BRAC University
Department of Mathematics and Natural Sciences
MAT 215: Machine Learning & Signal Processing
Assignment-02

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1. Using the definition **5×5=25**
   a) Find the derivative of $f(z) = \frac{9z-4}{9z+7i}$ at $z = i$.
   b) Find the derivative of $f(z) = \frac{4}{6z+7}$ at $z = z_0$.
   c) Show that $f(z) = 7z^2 + 6z - 7$ is differentiable at all points .
   d) Show that $f(z) = 7z\overline{z} - 6z + 7\overline{z}$ is not differentiable at $z = 0$.
   e) Find the derivative of $f(z) = \frac{8}{z^2}$ at $z = 5 + 6i$.

2. Using C-R equatins determine whether the functions are analytic or not. **5×5=25**
   a) $f(z) = 4\sinh(3z)$
   b) $f(z) = 2\cos(7z)$
   c) $f(z) = 2|z|^2 + 2z - 2\overline{z}$
   d) $f(z) = \frac{6}{z+9-4i}$
   e) $f(z) = 2z^2e^{7z}$

3. Show that the given function U (or V) is harmonic. Determine the harmonic conjugate V (or U) such that **U+iV** becomes analytic. **5×10=50**
   a) Given $V = 5\ln((x-6)^2 + (y-2)^2)$, show that V is harmonic and find U.
   b) Given $U = 9x^2y - 7x^2 - 3y^3 + 7y^2$, show that U is harmonic and find V.
   c) Given $V = 7e^{-9x}\cos(9y) - 7e^{5y}\sin(5x)$, show that V is harmonic and find U.
   d) Given $U = 5\sin(5x)\cosh(5y)$, show that U is harmonic and find V.
   e) Given $V = 2xe^{-7x}\cos(7y) + 2ye^{-7x}\sin(7y)$, show that V is harmonic and find U.

Assignment 02

1)

a) $f(z) = \dfrac{9z-4}{9z+7i}$; $a = i$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$\Rightarrow f'(i) = \lim_{\Delta z \to 0} \frac{\dfrac{9i+9\Delta z-4}{16i+9\Delta z} - \dfrac{9i-4}{16i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{144i^2+144\Delta z i-64i-(144i^2-64i+81\Delta z i-36\Delta z)}{16i(16i+9\Delta z)}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(63i+36)}{\Delta z(-256+144i\Delta z)}$$

$$= -\frac{36+63i}{256} \quad \underline{\text{As.}}$$

b)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{\frac{4}{6z_0+6\Delta z+7} - \frac{4}{6z_0+7}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{\cancel{24z_0}+\cancel{28}-\cancel{24z_0}-24\Delta z-\cancel{28}}{(6z_0+6\Delta z+7)(6z_0+7)}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-24\cancel{\Delta z}}{\cancel{\Delta z}(6z_0+6\Delta z+7)(6z_0+7)}$$

$$= \frac{-24}{(6z_0+7)^2}$$ Ans

c)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{7(\cancel{z^2} + \Delta z^2 + 2\Delta z z) + \cancel{6z} + 6\Delta z - \cancel{7} - \cancel{7z^2} - \cancel{6z} + \cancel{7}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{7\Delta z^2 + 14 z \Delta z + 6\Delta z}{\Delta z}$$

$= 14z + 6$ [limit exists for any point]

$\therefore f(z)$ differentiable at all points.

(8end)

d) $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$\Rightarrow f'(0) = \lim_{\Delta z \to 0} \dfrac{7\Delta z\,\overline{\Delta z} - 6\Delta z + 7\overline{\Delta z} - 0}{\Delta z}$

$= \lim_{\Delta z \to 0} 7\overline{\Delta z} - 6 + 7\dfrac{\overline{\Delta z}}{\Delta z}$

$\Delta z = \Delta x + i\Delta y$

$\Rightarrow \overline{\Delta z} = \Delta x - i\Delta y$

$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} 7(\Delta x - i\Delta y) - 6 + 7\dfrac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$

In $\Delta x = 0$ direction,

$\lim_{y \to 0} 7(-i\Delta y) - 6 + 7\dfrac{-i\Delta y}{i\Delta y}$

$= -13$

In $\delta y = 0$ direction,

$$\lim_{\delta x \to 0} 7\,\delta x - 6 + 7\frac{\delta x}{\delta x}$$

$= 1$

$\because -13 \neq 1 \quad \therefore$ limit doesn't exist

$\therefore f(z)$ is not differentiable at $z=0$ (Ans)

e) $f(z) = \frac{8}{z^2}$; $z = 5+6i$

$$\therefore f'(5+6i) = \lim_{\Delta z \to 0} \frac{\frac{8}{(5+6i+\Delta z)^2} - \frac{8}{(5+6i)^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{8(\cancel{25}+\cancel{60i}-\cancel{36}) - 8(\cancel{25}+\cancel{60i}-\cancel{36}+10\Delta z+12\Delta z i + \Delta z^2)}{(5+6i+\Delta z)^2 (5+6i)^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-8\Delta z^2 - 96\Delta z i - 80\Delta z}{(5+6i+\Delta z)^2 (5+6i)^2 \Delta z}$$

$$= \frac{-80-96i}{(5+6i)^4}$$ As.

