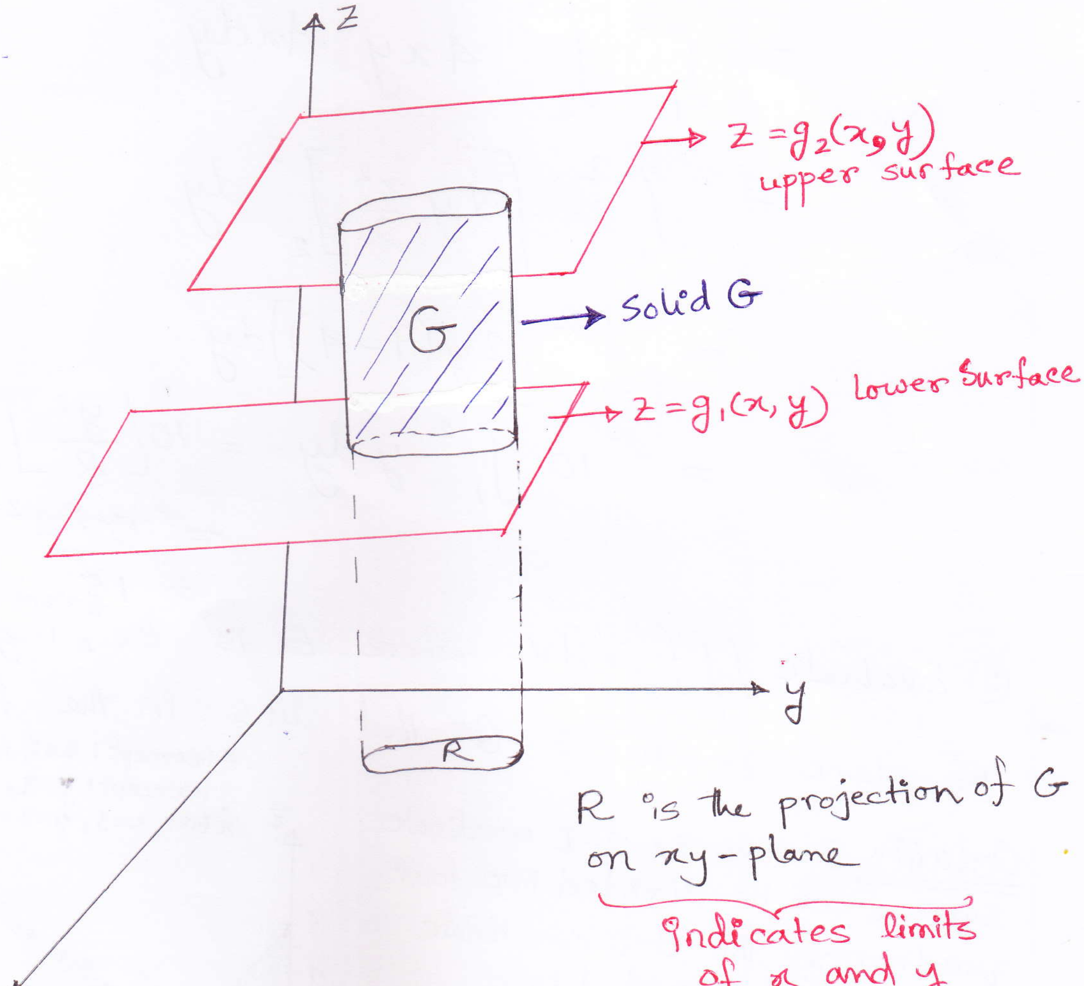
Triple Integral

Week 6

G is a simple xy-solid with $z = g_2(x,y)$ as upper surface and $z = g_1(x,y)$ as lower surface. R is the projection of G on xy-plane. If $f(x,y,z)$ is continuous on G then

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right] dA$$



R is the projection of G on xy-plane

$\underbrace{\phantom{R \text{ is the projection of G on xy-plane}}}_{\text{indicates limits of } x \text{ and } y \text{ on } xy\text{-plane.}}$

