MAT120 Week 6

TRIPLE INTEGRAL IN SPHERICAL COORDINATE

$$\iiint_G f(\rho,\theta,\varphi) = \iiint_{\substack{\text{appropriate} \\ \text{limits}}} f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$G \to$ Solid

ρ (rho) $\to$ constant that represents a sphere centered at the origin.

Eqn of sphere centered at the origin

$$x^2+y^2+z^2=\rho^2$$

$\theta \to$ constant, represents a half plane (height) [z-axis represents height].

$\varphi \to$ constant that represents a right circular cone with its' vertex at the origin and its' line of symmetry along the z-axis for $\varphi = \frac{\pi}{2}$ and in the $\underbrace{xy\text{-plane}}_{z=0}$ if $\varphi = \frac{\pi}{2}$.

A right circular cone is a circular cone whose altitude intersects the plane of the circle at the circle's center.

The height of an object or a point in relation to sea level or ground level is known as altitude.

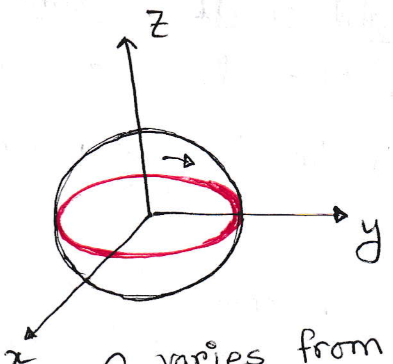
Relation

$$\underbrace{(\rho, \theta, \varphi)}_{\text{spherical coordinate}} \rightarrow \underbrace{(x, y, z)}_{\text{cartesian coordinate}}$$

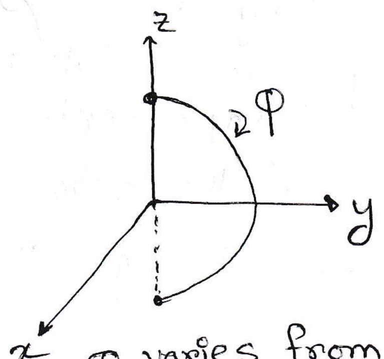
$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

$\therefore x^2 + y^2 + z^2 = \rho^2 \rightarrow$ Eqn of sphere centered at the origin

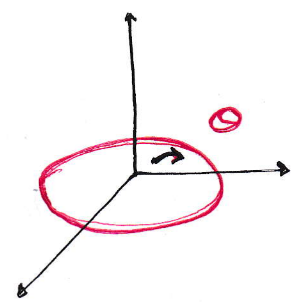
$\therefore \rho = \sqrt{x^2 + y^2 + z^2}$



ρ varies from 0 to ρ_0



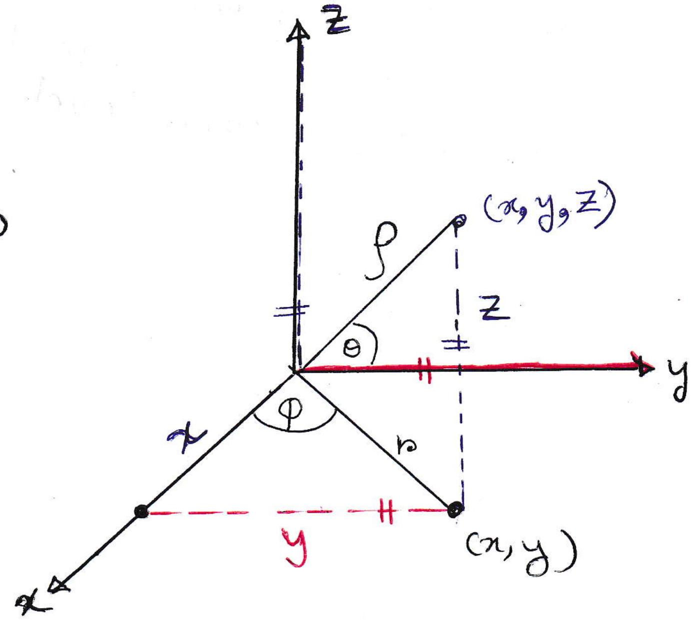
φ varies from 0 to π
(half circle)



θ varies from 0 to 2π
(full circle)

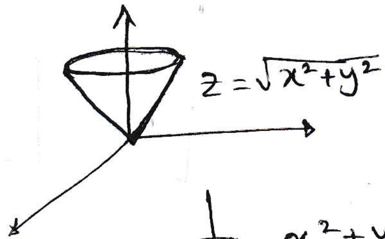
$\left.\begin{array}{l}\rho \rightarrow \text{radius of sphere} \\ \quad\text{in 3D-plane}\end{array}\right\} \rho \geq 0$

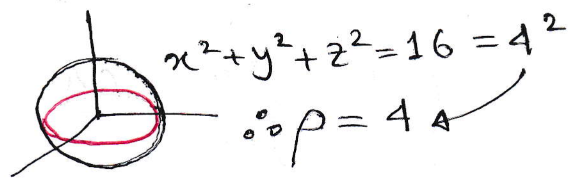
$\left.\begin{array}{l}r \rightarrow \text{radius of circle} \\ \quad\text{in 2D-plane}\end{array}\right\} r \geq 0$



Examples

(1) Use spherical coordinate to find the volume of the solid G bounded above by the sphere $x^2+y^2+z^2=16$ and below by the cone $z=\sqrt{x^2+y^2}$.





