

$$x^2 + y^2 = r^2$$

$$\tan \theta = \frac{y}{x} \Rightarrow \cot \theta = \frac{x}{y}$$

$$\sin \theta = \frac{y}{r} \Rightarrow \operatorname{cosec} \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \Rightarrow \sec \theta = \frac{r}{x}$$

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \Rightarrow 1 - \cos^2 \theta = \sin^2 \theta \\ 1 + \tan^2 \theta &= \sec^2 \theta \Rightarrow 1 - \sin^2 \theta = \cos^2 \theta \end{aligned}$$

$$\boxed{3} \quad \int \frac{x^2}{\sqrt{16-x^2}} dx$$

$$= \int \frac{x^2}{4\sqrt{1-\frac{1}{16}x^2}} dx$$

$$= \frac{1}{4} \int \frac{x^2}{\sqrt{1-\left(\frac{x}{4}\right)^2}} dx$$

$$= \int \frac{16 \sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta$$

$$= \int \frac{16 \sin^2 \theta \cos \theta}{\cos \theta} d\theta$$

$$= 16 \int \sin^2 \theta d\theta$$

$$= 16 \int \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$= 8 \left[\theta - \frac{\sin 2\theta}{2} \right] + C$$

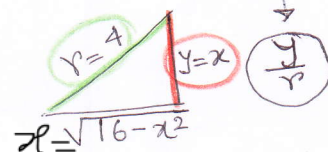
$$= 8 \left[\sin^{-1} \frac{x}{4} + \frac{2 \sin \theta \cos \theta}{2} \right] + C$$

$$= 8 \left[\sin^{-1} \frac{x}{4} + \sin \theta \cos \theta \right] + C$$

$$= 8 \left[\sin^{-1} \frac{x}{4} + \frac{x}{4} \cdot \frac{\sqrt{16-x^2}}{4} \right] + C$$

$$\begin{aligned} \text{let } \frac{x}{4} &= \sin \theta \Rightarrow \theta = \sin^{-1} \left(\frac{x}{4} \right) \\ \frac{x^2}{16} &= \sin^2 \theta \Rightarrow x^2 = 16 \sin^2 \theta \\ \frac{1}{4} dx &= \cos \theta d\theta \end{aligned}$$

$$\therefore \sin \theta = \frac{x}{4} \Rightarrow \cos \theta = \frac{x}{r} = \frac{\sqrt{16-x^2}}{4}$$



$$\begin{aligned} x &= \sqrt{r^2 - y^2} \\ &= \sqrt{16 - x^2} \end{aligned}$$

$$= 8 \sin^{-1} \frac{x}{4} + \frac{x \sqrt{16-x^2}}{2} + C$$

$$⑤ \int \frac{dx}{(4+4x^2)^2}$$

$$= \int \frac{dx}{[4(1+x^2)]^2}$$

$$= \frac{1}{16} \int \frac{dx}{(1+x^2)^2}$$

$$= \frac{1}{16} \int \frac{\sec^2 \theta d\theta}{(1+\tan^2 \theta)^2}$$

$$= \frac{1}{16} \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \frac{1}{16} \int \frac{1}{\sec^2 \theta} d\theta$$

$$= \frac{1}{16} \int \frac{1}{\cos^2 \theta} d\theta$$

$$= \frac{1}{16} \int \cos^2 \theta d\theta$$

$$= \frac{1}{16} \int \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{32} \left[\theta + \frac{\sin 2\theta}{2} \right] + C$$

$$= \frac{1}{32} \left[\theta + \frac{2 \sin \theta \cos \theta}{2} \right] + C$$

$$= \frac{1}{32} \left[\theta + \sin \theta \cos \theta \right] + C$$

$$= \frac{1}{32} \left[\tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right] + C$$