Examples ① Evaluate the following integral

$$\iiint_G 8xyz\,dV \qquad G=[2,3]\times[1,2]\times[0,1]$$

$$= \iint_R \left[\int_{z=0}^{1} 8xyz\,dz\right]dx\,dy$$

$$= \int_{y=1}^{2}\int_{x=2}^{3}\left[\overset{4}{\cancel{8}}\,xy\frac{z^2}{\cancel{2}}\right]_0^1 dx\,dy$$

$$= \int_1^2\int_2^3 4xy\,dx\,dy$$

$$= \int_1^2\left[\overset{2}{\cancel{4}}\,y\frac{x^2}{\cancel{2}}\right]_2^3 dy$$

$$= \int_1^2 \left[2y(9-4)\right]dy$$

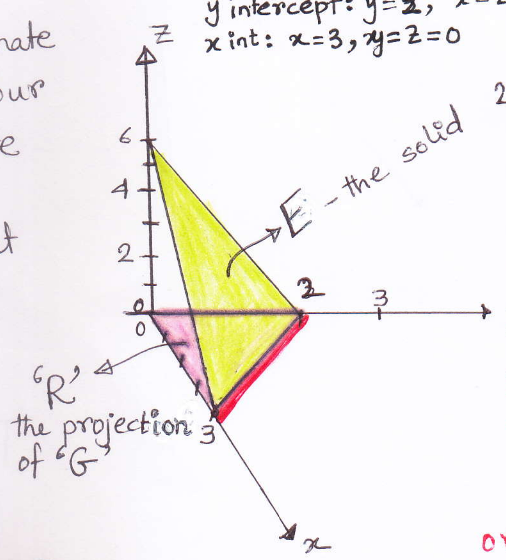
$$= 10\int_1^2 y\,dy = 10\left[\frac{y^2}{2}\right]_1^2$$

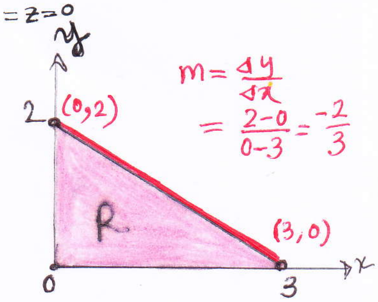
$$= 5\left[2^2-1^2\right]=15$$

② Evaluate $\iiint_E 2x\,dV$ where E is the triangular shape region under the plane $2x+3y+z=6$ that lies in the first octant

z intercept : $z=6$, $x=y=0$
y intercept: $y=2$, $x=z=0$
x int: $x=3$, $y=z=0$

Octant: Just as the 2-D coordinate system can be divided into four quadrants, the 3-D coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.





$m=\frac{dy}{dx}$
$=\frac{2-0}{0-3}=\frac{-2}{3}$

Eqn of the red line

$y-y_1=m(x-x_1)$

$y-2=-\frac{2}{3}(x-0)$

$y=-\frac{2}{3}x+2$

or $x=-\frac{3}{2}y+3$

We need to determine the region R in the xy-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region R in xy-plane.

So R will be the triangle with vertices at (0,0), (3,0) and (0,2).

Now we need the limits of integration. Since we are under the plane and in the 1^{st} octant (so we are above the plane $z=0$). $\therefore \quad 0 \le z \le 6-2x-3y$.

We can integrate the double integral over R using either of the following two sets of inequalities.

$$0 \le x \le 3 \qquad\qquad 0 \le x \le -\tfrac{3}{2}y+3$$
$$0 \le y \le -\tfrac{2}{3}x+2 \qquad \text{OR} \qquad 0 \le y \le 2$$

Note: If you consider your limits as $0 \le x \le 3$, $0 \le y \le 2$, you will end up considering a region of a rectangle rather than a triangle.

Recall <u>example</u> 4 from "double integral."

$\therefore$ neither region holds an advantange over the other, we will use the first one.

$$\iiint_E 2x\,dV = \iint_R \left[\int_0^{6-2x-3y} 2x\,dz\right] dA$$

$$= \iint_R 2x\left[z\right]_0^{6-2x-3y} dA$$

$$= \int_{x=0}^{3}\int_{y=0}^{-\frac{2}{3}x+2} 2x(6-2x-3y)\,dy\,dx$$

$$= \int_0^3 \left[12xy - 4x^2y - \cancel{6}^{\,3}x\frac{y^2}{\cancel{2}}\right]_0^{-2/3+2} dx$$

$$= \int_0^3 \left[12x\left(-\tfrac{2}{3}x+2\right) - 4x^2\left(-\tfrac{2}{3}x+2\right) - 3x\left(-\tfrac{2}{3}x+2\right)^2 \right] dx \quad - 0 + 0 + 0$$

$$= \int_0^3 \left(\frac{4}{3}x^3 - 8x^2 + 12x \right) dx$$

$$= \left[\frac{4}{3}\cdot\frac{x^4}{4} - 8\frac{x^3}{3} + 12\frac{x^2}{2} \right]_0^3$$

