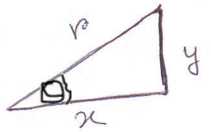
Integration by Trigonometric substitution

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \begin{cases} 1-\cos^2\theta = \sin^2\theta \\ 1-\sin^2\theta = \cos^2\theta \end{cases}$$

$$1 + \tan^2\theta = \sec^2\theta$$



$x^2+y^2=r^2$

$\tan\theta = \frac{y}{x} \Rightarrow \cot\theta = \frac{x}{y}$

$\sin\theta = \frac{y}{r} \Rightarrow \operatorname{cosec}\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r} \Rightarrow \sec\theta = \frac{r}{x}$

$\boxed{3}$ $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

let $\frac{x}{4} = \sin\theta$ $\quad\therefore \theta = \sin^{-1}\left(\frac{x}{4}\right)$

$\frac{x^2}{16} = \sin^2\theta$ $\quad\therefore x^2 = 16\sin^2\theta$

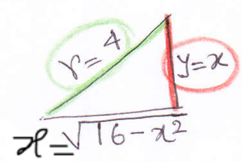
$\frac{1}{4}\,dx = \cos\theta\, d\theta$

$$= \int \frac{x^2}{4\sqrt{1-\frac{1}{16}x^2}}\,dx$$

$$= \frac{1}{4}\int \frac{x^2}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\,dx$$

$\because \sin\theta = \frac{x}{4}$, $x \to y$, $4 \to r$, $\frac{y}{r}$ $\quad\therefore \cos\theta = \frac{x}{r} = \frac{\sqrt{16-x^2}}{4}$

$$= \int \frac{16\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$$



$x = \sqrt{r^2-y^2}$

$= \sqrt{16-x^2}$

$$= \int \frac{16\sin^2\theta\cos\theta}{\cos\theta}\,d\theta$$

$$= 16\int \sin^2\theta\, d\theta$$

$$= 16\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= 8\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= 8\left[\sin^{-1}\frac{x}{4} - \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= 8\left[\sin^{-1}\frac{x}{4} - \sin\theta\cos\theta\right] + C$$

$$= 8\left[\sin^{-1}\frac{x}{4} - \frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}\right] + C$$

$$= 8\sin^{-1}\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C$$

(5) $\displaystyle\int \frac{dx}{(4+4x^2)^2}$ $\qquad 1 + \tan^2\theta = \sec^2\theta$

$$= \int \frac{dx}{\left[4(1+x^2)\right]^2} \qquad x = \tan\theta, \quad dx = \sec^2\theta\, d\theta$$

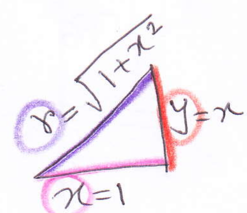
$$= \frac{1}{16}\int \frac{dx}{(1+x^2)^2}$$

$$= \frac{1}{16}\int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2} \qquad x = \tan\theta, \quad \theta = \tan^{-1}x$$

$$= \frac{1}{16}\int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta \qquad \therefore \tan\theta = \frac{y}{x}$$

$$= \frac{1}{16}\int \frac{1}{\sec^2\theta}\, d\theta \qquad \text{We have } \tan\theta = \frac{x}{1} \begin{matrix} \to y \\ \to x \end{matrix}$$

$$= \frac{1}{16}\int \frac{1}{\frac{1}{\cos^2\theta}}\, d\theta$$



$$= \frac{1}{16}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{16}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta \qquad r = \sqrt{x^2+y^2} = \sqrt{1+x^2}$$

$$= \frac{1}{32}\left[\theta + \frac{\sin 2\theta}{2}\right] + C \qquad \sin\theta = \frac{y}{r} = \frac{x}{\sqrt{1+x^2}} \qquad \cos\theta = \frac{x}{r} = \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{32}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{1}{32}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{32}\left[\tan^{-1}x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right] + C$$

$$= \frac{1}{32}\left[\tan^{-1}x + \frac{x}{1+x^2}\right] + C$$

7) $\int \frac{\sqrt{x^2-9}}{x}\,dx$

$= \int \frac{\sqrt{9\left(\frac{x^2}{9}-1\right)}}{x}\,dx$

$= 3\int \frac{\sqrt{\left(\frac{x}{3}\right)^2-1}}{x}\,dx$

$= \cancel{3}\int \frac{\sqrt{\sec^2\theta-1}}{\cancel{3\sec\theta}}\left(\cancel{3\sec\theta}\tan\theta\right)d\theta$

$= +3\int \tan\theta\left(\tan\theta\right)d\theta$

$= 3\int \tan^2\theta\, d\theta$

$= 3\int \left(\sec^2\theta-1\right)d\theta$

$= 3\left[\tan\theta-\theta\right]+C$

$= 3\left[\frac{\sqrt{x^2-9}}{3}-\sec^{-1}\left(\frac{x}{3}\right)\right]+C$

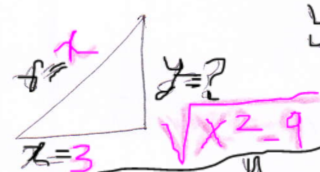
$\sec^2 x\theta - 1 = \tan^2 x\theta$

$\frac{x}{3} = \sec\theta \quad \left(\because \left(\frac{x}{3}\right)^2 = \sec^2\theta\right.$

