Integration & Antiderivative   Week 1

A function 'F' is called an antiderivative of the function 'f' on a given interval I if

$\boxed{F'(x) = f(x)}$ for all x in I or $\boxed{\int f(x)\,dx = F(x)}$

⸤(i) ⸤(ii)

(i) & (ii) represents same fact with different notation

$$F'(x) = f(x) \quad —(i)$$

$$\Rightarrow \frac{dF(x)}{dx} = f(x)$$

$$\Rightarrow dF(x) = f(x)\,dx \Rightarrow \int dF(x) = \int f(x)\,dx$$

$$F(x) = \int f(x)\,dx \quad —(ii)$$

if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again.

$$\frac{d}{dx}\left[\int \underset{\downarrow}{\boxed{f(x)\,dx}}\right] = f(x) + \underset{\downarrow}{C}$$

integrand   constant of integration

$\boxed{1}$

Some Fundamental Theorems of integral calculus:

i] If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)$$

ex $$\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}}$$

formula:

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|$$

$$= \int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1^2}}$$

$$= \left[\frac{1}{1}\sec^{-1}\frac{x}{1}\right]_{\sqrt{2}}^2$$
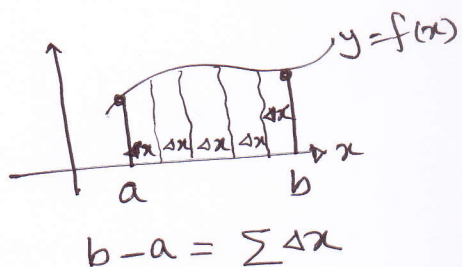
$$= \sec^{-1}(2) - \sec^{-1}(\sqrt{2})$$

ii] <u>Mean Value Theorem</u>

If f is continuous on a closed interval $[a, b]$, then there is at least one number x^* in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(x^*)(b-a) = f(x^*)\underset{\underset{\Delta x \text{ from } a \text{ to } b}{\longleftrightarrow}}{\Delta x}$$

$y = f(x)$

$a \quad b$

$b - a = \sum \Delta x$

Recall Riemann Sum & Riemann integral

$$\lim_{n \to \infty} \sum_{K=1}^{n} f(x_K^*)\,\Delta x = \int_a^b f(x)\, dx$$

$\boxed{2}$ Summation has been replaced by integration

Indefinite & Definite Integral

Some properties of integration

(i) $\int_a^a f(x)\, dx = 0$ (ii) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(iii) $\int_a^b c\, f(x)\, dx = c\int_a^b f(x)\, dx$

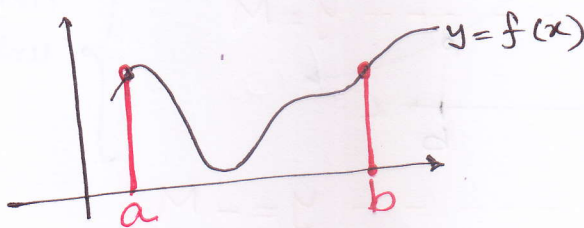(iv) $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x) \pm \int_a^b g(x)\, dx$

(v) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

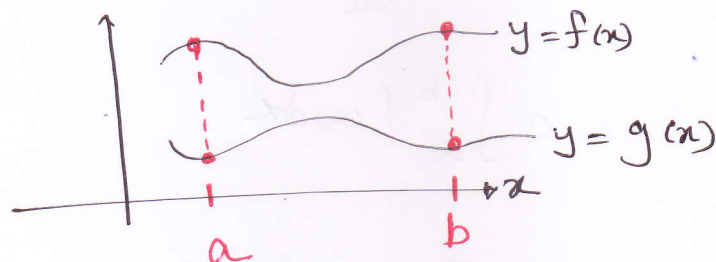while $f(x)$ is piecewise smooth curve

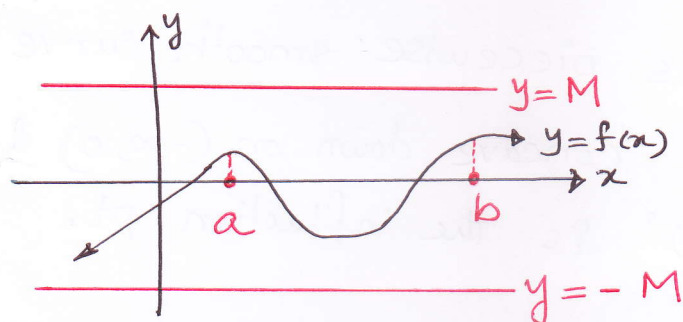Ex $y = x^3$ is concave down on $(-\infty, 0)$ & concave up on $(0, +\infty)$. '0' is the inflection pt.

