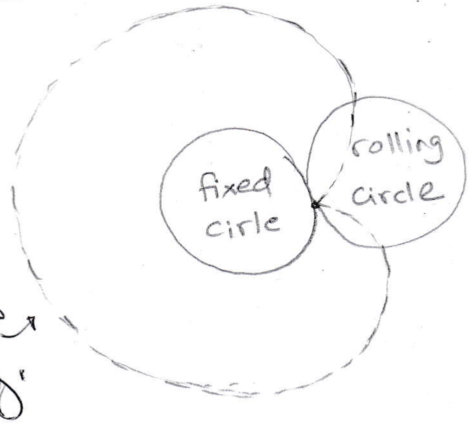
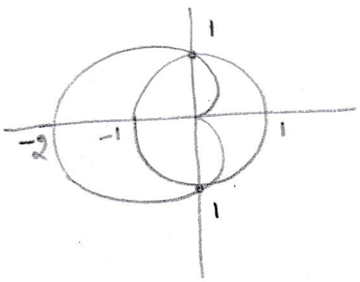
A <u>cardioid</u> is a plane curve traced by a pt on the perimeter of the circle that is rolling around a fixed circle of the same radius.

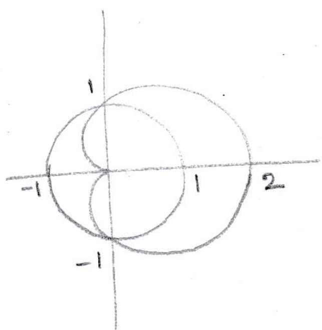


by completing the tour pt creates this pathway.

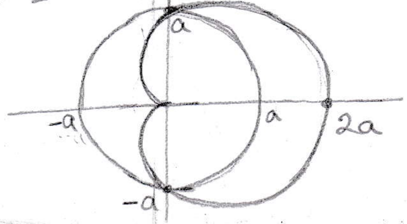
$r = 1 - \cos\theta$



$r = 1 + \cos\theta$



$r = a(1+\cos\theta) = a + a\cos\theta$



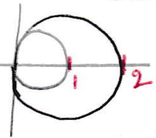
Rose petals

r = cos nθ → no. of petals

r = a cos θ

a=1, 2

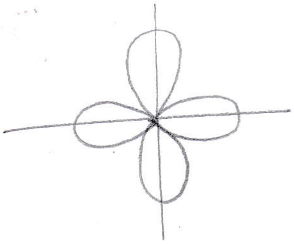
a=1



r = cos θ ✓

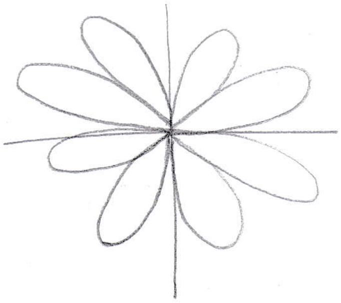
r=2cosθ

a=2



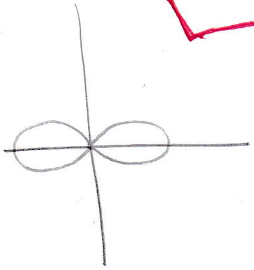
4

(c)



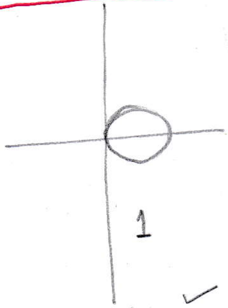
8

(c)



