In a matrix the first non zero element is called a pivot.

row echelon - if there is a zero row it will be at bottom and every elements below pivot will be zero.

$$\begin{pmatrix} a & 7 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

reduced row echelon - the matrix has to be row echelon. All the pivots will be one and every elements above them will be zero.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

No solution $\rightarrow$ $0 =$ non zero

Unique solution $\rightarrow$ # unknowns = # pivots

Many solution $\rightarrow$ # unknowns > # pivots

Row picture

$2x + 3y = 7$

$3x + 2y = 8$

take two random points for each equation and plot them as straight lines. (point out the solution) column picture

$x + 3y = -3$

$2x - y = 8$

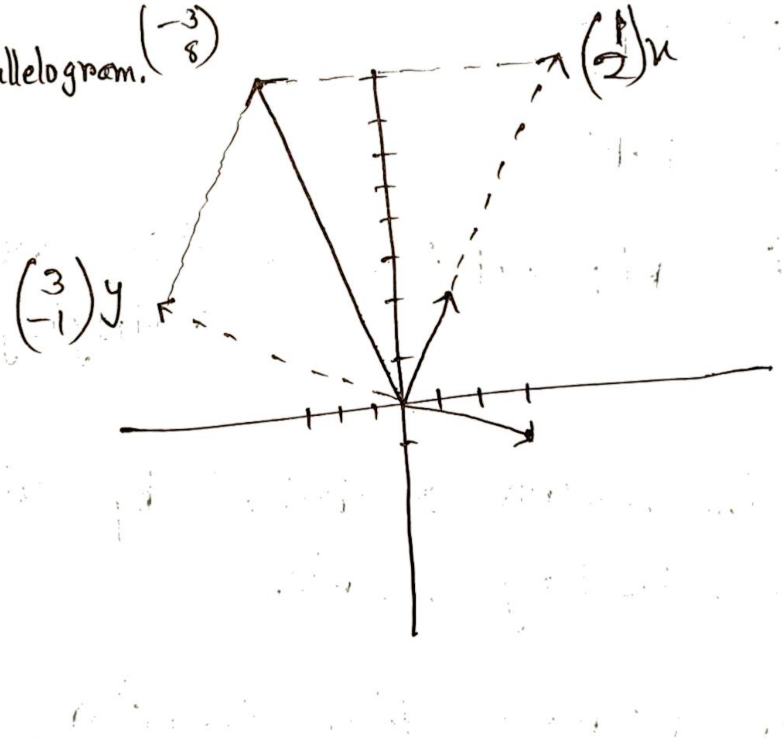
find the solution first $(3,-2)$

write the equations as vector,

$$\begin{pmatrix}1\\2\end{pmatrix}x + \begin{pmatrix}3\\-1\end{pmatrix}y = \begin{pmatrix}-3\\8\end{pmatrix}$$

plot the vector and multiply with solution.

Then make a parallelogram.



If a system has unique solution it can be solved with inverse matrix.

$$A \mid I$$

$A \cdot I = A$

make this (I)

this part will be A^{-1}

$AA^{-1} = I$

Elementary row operation

i. $R_i \longleftrightarrow R_j$

ii. $R_i' = k \cdot R_i$

iii. $R_i' = R_i \pm k \cdot R_j$ → the row we are modifying won't change

permutation matrix → obtained by doing One row/column swap on an identity matrix.

Elementary matrix → doing elementary operation once on an identity matrix.

$* \rightarrow$ binary operation

closure

G is closed under $*$ if $a*b \in G$ for all $a, b \in G$

Vector

vector + vector ✓

scalar . vector ✓

vector . vector ✗

- A vector will fullfill 10 axioms/statements

Vector Supspace

A4, A1, M1 axiom has to hold here

linear combination

show $\vec{v}$ can be expressed as linear com. of $\vec{v_1}$ & $\vec{v_2}$

Let, $\vec{v} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2}$

$(x_0, y_0) = (\alpha_1 x_1, \alpha_1 y_1) + (\alpha_2 x_2, \alpha_2 y_2)$

$(x_0, y_0) = (\alpha_1 x_1 + \alpha_2 x_2), (\alpha_1 y_1 + \alpha_2 y_2)$

now solve,

if
- → No. Sol^n — No linear com.
- → unique — " "
- → many — " " [write the ans. as $t = n$] $n \in \mathbb{R}$

Linear Independence/dependence

a set of vectors (v_1, v_2, v_3) is linearly ind. if $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \vec{0}$ has unique (only zero) solution. dep. is many solution exists.