2) a) $f(z) = 4\sinh(3z)$

$$= 4\,\frac{e^{3x+3iy} - e^{-3x-3iy}}{2}$$

$$= 2\left(e^{3x}e^{3iy} - e^{-3x}e^{-3iy}\right)$$

$$= 2\left(e^{3x}\cos 3y + i e^{3x}\sin 3y - e^{-3x}\cos 3y + i e^{-3x}\sin 3y\right)$$

$$= 2\left(\left(e^{3x} - e^{-3x}\right)\cos 3y + i\left(e^{3x} + e^{-3x}\right)\sin 3y\right)$$

$$= \underbrace{4\cosh(3x)\cos 3y}_{u} + i\,\underbrace{4\sinh(3x)\sin 3y}_{v}$$

$$\frac{\partial u}{\partial x} = 4\sinh(3x)\cdot 3\cos 3y$$

$$\frac{\partial u}{\partial y} = -4\cosh(3x)\sin 3y\cdot 3$$

$$\frac{\partial v}{\partial x} = 4\cosh(3x)\,3\sin 3y$$

$$\frac{\partial v}{\partial y} = 4\sinh(3x)\cos 3y\cdot 3$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

also they are continuous.

$\therefore f(z)$ is analytic.

b) $z = x+iy$

$f(z) = 2\cos(7z)$

$= 2\,\dfrac{e^{i7z} + e^{-i7z}}{2}$

$= e^{7ix-7y} + e^{-7ix+7y}$

$= e^{7ix} \cdot e^{-7y} + e^{-7ix}\, e^{7y}$

$= \cos(7x)\, e^{-7y} + i\sin(7x)\, e^{-7y} +$

$\cos(7x)\, e^{7y} - i\sin(7x)\, e^{7y}$

$= \cos(7x)\left(e^{7y} + e^{-7y}\right) - i\sin(7x)\left(e^{7y} - e^{-7y}\right)$

$= \underbrace{2\cos(7x)\cosh(7y)}_{u} - i\underbrace{\left(-2\sin(7x)\sinh(7y)\right)}_{v}$

$\dfrac{\partial u}{\partial x} = -14\sin(7x)\cosh(7y) \quad \dfrac{\partial v}{\partial y} = -14\sin(7x)\cosh(7y)$

$\dfrac{\partial u}{\partial y} = 14\cos(7x)\sinh(7y) \quad \dfrac{\partial v}{\partial x} = -14\cos(7x)\sinh(7y)$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ $\quad \therefore f(z)$ is analytic as all partial are continuous

c) $f(z) = 2|z|^2 + 2z - 2\bar{z}$

$= 2(x^2+y^2) + 2(x+iy) - 2(x-iy)$

$= 2x^2+2y^2 + \cancel{2x} + 2iy - \cancel{2x} + 2iy$

$= \underbrace{2x^2+2y^2}_{u} + i\underbrace{4y}_{v}$

$\frac{\partial u}{\partial x} = 4x \qquad \frac{\partial v}{\partial y} = 4$

$\frac{\partial u}{\partial y} = 4y \qquad \frac{\partial v}{\partial x} = 0$

$\therefore f(z)$ is not analytic.

d) $f(z) = \dfrac{6}{z+9-4i}$

$$= \frac{6}{x+9+i(y-4)}$$

$$= \frac{6(x+9-i(y-4))}{(x+9+i(y-4))(x+9-i(y-4))}$$

$$= \frac{6x+54-i6(y-4)}{x^2+18x+81+y^2-8y+16}$$

$$= \frac{6x+54}{x^2+y^2+18x-8y+97} + \frac{-6y+24}{x^2+y^2+18x-8y+97}$$

$$= \frac{6x+54}{(x+9)^2+(y-4)^2} + \frac{-6(y-4)}{(x+9)^2+(y-4)^2}$$

$$\frac{\partial u}{\partial x} = \frac{((x+9)^2+(y-4)^2)6-(6x+54)2(x+9)}{((x+9)^2+(y-4)^2)^2}$$