2

(a) ✓

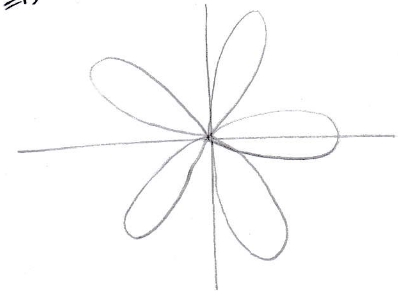


1

(a) ✓

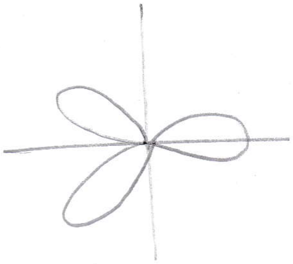
horizontal

Show up to cos3θ, sin3θ
n=1, 2, 3 only



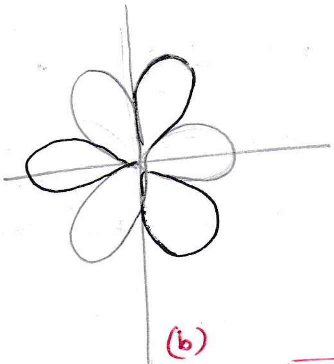
5

(b)



3

(b) ✓

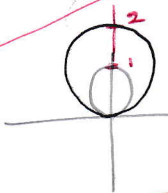


(b)

6

r=a sinθ ✓

a=1, 2



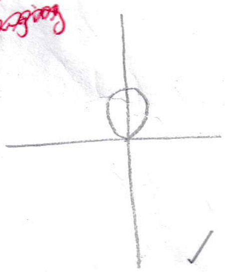
r=2sinθ

r=sinθ

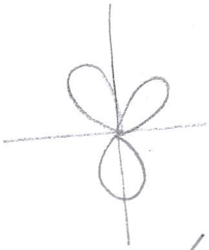
vertical

r = a sin nθ

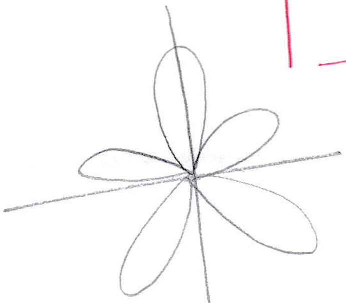
[illegible]



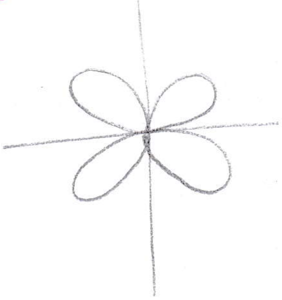
1 ✓



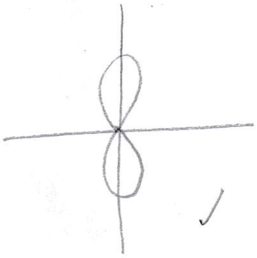
3 ✓



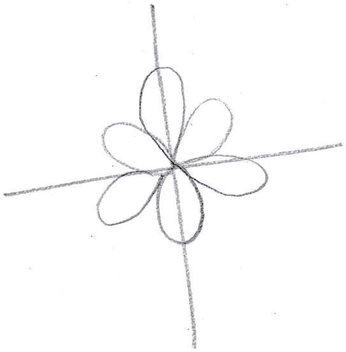
5



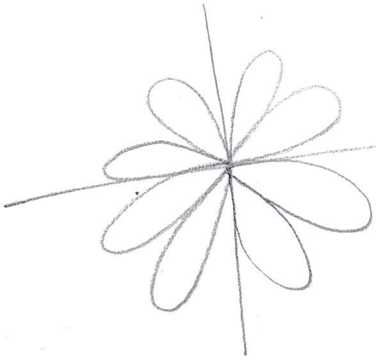
4



2 ✓



6

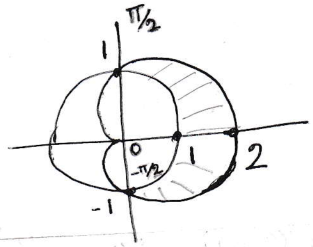


8

Double integrals in Polar form

① Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{r=1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta$$



$$=\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} 1\,r\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{1}^{1+\cos\theta} d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^2-1^2\right]d\theta$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[1+2\cos\theta+\cos^2\theta-1\right]d\theta$$

$$\cos^2\theta=\frac{1+\cos2\theta}{2}$$

$$=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[2\cos\theta+\frac{1+\cos2\theta}{2}\right]d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left(\cos\theta+\frac{1}{4}+\frac{\cos2\theta}{4}\right)d\theta$$