$$= \frac{1}{32} \left[\tan^{-1} x + \frac{x}{1+x^2} \right] + C$$

$$① \tan^2 \theta = \sec^2 \theta$$

$$x = \tan \theta$$

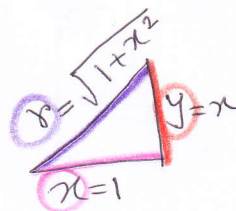
$$dx = \sec^2 \theta d\theta$$

$$x = \tan \theta$$

$$\theta = \tan^{-1} x$$

$$\because \tan \theta = \frac{y}{x}$$

$$\text{We have } \tan \theta = \frac{x}{1}$$



$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{1+x^2}$$

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r}$$

$$= \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$\begin{aligned}
 (7) \quad & \int \frac{\sqrt{x^2-9}}{x} dx \\
 &= \int \frac{\sqrt{9(\frac{x^2}{9}-1)}}{x} dx \\
 &= 3 \int \frac{\sqrt{(\frac{x}{3})^2-1}}{x} dx
 \end{aligned}$$

$$= 3 \int \frac{\sqrt{\sec^2 \theta - 1}}{3 \sec \theta} (3 \sec \theta \tan \theta) d\theta$$

$$= +3 \int \tan \theta (\tan \theta) d\theta$$

$$= 3 \int \tan^2 \theta d\theta$$

$$= 3 \int (\sec^2 \theta - 1) d\theta$$

$$= 3 [\tan \theta - \theta] + C$$

$$= 3 \left[\frac{\sqrt{x^2-9}}{3} - \sec^{-1} \left(\frac{x}{3} \right) \right] + C$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\begin{aligned}
 \frac{x}{3} &= \sec \theta \quad \left(\because \left(\frac{x}{3} \right)^2 = \sec^2 \theta \right) \\
 \frac{1}{3} dx &= \sec \theta \tan \theta d\theta \quad \left. \begin{aligned} x &= 3 \sec \theta \\ \therefore \theta &= \sec^{-1} \left(\frac{x}{3} \right) \end{aligned} \right\} \\
 \therefore dx &= 3 \sec \theta \tan \theta d\theta
 \end{aligned}$$

$$\sec \theta = \frac{x}{3} = \frac{r}{x}$$

$$\begin{aligned}
 y &= \sqrt{x^2 - 3^2} = \sqrt{x^2 - 9} \\
 \therefore \tan \theta &= \frac{y}{x} = \frac{\sqrt{x^2 - 9}}{3}
 \end{aligned}$$

$$(23) \int_{\sqrt{2}}^2 \frac{dx}{x^2 \sqrt{x^2-1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec \theta \tan \theta d\theta}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\tan \theta d\theta}{\sec \theta \sqrt{\tan^2 \theta}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec \theta} d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cos \theta d\theta$$

$$= \left[+ \sin \theta \right]_{\pi/4}^{\pi/3}$$

$$= + \left[\sin \frac{\pi}{4} - \sin \frac{\pi}{3} \right]$$

$$= + \left[\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \right]$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

let

$$x = \sec \theta \rightarrow \theta = \sec^{-1} x$$

$$x^2 = \sec^2 \theta$$

$$dx = \sec \theta \tan \theta d\theta$$

$$\begin{cases} x = \sqrt{2} \Rightarrow \theta = \sec^{-1} \sqrt{2} = \pi/4 \\ x = 2 \Rightarrow \theta = \sec^{-1} 2 = \pi/3 \end{cases}$$

$$\sec \theta = \frac{r}{1} \quad \left| \quad \sec \theta = \frac{r}{x} \right.$$

$$\begin{array}{c} r = x \\ \text{triangle} \\ x = 1 \end{array} \quad y = \sqrt{r^2 - x^2} = \sqrt{x^2 - 1}$$

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

$$\begin{cases} \sec^{-1} \sqrt{2} = \theta \\ \sec \theta = \sqrt{2} = \frac{r}{x} = \frac{\sqrt{2}}{1} \end{cases}$$

$$\begin{array}{c} r = \sqrt{2} \\ \text{triangle} \\ x = 1 \end{array} \quad \therefore y = 1 \quad \tan \theta = \frac{y}{x} = 1 \quad \therefore \theta = 45^\circ$$

$$\begin{cases} \sec^{-1} 2 = \theta \\ \sec \theta = 2 = \frac{r}{x} = \frac{2}{1} \end{cases}$$

$$\begin{array}{c} r = 2 \\ \text{triangle} \\ x = 1 \end{array}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \cos^{-1} \frac{1}{2} = 60^\circ = \frac{\pi}{3}$$

