fourth quadrant. This implies that only C and A are possible candidates. Since F lies outside the unit circle, its magnitude exceeds 1, so its reciprocal must have magnitude less than 1. This eliminates A, leaving C as the only possibility. □

Since no specific number in the first quadrant and outside the unit circle is given, you could also solve the problem by taking a specific example and seeing where its reciprocal lies. For example, if $z = 1 + i$, then

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{1-i}{2}.$$

So $1/z$ lies in the fourth quadrant and is inside the unit circle, since $|1-i| = \sqrt{2} < 2$.

Exercise 3 Answer (B) As with many sequence problems, we hope to find some repeating pattern. Computing the first few terms gives

$$z_1 = 0,\ z_2 = i,\ z_3 = i^2 + i = -1 + i,\ z_4 = (-1+i)^2 + i = -i,$$

and

$$z_5 = (-i)^2 + i = -1 + i = z_3.$$

So for $n \geq 3$, the odd terms of the sequence are $-1 + i = z_3$, and the even terms are $-i = z_4$. Since we are interested in an odd term, we have $|z_{2005}| = |-1+i| = \sqrt{2}$. □

Exercise 4 Answer (A) Suppose that $z = x + iy$ is a number in the set S. Then

$$(3+4i)z = (3+4i)(x+iy) = (3x+4y) + (4x+3y)i.$$

For this number to be real, we must have $0 = 4x + 3y$. This describes a line through the origin (with slope $-4/3$). □

Exercise 5 Answer (C) The argument of $i+1$ is $\pi/4$ and its magnitude is $|i+1| = \sqrt{2}$. The argument of $i-1$ is $3\pi/4$ and its magnitude is $|i-1| = \sqrt{2}$. Applying De Moivre's Theorem gives

$$\begin{aligned}(1+i)^{2008} &= \left(\sqrt{2}\right)^{2008}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2008}\\ &= 2^{1004}\left(\cos\left(\frac{2008}{4}\pi\right) + i\sin\left(\frac{2008}{4}\pi\right)\right)\\ &= 2^{1004}\left(\cos(502\pi) + i\sin(502\pi)\right)\\ &= 2^{1004}\end{aligned}$$

and

$$\begin{aligned}(1-i)^{2008} &= \left(\sqrt{2}\right)^{2008}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{2008}\\ &= 2^{1004}\left(\cos(3\cdot 502\pi) + i\sin(3\cdot 502\pi)\right)\\ &= 2^{1004}.\end{aligned}$$

Hence

$$(1+i)^{2008} - (1-i)^{2008} = 0. \qquad \square$$

We could also solve this problem by first noticing that both $1+i$ and $1-i$ are 8th roots of 16, so $(1+i)^8$ and $(1-i)^8$ are both 16. Since $2008 = 251\cdot 8$, $(1+i)^{2008}$ and $(1-i)^{2008}$ are also the same.

Exercise 6 Answer (D) It is probably best to first rewrite this series using radian notation instead of degrees. Then the value of the series is

$$\begin{aligned}S &= \sum_{n=0}^{40} i^n\cos(45+90n)^\circ = \sum_{n=0}^{40} i^n\cos\left(\frac{\pi}{4} + \frac{\pi}{2}n\right)\\ &= \sum_{n=0}^{40} i^n\cos\left(\frac{2n+1}{4}\pi\right).\end{aligned}$$

Now use the fact that

$$i^n = \begin{cases} 1, & \text{if } n \equiv 0 \bmod 4,\\ i, & \text{if } n \equiv 1 \bmod 4,\\ -1, & \text{if } n \equiv 2 \bmod 4,\\ -i, & \text{if } n \equiv 3 \bmod 4.\end{cases}$$

Also, $\cos\frac{n\pi}{4} = 0$ when n is even, and

$$\cos\frac{n\pi}{4} = \begin{cases} \frac{\sqrt{2}}{2}, & \text{if } n \equiv 1 \bmod 8, \\ -\frac{\sqrt{2}}{2}, & \text{if } n \equiv 3 \bmod 8, \\ -\frac{\sqrt{2}}{2}, & \text{if } n \equiv 5 \bmod 8, \\ \frac{\sqrt{2}}{2}, & \text{if } n \equiv 7 \bmod 8. \end{cases}$$

Gather together the real and imaginary parts of the series and simplify the result:

$$\begin{aligned} S &= \left(\cos\frac{\pi}{4} - \cos\frac{5\pi}{4} + \cos\frac{9\pi}{4} + \cdots - \cos\frac{81\pi}{4}\right) \\ &\quad + i\left(\cos\frac{3\pi}{4} - \cos\frac{7\pi}{4} + \cos\frac{11\pi}{4} + \cdots - \cos\frac{79\pi}{4}\right) \\ &= 21\left(\frac{\sqrt{2}}{2}\right) + 20i\left(-\frac{\sqrt{2}}{2}\right). \end{aligned}$$

So $S = \frac{\sqrt{2}}{2}(21 - 20i)$. □

It would be easy to miscount the number of terms in the sum that gives the real part and come up with the incorrect result $10\sqrt{2}(1-i)$. Note, however, that this is not one of the answer choices. This value has not been included as an answer choice because it would penalize students for making a minor error on a hard problem.

