— Exact Equations & Integrating Factors —

$M(x,y)\,dx + N(x,y)\,dy = 0$ is an Exact differential

eqn if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solve for $f(x,y) = c$ by the following steps:

$\underline{\underline{1\boxed{a}}}$ $\dfrac{\partial f}{\partial x} = M(x,y)$ or $1\boxed{b}$ $\dfrac{\partial f}{\partial y} = N(x,y)$

$\underline{\underline{2}}$ $\displaystyle\int \dfrac{\partial f}{\partial x} = \int M(x,y)$

$\displaystyle\int \partial f = \int M(x,y)\,\partial x$

$\underline{\underline{3}}$ $f = \underbrace{f(x,y)}_{\substack{\text{Result of} \\ \text{integration}}} + \underbrace{\phi(y)}_{\substack{\text{integration} \\ \text{constant} \\ \text{while} \\ x \to \text{variable} \\ y \to \text{constant}}}$

fundamental law of calculus: if we derive & integrate a particular function, then the function is unchanged

$\underline{\underline{4}}$ $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[f(x,y) + \phi(y)\right]$ derive step ③ with respect to 'y'

$N(x,y) = \dfrac{\partial}{\partial y}f(x,y) + \phi'(y)$ from $1\boxed{b}$ $\dfrac{\partial f}{\partial y} = N(x,y)$

$\phi'(y) = N(x,y) - \dfrac{\partial}{\partial y}f(x,y)$

$\phi(y) = \displaystyle\int \left[N(x,y) - \dfrac{\partial}{\partial y}f(x,y)\right]dy$

substitute $\phi(y)$ into step ③ and hence
solve $f(x,y) = c$.

$\boxed{1}$

Examples:

1. $(\sin y - y \sin x)dx + (\cos x + x \cos y - y)dy = 0$

$M dx + N dy = 0$

$M = \sin y - y \sin x$ $N = \cos x - x \cos y - y$

$\frac{\partial M}{\partial y} = \cos y - \sin x$ $\frac{\partial N}{\partial x} = -\sin x + \cos y$

$\therefore$ Exact DE

$\frac{\partial f}{\partial x} = M = \sin y - y \sin x$

$\int \partial f = \int (\sin y - y \sin x) \partial x$

$f = x \sin y + y \cos x + \phi(y)$ ———— (*)

$\frac{\partial f}{\partial y} = x \cos y + \cos x + \phi'(y)$ derived w.r.t. 'y'

$N = x \cos y + \cos x + \phi'(y)$

$\cancel{\cos x} + x \cancel{\cos y} - y = x \cancel{\cos y} + \cancel{\cos x} + \phi'(y)$

$\phi'(y) = -y$

$\phi(y) = -\int y \, dy = \frac{-y^2}{2} + C_0$

substitute $\phi(y)$ into (*)

$f(x,y) = x \sin y + y \cos x - \frac{y^2}{2} + C_0 = C$

$\therefore f(x,y) = C$

2

2 $(x - y^3 + y^2 \sin x) \, dx = (3xy^2 + 2y \cos x) \, dy$

$(x - y^3 + y^2 \sin x) \, dx - (3xy^2 + 2y \cos x) \, dy = 0$

$M = x - y^3 + y^2 \sin x$ $\qquad\qquad$ $N = -3xy^2 - 2y \cos x$

$\dfrac{\partial M}{\partial y} = -3y^2 + 2y \sin x$ $\qquad$ $\dfrac{\partial N}{\partial x} = -3y^2 + 2y \sin x$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\qquad$ It is exact DE.

$\dfrac{\partial f}{\partial x} = M = x - y^3 + y^2 \sin x$

$\int \partial f = \int (x - y^3 + y^2 \sin x) \, \partial x$

$f = \dfrac{x^2}{2} - xy^3 - y^2 \cos x + \phi(y)$ ———— ⊛

$\dfrac{\partial f}{\partial y} = 0 - x \cdot 3y^2 - 2y \cdot \cos x + \phi'(y)$ $\quad$ derived w.r.t. 'y'

$N = -3xy^2 - 2y \cos x + \phi'(y)$

$-3xy^2 - 2y \cos x = -3xy^2 - 2y \cos x + \phi'(y)$ $\quad$ ∵ $N = \dfrac{\partial f}{\partial y}$

$\phi'(y) = 0$

$\phi(y) = \int 0 \, dy = C_0$

substitute $\phi(y)$ into ⊛

$f(x,y) = \dfrac{x^2}{2} - xy^3 - y^2 \cos x + C_0 = C$

∴ $f(x,y) = C$.