$$=\left[\sin\theta+\frac{\theta}{4}+\frac{\sin2\theta}{8}\right]_{-\pi/2}^{\pi/2}$$

$$=\sin\frac{\pi}{2}+\frac{\pi/2}{4}+\frac{\sin2(\pi/2)}{8}-\sin\left(-\frac{\pi}{2}\right)-\frac{\left(-\frac{\pi}{2}\right)}{4}-\frac{\sin2(-\pi/2)}{8}$$

$$=1+\frac{\pi}{8}+0-(-1)+\frac{\pi}{8}-0$$

$$=2+\frac{\pi}{4}.$$

② $\iint_R e^{-(x^2+y^2)}\,dA$, where R is the region bounded by the circle $x^2+y^2=1$

$x^2+y^2=r^2$ eqn of a circle

Let

$r^2=z$

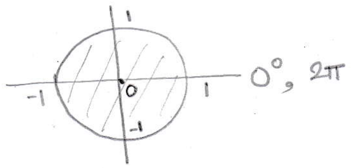
$2r\,dr=dz$

$r\,dr=\frac{1}{2}dz$

$r=0\rightarrow z=0$

$r=1\rightarrow z=1$

$$\iint_R e^{-(x^2+y^2)}\,dA=\int_0^{2\pi}\int_0^1 e^{-r^2}\,r\,dr\,d\theta$$



$$\int_0^{2\pi}\int_0^1 e^{-z}\,\frac{1}{2}dz\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left[\frac{e^{-z}}{-1}\right]_0^1 d\theta$$

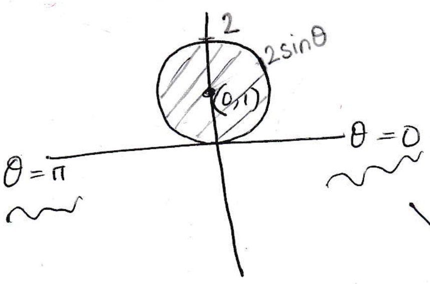
$$=\frac{1}{2}\int_0^{2\pi}\left(-e^{-1}+1\right)d\theta=\frac{1}{2}\,2\pi\left(1-e^{-1}\right)=\pi\left(1-e^{-1}\right)$$

③ Find the ~~volume~~ area under the plane $6x+4y+z=12$ above the disk $x^2+y^2=2y$

$x^2+y^2=2y$

$x^2+y^2-2y+1=1$

$x^2+(y-1)^2=1^2$



$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta$ → above x axis $\therefore y \geq 0$

$0 \leq \theta \leq \pi$ (1^{st} & 2^{nd} quadrant)

$0 \leq r \leq 2\sin\theta$

Now,

$6x+4y+z=12$

$z = 12-6x-4y$

$= 12-6r\cos\theta-4r\sin\theta$

$$I = \int_0^{\pi}\int_0^{2\sin\theta} (12-6r\cos\theta-4r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2\sin\theta} (12r-6r^2\cos\theta-4r^2\sin\theta)\,dr\,d\theta$$

$\sin^2\theta = \frac{1-\cos2\theta}{2}$

$\cos^2\theta = \frac{1+\cos2\theta}{2}$

$$= \int_0^{\pi}\left[6r^2-2r^3\cos\theta-\tfrac{4}{3}r^3\sin\theta\right]_0^{2\sin\theta} d\theta$$

$$= \int_0^{\pi}\left[24\sin^2\theta-16\sin^3\theta\cos\theta-\tfrac{4}{3}\cdot 8\sin^4\theta\right]d\theta$$

$$= \int_0^{\pi}\left[24\sin^2\theta-16\sin^3\theta\cos\theta-\frac{32}{3}\overset{(\sin^2\theta)^2}{\sin^4\theta}\right]d\theta$$

$$= \int_0^{\pi} 24\cdot\tfrac{1}{2}(1-\cos2\theta)\,d\theta - 16\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta - \frac{32}{3}\int_0^{\pi}\left\{\tfrac{1}{2}(1-\cos2\theta)\right\}^2 d\theta$$

$$= 12\int_0^{\pi}(1-\cos2\theta)\,d\theta - 16\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta - \frac{32}{3}\int_0^{\pi}\tfrac{1}{4}(1-2\cos2\theta+\cos^2 2\theta)\,d\theta$$

$$= 12\int_0^{\pi}(1-\cos2\theta)\,d\theta - 16\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta - \tfrac{8}{3}\int_0^{\pi}\left(1-2\cos2\theta+\frac{(1+\cos4\theta)}{2}\right)d\theta$$