Examples.

$$\boxed{1} \int \frac{dx}{x^2 \sqrt{4-x^2}}$$

$$= \int \frac{dx}{x^2 \sqrt{4(1-\frac{x^2}{4})}}$$

$$= \frac{1}{2} \int \frac{dx}{x^2 \sqrt{1-(\frac{x}{2})^2}}$$

$$= \frac{1}{2} \int \frac{2 \cos \theta d\theta}{4 \sin^2 \theta \sqrt{1-\sin^2 \theta}}$$

$$= \int \frac{\cos \theta d\theta}{4 \sin^2 \theta \sqrt{\cos^2 \theta}}$$

$$= \frac{1}{4} \int \frac{1}{\sin^2 \theta} d\theta$$

$$= \frac{1}{4} \int \csc^2 \theta d\theta$$

$$= \frac{1}{4} (-\cot \theta) + C$$

$$= -\frac{1}{4} \cot \theta + C$$

Solve $\int_1^{\sqrt{2}} \frac{dx}{x^2 \sqrt{4-x^2}}$

$$= \frac{1}{4} \int_{\pi/6}^{\pi/4} \csc^2 \theta d\theta$$

$$= -\frac{1}{4} [\cot \theta]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{4} \left[\frac{1}{\tan \theta} \right]_{\pi/6}^{\pi/4}$$

$$\begin{aligned} \frac{x}{2} &= \sin \theta & \therefore x &= 2 \sin \theta \\ \frac{1}{2} dx &= \cos \theta d\theta & x^2 &= 4 \sin^2 \theta \\ dx &= 2 \cos \theta d\theta & \therefore \theta &= \sin^{-1} \left(\frac{x}{2} \right) \end{aligned}$$

$$\operatorname{cosec} = \csc$$

$$x=1 \Rightarrow \theta = \sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6} = 30^\circ$$

$$x=\sqrt{2} \Rightarrow \theta = \sin^{-1} \left(\frac{\sqrt{2}}{2} \right)$$

$$= \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4} = 45^\circ$$

$$= -\frac{1}{4} \left[\frac{1}{\tan \frac{\pi}{4}} - \frac{1}{\tan \frac{\pi}{6}} \right]$$

$$= -\frac{1}{4} \left[\frac{1}{1} - \frac{1}{\frac{1}{\sqrt{3}}} \right]$$

$$= -\frac{1}{4} [1 - \sqrt{3}]$$

$$= \frac{\sqrt{3}-1}{4}$$

$$\boxed{2} \int \frac{x}{x^2-4x+8} dx$$

$$= \int \frac{x}{x^2 - 2 \cdot x \cdot 2 + 2^2 + 4} dx$$

$$= \int \frac{x}{(x-2)^2 + 4} dx$$

$$\begin{aligned} \text{let } u &= x-2 \\ du &= dx \end{aligned}$$

$$\therefore x = u+2$$

$$= \int \frac{u+2}{u^2+4} du$$

$$= \int \frac{u}{u^2+4} du + 2 \int \frac{1}{u^2+4} du$$

$$= \int \frac{\frac{1}{2} dz}{z} + 2 \int \frac{du}{4(\frac{u^2}{4}+1)}$$

$$\begin{aligned} u^2+4 &= z \\ 2u du &= dz \\ u du &= \frac{1}{2} dz \end{aligned}$$

$$= \frac{1}{2} \int \frac{1}{z} dz + \frac{1}{2} \int \frac{du}{(\frac{u}{2})^2+1}$$

$$= \frac{1}{2} \ln z + \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2} \ln(u^2+4) + \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2} \ln[(x-2)^2+4] + \frac{1}{2} \tan^{-1}\left(\frac{x-2}{2}\right) + C$$