3 $x\dfrac{dy}{dx} = 2xe^{x} - y + 6x^{2}$

$x\,dy = (2xe^{x} - y + 6x^{2})\,dx$

$(2xe^{x} - y + 6x^{2})\,dx - x\,dy = 0$

$M = 2xe^{x} - y + 6x^{2}$; $N = -x$

$\dfrac{\partial M}{\partial y} = -1$

$\dfrac{\partial N}{\partial x} = -1$

∴ Exact DE

$\dfrac{\partial f}{\partial x} = M = 2xe^{x} - y + 6x^{2}$

$\int \partial f = \int (2xe^{x} - y + 6x^{2})\,\partial x$

$f = 2\left[x\int e^{x}\,dx - \int\left[\dfrac{\partial}{\partial x}(x)\int e^{x}\,dx \right]dx \right] - xy$

$\qquad + 6\cdot\dfrac{x^{3}}{3} + \phi(y)$

$\qquad = 2xe^{x} - 2\int e^{x}\,dx - xy + 2x^{3} + \phi(x)$

$f = 2xe^{x} - 2e^{x} - xy + 2x^{3} + \phi(y) \longrightarrow \circledast$

$\dfrac{\partial f}{\partial y} = 0 - 0 - x + 0 + \phi'(y)$ derived w.r.t. 'y'

$\qquad\qquad\qquad\qquad ∴ N = \dfrac{\partial f}{\partial y}$

$N = -x + \phi'(y)$

$-x = -x + \phi'(y)$

$\phi'(y) = 0$

$\phi(y) = C_{0}$

substitute $\phi(y)$ into $\circledast$

$f(x,y) = 2xe^{x} - 2e^{x} - xy + 2x^{3} + C_{0} = C_{1}$

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4. $(\tan x - \sin x \sin y)\,dx + \cos x \cdot \cos y\,dy = 0$

$M = \tan x - \sin x \sin y \;;\quad N = \cos x \cdot \cos y$

$\dfrac{\partial M}{\partial y} = -\sin x \cos y \qquad\qquad \dfrac{\partial N}{\partial x} = -\sin x \cdot \cos y$

$\therefore$ Exact DE

$\dfrac{\partial f}{\partial x} = M = \tan x - \sin x \cdot \sin y$

$\displaystyle \int \partial f = \int (\tan x - \sin x \cdot \sin y)\,\partial x$

$f = \ln|\sec x| + \cos x \cdot \sin y + \phi(y) \longrightarrow \circledast$

$\dfrac{\partial f}{\partial y} = 0 + \cos x \cos y + \phi'(y) \qquad$ derived w.r.t. 'y'

$N = \cos x \cos y + \phi'(y) \qquad\qquad \therefore N = \dfrac{\partial f}{\partial y}$

$\cos x \cos y = \cos x \cos y + \phi'(y)$

$\phi'(y) = 0$

$\phi(y) = \displaystyle\int 0\,dy = c_0$

substitute $\phi(y)$ into $\circledast$

$f(x,y) = \ln|\sec x| + \cos x \cdot \sin y + c_0 = c.$

$\boxed{5}$ $\left(e^{x}+y\right)dx+\left(2+x+ye^{y}\right)dy=0$; $y(0)=1$

$M=e^{x}+y$; $N=2+x+ye^{y}$

$\dfrac{\partial M}{\partial y}=1$ $\qquad$ $\dfrac{\partial N}{\partial x}=1$

$x=0$

$y=1$

$\therefore$ exact D.E.