Exercise 7 Answer (E) If we write z in terms of its real and imaginary parts, x and y, we see that

$$2 + 8i = x + iy + \sqrt{x^2+y^2} = \left(x + \sqrt{x^2+y^2}\right) + iy,$$

so

$$8 = y \quad \text{and} \quad 2 = x + \sqrt{x^2+y^2} = x + \sqrt{x^2+64}.$$

Squaring each side of the equation $2 - x = \sqrt{x^2+64}$ gives

$$4 - 2x + x^2 = x^2 + 64 \quad \text{so} \quad x = -15.$$

The squaring operation could have produced an *extraneous* solution, so we need to check that -15 is a solution to the original equation, which it is, since both

$$2-(-15)=17 \quad \text{and} \quad \sqrt{(-15)^2+64}=\sqrt{225+64}=\sqrt{289}=17.$$

Hence

$$|z|^2=x^2+y^2=(-15)^2+8^2=225+64=289. \qquad \square$$

Exercise 8 Answer (D) The reciprocal of a complex number z is

$$\frac{1}{z}=\frac{\overline{z}}{|z|^2}.$$

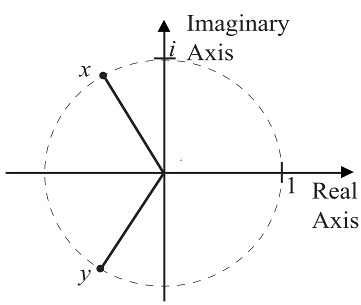
Since all the zeros of $P(x)$ lie on the unit circle, each of their magnitudes is 1. This implies that the reciprocal of each zero is simply its complex conjugate. Also, the sum of the zeros is the real number $-a$, so the sum of the imaginary parts of the zeros is 0. As a consequence, the sum of the complex conjugates of the zeros is the same as the sum of the zeros, which is $-a$. $\square$

Exercise 9 Answer (C) Since x and y are complex conjugates, their sum is just the sum of their real parts, -1. Write x and y in polar form as

$$x=\frac{-1+i\sqrt{3}}{2}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

and

$$y=\frac{-1-i\sqrt{3}}{2}=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}.$$



Now use De Moivre's formula to conclude that

$$\begin{aligned} x^2 &= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = y, \\ y^2 &= \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = x, \\ x^3 &= \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} = 1, \text{ and} \\ y^3 &= \cos\frac{12\pi}{3} + i\sin\frac{12\pi}{3} = 1. \end{aligned}$$

As a consequence, $x^n + y^n$ is the same as $x + y = -1$ unless n is a multiple of 3, in which case the sum is 2. Hence only (C) is incorrect. □

Exercise 10 Answer (B) First write the roots of this quadratic equation as

$$r_1 = a_1 + ib_1 \quad \text{and} \quad r_2 = a_2 + ib_2.$$

Then

$$0 = z^2 - z + (-5 + 5i) = (z - (a_1 + ib_1))(z - (a_2 + ib_2)).$$

The fact that the sum of the roots is the negative of the linear term implies that

$$-(-1) = 1 = a_1 + ib_1 + a_2 + ib_2 = (a_1 + a_2) + i(b_1 + b_2).$$

Comparing the real and imaginary parts gives

$$1 = a_1 + a_2 \quad \text{and} \quad 0 = b_1 + b_2.$$

Also, the constant term is the product of the roots, and using the fact that $b_2 = -b_1$ gives

$$\begin{aligned} -5 + 5i &= (a_1 + ib_1)(a_2 + ib_2) = (a_1 + ib_1)(a_2 - ib_1) \\ &= a_1 a_2 + b_1^2 + i(a_2 - a_1)b_1. \end{aligned}$$

