Beta Gamma Function

Beta function is also known as the Euler's integral defined by:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx;$$

$$m > 0, n > 0.$$

It is a useful distribution to evaluate

- Laplace transformation
- Probability density function in statistics
- Quantum field theory in physics
- MATLAB programing

Gamma function (Γ)is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers.

$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}dx = (n-1)!$

Gamma Beta function can be related to each other: $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Trigonometric Function: $\int_0^{\pi/2} sin^p x cos^q x\; dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$.

OR: $\int_0^{\pi/2} 2sin^{2x-1}(t)cos^{2y-1}(t)dt = \beta(x,y)$

2. (ix) $\int_0^{\infty} \frac{1}{1+x^4}\,dx$

Trigonometric Substitution:

$x^2 = tan\theta; \therefore x^4 = tan^2\theta; 1 + x^4 = 1 + tan^2\theta = sec^2\theta.$

$$x = \sqrt{tan\theta}$$

$$dx = \frac{1}{2\sqrt{tan\theta}} sec^2\theta\, d\theta$$

Limits: $x = 0 => \theta = 0$

$$x = \infty => \theta = \frac{\pi}{2}.$$

$\int_0^{\infty} \frac{1}{1+x^4}\,dx$

$= \int_0^{\pi/2} \frac{1}{1+tan^2\theta} . \frac{1}{2\sqrt{tan\theta}} sec^2\theta d\theta$

$= \int_0^{\pi/2} \frac{sec^2\theta}{sec^2\theta} . \frac{1}{2\sqrt{tan\theta}} d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \sqrt{cot\theta}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sqrt{\frac{cos\theta}{sin\theta}}\, d\theta = \frac{1}{2}\int_0^{\pi/2} sin^{-1/2}\theta\, cos^{1/2}\theta\, d\theta$

Alternative Method

OR

$= \frac{1}{4}\int_0^{\pi/2} 2\sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$

$= \frac{1}{4}\,\beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$x \quad y$

$2y - 1 = \frac{1}{2}$

$2y = \frac{1}{2} + 1 = \frac{3}{2}$

$y = \frac{3}{4}$

$2x - 1 = -\frac{1}{2}$

$2x = -\frac{1}{2} + 1$

$2x = \frac{1}{2}$

$x = \frac{1}{4}$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}$$

$$= \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right).$$

$$\int_0^{\pi/2} sin^p x cos^q x \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

3(i)$\int_0^{\pi} sin^5\theta \, cos^4 \theta \, d\theta$

$=2\int_0^{\pi/2} sin^5\theta \, cos^4 \theta \, d\theta$

$=2\,\frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{5+4+2}{2}\right)}$

$=2\,\frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)}$

$=\frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(3+\frac{5}{2}\right)} = \beta\left(3,\frac{5}{2}\right).$

$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ when $f(2a-x) = -f(x)$

$\therefore \int_0^{\pi} sin^5\theta \, cos^4 \theta \, d\theta = 2\int_0^{\pi/2} sin^5\theta \, cos^4 \theta \, d\theta \; iff \; f(\pi-\theta) = -f(\theta).$

$l.s. = f(\pi-\theta) = sin^5(\pi-\theta)cos^4(\pi-\theta) = (sin^5\theta)(-cos^4\theta)$

$= -sin^5\theta cos^4\theta = -f(\theta)$

$\therefore \int_0^{\pi} sin^5\theta \, cos^4 \theta \, d\theta = 2\int_0^{\pi/2} sin^5\theta \, cos^4 \theta \, d\theta.$ 3(ii) is similar to 3(i)

Alternative Method: $\int_0^{\pi/2} 2sin^{2x-1}(t)cos^{2y-1}(t)dt = \beta(x,y)$

3(i)$\int_0^{\pi} sin^5\theta \, cos^4 \theta \, d\theta$

$=2\int_0^{\pi/2} sin^5\theta \, cos^4 \theta \, d\theta$

$= \int_0^{\pi/2} 2sin^5\theta \, cos^4 \theta \, d\theta$

$= \int_0^{\pi/2} 2sin^{2x-1}(\theta)cos^{2y-1}(\theta)d\theta, \quad where \; 2x-1 = 5, 2y-1 = 4 \; hence \; x = 3, y = \frac{5}{2}$

$= \beta(x,y) = \beta(3,\frac{5}{2})$