Reference Book:
Chapter 14.1 & 14.2 from
Anton's Calculus 10th Ed.

Double Integral

Week 5

Iterated Integral → function more than one variables.

Examples: (1) $\boxed{a}$ $\displaystyle\int_{\pi/2}^{\pi}\int_{0}^{x^2}\frac{1}{x}\cos\frac{y}{x}\,dy\,dx$

$$= \int_{\pi/2}^{\pi}\int_{0}^{x^2}\frac{1}{x}\cos\left(\frac{1}{x}\cdot y\right)dy\,dx$$

$$= \int_{\pi/2}^{\pi}\frac{1}{x}\int_{0}^{x^2}\cos\left[\frac{1}{x}\cdot y\right]dy\,dx$$

$$= \int_{\pi/2}^{\pi}\frac{1}{x}\left[\frac{\sin\frac{1}{x}\cdot y}{\frac{1}{x}}\right]_{0}^{x^2}dx$$

$$= \int_{\pi/2}^{\pi}\left[\sin\frac{1}{x}\cdot x^2 - \sin\frac{1}{x}\cdot 0\right]dx$$

$$= \int_{\pi/2}^{\pi}\left[\sin x - \sin 0\right]dx$$

$$= \int_{\pi/2}^{\pi}\sin x\,dx$$

$$= \left[-\cos x\right]_{\pi/2}^{\pi}$$

$$= -\left(\cos\pi - \cos\frac{\pi}{2}\right)$$

$$= -(-1-0)$$

$$= 1$$

① (c) $\int_1^2 \int_0^{y^2} e^{x/y^2}\, dx\, dy$

$$= \int_1^2 \left[\frac{e^{\frac{1}{y^2}\cdot x}}{\frac{1}{y^2}}\right]_0^{y^2} dy$$

$$= \int_1^2 y^2 \left[e^{\frac{1}{y^2}\cdot y^2} - e^{\frac{1}{y^2}\cdot 0}\right] dy$$

$$= \int_1^2 y^2 \left[e - 1\right] dy \quad \because e^0 = 1$$

$$= \int_1^2 (e-1)\, y^2\, dy$$

$$= (e-1)\left[\frac{y^3}{3}\right]_1^2$$

$$= (e-1)\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = (e-1)\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3}(e-1)$$