Matching the real and imaginary parts in this result gives the two equations

$$a_1 a_2 = -5 - b_1^2 \quad \text{and} \quad b_1 = \frac{5}{a_2 - a_1}.$$

We will square this second equation and substitute the resulting value of b_1^2 into the first equation. At first this might not seem promising, but recall that we also know that $a_1 + a_2 = 1$. We have

$$\begin{aligned} b_1^2 &= \frac{5^2}{(a_2 - a_1)^2} \\ &= \frac{25}{a_2^2 - 2a_1a_2 + a_1^2} \\ &= \frac{25}{(a_2^2 + 2a_1a_2 + a_1^2) - 4a_1a_2} = \frac{25}{(a_2 + a_1)^2 - 4a_1a_2} = \frac{25}{1 - 4a_1a_2}. \end{aligned}$$

Hence we have

$$a_1a_2 = -5 - b_1^2 = -5 - \frac{25}{1 - 4a_1a_2},$$

which reduces to the quadratic equation

$$0 = 4(a_1a_2)^2 + 19(a_1a_2) - 30 = (a_1a_2 + 6)(4a_1a_2 - 5).$$

So either $a_1a_2 = -6$ or $a_1a_2 = 5/4$. The latter solution, however, is *extraneous* since we know that $a_1a_2 = -5 - b_1^2 < 0$. Hence $a_1a_2 = -6$. $\square$

Epilogue

This brief section contains some references to problem-solving material for students on the high school level that either supplements what I have given here or goes beyond the techniques I have described. There is a vast amount of problem-solving material, and if you are seriously interested in this activity you will no doubt find a favorite that I have not included here. These are simply a few of my personal favorites. The goal is not to be inclusive, but simply to provide paths for you to begin your journey.

Many of the books I will list are published by the Mathematical Association of America (MAA), which administers the AMC. Any of these can be obtained from the MAA Bookstore. Simply go to the web site

www.maa.org

and search under Problem Solving. They can generally also be purchased from Amazon, whose web address is

www.amazon.com

The first books to consider if you are interested in doing well on the AMC are the various volumes of *The Contest Problem Book*. These contain all the past AHSME and AMC contests, the published solutions, and often some additional problem-solving information and advice. They have been written by the Contest Directors, so they can provide good insight to the manner in which the contests are constructed.

The next source that I recommend is the *Art of Problem Solving, Volumes I and II* by Sandor Lehoczky and Richard Rusczyk. These books contain great techniques and insights into problem solving and can be obtained directly from Art of Problem Solving, whose web address is

www.artofproblemsolving.com

You will likely find other items of interest at this site, since the authors are interested in problem solving at many levels.

The MAA has many problem-solving books in addition to the Contest Books. They range from moderately difficult to extremely challenging. For example, if you are interested in sources of problems that have been given at the highest international level, the books by Titu Andreescu and his coauthors contain material that will challenge almost anyone.

If some particular problem-solving topic particularly interests you, there is plenty of material to choose from. Dover Publications, whose web address is

store.doverpublications.com

has many books on problems, puzzles, and logic. The books from Dover are generally very reasonably priced and are often classics, which though old, contain excellent material. Some of my favorites from this source are *Challenging Problems in Geometry* by Alfred Posamentier and Charles Salkind, *The Red Book of Mathematical Problems* by Kenneth Hardy and Kenneth Williams, and *Mathematical Quickies: 270 Stimulating Problems with Solutions* by Charles Trigg. Dover also publishes many of the puzzle books by Henry Earnest Dudeney and Sam Loyd, which I particularly like.

Finally, there are some books that I would feel remiss not to mention, because I feel that they are classics and the best of their kind. The books *Mathematics of Choice: How to Count Without Counting* and *Numbers: Rational and Irrational*, written by Ivan Niven and published by the MAA, are wonderful examples of how mathematics should be written. The book *Problem Solving Through Recreational Mathematics* by Bonnie Averbach and Orin Chein is available from Dover and contains clever topics and ideas with extensions that provide many challenges. *Problem Solving through Problems* written by Loren Larson and published by Springer-Verlag, is available from Amazon and is a classic in the area. Any of the MAA books written by Ross Honsberger are delightful, and the volumes of *Winning ways for your Mathematical Plays* by Berlekamp, Conway, and Guy. However, my overall favorite is *Concrete Mathematics*, which is written by Ronald Graham, Donald Knuth, and Oren Patashnik. It is published by Addison-Wesley and has more clever ideas per page than any other book in my library. Well, at least than any book that has not been written by Donald Knuth.

I hope you find this information useful, but as I stated in the opening paragraph, it is a personal choice. I hope that you have sufficient interest to make your own favorites. I will be creating a web site for this book at

www.as.ysu.edu/~faires/AMCBook/

At this site I will be placing additional problems when new exams have been given. If you have problem-solving sources that you feel are particularly valuable, send me an e-mail and I will consider posting them on this site.

