

# Infinite Integrals (Improper Integrals)

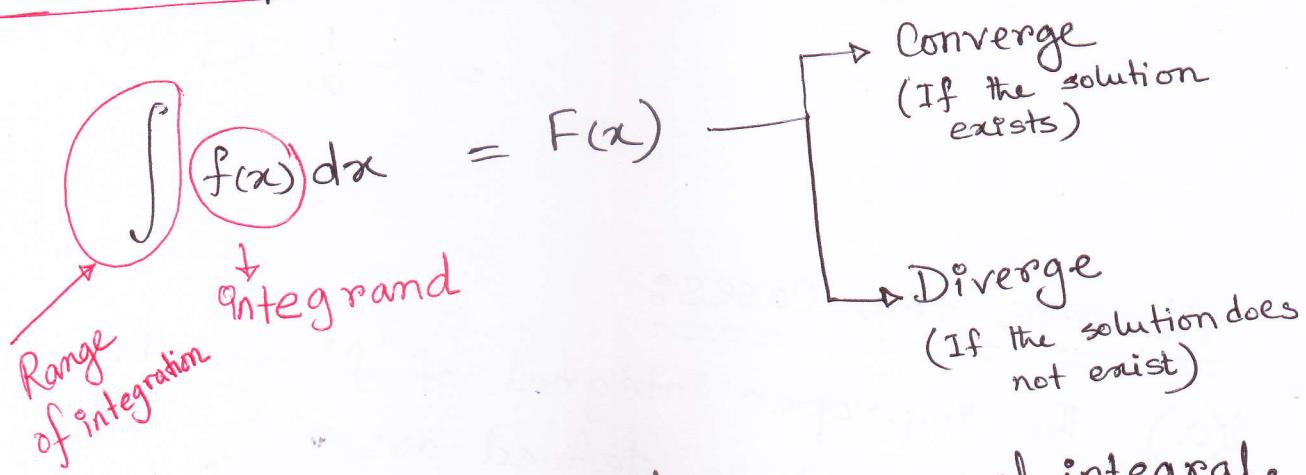
Week 1

Consider the followings:

$$\int_c^{\infty} (ax+b) dx ; \int_{-\infty}^c (ax+b) dx$$

$$\int_{-\infty}^{\infty} (ax+b) dx ; \int_a^3 \frac{1}{x-3} dx$$

An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinitely at one or more points in the range of integration.



It cannot be computed using normal integral.  
So we introduce limit.

Consider:

$$\int_1^\infty x^{-2} dx$$

$$y = \frac{1}{x^2}, x \in [1, \infty)$$



It can be computed by

replacing infinite

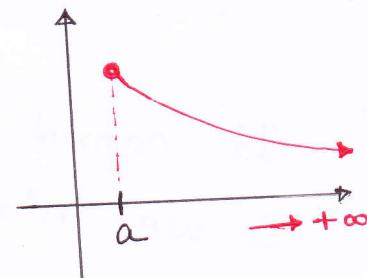
limits with finite values:

$$\begin{aligned}\int_1^\infty x^{-2} dx &= \lim_{l \rightarrow \infty} \int_1^l x^{-2} dx = \lim_{l \rightarrow \infty} \left[ \frac{x^{-2+1}}{-2+1} \right]_1^l \\ &= \lim_{l \rightarrow \infty} \left[ \frac{-1}{x} \right]_1^l \\ &= \lim_{l \rightarrow \infty} \left[ -\frac{1}{l} + 1 \right] \\ &= -\frac{1}{\infty} + 1 = 0 + 1 = 1\end{aligned}$$

Observe few cases:

1(a) The improper integral of 'f' over the interval  $[a, +\infty)$  is defined as:

$$\int_a^{+\infty} f(x) dx = \lim_{l \rightarrow +\infty} \int_a^l f(x) dx$$

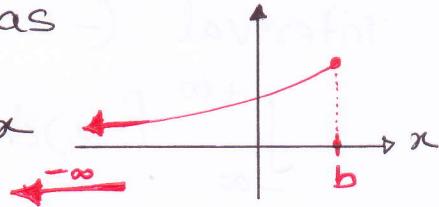


The integral is said to converge if the limit exists and diverge if it does not.