Span

take a random $\vec{v}$ $\boxed{\vec{v} = (a, b)}$

given $\vec{v_1}, \vec{v_2}$ — do they span over W

$\therefore \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} = \vec{v}$

find α_1, α_2

if no sol^n $\rightarrow$ do not span

> A set of vectors $S = \{u_1, u_2, \ldots, u_k\} \subset \mathbb{R}^n$ can not span $\mathbb{R}^n$ if $k < n$.

Basis → minimum spanning set

S will be basis for V

if
- → S is linearly independent
- → S spans V

Rowspace / col space

$$A = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

do row ~~e~~ ecelon.

row that has non zero element is on Basis of row space

col that has pivot is on Basis of col space.

Row(A) = Dimension of Basis of row space.

Null space

$$A = \begin{pmatrix} x_a & x_b & x_c \\ x_d & x_e & f \\ x_g & h & i \end{pmatrix} \begin{pmatrix} \alpha_1 x_1 & \beta_1 x_2 & \gamma x_3 \\ \alpha_2 x_1 & \beta_2 x_2 & \gamma x_3 \\ \alpha_3 x_1 & x_2 & x_3 \end{pmatrix}$$

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$ Null space (T)

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \end{array}\right)$$

$$\therefore \left(\begin{array}{ccc|c} \alpha_1 x_1 & \beta_1 x_2 & \gamma x_3 & 0 \\ \alpha_2 x_1 & \beta_2 x_2 & \gamma x_3 & 0 \\ \alpha_3 x_1 & x_2 & x_3 & 0 \end{array}\right)$$

Basis of null space $= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Nullity = dimension of null space

left Null space

$A = M_{3\times3}$

Let, $(x,y,z) \in$ Left Null space (T)

$\therefore (x,y,z)\, A = (0,0,0)$

if $sol^n = 0$

then Basis $= \{\ \}$

$$\boxed{\begin{aligned} &\text{Row}(A) + \text{Null}(A) = \#\text{ columns} \\ &\text{col}(A) + \text{lefNull}(A) = \#\text{ Rows} \end{aligned}}$$

Linear Transformation

$T: m \longrightarrow n$

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
$T(\alpha \vec{u}) = \alpha T(\vec{u})$

if a transformation follows these for all vectors than that is a linear transformation.

If $T: V \to V$ is a linear transformation then T is a linear operator.

$T: R^2 \to R^3$

$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\7\\4\end{pmatrix}$, $T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}$, find $T\begin{pmatrix}x\\y\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix} = \alpha\begin{pmatrix}2\\1\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix} \longrightarrow (1)$$

$$\Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\alpha+\beta\\ \alpha+2\beta\end{pmatrix}$$

$$\therefore \left(\begin{array}{cc|c} 2 & 1 & \alpha \\ 1 & 2 & \beta \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|c} 2 & 1 & \alpha \\ 0 & 3 & \beta \end{array}\right)$$

$\therefore \alpha = \boxed{a_1}$, $\beta = \boxed{a_2}$

$$(1) \Rightarrow T\begin{pmatrix}x\\y\end{pmatrix} = \alpha\, T\begin{pmatrix}2\\1\end{pmatrix} + \beta\, T\begin{pmatrix}1\\2\end{pmatrix}$$

$$= a_1\begin{pmatrix}3\\7\\4\end{pmatrix} + a_2\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3a_1+a_2\\7a_1+2a_2\\4a_1+a_2\end{pmatrix}$$

Matrix representation

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

find $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ → Standard Basis

and write the side by side $\begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$

axis

$$\begin{array}{ccc} x \rightarrow & a & b \\ y \rightarrow & c & d \end{array}$$

b, c → shear

a, d → scaling

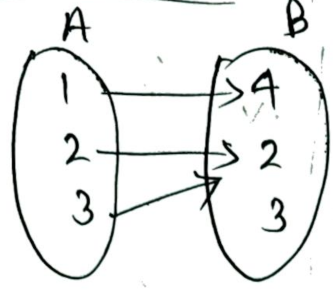
$x_1 + 2x_2 + 3x_3$

$2x_2 + 4x_3$

$4x_1 + x_2 + x_3$

Mat rep → $\begin{matrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 4 & 1 & 1 \end{matrix}$