Good luck and have fun.

Doug Faires
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April 3, 2006

Sources of the Exercises

As mentioned in the Preface, all the Examples and Exercises have been taken from past AHSME and AMC competitions. The specific competitions were noted in the text for the Examples but not for the Exercises. The omission for the Exercises was done so that students can attempt the problems without any preconceived notion of the level of difficulty, although it becomes quite clear to any observant student that the level of difficulty increases with the number of the exercise.

This section lists the source of all the problems used for the Exercises. If you consult the relevant *Contest Problem Book* for each competition you might find alternate solutions to the exercises.

Exercises for Chapter 1: Arithmetic Ratios

1. 2000 AMC 10 #3 and 12 #3
2. 2001 AMC 10 #8
3. 1986 AHSME #14
4. 2004 AMC 10A #11 and 12A #9
5. 2002 AMC 10A #12 and 12A #11
6. 2003 AMC 10B #17 and 12B #13
7. 2003 AMC 12B #11
8. 1991 AHSME #11
9. 2002 AMC 10A #17 and 12A #10
10. 1998 AHSME #21

Exercises for Chapter 2: Polynomials and Their Zeros

1. 2003 AMC 12B #9
2. 1974 AHSME #2
3. 1974 AHSME #4
4. 1999 AHSME #12
5. 2001 AMC 12 #13
6. 1999 AHSME #17
7. 2001 AMC 12 #19
8. 2000 AMC 10 #24
9. 1977 AHSME #21
10. 1977 AHSME #23

Exercises for Chapter 3: Exponentials and Radicals

1. 1994 AHSME #1
2. 1993 AHSME #3
3. 1998 AHSME #5
4. 1992 AHSME #6
5. 1993 AHSME #6
6. 1996 AHSME #6
7. 2003 AMC 10B #9
8. 1993 AHSME #10
9. 1991 AHSME #20
10. 1983 AHSME #25

Exercises for Chapter 4: Defined Functions and Operations

1. 1993 AHSME #4
2. 1998 AHSME #4
3. 1982 AHSME #7
4. 2003 AMC 12A #6
5. 2001 AMC 12 #2
6. 2003 AMC 10B #13 and 12B #8
7. 2001 AMC 12 #9
8. 1988 AHSME #14
9. 1977 AHSME #22
10. 1975 AHSME #21

Exercises for Chapter 5: Triangle Geometry

1. 1986 AHSME #3
2. 1991 AHSME #5
3. 2002 AMC 10A #13
4. 1992 AHSME #9
5. 1989 AHSME #15
6. 1995 AHSME #19
7. 1984 AHSME #17
8. 2003 AMC 10A #22
9. 2002 AMC 10B #22 and 12B #20
10. 1983 AHSME #19

Exercises for Chapter 6: Circle Geometry

1. 1985 AHSME #2
2. 1977 AHSME #9
3. 1995 AHSME #26
4. 1991 AHSME #22
5. 1992 AHSME #11
6. 2003 AMC 12A #15
7. 1985 AHSME #22
8. 2000 AMC 12 #24
9. 1997 AHSME #26
10. 1992 AHSME #27

Exercises for Chapter 7: Polygons

1. 2001 AMC 10 #15
2. 1990 AHSME #4
3. 2002 AMC 10A #19
4. 1994 AHSME #7
5. 2002 AMC 10A #25
6. 2003 AMC 12A #14
7. 1990 AHSME #20
8. 1994 AHSME #26
9. 2002 AMC 12B #24
10. 2003 AMC 12B #22

Exercises for Chapter 8: Counting

1. 2004 AMC 10A #13
2. 2004 AMC 10A #12
3. 2004 AMC 10B #14
4. 2003 AMC 10B #10
5. 2002 AMC 10B #9
6. 2001 AMC 10 #19
7. 2003 AMC 10A #25 and 12A #18
8. 1998 AHSME #24
9. 1994 AHSME #22
10. 2003 AMC 12A #20

Exercises for Chapter 9: Probability

1. 2003 AMC 10A #8 and 12A #8
2. 2004 AMC 10B #11
3. 2003 AMC 10A #12
4. 2004 AMC 10B #23 and 12B #20
5. 2001 AMC 12 #11
6. 2002 AMC 12B #16
7. 2001 AMC 12 #17
8. 2003 AMC 12B #19
9. 2005 AMC 12A #14
10. 2003 AMC 12A #16