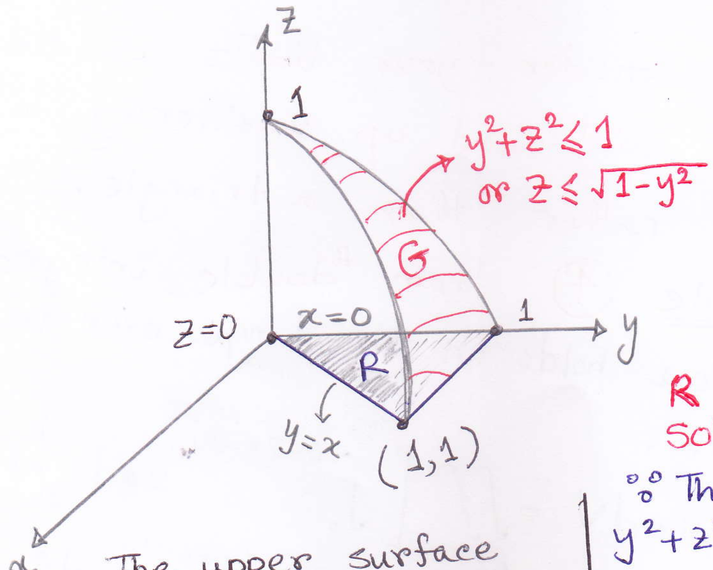
$$= \frac{(3)^4}{3} - \frac{8}{3}(3)^3 + 6(3)^2 - 0 + 0 - 0$$

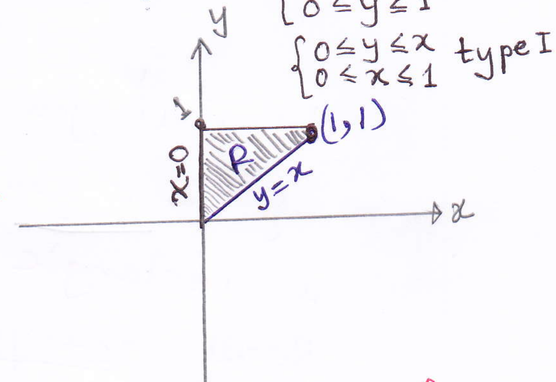
$$= 9.$$

$\boxed{3}$ Let G be the wedge in the 1^{st} Octant cut from the cylindrical solid $y^2+z^2 \le 1$ by the planes $y=x$ & $x=0$. Evaluate $\iiint_G z\,dv$.

For double integral over R: $\begin{cases} 0 \le x \le y \\ 0 \le y \le 1 \end{cases}$ type II

$\begin{cases} 0 \le y \le x \\ 0 \le x \le 1 \end{cases}$ type I





R is the pojection of solid G on xy-plane.

The upper surface of the solid is formed by cylinder (i.e. $y^2+z=1$) and lower surface by xy plane.

$\therefore$ The portion of the cylinder $y^2+z^2=1$ lies above xy-plane has eqn $z=\sqrt{1-y^2}$ and xy-plane has eqn $z=0$.

$$\iiint_G z\,dv = \iint \left\{ \int_{z=0}^{\sqrt{1-y^2}} z\,dz \right\} dA$$

$$\iiint_G z\,dV = \int_{y=0}^{1} \int_{x=0}^{y} \int_{z=0}^{\sqrt{1-y^2}} z \, dz\, dx\, dy$$

$$= \int_0^1 \int_0^y \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^y \left[1-y^2-0\right] dx\, dy$$

$$= \frac{1}{2}\int_0^1 \left[x - xy^2\right]_0^y dy$$

$$= \frac{1}{2}\int_0^1 \left[y - y^3\right] dy$$

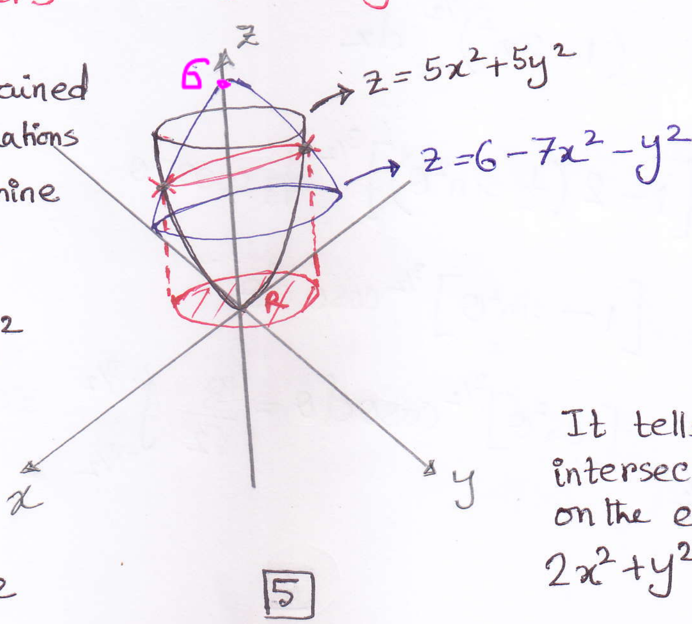
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