Solution: In the spherical coordinates:

the eqn $x^2+y^2+z^2=16$ is $\rho=4$ and

the eqn of the cone $z=\sqrt{x^2+y^2}$

$$\Rightarrow \rho\cos\varphi=\sqrt{\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta}$$

$$\rho\cos\varphi=\sqrt{\rho^2\sin^2\varphi\,\underbrace{(\cos^2\theta+\sin^2\theta)}_{\to 1}}$$

$$\rho\cos\varphi=\rho\sin\varphi$$

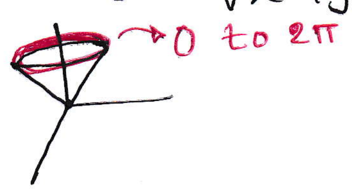
$$1=\frac{\sin\varphi}{\cos\varphi} \Rightarrow \tan\varphi=1$$

$$\Rightarrow \varphi=\pi/4$$

$\therefore \rho\in[0,4]$

$\varphi\in[0,\pi/4]$

$\theta\in[0,2\pi]$ $\therefore$ the cone is given by $z=\sqrt{x^2+y^2}$



$$\text{Volume} = \iiint dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/4} \int_{\rho=0}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi/4} \left[\frac{\rho^3}{3}\right]_0^4 \sin\varphi \, d\varphi \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi/4} \frac{64}{3} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{64}{3} \int_{0}^{2\pi} \left[-\cos\varphi\right]_0^{\pi/4} d\theta$$

$$= -\frac{64}{3} \int_{0}^{2\pi} \left[\cos\frac{\pi}{4} - \cos 0\right] d\theta$$

$$= -\frac{64}{3} \int_{0}^{2\pi} \left[\frac{1}{\sqrt{2}} - 1\right] d\theta$$

$$= -\frac{64}{3}\left(\frac{1}{\sqrt{2}} - 1\right) \left[\theta\right]_0^{2\pi}$$

$$= -\frac{64}{3}\left(\frac{1}{\sqrt{2}} - 1\right)(2\pi)$$

$$= \frac{64\pi}{3}\left(2 - \sqrt{2}\right)$$

② The solid bounded by the sphere $\rho = 4$ and below by the cone $\varphi = \frac{\pi}{3}$.

Solution: $V = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/3} \int_{\rho=0}^{4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin\varphi \left[\frac{\rho^3}{3}\right]_0^4 d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \sin\varphi \,(64/3)\, d\varphi \, d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} \left[-\cos\varphi\right]_0^{\pi/3} d\theta$$

$$= -\frac{64}{3} \int_0^{2\pi} \left[\cos\frac{\pi}{3} - \cos 0\right] d\theta$$

$$= -\frac{64}{3} \int_0^{2\pi} \left[\frac{1}{2} - 1\right] d\theta$$

$$= -\frac{64}{3}\left(-\frac{1}{2}\right) \left[\theta\right]_0^{2\pi}$$

$$= \frac{32}{3}\left[2\pi - 0\right]$$

$$= \frac{64\pi}{3}.$$

(3) The solid enclosed by the sphere $x^2+y^2+z^2=4a^2$ and the planes $z=0$ and $z=a$.

Solution: In spherical coordinates the sphere and the plane $z=a$

Consider $x^2+y^2+z^2=4a^2=(2a)^2$

$\therefore \rho = 2a$

We know $z=\rho\cos\varphi$

$\therefore a = 2a\cos\varphi$

$\because z=a, \rho=2a$

$\frac{a}{2a} = \cos\varphi$

$\cos\varphi = \frac{1}{2}$

$\varphi = \cos^{-1}\frac{1}{2}$

$\varphi = \frac{\pi}{3}$

$$V = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/3}\int_{\rho=0}^{2a}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^{2a}\sin\varphi\, d\varphi\, d\theta$$

$$= \frac{8a^3}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/3} d\theta$$

$$= -\frac{8a^3}{3}\int_0^{2\pi}\left[\cos\frac{\pi}{3}-\cos 0\right] d\theta$$

$$= -\frac{8a^3}{3}\int_0^{2\pi}\left(\frac{1}{2}-1\right) d\theta$$

$$= -\frac{8a^3}{3}\left(-\frac{1}{2}\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{4a^3}{3}\left[2\pi - 0\right]$$

$$= \frac{8\pi a^3}{3}.$$