$\dfrac{\partial f}{\partial x}=M=e^{x}+y$

$\int \partial f = \int\left(e^{x}+y\right)\partial x$

$f=e^{x}+xy+\phi(y)$ $\qquad$ $\circledast$

$\dfrac{\partial f}{\partial y}=x+\phi'(y)$ $\qquad$ derived w.r.t. 'y'

$N=x+\phi'(y)$ $\qquad$ $\therefore N=\dfrac{\partial f}{\partial y}$

$2+x+ye^{y}=x+\phi'(y)$ $\qquad \rightarrow \int ye^{y}dy$

$\phi'(y)=2+ye^{y}$ $\qquad = y\int e^{y}dy - \int\left\{\frac{d}{dy}y\int e^{y}dy\right\}dy$

$\phi(y)=\int\left(2+ye^{y}\right)dy$ $\qquad = ye^{y}-\int 1\, e^{y}dy$

$\qquad\qquad\qquad = ye^{y}-e^{y}$

$\phi(y)=2y+\underline{ye^{y}-e^{y}}+C_{0}$

substitute $\phi(y)$ into $\circledast$

$f(x,y)=e^{x}+xy+2y+ye^{y}-e^{y}+C_{0}=C$

$\Rightarrow e^{x}+xy+2y+ye^{y}-e^{y}=C-C_{0}=C$

$\qquad\qquad\qquad\qquad\qquad$ Relabel the
$\qquad\qquad\qquad\qquad\qquad$ Constant $C-C_{0}=C$

$e^{x}+xy+2y+ye^{y}-e^{y}=C$ $\quad$ $\circledast\circledast$

$\boxed{6}$

substitute $x=0$, $y=1$ into $(**)$

$e^0 + (0)(1) + 2(1) + (1)e^1 - e^1 = C$

$1 + 0 + 2 + e - e = C$

$C = 3$

$\therefore f(x,y) = e^x + xy + 2y + ye^y - e^y = C$

$\Rightarrow e^x + xy + 2y + ye^y - e^y = 3.$

[6] $\left(\dfrac{1}{1+y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y+\sin x); \quad y(0) = 1$

$x=0$

$y=1$

$\left(\dfrac{1}{1+y^2} + \cos x - 2xy\right)dy = y(y+\sin x)\,dx$

$y(y+\sin x)\,dx - \left(\dfrac{1}{1+y^2} + \cos x - 2xy\right)dy = 0$

$M = y(y+\sin x) = y^2 + y\sin x$

$\begin{cases} N = -\dfrac{1}{1+y^2} - \cos x + 2xy \\ \dfrac{\partial N}{\partial x} = -0 + \sin x + 2y \end{cases}$

$\dfrac{\partial M}{\partial y} = 2y + \sin x$

$\therefore$ exact D.E.

$\dfrac{\partial f}{\partial x} = M = y^2 + y\sin x$

$\int \partial f = \int (y^2 + y\sin x)\,\partial x$

$f = xy^2 - y\cos x + \phi(y) \quad\quad (*)$

$\dfrac{\partial f}{\partial y} = 2xy - \cos x + \phi'(y)$ derived w.r.t. 'y'

$N = 2xy - \cos x + \phi'(y)$ $\therefore N = \dfrac{\partial f}{\partial y}$

[7]

$$N = 2xy - \cos x + \phi'(y)$$

$$-\frac{1}{1+y^2} - \cos x + 2xy = 2xy - \cos x + \phi'(y)$$

$$\phi'(y) = -\frac{1}{1+y^2}$$

$$\phi(y) = -\int \frac{1}{1+y^2}dy = -\tan^{-1}y + C_0$$

substitute $\phi(y)$ into ⊛

$$f = xy^2 - y\cos x - \tan^{-1}y + C_0 = C \qquad \because f(x,y) = C$$

$$\Rightarrow xy^2 - y\cos x - \tan^{-1}y = C - C_0 = C \quad \text{Relabel}$$
$$\underbrace{\qquad\qquad\qquad}_{⊛⊛} \qquad\qquad \text{Constant } C - C_0 = C$$

substitute $x=0, y=1$ into ⊛⊛

$$(0)(1)^2 - 1\cos(0) - \tan^{-1}(1) = C$$

$$0 - 1 - \frac{\pi}{4} = C$$

$$C = -1 - \frac{\pi}{4}$$

Substitute C into ⊛

$$\therefore f(x,y) = xy^2 - y\cos x - \tan^{-1}y = C$$
$$\Rightarrow xy^2 - y\cos x - \tan^{-1}y = -1 - \frac{\pi}{4}.$$