Differential Equation (DE)

↳ An eqn involves derivative of a function

↳ A DE relates some function of one or more variables with its derivatives.

Initial Value Problems:

In initial value problem, we are given the value of function $\underbrace{f(a)}_{\hookrightarrow y}$ and its derivative $\underbrace{f'(x)}_{y'}$ at the same point (initial point), say at $x=0$ along with $f(0)=x_1$ and $f'(0)=x_2$

Boundary Value Problem:

In boundary value problem, we are given the value of function $f(x)$, at two different points, say at $f(a)=x_1$ and $f(b)=x_2$.

Separable Variables

Examples:

$\boxed{1}$ $$x\frac{dy}{dx} = 4y$$

$$\int \frac{dy}{y} = \int \frac{4}{x}\,dx$$

$$\ln y = 4\ln x + c$$

$$\log_e y = \ln x^4 + c$$

$$y = e^{\ln x^4 + c}$$

$$= e^{\ln x^4}\, e^{c}$$

$$= x^4 c \qquad \text{Relabel the constant } e^c = c$$

$$y = cx^4$$

$\boxed{2}$ $$y\ln x\frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$$

$$= \frac{(y+1)^2}{x^2} = \frac{y^2+2y+1}{x^2}$$

$$y\ln x\,dx = \frac{y^2+2y+1}{x^2}\,dy$$

$$x^2\ln x\,dx = \frac{y^2+2y+1}{y}\,dy$$

$$\int x^2\ln x\,dx = \int\left(y+2+\frac{1}{y}\right)dy$$

$$\ln x\int x^2dx - \int\left[\left\{\frac{d}{dx}\ln x\right\}\int x^2dx\right]dx = \int\left(y+2+\frac{1}{y}\right)dy$$

$$\frac{x^3}{3}\ln x - \int\frac{1}{x}\cdot\frac{x^3}{3}\,dx = \frac{y^2}{2} + 2y + \ln y$$

$$\frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{y^2}{2} + 2y + \ln y$$

$$\frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C = \frac{y^2}{2} + 2y + \ln y$$

$$\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C = \frac{y^2}{2} + 2y + \ln y$$

$\boxed{3}$ $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$

$$\sin 3x\,dx = -2y\cos^3 3x\,dy$$

$$\frac{\sin 3x}{\cos^3 3x}\,dx = -2y\,dy$$

$$\int \frac{\sin 3x}{\cos 3x}\cdot\frac{1}{\cos^2 3x}\,dx = -2\int y\,dy$$

$$\int \tan 3x \sec^2 3x\,dx = -2\,\frac{y^2}{2}$$

Let

$\tan 3x = z$

$\sec^2 3x(3)dx = dz$

$\sec^2 3x\,dx = \frac{1}{3}dz$

$$\frac{1}{3}\int z\,dz = -y^2$$

$$\frac{1}{3}\cdot\frac{z^2}{2} = -y^2$$

$$\frac{1}{6}\tan^2 3x = -y^2$$

$$y^2 = -\frac{1}{6}\tan^2 3x + C$$

$\boxed{4}$ $$\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$$

$$= \frac{xy-y+3x-y}{xy+4y-2x-8}$$

$$= \frac{y(x-1)+3(x-1)}{y(x+4)-2(x+4)}$$

$$\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx$$

$$\int \frac{y+3-5}{y+3}\,dy = \int \frac{x+4-5}{x+4}\,dx$$

$$\int \left(1-\frac{5}{y+3}\right)dy = \int \left(1-\frac{5}{x+4}\right)dx$$

$$y - 5\ln(y+3) = x - 5\ln(x+4) + c$$

$$y - x - c = \ln(y+3)^5 - \ln(x+4)^5$$

$$\ln\left[\frac{(y+3)^5}{(x+4)^5}\right] = y - x - c$$

$$\frac{(y+3)^5}{(x+4)^5} = e^{y-x-c} = \frac{e^y}{e^x e^c}$$

$$\rightarrow e^y . e^{-x} . e^{-c}$$

$$\frac{e^x e^c}{(x+4)^5} = \frac{e^y}{(y+3)^5}$$

$$Ce^x (x+4)^{-5} = e^y (y+3)^{-5}$$

Relabel the constant $e^c = C$

Boundary value problem

$\boxed{5}$ $\frac{dx}{dt} = 4(x^2+1)$; $\underbrace{x\left(\frac{\pi}{4}\right) = 1}_{t = \frac{\pi}{4},\ x = 1}$

$$\int \frac{dx}{x^2+1} = \int 4\, dt$$

$$\tan^{-1} x = 4t + c \quad \text{——— } (1)$$

Substitute $t = \frac{\pi}{4}$, $x = 1$ into (1)

$$\tan^{-1} 1 = 4 \cdot \frac{\pi}{4} + c$$

$$\tan^{-1} \tan\frac{\pi}{4} = \pi + c$$

$$\frac{\pi}{4} = \pi + c$$

$$c = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$$

Substitute c into (1)

$$\tan^{-1} x = 4t - \frac{3\pi}{4}$$

$$x = \tan\left(4t - \frac{3\pi}{4}\right)$$

$\boxed{6}$ $\dfrac{dy}{dx} = \dfrac{y^2-1}{x^2-1}$, $\underbrace{y(2) = 2}_{x=2,\ y=2}$

$$\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1}$$

$$\frac{1}{2} \ln \frac{y-1}{y+1} = \frac{1}{2} \ln \frac{x-1}{x+1}$$

$$\ln \frac{y-1}{y+1} = \ln \frac{x-1}{x+1} + C$$

$$\log_e \left(\frac{y-1}{y+1}\right) = \ln \left(\frac{x-1}{x+1}\right) + c$$

$$\frac{y-1}{y+1} = e^{\ln\left(\frac{x-1}{x+1}\right) + c}$$

$$= e^{\ln\left(\frac{x-1}{x+1}\right)} e^{c}$$

$$\frac{y-1}{y+1} = \left(\frac{x-1}{x+1}\right) C \quad (1)$$

Relabel the constant $e^c = C$

Substitute $x = 2$, $y = 2$ into (1)

$$\frac{2-1}{2+1} = \frac{2-1}{2+1} C$$

$$\frac{1}{3} = \frac{1}{3} C$$

$$C = 1$$

Substitute $C = 1$ into (1)

$$\frac{y-1}{y+1} = \frac{x-1}{x+1}.$$