$$= \frac{6(y-4)^2-6(x+9)^2}{((x+9)^2+(y-4)^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{12(y-4)(x+9)}{\left[(x+9)^2+(y-4)^2\right]^2}$$

$$\frac{\partial u}{\partial y} = \frac{(-12)(y-4)(x+9)}{\left((x+9)^2+(y-4)^2\right)^2}$$

$$\frac{\partial v}{\partial y} = \frac{6(y-4)^2-6(x+9)^2}{\left((x+9)^2+(y-4)^2\right)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$$

all are continuous except $(x,y) = (-9,4)$

$\therefore f(z)$ is analytic except $(z = -9+4i$

e)

$$f(z) = 2z^2 e^{7z}$$

$$= 2\left(x^2 - y^2 + i2xy\right) e^{7x+i7y}$$

$$= 2e^{7x}\left(x^2 - y^2 + i2xy\right)\left(\cos 7y + i\sin 7y\right)$$

$$= 2e^{7x}\Big(\left((x^2-y^2)\cos 7y - 2xy\sin 7y\right) +$$
$$i\left(2xy\cos 7y + (x^2-y^2)\sin 7y\right)\Big)$$

$$\frac{\partial u}{\partial x} = 2\frac{\partial}{\partial x}\left(e^{7x}(x^2-y^2)\cos 7y - 2e^{7x}xy\sin 7y\right)$$

$$= 2\Big(\cos 7y\left(-7y^2e^{7x} + 7e^{7x}x^2 + 2xe^{7x}\right) -$$
$$2y\sin 7y\left(e^{7x} + 7xe^{7x}\right)\Big)$$

$$= 2e^{7x}\left(\cos 7y\left(7x^2 + 2x - 7y^2\right) - \sin 7y\left(14xy + 2y\right)\right)$$

$$\frac{\partial u}{\partial y} = 2e^{7x}\left(-7x^2 \sin 7y + y^2 \sin 7y \, 7 - \cos 7y \, 2y - 2x(y \cos 7y \, 7 + \sin 7y)\right)$$

$$= 2e^{7x}\left(\cos 7y(-2y - 14xy) + \sin 7y(-7x^2 + 7y^2 - 2x)\right)$$

$$\frac{\partial v}{\partial x} = 2e^{7x}\left(\cos 7y(2y + 14xy) + \sin 7y(7x^2 + 2x - 7y^2)\right)$$

$$\frac{\partial v}{\partial y} = 2e^{7x}\left(\cos 7y(7x^2 + 2x - 7y^2) - \sin 7y(14x + 2y)\right)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

all are continuous

$\therefore f(z)$ is analytic.

3)

a) $v = 5\ln\left((x-6)^2 + (y-2)^2\right)$

$$\frac{\partial v}{\partial x} = \frac{10(x-6)}{(x-6)^2 + (y-2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{10\left((y-2)^2 - (x-6)^2\right)}{\left((x-6)^2 + (y-2)^2\right)^2}$$

$$\frac{\partial v}{\partial y} = \frac{10(y-2)}{(x-6)^2 + (y-2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{10\left((x-6)^2 - (y-2)^2\right)}{\left((x-6)^2 + (y-2)^2\right)^2}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad [v \text{ is harmonic}]$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow u = 10(y-2) \int \frac{1}{(x-6)^2+(y-2)^2} \, dx$$

$$= 10 \cancel{(y-2)} \frac{1}{\cancel{y-2}} \tan^{-1} \frac{x-6}{y-2} + g(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow 10 \frac{-1}{\frac{(x-6)^2}{(y-2)^2}+1} \frac{x-6}{(y-2)^2} + g'(y) = \frac{-10(x-6)}{(x-6)^2+(y-2)^2}$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = c$$

$$u = 10 \tan^{-1} \frac{x-6}{y-2} + c$$ Ans.

b) $u = 9x^2y - 7x^2 - 3y^3 + 7y^2$

$$\Rightarrow \frac{\partial u}{\partial x} = 18xy - 14x$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 18y - 14$$

$$\therefore \frac{\partial u}{\partial y} = 9x^2 - 9y^2 + 14y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -18y + 14$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0$$ [u is harmonic]