[2]

1(b) The improper integral of  $f$  over the integral  $(-\infty, b]$  is defined as

$$\int_{-\infty}^b f(x) dx = \lim_{K \rightarrow -\infty} \int_K^b f(x) dx$$



The integral is said to converge if the limit exists and diverge if it does not.

**Example**  $\int_4^{+\infty} \left( \frac{1}{x-1} - \frac{1}{x+1} \right) dx$

$$= \lim_{l \rightarrow +\infty} \int_4^l \left( \frac{1}{x-1} - \frac{1}{x+1} \right) dx$$

$$= \lim_{l \rightarrow +\infty} \left[ \ln(x-1) - \ln(x+1) \right]_4^l$$

$$= \lim_{l \rightarrow +\infty} \left[ \ln \left( \frac{x-1}{x+1} \right) \right]_4^l$$

$$= \lim_{l \rightarrow +\infty} \left[ \ln \left( \frac{l-1}{l+1} \right) - \ln \left( \frac{3}{5} \right) \right]$$

$$= \lim_{l \rightarrow +\infty} \ln \frac{l-1}{l+1} - \lim_{l \rightarrow +\infty} \ln \left( \frac{3}{5} \right)$$

$$= \lim_{l \rightarrow +\infty} \ln \frac{l+1-2}{l+1} - \lim_{l \rightarrow \infty} \ln \left( \frac{3}{5} \right)$$

$$= \lim_{l \rightarrow \infty} \ln \left( 1 - \frac{2}{l+1} \right) - \ln \left( \frac{3}{5} \right)$$

$$= \ln \left( 1 - \frac{2}{\infty+1} \right) - \ln \left( \frac{3}{5} \right)$$

$$= \ln(1-0) - \ln \left( \frac{3}{5} \right) = 0 - \ln \left( \frac{3}{5} \right) = \ln \left( \frac{5}{3} \right)$$

$\ln 1 = 0$

Converges.

2 The improper integral of  $f$  over the interval  $(-\infty, +\infty)$  is defined as

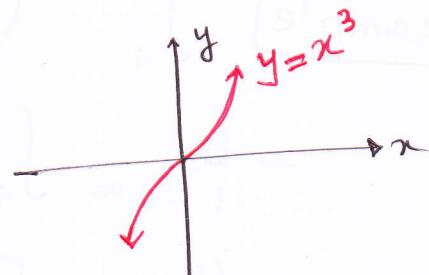
$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx$$

piecewise continuous function from  $(-\infty, c)$  &  $(c, +\infty)$ .

where 'c' is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

**Example**

$$\int_{-\infty}^{+\infty} x^3 dx$$



$$= \int_{-\infty}^0 x^3 dx + \int_0^{+\infty} x^3 dx$$

$$= \lim_{K \rightarrow -\infty} \int_K^0 x^3 dx + \lim_{l \rightarrow +\infty} \int_0^l x^3 dx$$

$$= \lim_{K \rightarrow -\infty} \left[ \frac{x^4}{4} \right]_K^0 + \lim_{l \rightarrow +\infty} \left[ \frac{x^4}{4} \right]_0^l$$

$$= \lim_{K \rightarrow -\infty} \left[ 0 - \frac{K^4}{4} \right] + \lim_{l \rightarrow +\infty} \left[ \frac{l^4}{4} - 0 \right]$$

$$= -\frac{1}{4} \lim_{K \rightarrow -\infty} K^4 + \frac{1}{4} \lim_{l \rightarrow +\infty} l^4$$

$$= \frac{1}{4} \left[ \lim_{l \rightarrow +\infty} l^4 - \lim_{K \rightarrow -\infty} K^4 \right]$$

$$= \frac{1}{4} [\infty - \infty] = \infty \text{ diverges.}$$

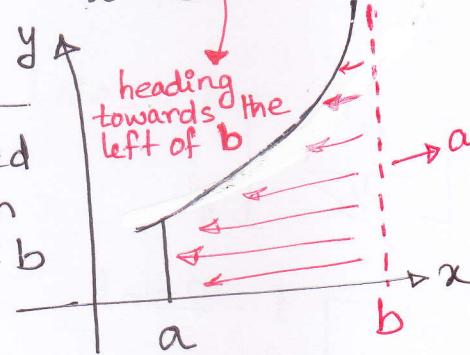
**4**

3 If  $f$  is continuous on the interval  $[a, b]$  except for an infinite discontinuity at  $b$ , then the improper integral of  $f$  over the interval  $[a, b]$  is defined as

$$\int_a^b f(x) dx = \lim_{l \rightarrow b^-} \int_a^l f(x) dx$$

The integral converges if the limit exists and diverges if it does not.