Image / Range



$f : A \to B$

Domain $= \{1, 2, 3\}$

codomain $= \{4, 2, 3\}$

Image $= \{4, 2\}$

$Im(T)$ is the column space of T

$Rank(T) = Dim(Image)$

kernel (nullspace)

those set whose output is 0.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrix representation

$$\Rightarrow A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$T : U \to V$

$\text{Rank}(T) + \text{Nullity}(T) = \text{Dim}(U)$

Inverse

One - one and onto ?

codomain = Range

$\downarrow$

1 input
1 output

$T : U \to V$

T is one-one if $\ker(T) = 0$ / $\text{Nullity}(T) = 0$

T is onto if $\text{Rank}(T) = \text{Dim}(V)$

$R^5 \to R^3$ can not be one-one

$R^3 \to R^5$ " " " onto

check Assigement

To find eigenvector, first find the eigen values. Then find $\vec{v}$ for every eigen value.

$$(A-\lambda I)\vec{v}=0$$
$$\downarrow$$
$$S\vec{v}=0$$

find null vector of S.

If eigen value is same we will find the eigenvector once. ~~(we still will get same amount of eigenvector)~~

Diagonalization

Similar matrices

$A, B \in n \times n$

$P \in n \times n$

A & B are similar if,

$$\cancel{A = PBP^{-1}}$$

Also,

$A, B \Rightarrow$ same eigen value w/ main

Determinant same

Trace same

$A = PDP^{-1}$

$(n \times n)$ ↳ Diagonal (non zero in diagonal)

A is diagonalizable if it has n linearly independent eigenvectors.

n ↳ dimention

D → eigenvalues in main diagonal

P → corresponding eigenvectors in columns.

$A = PDP^{-1}$

$A^2 = P D \underbrace{P^{-1} \cdot P}_{I} D P^{-1} = P D I D P^{-1} = P D^2 P^{-1}$

$A^3 = A^2 A = P D^2 P^{-1} \cdot P D P^{-1} = P D^3 P^{-1}$

$\therefore A^n = P D^n P^{-1}$

Inner Product

is like dot product

$\vec{u} \cdot \vec{v} \equiv \langle \vec{u}, \vec{v} \rangle$

$V \in \mathbb{R}$

Inner product of V is a function that takes two vector as input and gives a number.

$\langle \vec{u}, \vec{v} \rangle \in \mathbb{R}$

Inner product has following properties,

$\langle \vec{u}, \vec{u} \rangle \geq 0$

$\langle \vec{u}, \vec{u} \rangle = 0$ if $\vec{u} = 0$

$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$\langle \lambda \vec{u}, \vec{v} \rangle = \lambda \langle \vec{u}, \vec{v} \rangle$ $\qquad \lambda \in \mathbb{R}/\mathbb{C}$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ $\qquad$ for $\mathbb{C}$, $\mathbb{R}$ will be replaced

$\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ for complex field standard inner product in $\mathbb{R}^n$

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

standard inner product in $\mathbb{C}^2$

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$$

For Matrix,

Let $A, B \in m \times n$

$$\langle A, B \rangle = \text{Trace}(A^{\dagger} B)$$

for function,

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, g(x)\, dx$$ — will be given

Norm

Generalization of length

Eucledean Norm

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

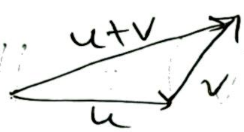
A function that assign a length to a vector.

Properties:

i. $\|\vec{u}\| \geq 0$

ii. $\|\vec{u}\| = 0$ if $\vec{u} = 0$

iii. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \longrightarrow$



iv. $\|\lambda \vec{u}\| = |\lambda| \, \|\vec{u}\|$

$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$$

Metric

Abstraction of distance

$$d((a,b),(c,d)) = \sqrt{(a-c)^2 + (b-d)^2}$$

properties:

i. $d(\vec{u},\vec{v}) \geq 0$

ii. $d(\vec{u},\vec{v}) = 0$ if $\vec{u} = \vec{v}$

iii. $d(\vec{u},\vec{w}) \leq d(\vec{u},\vec{v}) + d(\vec{v},\vec{w}) \rightarrow \triangle$

iv. $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$

$$d(\vec{u},\vec{v}) = \| \vec{u} - \vec{v} \|$$

Orthogonality

measuring if two vectors are at 90°.