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow v(x,y) = \int (18xy - 14x)\, dy$$

$$= 9xy^2 - 14xy + g(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(9xy^2 - 14xy + g(x)\right) = -9x^2 + 9y^2 - 14y$$

$$\Rightarrow \cancel{9y^2} - \cancel{14y} + g'(x) = \cancel{9y^2} - \cancel{14y} - 9x^2$$

$$\Rightarrow g'(x) = -9x^2$$

$$\Rightarrow x = -3x^3 + C$$

$$\therefore v(x,y) = 9xy^2 - 14xy - 3x^3 + C \quad \text{As.}$$

9) $v = 7e^{-9x}\cos(9y) - 7e^{5y}\sin(5x)$

$$\frac{\delta v}{\delta x} = 7\cos 9y\,(-9)e^{-9x} - 7.5e^{5y}\cos 5x$$

$$= -63e^{-9x}\cos 9y - 35e^{5y}\cos 5x$$

$$\frac{\delta^2 v}{\delta x^2} = 567e^{-9x}\cos 9y + 175e^{5y}\sin 5x$$

$$\frac{\delta v}{\delta y} = -63e^{-9x}\sin 9y - 35e^{5y}\sin 5x$$

$$\frac{\delta^2 v}{\delta y^2} = -567e^{-9x}\cos 9y - 175e^{5y}\sin 5x$$

$\therefore \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \quad \therefore$ v is harmonic

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$$

$$\Rightarrow u = \int \left(-63 e^{-9x} \sin 9y - 35 e^{5y} \sin 5x\right) \delta x$$

$$\Rightarrow u = 7 e^{-9x} \sin 9y + 7 e^{5y} \cos 5x + g(y)$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$\Rightarrow 7 e^{-9x} \cos 9y \cdot 9 + 7 \cdot 5\, e^{5y} \cos 5x + g'(y) = 63 e^{-9x} \cos 9y + 35 e^{5y} \cos 5x$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$$

$$\therefore u = 7 e^{-9x} \sin 9y + 7 e^{5y} \cos 5x + c$$

d) $u = 5 \sin 5x \cosh(5y)$

$$\Rightarrow \frac{\partial u}{\partial x} = 25 \cos 5x \cosh(5y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -125 \sin 5x \cosh(5y)$$

$$\frac{\partial u}{\partial y} = 25 \sin 5x \sinh(5y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = 125 \sin 5x \cosh(5y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (u \text{ is harmonic})$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow v = 25 \cos 5x \int \cosh(5y)\, dy$$

$$= 5 \cos 5x \sinh(5y) + g(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow 25 \sin 5x \sinh(5y) = 25 \sin 5x \sinh(5y) - g'(x)$

$\Rightarrow g'(x) = 0$

$\Rightarrow g(x) = c$

$\therefore v = 5 \cos 5x \sinh(5y) + c$

e) $v = 2ne^{-7u}\cos 7y + 2ye^{-7u}\sin 7y$

$\Rightarrow \frac{\delta v}{\delta u} = 2\cos 7y\left(e^{-7u} + 7ue^{-7u}\right) - 14ye^{-7u}\sin 7y$

$= -14ue^{-7u}\cos 7y + 2e^{-7u}\cos 7y - 14ye^{-7u}\sin 7y$

$\Rightarrow \frac{\delta^2 v}{\delta u^2} = -14\cos 7y\left(ue^{-7u}(-7) + e^{-7u}\right) + 2\cos 7y\, e^{-7u}$

$(-7) - 14y\sin 7y\, e^{-7u}(-7)$

$= 98ue^{-7u}\cos 7y - 28e^{-7u}\cos 7y + 98ye^{-7u}\sin 7y$

$\frac{\delta v}{\delta y} = -14ue^{-7u}\sin 7y + 14e^{-7u}y\cos 7y + 2e^{-7u}\sin 7y$

$\Rightarrow \frac{\delta v^2}{\delta y} = -98ue^{-7u}\cos 7y - 98ye^{-7u}\sin 7y + 28e^{-7u}\cos 7y$

$\therefore \frac{\delta^2 v}{\delta u^2} + \frac{\delta^2 v}{\delta y^2} = 0$ [v is harmonic]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow u = -14\sin 7y \int xe^{-7x}dx + 14y\cos 7y \int e^{-7x}dx + 2\sin 7y \int e^{-7x}dx$$

$$= -14\sin 7y\left[\frac{xe^{-7x}}{-7} + \frac{1}{49}e^{-7x}\right] - \frac{14}{7}y\cos y\, e^{-7x} + 2\sin 7y \frac{e^{-7x}}{-7} + g(y)$$

$$= 2xe^{-7x}\sin 7y - 2ye^{-7x}\cos 7y + g(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow 2xe^{-7x}\cos 7y \cdot 7 - 2e^{-7x}(-y\sin 7y \cdot 7 + \cos 7y) + g'(y)$$

$$= 14xe^{-7x}\cos 7y - 2e^{-7x}\cos 7y + 14ye^{-7x}\sin 7y$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = c$$

$$u = 2xe^{-7x}\sin 7y - 2ye^{-7x}\cos 7y + c$$