Exercises for Chapter 10: Prime Decomposition

1. 2004 AMC 10B #4
2. 1999 AHSME #6
3. 2002 AMC 10B #14
4. 2002 AMC 10A #14 and 12A #12
5. 1986 AHSME #23
6. 2003 AMC 12A #23
7. 1990 AHSME #11
8. 1993 AHSME #15
9. 2002 AMC 12A #20
10. 1996 AHSME #29

Exercises for Chapter 11: Number Theory

1. 1972 AHSME #31
2. 2003 AMC 10A #20
3. 1992 AHSME #17
4. 1987 AHSME #16
5. 1982 AHSME #26
6. 2000 AMC 10 #25
7. 1991 AHSME #15
8. 1986 AHSME #17
9. 1994 AHSME #19
10. 1992 AHSME #23

Exercises for Chapter 12: Sequences and Series

1. 2004 AMC 10B #10 and 12B #8
2. 2003 AMC 10B #8 and 12B #6
3. 2000 AMC 12 #8
4. 2004 AMC 10B #21
5. 1993 AHSME #21
6. 2004 AMC 10A #18 and 12A #14
7. 2002 AMC 12A #21
8. 1981 AHSME #26
9. 1984 AHSME #12
10. 1992 AHSME #18

Exercises for Chapter 13: Statistics

1. 1996 AHSME #4
2. 1997 AHSME #6
3. 1998 AHSME #9
4. 2004 AMC 10A #14 and 12A #11
5. 2004 AMC 12B #11
6. 1997 AHSME #11
7. 2001 AMC 12 #4
8. 2002 AMC 10A #21 and 12A #15
9. 1999 AHSME #20
10. 1997 AHSME #18

Exercises for Chapter 14: Trigonometry

1. 1983 AHSME #11
2. 1989 AHSME #13
3. 1988 AHSME #13
4. 1995 AHSME #18
5. 1980 AHSME #23
6. 1993 AHSME #23
7. 1999 AHSME #27
8. 2003 AMC 12B #21
9. 2004 AMC 12A #21
10. 2002 AMC 12A #23

Exercises for Chapter 15: Three-Dimensional Geometry

1. 1995 AHSME #6
2. 2001 AMC 12 #8 and 2001 AMC 10 #17
3. 1994 AHSME #11
4. 1990 AHSME #10
5. 1982 AHSME #18
6. 1985 AHSME #20
7. 1997 AHSME #23
8. 2000 AMC 12 #25
9. 1996 AHSME #28
10. 2004 AMC 10A #25 and 12A #22

Exercises for Chapter 16: Functions

1. 1993 AHSME #12
2. 1996 AHSME #12
3. 1982 AHSME #12
4. 1984 AHSME #16
5. 1980 AHSME #14
6. 1983 AHSME #18
7. 1996 AHSME #25
8. 1981 AHSME #18
9. 2000 AMC 12 #22
10. 2002 AMC 12B #25

Exercises for Chapter 17: Logarithms

1. 1998 AHSME #12
2. 1993 AHSME #11
3. 2000 AMC 12 #7
4. 2002 AMC 12A #14
5. 2004 AMC 12B #17
6. 1997 AHSME #21
7. 1998 AHSME #22
8. 1991 AHSME #24
9. 1987 AHSME #20
10. 2003 AMC 12A #24

Exercises for Chapter 18: Complex Numbers

1. 1984 AHSME #10
2. 1983 AHSME #17
3. 1992 AHSME #15
4. 1991 AHSME #18
5. 1974 AHSME #17
6. 1977 AHSME #16
7. 1988 AHSME #21
8. 1987 AHSME #28
9. 1985 AHSME #23
10. 1992 AHSME #28

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About the Author

J. Douglas Faires received his BS in mathematics from Youngstown University in 1963. He earned his PhD in mathematics from the University of South Carolina in 1970. Faires has been a Professor at Youngstown State University since 1980. He has been actively involved in the MAA for many years. For example, he was Governor of the Ohio Section from 1997–2000. He is currently a member of the MAA's Strategic Planning Committee for the AMC. Faires is a past President of Pi Mu Epsilon and he was a member of the Council for many years. He has been the National Director of the AMC-10 Competition of the American Mathematics Competitions since 1999. Faires has been the recipient of many awards and honors. He was named the Outstanding College-University Teacher of Mathematics by the Ohio Section of the MAA in 1996; he has also received five Distinguished Professorship awards from Youngstown State University and an honorary Doctor of Science degree in May 2006. Faires has authored and coauthored numerous books including *Numerical Analysis* (now in its eighth edition!), *Numerical Methods* (third edition), and *Precalculus* (fourth edition).