$= 12\int_0^{\pi}$

Now (a) $\int_0^\pi (1-\cos 2\theta)\, d\theta = \theta - \frac{1}{2}\sin 2\theta \Big|_0^\pi = \pi - 0 = \pi$

(b) $\int_0^\pi \sin^3\theta \cos\theta d\theta$

let
$\sin\theta = u$
$\cos\theta d\theta = du$
$\theta = 0 \Rightarrow u = 0$
$\theta = \pi \Rightarrow u = 0$

$= \int_0^0 u^3 du$

$= 0$

(c) $\int_0^\pi \left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$

$= \int_0^\pi \left(3/2 - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$

$= \left[3/2\,\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^\pi$

$= 3/2\,\pi$

$I = 12\pi - 16 \times 0 - 8/3 \times \frac{3}{2}\pi$

$= 12\pi - 4\pi = 8\pi$

Examples

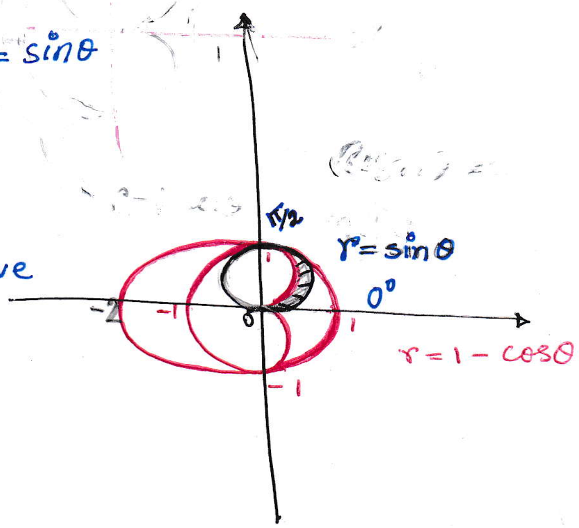
Double Integral

④ Find the area of the region enclosed by the inside of a circle $r = \sin\theta$ & outside of a Cardioid $r = 1 - \cos\theta$

$\theta \in [0, \frac{\pi}{2}]$

Comparing the curves $r = \sin\theta$ & $r = 1 - \cos\theta$

$r \in [\underset{\substack{\uparrow \\ \text{inner curve}}}{1 - \cos\theta}, \underset{\substack{\uparrow \\ \text{outer curve}}}{\sin\theta}]$



$$A = \int_0^{\pi/2} \int_{1-\cos\theta}^{\sin\theta} r \, dr d\theta$$

$$= \int_0^{\pi/2} \left. \frac{r^2}{2} \right]_{1-\cos\theta}^{\sin\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\sin^2\theta - (1 - \cos^2\theta) \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[-\cos 2\theta - 1 + 2\cos\theta \right] d\theta$$

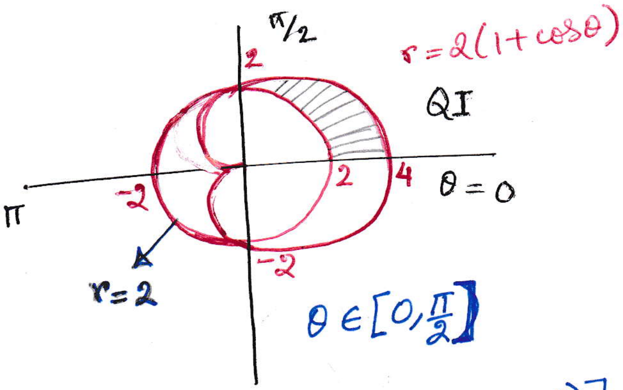
$$= \frac{1}{2} \left[-\frac{\sin 2\theta}{2} - \theta + 2\sin\theta \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left[-\frac{0}{2} - \frac{\pi}{2} + 2(1) - 0 \right]$$