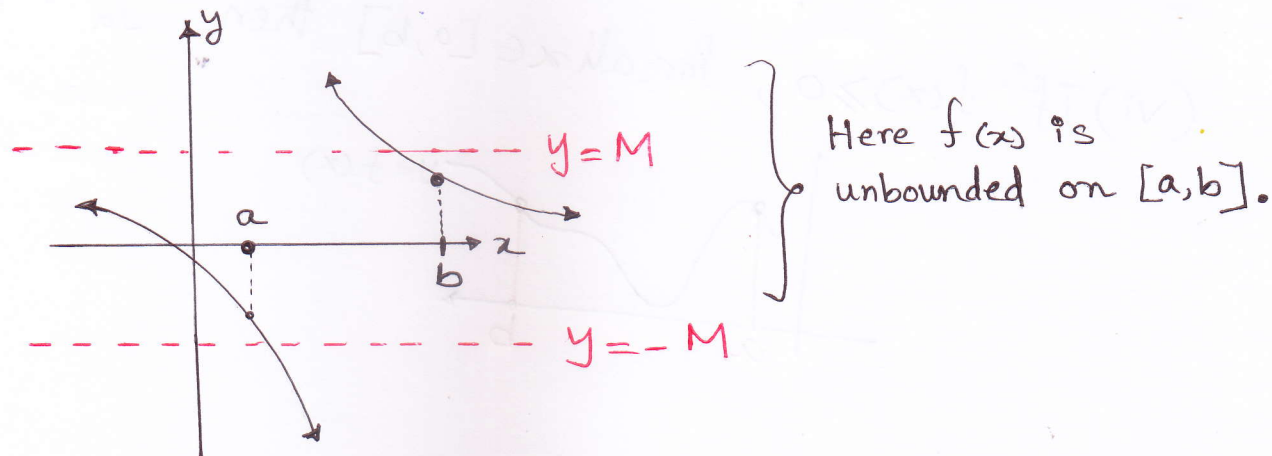(vi) If $f(x) \geqslant 0$, for all $x \in [a,b]$ then $\int_a^b f(x)\, dx \geqslant 0$

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If

(vii) $f(x) \geqslant g(x)$ for all $x \in [a,b]$

Then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$

(viii) If $f(x)$ is bounded, for example $-M \leq f(x) \leq M$ for all $x \in [a,b]$, then $f(x)$ is integrable.

(ix) If $f(x)$ is not bounded on $[a,b]$ then $f(x)$ is not integrable on $[a,b]$.

Here $f(x)$ is unbounded on $[a,b]$.

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Examples

① $\int \sin 2x \cos x \, dx$

$= \int \frac{1}{2} \left[\sin(2x+x) + \sin(2x-x) \right] dx$

$= \frac{1}{2} \int (\sin 3x + \sin x) \, dx$

$= \frac{1}{2} \left[-\frac{\cos 3x}{3} - \cos x \right] + c$

$\color{red}{\sin a \cos b}$
$\color{red}{= \frac{1}{2} \left[\sin(a+b) + \sin(a-b) \right]}$

② $\int \frac{dx}{\sqrt{1-4x^2}}$

$= \int \frac{dx}{\sqrt{1-(2x)^2}}$

$= \frac{1}{2} \int \frac{dz}{\sqrt{1-z^2}}$

$= \frac{1}{2} \sin^{-1}(z) + c$

$= \frac{1}{2} \sin^{-1}(2x) + c$

let $\quad 2x = z$
$2dx = dz$
$dx = \frac{1}{2} dz$

we know
$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$

③ $\int x \sqrt{1-x} \, dx$

$= \int (1-z) \sqrt{z} \, (-dz)$

$= -\int (\sqrt{z} - z\sqrt{z}) \, dz$

$= \int (z\sqrt{z} - \sqrt{z}) \, dz$

$= \int (z^{3/2} - z^{1/2}) \, dz = \frac{z^{3/2+1}}{\frac{3}{2}+1} - \frac{z^{1/2+1}}{\frac{1}{2}+1} + c = \frac{2}{5}z^{5/2} - \frac{2}{3}z^{3/2} + c$

let $\quad 1-x = z$
$-dx = dz$
$dx = -dz$

$\therefore 1-x = z$
$-x = z-1$
$x = 1-z$

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + c$

$\boxed{5}$

$\boxed{4}$ $\int x^2 e^{-2x^3} dx$

$$2x^3 = z$$
$$6x^2 \, dx = dz$$
$$x^2 \, dx = \frac{1}{6} dz$$

$= \frac{1}{6} \int e^{-z} dz$

$= \frac{1}{6} \left[\dfrac{e^{-z}}{-1} \right] + c$

$= -\frac{1}{6} e^{-z} + C = -\frac{1}{6} e^{-2x^3} + C$

$\boxed{5}$ $\int \dfrac{e^x + e^{-x}}{e^x - e^{-x}} dx$

Let
$$e^x - e^{-x} = z$$
$$(e^x + e^{-x}) dx = dz$$

$= \int \dfrac{dz}{z}$

$= \ln z + c = \ln (e^x - e^{-x}) + c$

$\boxed{6}$ $\int \dfrac{e^x}{1 + e^{2x}} dx$

Let
$$e^x = z$$
$$e^x \, dx = dz$$

$= \int \dfrac{e^x}{1 + (e^x)^2} dx$

$= \int \dfrac{dz}{1 + z^2} = \tan^{-1} z + c = \tan^{-1}(e^x) + c$

$\boxed{6}$

7 $\displaystyle\int \frac{\sec^2 x \, dx}{\sqrt{1-\tan^2 x}}$ $\tan x = z$

$\sec^2 x \, dx = dz$

$= \displaystyle\int \frac{dz}{\sqrt{1-z^2}} = \sin^{-1} z + C = \sin^{-1}(\tan x) + C$

8 $\displaystyle\int \frac{\sin x}{\cos^2 x + 1} \, dx$ Hint

$\displaystyle\int \frac{1}{1+x^2} \, dx = \tan^{-1} x$

⋮

substitute $\cos x = z$

⋮

$= -\tan^{-1}(\cos x) + C$