Week 3

Beta Function

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$m>0,\ n>0$

Exercise:

2 iii) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx$

$$=\int_0^1 (1-x^3)^{-1/2}\,dx$$

$$=\int_0^1 \frac{1}{3} z^{\overset{m-1}{-2/3}}(1-z)^{\overset{n-1}{-1/2}}\,dz$$

$$=\frac{1}{3}\int_0^1 z^{1/3-1}(1-z)^{1/2-1}\,dz$$

$$=\frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right)$$

$x^3=z$

$3x^2\,dx=dz$

$dx=\dfrac{1}{3x^2}dz$

$=\dfrac{1}{3z^{2/3}}dz$

$x^3=z$

$x=z^{1/3}$

$\therefore x^2=z^{2/3}$

limits

$x=0\rightarrow z=0$

$x=1\rightarrow z=1$

$\rightarrow m-1=-2/3$

$m=1-2/3=1/3$

$\rightarrow n-1=-1/2$

$n=\frac{1}{2}$

2vi) $\int_0^a y^7 \sqrt{a^4 - y^4}\, dy$

$a^4 y^4 \to$ factor out a^4

let
$y = az$
$dy = a\,dz$

$= \int_0^1 a^7 z^7 \sqrt{a^4 - a^4 z^4}\; a\,dz$

limits
$y = 0 \Rightarrow az = 0$
$z = 0$
$y = a \Rightarrow az = a$
$z = 1$

$= a^8 \int z^7 a^2 \sqrt{1 - z^4}\, dz$

$= a^{10} \int_0^1 z^7 (1 - z^4)^{1/2}\, dz$

let
$z^4 = x$
$4z^3 dz = dx$
$z^3 dz = \frac{1}{4} dx$

$= a^{10} \int_0^1 z^4 \cdot \underbrace{z^3}\, (1 - z^4)^{1/2} \underbrace{dz}$

$= a^{10} \int_0^1 x \frac{1}{4} (1 - x)^{1/2} dx$

limits
$z = 0 \to x = 0$
$z = 1 \to x = 1$

$= \frac{a^{10}}{4} \int_0^1 x (1 - x)^{1/2} dx$

$= \frac{a^{10}}{4} \int_0^1 x^{2-1} (1 - x)^{3/2 - 1} dx$

$= \frac{a^{10}}{4} \beta\left(2, \frac{3}{2}\right)$

Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \; ; \; n > 0$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \; ; \; \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$m>0, n>0$

1i) $\int_0^4 x^{3/2} (\underset{1}{4} - x)^{5/2}\, dx$

$x = 4y$
$dx = 4dy$

limits
$x = 0 \to y = 0$
$x = 4 \to y = 1$

$$= \int_0^1 4^{3/2} y^{3/2} (4-4y)^{5/2}\, 4dy$$

$$= 4^{3/2+1} \int_0^1 y^{3/2}\, 4^{5/2} (1-y)^{5/2}\, dy$$

$$= 4^{\frac{3}{2}+1+\frac{5}{2}} \int_0^1 y^{3/2} (1-y)^{5/2}\, dy$$

$$= 4^{\frac{10}{2}} \int_0^1 y^{5/2-1} (1-y)^{7/2-1}\, dy$$

$$= 4^5 \; \beta\left(\frac{5}{2}, 7/2\right)$$

$$= 1024 \frac{\Gamma(\frac{5}{2})\Gamma(7/2)}{\Gamma(\frac{5}{2}+\frac{7}{2})} = 1024 \frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(6)}.$$

$$1(ii) \int_0^b y^5 \sqrt{b^2-y^2}\, dy$$

let $y = bz$

$dy = b\,dz$

limits

$y=0 \rightarrow z=0$

$y=b \rightarrow z=1$

$$= \int_0^1 b^5 z^5 \sqrt{b^2-b^2z^2}\, b\, dz$$

$$= b^6 \int_0^1 z^5\, b\sqrt{1-z^2}\, dz$$

$$= b^7 \int_0^1 z^5 (1-z^2)^{1/2} dz$$

$z^2 = x \rightarrow z^4 = x^2$

$2z\,dz = dx$

$z\,dz = \frac{1}{2} dx$

limits:

$z=0 \rightarrow x=0$

$z=1 \rightarrow x=1$

$$= b^7 \int_0^1 z \cdot z^4 (1-z^2)^{1/2} dz$$

$$= b^7 \int_0^1 \frac{1}{2} \cdot x^2 (1-x)^{1/2}\, dx$$

$$= \frac{b^7}{2} \int_0^1 x^{3-1} (1-x)^{3/2-1}\, dx$$

$$= \frac{b^7}{2} \beta\left(3, 3/2\right)$$

$$= \frac{b^7}{2} \frac{\Gamma(3)\,\Gamma(3/2)}{\Gamma(3+3/2)}$$

$$= \frac{b^7}{2} \frac{\Gamma(3)\,\Gamma(3/2)}{\Gamma(9/2)}$$

1 vii) $\int_0^\infty x^6 e^{-3x}\,dx$

$3x = z \longrightarrow \frac{z}{3} = x$

$3dx = dz$

$dx = \frac{1}{3}dz$

limits:

$x=0 \to z=0$

$x=\infty \to z=\infty$

$= \int_0^\infty \left(\frac{z}{3}\right)^6 e^{-z} \frac{1}{3}dz$

$= \frac{1}{3^7}\int_0^\infty e^{-z} z^{7-1}\,dz$

$= \frac{1}{3^7}\ \Gamma(7)$

1 ix) $\int_0^\infty \sqrt{x}\, e^{-x^2}\,dx$

$x^2 = z \longrightarrow x = \sqrt{z}$

$\therefore \sqrt{x} = z^{1/4}$

$2x\,dx = dz$

$dx = \frac{1}{2x}dz$

$= \frac{1}{2\sqrt{z}}$

limits

$x=0 \to z=0$

$x=\infty \to z=\infty$

$= \int_0^\infty z^{1/4} e^{-z} \frac{1}{2\sqrt{z}}\,dz$

$= \frac{1}{2}\int_0^\infty z^{\frac{1}{4}-\frac{1}{2}} e^{-z}\,dz$

$= \frac{1}{2}\int_0^\infty z^{-1/4} e^{-z}\,dz$

$= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{3}{4}-1}\,dz$

$= \frac{1}{2}\ \Gamma(3/4)$

1 xi) $\int_0^1 \frac{1}{\sqrt{x \ln\left(\frac{1}{x}\right)}} dx$

$\ln\left(\frac{1}{x}\right) = z$

$\ln 1 - \ln x = z$

$0 - \ln x = z$

$\ln x = -z$

$\log_e x = -z$

$x = e^{-z}$

$dx = -e^{-z} dz$

limits:

$x = 0 \to z = \infty$

$x = 1 \to z = 0$

$$= \int_{\infty}^{0} \frac{-e^{-z} dz}{\sqrt{e^{-z} z}}$$

$$= -\int_{\infty}^{0} \frac{e^{-z} dz}{e^{-z/2} z^{1/2}}$$

$$= \int_0^{\infty} e^{-z-(-z/2)} z^{-1/2} dz$$

$$= \int_0^{\infty} e^{-z/2} z^{-1/2} dz \longrightarrow$$

$\frac{z}{2} = y$

$z = 2y$

$dz = 2dy$

limits

$z = 0 \to y = 0$

$z = \infty \to y = \infty$

$$= \int_0^{\infty} e^{-y} (2y)^{-1/2} \, 2 \, dy$$

$$= 2^{-1/2+1} \int_0^{\infty} e^{-y} y^{-1/2} dy$$

$$= \sqrt{2} \int_0^{\infty} e^{-y} y^{\frac{1}{2}-1} dy$$

$$= \sqrt{2} \; \Gamma\left(\frac{1}{2}\right)$$

1(xii) $\int_0^1 \left(1 - \frac{1}{x}\right)^{1/3} dx$

$$= \int_0^1 \left(\frac{x-1}{x}\right)^{1/3} dx = \int_0^1 -\left(\frac{1-x}{x}\right)^{1/3} dx$$

$$= -\int_0^1 x^{-1/3} (1-x)^{1/3} dx$$

⋮ continue

Formula:

OR

$\boxed{1}$ $\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$.

$\boxed{2}$ $\int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)\,dt = \beta(x,y)$.

3 iii) $\int_0^{\pi/6} \sin^2 6x \cos^4 3x\,dx$

$= \int_0^{\pi/6} (\sin 2\cdot 3x)^2 \cos^4 3x\,dx$

$= \int_0^{\pi/6} (2\sin 3x \cdot \cos 3x)^2 \cos^4 3x\,dx$

$= 4\int_0^{\pi/6} \sin^2 3x \cos^2 3x \cos^4 3x\,dx$

$= 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x\,dx$

$3x = z$
$3\,dx = dz$
$dx = \frac{1}{3}dz$

$= 4\cdot\frac{1}{3}\int_0^{\pi/2} \sin^2 z \cos^6 z\,dz$

limits:
$x=0 \to z=0$
$x=\frac{\pi}{6} \to z=\pi/2$

$= \frac{4}{3}\dfrac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}$

$= \frac{4}{3}\dfrac{\Gamma(3/2)\Gamma(7/2)}{2\Gamma(5)} = \frac{2}{3}\dfrac{\Gamma(3/2)\Gamma(7/2)}{\Gamma(5)}$.

Alternative Method:

$= \frac{2}{3}\int_0^{\pi/2} 2\sin^2 z \cos^6 z\,dz$

$2x-1=2$
$2x = 2+1 = 3$
$x = 3/2$

$2y-1=6$
$2y=7$
$y=7/2$

$= \frac{2}{3}\beta(x,y) = \frac{2}{3}\beta\left(\frac{3}{2},\frac{7}{2}\right)$

$\boxed{7}$