$$= \frac{1}{2} \left(2 - \frac{\pi}{2} \right)$$

$$= 1 - \frac{\pi}{4}$$

⑤ Evaluate $\iint_R \sin\theta \, dA$ where R is the region in the 1st quadrant that is outside the circle r=2 & inside the cardioid $r = 2(1+\cos\theta)$



$\theta \in [0, \frac{\pi}{2}]$

$\rho \in [2, 2(1+\cos\theta)]$

$$\int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left([2(1+\cos\theta)]^2 - 4 \right) \sin\theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[4(1+2\cos\theta+\cos^2\theta)\sin\theta - 4\sin\theta \right] d\theta$$

$$= \frac{1}{2} \times 4 \int_0^{\pi/2} \left(\cancel{\sin\theta} + 2\sin\theta\cos\theta + \sin\theta\cos^2\theta - \cancel{\sin\theta} \right) d\theta$$

$$= 2 \int_0^{\pi/2} \left(\sin 2\theta + \cos^2\theta \sin\theta \right) d\theta$$

$$= 2 \left[\int_0^{\pi/2} \sin 2\theta \, d\theta + \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta \right]$$

Let $\cos\theta = u$

$-\sin\theta d\theta = du$

$\sin\theta d\theta = -du$

$\theta = 0 \rightarrow u = 1$

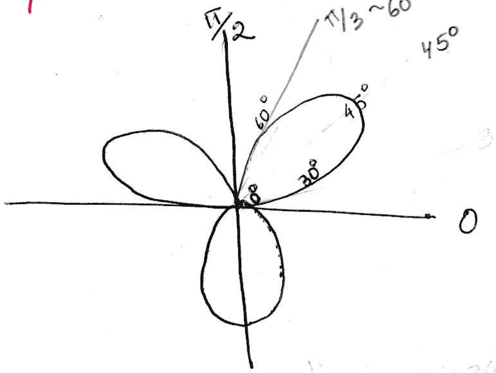
$\theta = \pi/2 \rightarrow u = 0$

$$= 2 \left[-\frac{\cos 2\theta}{2} \Big|_0^{\pi/2} + \int_1^0 (-u^2) \, du \right]$$

$$= 2 \left[-\left(\frac{-1}{2} - \frac{1}{2} \right) + \left[-\frac{u^3}{3} \right]_1^0 \right]$$

$$= 2 \left[+1 + \left(-\frac{0}{3} + \frac{1}{3} \right) \right] = 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$$

⑥ Use a polar double integral to find the area enclosed by the three petaled rose $r = \sin 3\theta$.



$$A = 3 \iint_R dA$$

$$= 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r\,dr\,d\theta$$

3 petals

$$= 3 \int_0^{\pi/3} \frac{r^2}{2}\Big|_0^{\sin 3\theta} d\theta$$

$$= 3/2 \int_0^{\pi/3} \sin^2 3\theta \; d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} \left(\frac{1 - \cos 6\theta}{2}\right) d\theta$$

$$= \frac{3}{4} \int_0^{\pi/3} [1 - \cos 6\theta]\, d\theta$$

$$= 3/4 \left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$$

$$= 3/4 \left[\pi/3 - \frac{\sin 6\left(\frac{\pi}{3}\right)}{6}\right]$$

$$= 3/4 \left[\pi/3 - \frac{\sin 2\pi}{6} - 0\right]$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$r = 0,\ r = \sin 3\theta$

$\sin 3\theta = 0$

$3\theta = \sin^{-1} 0$

$= 0, 180, 360, \cdots$

$\theta = \underbrace{0, 60}_{QI}, \underbrace{120,}_{QII}$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$\therefore \theta \in [0, 60°]$