The required region is on the left of  $b$



asymptote  
(It means the curve will not touch this line or exit this line)

Example

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{\sqrt{1-2\sin x}} dx$$

$$= \lim_{l \rightarrow \frac{\pi}{6}^-} \int_0^l \frac{\cos x}{\sqrt{1-2\sin x}} dx$$

$$= \lim_{l \rightarrow \frac{\pi}{6}^-} \int_0^{\sin l} \frac{dz}{\sqrt{1-2z}}$$

Observe  
 $x \neq \frac{\pi}{6}$   $\therefore \sin \frac{\pi}{6} = \frac{1}{2}$

$$\therefore \sqrt{1-2\sin x} = \sqrt{1-2\sin \frac{\pi}{6}}$$

$$= \sqrt{1-2\left(\frac{1}{2}\right)}$$

$$= \sqrt{1-1}$$

$$= 0.$$

$$\therefore f(x) = \frac{\cos x}{0} = \infty$$

let  $\sin x = z$   
 $\cos x dx = dz$

limits:

$$x=0 \rightarrow z=\sin 0=0$$

$$x=l \rightarrow z=\sin l$$

$$= \lim_{l \rightarrow \frac{\pi}{6}^-} \int_0^{\sin l} \frac{dz}{\sqrt{1-2z}}$$

let  $1-2z = p$   
 $-2dz = dp$   
 $dz = -\frac{1}{2} dp$

$$= \lim_{l \rightarrow \frac{\pi}{6}^-} \int_1^{1-2\sin l} -\frac{1}{2} \frac{dp}{\sqrt{p}}$$

limits  
 $z=0 \rightarrow p=1$   
 $z=\sin l \rightarrow p=1-2\sin l$

$$= -\frac{1}{2} \lim_{l \rightarrow \frac{\pi}{6}^-} \int_1^{1-2\sin l} p^{-\frac{1}{2}} dp$$

$$= -\frac{1}{2} \lim_{l \rightarrow \frac{\pi}{6}^-} \left[ \frac{p^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^{1-2\sin l}$$

$$= -\lim_{l \rightarrow \frac{\pi}{6}^-} \left[ p^{\frac{1}{2}} \right]_1^{1-2\sin l}$$

$$= -\lim_{l \rightarrow \frac{\pi}{6}^-} \left[ \sqrt{1-2\sin l} - \sqrt{1} \right]$$

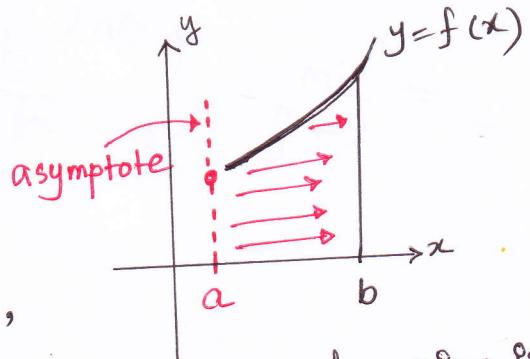
$$= -\sqrt{1-2\left(\frac{1}{2}\right)} + 1$$

$$= 1 \quad \text{Converges}$$

4 If  $f$  is continuous on the interval  $[a, b]$ , except for an infinite discontinuity at  $a$ , then the improper integral of  $f$  over the interval  $[a, b]$  is

$$\int_a^b f(x) dx = \lim_{K \rightarrow a^+} \int_K^b f(x) dx$$

The integral converges if the limit exists & diverges if it does not.



required region is  
on the right of 'a'