1 (b) $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$

let $xy + 1 = u$

$x\, dy = du$ {x constant, y variable}

$dy = \frac{1}{x} du$

$y = 0 \rightarrow u = 1$

$y = 1 \rightarrow u = 1 + x$

$$= \int_0^1 x \int_0^1 \frac{1}{(xy+1)^2}\, dy\, dx$$

$$= \int_0^1 x \int_1^{1+x} \frac{1}{u^2} \cdot \frac{1}{x}\, du\, dx$$

$$= \int_0^1 x \cdot \frac{1}{x} \int_1^{1+x} u^{-2}\, du\, dx$$

$$= \int_0^1 \left[\frac{u^{-2+1}}{-2+1}\right]_1^{1+x} dx$$

$$= -\int_0^1 \left[u^{-1}\right]_1^{1+x} dx$$

$$= -\int_0^1 \left[\frac{1}{1+x} - \frac{1}{1}\right] dx$$

$$= \int_0^1 \left(1 - \frac{1}{1+x}\right) dx$$

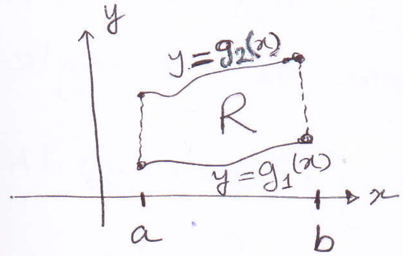
$$= \left[x - \ln|1+x|\right]_0^1$$

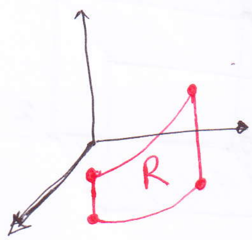
$$= 1 - \ln 2 - 0 - \ln 1$$

$$= 1 - \ln 2.$$

A $\boxed{\text{type I}}$ region 'R' bounded by

$$\begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases}$$

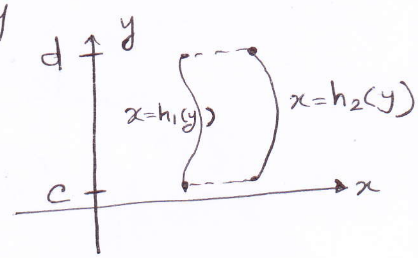




$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

A $\boxed{\text{type II}}$ region 'R' bounded by

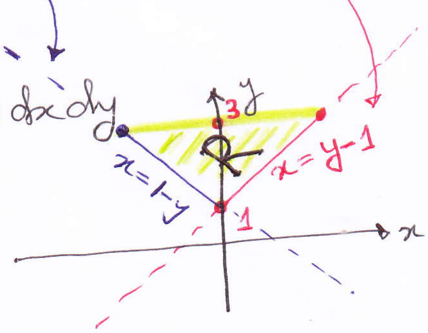
$$\begin{cases} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{cases}$$



Evaluate $\iint (2x - y^2)\,dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, $y = 3$.

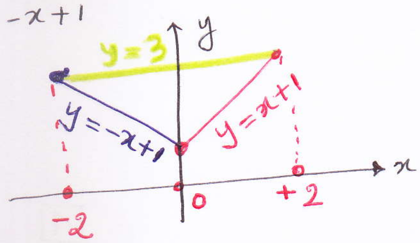
Using type II region:

$$\iint_R (2x - y^2)\,dA = \int_{y=1}^{3} \int_{x=1-y}^{y-1} (2x - y^2)\,dx\,dy$$



using type I region:

$$\iint_R (2x - y^2)\,dA = \int_{x=-2}^{0} \int_{y=-x+1}^{3} (2x - y^2)\,dy\,dx + \int_{x=0}^{2} \int_{y=x+1}^{3} (2x - y^2)\,dy\,dx$$



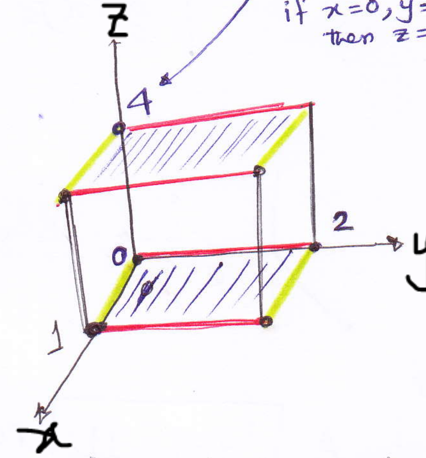
Examples ④ Use double integral to find the volume of the solid that is bounded by the plane $z = 4 - x - y$ and below the rectangle $R = [0, 1] \times [0, 2]$

if $x=0, y=0$ then $z=4$

$$V = \iint (4 - x - y)\, dA$$

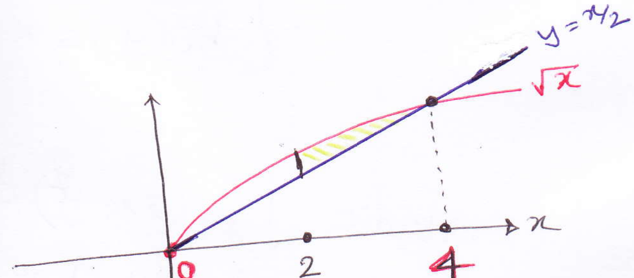
$$= \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

$$= 5$$



⑤ Evaluate $\iint_R xy\, dA$ over the region bounded by $y = \frac{x}{2}$, $y = \sqrt{x}$, $x = 2$, $x = 4$