$\frac{1}{3}dx = \sec\theta\tan\theta\, d\theta \quad x = 3\sec\theta$

$\therefore dx = 3\sec\theta\tan\theta\, d\theta \quad \therefore \theta = \sec^{-1}\left(\frac{x}{3}\right)$

$\sec\theta = \frac{x}{3} = \frac{r}{x}$



$y = \sqrt{r^2-x^2}$

$= \sqrt{x^2-9}$

$\boxed{\therefore \tan\theta = \frac{y}{x} = \frac{\sqrt{x^2-9}}{3}}$

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$$\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\tan\theta\, d\theta}{\sec\theta\sqrt{\tan^2\theta}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec\theta}\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cos\theta\, d\theta$$

$$= \Big[+\sin\theta\Big]_{\pi/4}^{\pi/3}$$

$$= +\left[\sin\frac{\pi}{4} - \sin\frac{\pi}{3}\right]$$

$$= +\left[\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\right]$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

$\sec^2\theta - 1 = \tan^2\theta$

let

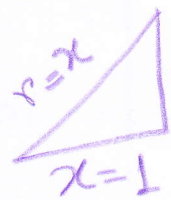
$x = \sec\theta \rightarrow \theta = \sec^{-1} x$

$x^2 = \sec^2\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$$\begin{cases} x=\sqrt{2} \Rightarrow \theta = \sec^{-1}\sqrt{2} = \pi/4 \\ x = 2 \Rightarrow \theta = \sec^{-1} 2 = \pi/3 \end{cases}$$

$\sec\theta = \frac{x}{1}$ | $\sec\theta = \frac{r}{x}$



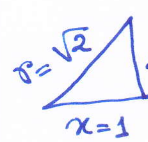
$y = \sqrt{r^2 - x^2}$

$= \sqrt{x^2-1}$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{x^2-1}}{1}$

$= \sqrt{x^2-1}$

$$\begin{cases} \sec^{-1}\sqrt{2} = \theta \\ \sec\theta = \sqrt{2} = \frac{r}{x} = \frac{\sqrt{2}}{1} \end{cases}$$

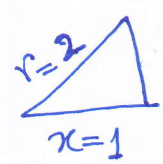


$\therefore y = 1$

$\tan\theta = \frac{y}{x} = 1$

$\therefore \theta = 45^\circ$

$$\begin{cases} \sec^{-1} 2 = \theta \\ \sec\theta = 2 = \frac{r}{x} = \frac{2}{1} \end{cases}$$



$\cos\theta = \frac{1}{\sec\theta}$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\frac{1}{2}$

$= 60^\circ = \frac{\pi}{3}$

Examples

$\boxed{1}$ $\int \frac{dx}{x^2\sqrt{4-x^2}}$

$= \int \frac{dx}{x^2\sqrt{4(1-\frac{x^2}{4})}}$

$= \frac{1}{2}\int \frac{dx}{x^2\sqrt{1-(\frac{x}{2})^2}}$

$= \frac{1}{2}\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{1-\sin^2\theta}}$

$= \int \frac{\cos\theta\, d\theta}{4\sin^2\theta\sqrt{\cos^2\theta}}$

$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \csc^2\theta\, d\theta$

$= \frac{1}{4}(-\cot\theta) + C$

$= -\frac{1}{4}\cot\theta + C$

$\frac{x}{2} = \sin\theta$

$\frac{1}{2}dx = \cos\theta\, d\theta$

$dx = 2\cos\theta\, d\theta$

$\therefore x = 2\sin\theta$

$x^2 = 4\sin^2\theta$

$\therefore \theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\text{cosec} = \csc$

$x=1 \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} = 30^\circ$

$x=\sqrt{2} \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} = 45^\circ$

Solve $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

$= \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta$

$= -\frac{1}{4}\left[\cot\theta\right]_{\pi/6}^{\pi/4}$

$= -\frac{1}{4}\left[\frac{1}{\tan\theta}\right]_{\pi/6}^{\pi/4}$

$= -\frac{1}{4}\left[\frac{1}{\tan\frac{\pi}{4}} - \frac{1}{\tan\frac{\pi}{6}}\right]$

$= -\frac{1}{4}\left[\frac{1}{1} - \frac{1}{\frac{1}{\sqrt{3}}}\right]$

$= -\frac{1}{4}\left[1 - \sqrt{3}\right]$

$= \frac{\sqrt{3}-1}{4}$

$\boxed{2}$ $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$

$$= \int \frac{x}{x^2-2\cdot x\cdot 2+2^2+4}\,dx$$

let $u = x-2$

$du = dx$

$$= \int \frac{x}{(x-2)^2+4}\,dx$$

$\therefore x = u+2$

$$= \int \frac{u+2}{u^2+4}\,du$$

$$= \int \frac{u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du$$

$u^2+4 = z$

$2u\,du = dz$

$u\,du = \frac{1}{2}dz$

$$= \int \frac{\frac{1}{2}dz}{z} + 2\int \frac{du}{4\left(\frac{u^2}{4}+1\right)}$$

$$= \frac{1}{2}\int \frac{1}{z}\,dz + \frac{1}{2}\int \frac{du}{\left(\frac{u}{2}\right)^2+1}$$

$$= \frac{1}{2}\ln z + \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln(u^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln\left[(x-2)^2+4\right] + \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$$