Two vectors are called orthogonal if,

$$\langle \vec{u}, \vec{v} \rangle = 0$$

~~Orthonormal set~~

Orthogonal vectors / set

$S = \{ \vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \}$ is orthogonal if,

$$\langle \vec{v_m}, \vec{v_n} \rangle = 0 \quad \text{for all } m \neq n$$

every vector of S.

Orthonormal set

$$\langle \vec{v_m}, \vec{v_n} \rangle = 0 \quad \text{for all } m \neq n$$

$$\langle \vec{v_m}, \vec{v_n} \rangle = 1 \quad \text{for all } m = n$$

norm / length

Orthogonal → Orthonormal

$$\frac{\vec{V_k}}{\|\vec{V_k}\|} = \vec{V_k'}$$

for orthonormality / orthogonality of a set we will chose 2 vectors at a time from that set. We have to make combination of one vector with all the others.

$\sin(n\pi) = 0 \qquad n \in \mathbb{Z}$

$\cos(n\pi) = (-1)^n$

Orthogonal complement

ortho. com. of set S is $S^{\perp}$

$$\langle \vec{u}, \vec{v} \rangle = 0 \text{ for all } \vec{u} \in S \text{ \& } \vec{v} \in S^{\perp}$$

all vectors of $S^{\perp}$ will be perpendicular to all vectors of S

$$W = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$$

Let, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in W^{\perp}$

$$\therefore \left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle = 0 \Rightarrow x_1 + 2x_2 - x_3 = 0$$

$$\therefore \left\langle \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle = 0 \Rightarrow 2x_1 + 4x_2 + x_3 = 0$$

solve

Orthogonal projection

Or. pro. of $\vec{u}$ onto $\vec{v}$,

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

$$\vec{u} \cdot \vec{v} = |\vec{u}| \, |\vec{v}| \cos\theta$$

$$\Rightarrow |\vec{u}| \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

$$\Rightarrow \text{Projection of } u = \frac{\langle \vec{u}, \vec{v} \rangle}{\sqrt{\langle \vec{v}, \vec{v} \rangle}}$$

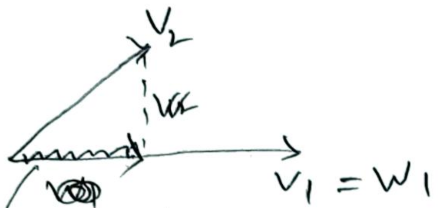
unique vec.

$$\Rightarrow \text{'' (with dir.)} = \frac{\langle \vec{u}, \vec{v} \rangle}{\sqrt{\langle \vec{v}, \vec{v} \rangle}} \cdot \frac{\vec{v}}{\sqrt{\langle \vec{v}, \vec{v} \rangle}}$$

$$= \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

Gram-Schmidt Procedure

Basis $\xrightarrow{\text{to}}$ orthogonal basis



$$\frac{\langle \vec{v_2}, \vec{w_1} \rangle}{\langle \vec{w_1}, \vec{w_1} \rangle} \vec{w_1}$$

$$w_2 = \vec{v_2} - \frac{\langle \vec{v_2}, \vec{w_1} \rangle}{\langle \vec{w_1}, \vec{w_1} \rangle} \vec{w_1}$$

$$w_3 = \vec{v_3} - \frac{\langle \vec{v_3}, \vec{w_1} \rangle}{\langle \vec{w_1}, \vec{w_1} \rangle} \vec{w_1} - \frac{\langle \vec{v_3}, \vec{w_2} \rangle}{\langle \vec{w_2}, \vec{w_2} \rangle} \vec{w_2}$$

proj of $\vec{v_3}$ on $\vec{w_1}$

" " " " $\vec{w_2}$

for orthonormal divide each of the with value.

Consider orthogonal basis $B = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

Expressing $\vec{v}$ as a linear com. of B

$$\vec{v} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3}$$

$$\Rightarrow \langle \vec{v}, \vec{v_1} \rangle = \alpha_1 \langle \vec{v_1}, \vec{v_1} \rangle + \alpha_2 \underbrace{\langle \vec{v_2}, \vec{v_1} \rangle}_{0} + \alpha_3 \underbrace{\langle \vec{v_3}, \vec{v_1} \rangle}_{0}$$

as orthogonal

$$\Rightarrow \alpha_1 = \frac{\langle \vec{v}, \vec{v_1} \rangle}{\langle \vec{v_1}, \vec{v_1} \rangle}$$

& do it for α_2, α_3.