Example

$$\int_3^4 \frac{dx}{(x-3)^2}$$

$x \neq 3$   
otherwise  $f(x)$  is undefined.

$$= \lim_{l \rightarrow 3^+} \int_l^4 \frac{dx}{(x-3)^2}$$

$$\frac{1}{(x-3)^2} = \frac{1}{(3-3)^2} = \frac{1}{0} = \infty$$

$$= \lim_{l \rightarrow 3^+} \int_{l-3}^1 \frac{dz}{z^2}$$

let  $x-3 = z$   
 $dx = dz$

limits:  
 $x = l \rightarrow z = l-3$   
 $x = 4 \rightarrow z = 1$

$$= \lim_{l \rightarrow 3^+} \int_{l-3}^1 z^{-2} dz$$

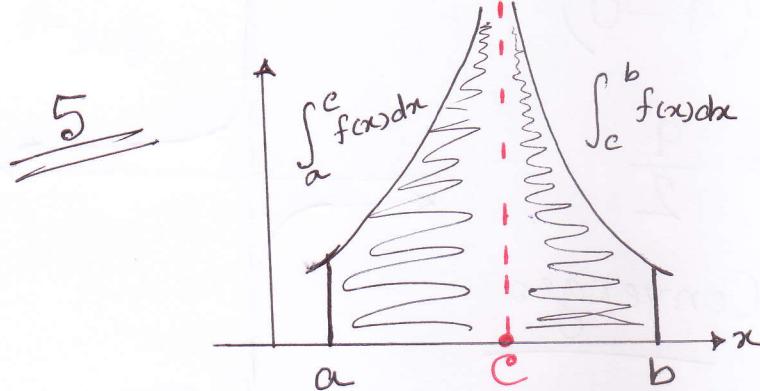
$$= \lim_{l \rightarrow 3^+} \left[ \frac{z^{-1}}{-1} \right]_{l-3}^1$$

$$= - \lim_{l \rightarrow 3^+} \left[ \frac{1}{z} \right]_{l-3}^1$$

$$= - \lim_{l \rightarrow 3^+} \left[ 1 - \frac{1}{l-3} \right] = -1 + \lim_{l \rightarrow 3^+} \frac{1}{l-3} = -1 + \frac{1}{3-3}$$

$\rightarrow$  asymptote line  
 $= -1 + \infty = \infty$

diverges.



If  $f$  is continuous on the interval  $[a, b]$ , except for an infinite discontinuity at 'c' in  $(a, b)$ , then the improper integral of 'f' over the interval  $[a, b]$  is defined as:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

The integral converges if both terms converge and diverges if either term diverges.

Example

$$\int_{-1}^8 x^{-1/3} dx$$

$$y = x^{-1/3} = \frac{1}{\sqrt[3]{x}}$$

$\therefore x \neq 0$

$$= \int_{-1}^0 x^{-1/3} dx + \int_0^8 x^{-1/3} dx$$

Hence an infinite discontinuity at  $x=0$

$$= \lim_{l \rightarrow 0^-} \int_{-1}^l x^{-1/3} dx + \lim_{K \rightarrow 0^+} \int_K^8 x^{-1/3} dx$$

$$= \lim_{l \rightarrow 0^-} \left[ \frac{x^{-1/3} + 1}{-\frac{1}{3} + 1} \right]_{-1}^l + \lim_{K \rightarrow 0^+} \left[ \frac{x^{-1/3} + 1}{-\frac{1}{3} + 1} \right]_K^8$$

$$= \lim_{l \rightarrow 0^-} \left[ \frac{x^{2/3}}{\frac{2}{3}} \right]_{-1}^l + \lim_{K \rightarrow 0^+} \left[ \frac{x^{2/3}}{\frac{2}{3}} \right]_K^8$$

$$= \frac{3}{2} \lim_{l \rightarrow 0^-} \left[ l^{2/3} - (-1)^{2/3} \right] + \frac{3}{2} \lim_{K \rightarrow 0^+} \left[ 8^{2/3} - K^{2/3} \right]$$

$$= \frac{3}{2} (0 - 1) + \frac{3}{2} (4 - 0)$$

$$= \frac{3}{2} (-1 + 4) = \frac{9}{2}$$

Converges

