MAT 120
DIFFERENTIAL EQN (DE)

> **Homogeneous Linear Eqn**
> **with constant coefficients**

Consider the nth order DE below:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_2 y'' + a_1 y' + a_0 y = g(x) \quad \text{——} \textcircled{i}$$

if $g(x) = 0$ then $\textcircled{i}$ is a homogenous DE

if $g(x) \neq 0$ then $\textcircled{i}$ is a non-homogeneous DE

<u>Consider</u>

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \quad \text{——} \textcircled{ii}$$

(homogeneous DE)

<u>Recall 1st order DE</u>

$$a_1 y' + P(x) y = f(x) \quad \text{——} \textcircled{iii}$$

if $\textcircled{iii}$ is homogeneous DE

then $a_1 y' + P(x) y = 0$

Replace $a_1 = a$ & $P(x) = b$

Now we have $ay' + by = 0$

$$\frac{ay'}{a} + \frac{b}{a} y = 0$$

$$y' + \frac{b}{a} y = 0$$

$$\textcircled{iv} \longrightarrow y' + (m) y = 0 \left\{ \begin{array}{l} \text{Let} \quad \frac{b}{a} = m \\ \text{Relabel constant} \end{array} \right\}$$

I.F. $e^{\int P(x) dx} = e^{\int m \, dx} = e^{mx}$

$$e^{mx} [y' + my] = e^{mx} (0) \quad \text{multiply } \textcircled{iv} \text{ with I.F.}$$

$$\int e^{mx} [y' + my] = \int 0 \, dx$$

$$y e^{mx} = c$$

$$y = c e^{-mx}$$

$$y = e^{mx} \longrightarrow \begin{array}{l} \text{Relabel constants:} \\ \text{Consider } c = 1, \quad \dfrac{-m = +m}{\text{Replace}} \end{array}$$

$\therefore$ we found $y = e^{mx}$

$\therefore y' = me^{mx}$

$y'' = m^2 e^{mx}$

$y''' = m^3 e^{mx}$

$\vdots$

$y^{(n-1)} = m^{n-1} e^{mx}$

$y^{(n)} = m^n e^{mx}$

Substitute $y, y', y'', \dots, y^{(n)}$ into $\textcircled{ii}$

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \dots + a_2 m^2 e^{mx} + a_1 m e^{mx} + a_0 e^{mx} = 0$$

$\textcircled{v}$

eqn $\textcircled{v}$ may have distinct roots, repeating roots or complex roots.

Note eqn $\textcircled{v}$ is known as <u>Auxiliary Equation.</u>

<u>Roots</u>

Consider the following quadratic eqns:

(a) $2m^2 - 5m - 3 = 0$

$\Rightarrow (2m+1)(m-3) = 0$

$\Rightarrow 2m+1 = 0 \; ; \; m-3 = 0$

$\Rightarrow m_1 = -\frac{1}{2}, \; m_2 = 3$

here the solutions m_1, m_2 are distinct roots

(b) $m^2 - 10m + 25 = 0$

$\Rightarrow (m-5)^2 = 0$

$\Rightarrow (m-5)(m-5) = 0$

$\Rightarrow m_1 = 5, \; m_2 = 5$

here the solutions m_1, m_2 are repeating roots

$\textcircled{2}$

Considering (a)

$$2m^2 - 5m - 3 = 0 \quad \text{------ Auxiliary eqn (a)}$$

$$m_1 = -\tfrac{1}{2}, \quad m_2 = 3$$

∴ the solutions are unique

∴ the solution of Auxiliary eqn (a) can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{-\tfrac{1}{2}x} + c_2 e^{3x}$$

$$= c_1 f_1(x) + c_2 f_2(x) \quad \begin{bmatrix} \text{observe } f_1(x) \, \& \, f_2(x) \\ \text{are independent} \end{bmatrix}$$

Considering (b)

$$m^2 - 10m + 25 = 0 \quad \text{------ Auxiliary eqn (b)}$$

$$m_1 = 5, \quad m_2 = 5$$

∴ the solutions are repeating

∴ the solution of Auxiliary eqn (b) can be written as

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

$$= c_1 e^{5x} + c_2 x e^{5x}$$

$$= c_1 f_1(x) + c_2 f_2(x) \quad [\text{observer } f_1(x) \, \& \, f_2(x) \text{ are}$$

now independent since
we introduced x along with
c_2, otherwise the solution
will look like $\underbrace{c_1 e^{5x} + c_2 e^{5x}}_{\text{linearly dependent}}$

But we are looking forward
to the independent set of solutions
hence introduced x with c_2.

③

Going back to eqn $\textcircled{v}$

→ if the roots are <u>non repeating</u> : $m_1 \neq m_2 \neq m_3 \cdots \neq m_n$

then solution of $\textcircled{v}$ will be:

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$$

→ if the roots are <u>repeating</u> : $m_1 = m_2 = m_3 \cdots = m_n$

then the solution of $\textcircled{v}$ will be:

$$y = C_1 e^{m_1 x} + C_2 x e^{m_2 x} + C_3 x^2 e^{m_3 x} + C_4 x^3 e^{m_4 x} + \cdots$$
$$\cdots + C_n x^{n-1} e^{m_n x}$$

→ if the roots are <u>complex number</u>:

$$m = a \pm bi \Rightarrow \begin{matrix} m_1 = a + bi \\ m_2 = a - bi \end{matrix}$$

then the solution of $\textcircled{v}$ will be:

$$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx$$

$\boxed{\text{Exercise}}$

Find the General Solution of the given 2nd order differential eqn:

$\textcircled{1}$ $3y'' + 2y' + y = 0$ —— $\textcircled{e}$

substitute $y = e^{mx}$, $y' = m e^{mx}$, $y'' = m^2 e^{mx}$ into $\textcircled{1}$

$$3m^2 e^{mx} + 2m e^{mx} + e^{mx} = 0$$

$$3m^2 + 2m + 1 = 0 \quad \left(\div \text{ by } e^{mx}\right)$$

$$m = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

$$= \frac{-2 \pm \sqrt{-8}}{6} = \frac{-2 \pm i2\sqrt{2}}{6} = \frac{-1 \pm i\sqrt{2}}{3} = -\frac{1}{3} \pm \frac{i\sqrt{2}}{3}$$

$$\therefore y = C_1 e^{-\frac{1}{3}x} \cos\left(\frac{\sqrt{2}}{3}\right)x + C_2 e^{-\frac{1}{3}x} \sin\left(\frac{\sqrt{2}}{3}\right)x$$

(2) $y'' - 3y' + 2y = 0$ —①

substitute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$ into ①

$m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$m^2 - 3m + 2 = 0$ ($\div$ by e^{mx})

$m^2 - 2m - m + 2 = 0$

$(m-2)(m-1) = 0$

$m_1 = 2, \quad m_2 = 1$

$\therefore y = C_1 e^{2x} + C_2 e^{x}$

(3) $y'' + 8y' + 16y = 0$ —①

substitute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$ into ①

$m^2 e^{mx} + 8me^{mx} + 16e^{mx} = 0$

$m^2 + 8m + 16 = 0$ ($\div$ by e^{mx})

$m^2 + 2 \cdot m \cdot 4 + 4^2 = 0$

$(m+4)^2 = 0$

$(m+4)(m+4) = 0 \Rightarrow m_1 = m_2 = -4$

$\therefore y = C_1 e^{-4x} + C_2 x e^{-4x}$

Find the General solution of higher order DE:

(4) $16 \dfrac{d^4 y}{dx^4} + 24 \dfrac{d^2 y}{dx^2} + 9y = 0$ —①

$16 y^{(4)} + 24 y'' + 9y = 0$

substitute $y = e^{mx}$, $y'' = m^2 e^{mx}$, $y^{(4)} = m^4 e^{mx}$ into ①

$16 m^4 e^{mx} + 24 m^2 e^{mx} + 9 e^{mx} = 0$

$16 m^4 + 24 m^2 + 9 = 0$

$(4m^2)^2 + 2 \cdot 4m^2 \cdot 3 + 3^2 = 0 \Rightarrow (4m^2 + 3)^2 = 0$

$\Rightarrow (4m^2 + 3)(4m^2 + 3) = 0$

$4m^2+3=0$

$m_1=m_2=\pm\sqrt{\frac{-3}{4}}$

$4m^2+3=0$

$m_3=m_4=\pm\sqrt{\frac{-3}{4}}$

$=\pm\frac{i\sqrt{3}}{2}$

$=0\pm i\sqrt{3}/2$

$=\pm\frac{i\sqrt{3}}{2}$

$=0\pm i\sqrt{3}/2$

m_3, m_4 are repeating roots of m_1, m_2

$\therefore y = C_1 e^{0\cdot x}\cos\left(\frac{\sqrt{3}}{2}\right)x+C_2 e^{0\cdot x}\sin\left(\frac{\sqrt{3}}{2}\right)x+C_3 xe^{0\cdot x}\cos\left(\frac{\sqrt{3}}{2}\right)x$

$+C_4 xe^{0\cdot x}\sin\left(\frac{\sqrt{3}}{2}\right)x$

$=C_1\cos\left(\frac{\sqrt{3}}{2}\right)x+C_2\sin\left(\frac{\sqrt{3}}{2}\right)x+C_3 x\cos\left(\frac{\sqrt{3}}{2}\right)x+C_4 x\sin\left(\frac{\sqrt{3}}{2}\right)x$

Solve the given initial value problem:

⑤ $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=0$; $y(1)=0$; $y'(1)=2$

$y''-4y'-5y=0$ —①

substitute $y=e^{mx}$, $y'=me^{mx}$, $y''=m^2e^{mx}$ into ①

$m^2e^{mx}-4me^{mx}-5e^{mx}=0$

$m^2-4m-5=0$ $\quad(\div$ by $e^{mx})$

$m^2-5m+m-5=0$

$(m-5)(m+1)=0 \Rightarrow m_1=5, m_2=-1$

$\therefore \boxed{y=C_1e^{5x}+C_2e^{-x}}$ —②

$y(1)=C_1e^{5(1)}+C_2e^{-(1)}$

$0=C_1e^5+C_2\frac{1}{e}$ —ⓐ

& $y'=5C_1e^{5x}-C_2e^{-x}$

$y'(1)=5C_1e^{5(1)}-C_2e^{-(1)}$

$2=5C_1e^5-C_2\frac{1}{e}$ —ⓑ

ⓐ+ⓑ ⇒ $6C_1e^5=2$

$C_1=\frac{1}{3e^5}$

substitute C_1 into ⓐ

$\frac{1}{3e^5}\cdot e^5+C_2\frac{1}{e}=0$

$\frac{1}{3}+C_2\frac{1}{e}=0$

$\frac{C_2}{e}=-\frac{1}{3} \Rightarrow C_2=-\frac{e}{3}$

substitute C_1 & C_2 into ②

$y=\frac{1}{3e^5}\cdot e^{5x}-\frac{e}{3}\cdot e^{-x}$

$=\frac{1}{3}\left[e^{5(x-1)}-e^{(1-x)}\right]$