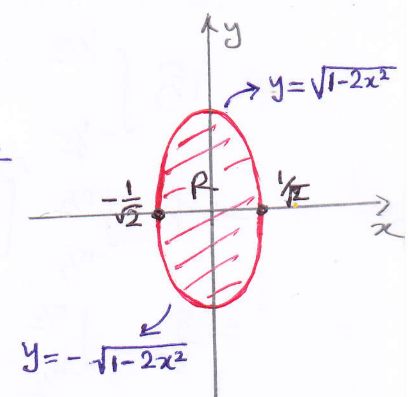
$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{8}$$

$\boxed{4}$ Find the volume of the solid enclosed between the paraboloids $z = 5x^2+5y^2$ & $z = 6-7x^2-y^2$.

The projection R is obtained by solving the given equations simultaneously to determine whether the paraboloids intersect.

$$5x^2+5y^2 = 6-7x^2-y^2$$
$$12x^2+6y^2 = 6$$
$$2x^2+y^2 = 1 \quad \text{eqn of ellipse}$$





It tells the paraboloids intersect in a curve on the elliptic cylinder $2x^2+y^2=1$

$$V = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \left[6 - 12x^2 - 6y^2\right] dy\, dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[6y - 12x^2 y - 2y^3\right]_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[6y(1-2x^2) - 2y^3\right]_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{1/\sqrt{2}} \left[6\sqrt{1-2x^2}(1-2x^2) - 2\left(\sqrt{1-2x^2}\right)^3 - 6\left(-\sqrt{1-2x^2}\right)(1-2x^2) + 2\left(-\sqrt{1-2x^2}\right)^3\right] dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[6(1-2x^2)^{3/2} \cancel{- 2(1-2x^2)^{3/2}} + 6(1-2x^2)^{3/2} \cancel{- 2(1-2x^2)^{3/2}}\right] dx$$

let $x = \frac{1}{\sqrt{2}} \sin\theta$

$dx = \frac{1}{\sqrt{2}} \cos\theta\, d\theta$

$$= 8 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx$$

limits:

$x = -\frac{1}{\sqrt{2}} \rightarrow \theta = -90^\circ$

$x = \frac{1}{\sqrt{2}} \rightarrow \theta = 90^\circ$

$$= 8 \int_{-\pi/2}^{\pi/2} \left[1 - 2\left(\tfrac{1}{2}\sin^2\theta\right)\right]^{3/2} \frac{1}{\sqrt{2}} \cos\theta\, d\theta$$

$$= \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \left[1 - \sin^2\theta\right]^{3/2} \cos\theta\, d\theta$$

$$= \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \left[\cos^2\theta\right]^{3/2} \cos\theta\, d\theta = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{8}{\sqrt{2}} \int [\cos^2\theta]^2 \, d\theta$$

$$= \frac{8}{\sqrt{2}} \int \left[\frac{1+\cos 2\theta}{2}\right]^2 d\theta$$

$$= \frac{8}{\sqrt{2}} \int \frac{1}{4} \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{2}{\sqrt{2}} \int \left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right] d\theta$$

$$= \sqrt{2} \int \left[1 + 2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right] d\theta$$

$$= \sqrt{2} \int \left[\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right] d\theta$$

$$= \sqrt{2} \left[\frac{3\theta}{2} + \frac{2\sin 2\theta}{2} + \frac{1}{2}\,\frac{\sin 4\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