$$\int_{x=2}^{4} \int_{y=\frac{x}{2}}^{\sqrt{x}} xy\, dy\, dx = \frac{11}{6}$$



$$\sqrt{x} = \frac{x}{2}$$

$$x = \frac{x^2}{4}$$

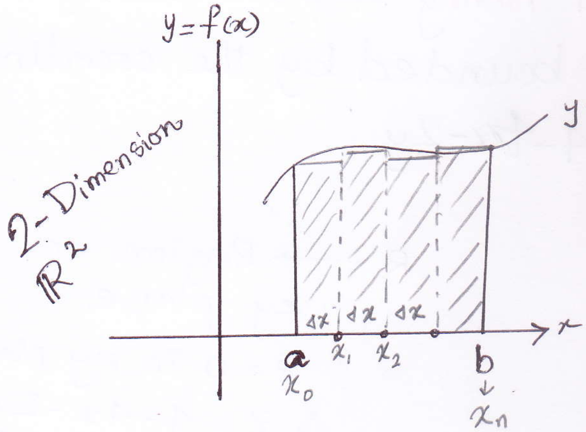
$$x - \frac{x^2}{4} = 0$$

$$x\left(1 - \frac{x}{4}\right) = 0$$

$x = 0 \qquad 1 - \frac{x}{4} = 0$

$\boxed{x=0}, \quad \boxed{x=4}$

Concept of Double Integral:



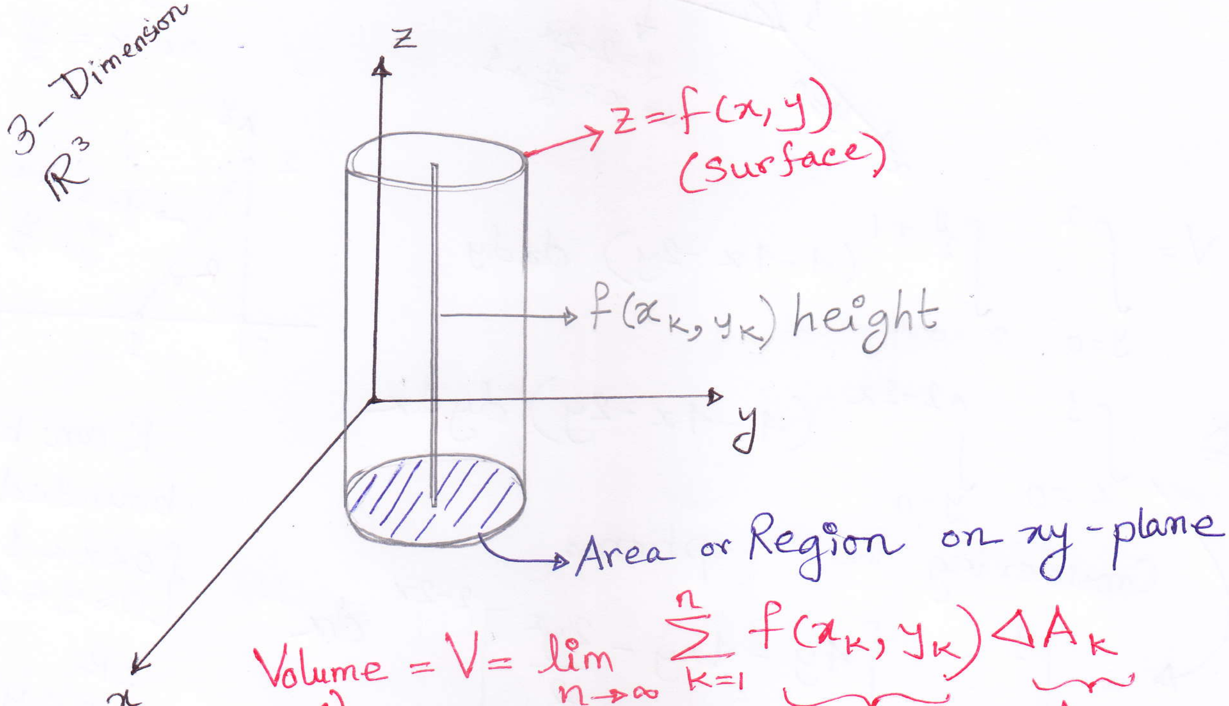
Riemann Sum:

area

$$A = \int_a^b f(x)\,dx \quad (\text{length} \times \underbrace{\text{height}}_{\text{width/heigth}})$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

$$\Delta x_k = \frac{b-a}{n}$$



$$\text{Volume} = V = \lim_{n \to \infty} \sum_{k=1}^{n} \underbrace{f(x_k, y_k)}_{\text{height}} \underbrace{\Delta A_k}_{\text{Area}}$$

(thickness)

$$= \iint_R f(x,y)\,dA$$

$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\,dx\,dy$$

$$dA = dx\,dy = dy\,dx$$

OR

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

Example Use double integral to find the volume of the tetrahedron (A solid having 4 plane triangular faces. A triangular pyramid) bounded by the coordinate planes and the plane $z = 4-4x-2y$

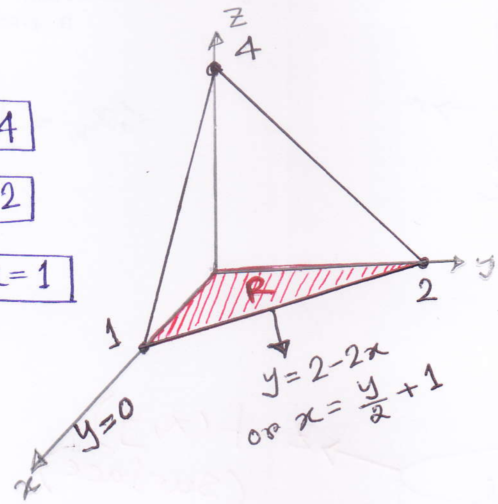
Planes

$z=4-4x-2y$

$x=0, y=0 \Rightarrow \boxed{z=4}$

$x=0, z=0 \Rightarrow \boxed{y=2}$

$y=0, z=0 \Rightarrow \boxed{x=1}$



R → Region on xy plane

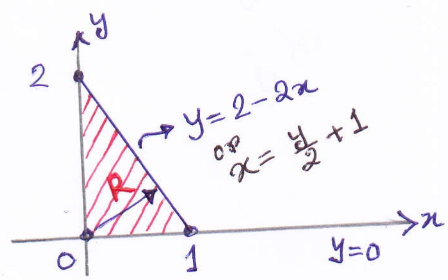
$z=0$ in xy plane

$\therefore z = 4-4x-2y$

$\Rightarrow 0 = 4-4x-2y$

$\Rightarrow y = 2-2x$

or $x = \frac{y}{2}+1$



$$V = \int_{y=0}^{2}\int_{x=0}^{\frac{y}{2}+1} (4-4x-2y)\,dx\,dy$$

OR

$$\int_{x=0}^{1}\int_{y=0}^{2-2x} (4-4x-2y)\,dy\,dx$$

R can be bounded by

$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2-2x \end{cases}$$

OR

$$\begin{cases} 0 \le y \le 2 \\ 0 \le x \le \frac{y}{2}+1 \end{cases}$$

Considering the 2^{nd} option:

$$= \int_{x=0}^{1} \left[4y - 4xy - \frac{2y^2}{2}\right]_0^{2-2x} dx$$

$$= \int_0^1 \left[4(2-2x) - 4x(2-2x) - (2-2x)^2\right] dx$$

$$= \int_0^1 \left(8-8x-8x+8x^2-4+8x-4x^2\right) dx$$

$$= \int_0^1 \left(4-8x+4x^2\right) dx = \left[4x - \frac{8x^2}{2} + \frac{4x^3}{3}\right]_0^1$$

$$= 4-4+\frac{4}{3} = \frac{4}{3}$$