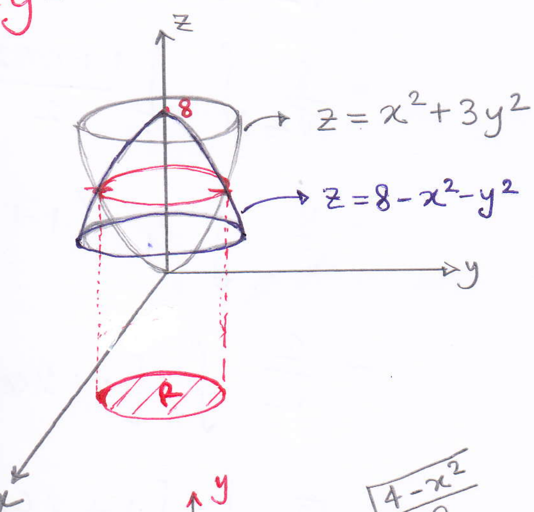
$$= \sqrt{2} \left[\frac{3}{2}\left(\frac{\pi}{2}\right) + \sin 2\left(\frac{\pi}{2}\right) + \frac{1}{8}\sin 4\left(\frac{\pi}{2}\right) - \frac{3}{2}\left(-\frac{\pi}{2}\right) - \sin 2\left(-\frac{\pi}{2}\right) - \frac{1}{8}\sin 4\left(-\frac{\pi}{2}\right)\right]$$

$$= \sqrt{2}\left[\frac{3\pi}{4} + \sin\pi + \frac{\sin 2\pi}{8} + \frac{3\pi}{4} - \sin\pi - \frac{\sin 2\pi}{4}\right]$$

$$= \sqrt{2}\left[\frac{6\pi}{4}\right] = \sqrt{2}\left[\frac{3\pi}{2}\right] = \frac{3\pi}{\sqrt{2}}.$$

$\boxed{5}$ Find the volume of the solid enclosed by the surfaces $z = x^2+3y^2$ and $z = 8 - x^2 - y^2$

Similar to example ④



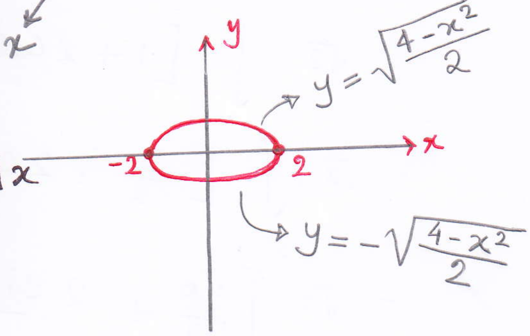
$$x^2+3y^2 = 8 - x^2 - y^2$$

$$x^2 + 2y^2 = 4 \longrightarrow \text{ellipse}$$

$\hookrightarrow R$

$$y = \pm\sqrt{\frac{4-x^2}{2}}$$

if $y = 0 \Rightarrow x = \pm 2$



$$V = \int_{x=-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left[8-2x^2-4y^2\right] dy\,dx$$

$$= \int_{x=-2}^{2} \left[8y - 2x^2y - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[2y(4-x^2) - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[2\sqrt{\frac{4-x^2}{2}}(4-x^2) - \frac{4}{3}\left(\sqrt{\frac{4-x^3}{2}}\right)^3 - 2\left(-\sqrt{\frac{4-x^2}{2}}\right)(4-x^2) + \frac{4}{3}\left(-\sqrt{\frac{4-x^2}{2}}\right)^3\right] dx$$

$$= \int_{-2}^{2} \left[\frac{1}{\sqrt{2}}(4-x^2)^{3/2} - \frac{\cancel{4}^{\,2}}{3}\cdot\frac{(4-x^2)^{3/2}}{\cancel{2}\sqrt{2}} + \frac{1}{\sqrt{2}}(4-x^2)^{3/2} - \frac{\cancel{4}^{\,2}}{3}\cdot\frac{(4-x^2)^{3/2}}{\cancel{2}\sqrt{2}}\right] dx$$

$$= \int_{-2}^{2} \left[\frac{2}{\sqrt{2}}(4-x^2)^{3/2} - \frac{4}{3\sqrt{2}}(4-x^2)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left[\frac{6(4-x^2)^{3/2} - 4(4-x^2)^{3/2}}{3\sqrt{2}} \right] dx$$

$$= \frac{2}{3\sqrt{2}} \int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = -2 \rightarrow \theta = -\pi/2$

$x = 2 \rightarrow \theta = \pi/2$

$$= \frac{\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} (4 - 4\sin^2\theta)^{3/2}\, 2\cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int \left[4(1-\sin^2\theta)\right]^{3/2} \cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int 4^{3/2} (1-\sin^2\theta)^{3/2} \cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int (2^2)^{3/2} (\cos^2\theta)^{3/2} \cos\theta\, d\theta$$

$$= \frac{16\sqrt{2}}{3} \underbrace{\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta}_{\text{see example } 4}$$

$$= \frac{\overset{4}{\cancel{16}}\sqrt{2}}{\underset{1}{\cancel{3}}} \left[\frac{\overset{2}{\cancel{6}}\pi}{\underset{1}{\cancel{4}}} \right] = 8\sqrt{2}\pi.$$