BRAC University

Department of Mathematics & Natural Sciences

MAT215: Complex Variables & Laplace Transformations

Assignment-01

Deadline: Summer 2025 Total Marks: 200

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Section: 27

Use this page as the cover page of your assignment. No late submission will be graded

1. Express each of the following complex numbers in polar form. **(5×2=10)**

   (a) $2+\sqrt{3}i$ **(b)** $-5+5i$ **(c)** $-3i$ **(d)** $1-i$ **(e)** $-6-\sqrt{2}i$

2. Solve for x and y, given that $\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)^{50}=3^{25}(x+iy)$ **(1×5=5)**

3. Find each of the indicated roots and locate them graphically. **(4×5=20)**

   (a) $(-1+i)^{\frac{1}{3}}$ **(b)** $(-2\sqrt{3}-2i)^{\frac{1}{3}}$ **(c)** $(-32\sqrt{-1})^{\frac{1}{5}}$ **(d)** $(-32\sqrt{3}+32i)^{1/6}$

4. Find all the roots of each equation and locate them graphically. **(4×5=20)**

   (a) $z^4=i$ **(b)** $z^6=-64i$ **(c)** $z^3+z^2+z+1=0$ **(d)** $z^5=-16\sqrt{2}+16\sqrt{-2}$

5. Describe the following locus in the complex plane **(10×4=40)**

   (a) $|z+3i|=2$ **(b)** $\left|\frac{z+3i}{z-3i}\right|=5$ **(c)** $|z+4i|+|z-4i|=10$ **(d)** $Re\{z^2\}=9$ **(e)** $Im\{z^2\}=1$

   (f) $Re\left\{\frac{1}{z}\right\}=4$ **(g)** $|z-3|-|z+3|=4$ **(h)** $Re\{z\}+Im\{z\}=0$ **(i)** $Arg\{z\}=\frac{\pi}{4}$ **(j)** $z\cdot\bar{z}=4$

6. Describe the following locus in the complex plane **(8×5=50)**

   (b) $1\le|z|<2$ **(b)** $\left|\frac{z-3}{z+3}\right|<5$ **(c)** $|z+4|+|z-4|>10$ **(d)** $Re\{z^2\}<9$ **(e)** $Im\{z^2\}\ge1$

   (g) $Re\left\{\frac{1}{z}\right\}<4$ **(g)** $|z-3|-|z+3|\le4$ **(h)** $\frac{-2\pi}{3}\le Arg\{z\}<\frac{\pi}{4}$

7. Solve the equations. **(4×5=20)**

(b) $z^4 = i$ **(b)** $z^6 = -64i$ **(c)** $z^3 + z^2 + z + 1 = 0$ **(d)** $z^5 = -16\sqrt{2} + 16\sqrt{-2}$

8. Prove that **(5×5=25)**

a. $\sin(iz) = i\sinh(z)$

b. $\sin^{-1} z = -i\ln(iz \pm \sqrt{1 - z^2})$

c. $\cos^{-1} z = -i\ln(z + \sqrt{z^2 - 1})$

d. $\cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$

e. $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$

9. Evaluate the following limits using L'Hospital's rule: **(4×5=20)**

a. $\lim_{z\to 0} \frac{z - \sin z}{z^3}$

b. $\lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$

c. $\lim_{z\to 0} \left(\frac{\tan z}{z}\right)^{\frac{1}{z^2}}$

d. $\lim_{z\to 0} (sec)^{\frac{1}{z^2}}$

10. Let $f(z) = \frac{2z-1}{3z+2}$. For $z_0 \neq -\frac{2}{3}$ prove that **(1×5=5)**

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{7}{(3z_0 + 2)^2}$$

11. Consider the function $f: \mathbb{C} \to \mathbb{C}$ defined by **(1×5=5)**

$$f(z) = \begin{cases} \dfrac{1 - \cos(a \cdot z)}{z^2} & when\ z \neq 0 \\ 1 & when\ z = 0 \end{cases}$$

Find the all-possible values of $\boldsymbol{a}$ such that $f(z)$ becomes continuous at $z = 0$.

1)

a) $r = \sqrt{2^2 + 3} = \sqrt{10}$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2n\pi = 40.89 + 2n\pi$

$\therefore 2 + \sqrt{3}i = \sqrt{10}\, e^{i(40.89 + 2n\pi)}$

b) $r = 5\sqrt{2}$ $\quad \theta = \tan^{-1}(1) + 2n\pi = \frac{\pi}{4} + 2n\pi$

$\therefore -5 + 5i = 5\sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2n\pi\right)}$

c) $r = 3$ $\quad \theta = -\frac{\pi}{2} + 2n\pi$

$\therefore -3i = 3\, e^{i\left(-\frac{\pi}{2} + 2n\pi\right)}$

d) $r = \sqrt{2}$ $\quad \theta = -\frac{\pi}{2} + 2n\pi$

$\therefore 1 - i = \sqrt{2}\, e^{i\left(-\frac{\pi}{2} + 2n\pi\right)}$

e) $r = \sqrt{38}$ $\quad \theta = -13.26 + 2n\pi$

$\therefore -6 - \sqrt{2}i = \sqrt{38}\, e^{i(-13.26 + 2n\pi)}$

2)

$$\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)^{50} = 3^{25}\,(x+iy)$$

$$\Rightarrow (\sqrt{3})^{50}\, e^{i\left(\frac{\pi}{6}+2n\pi\right)} = 3^{25}\,(x+iy)$$

$$\Rightarrow e^{i\left(\frac{\pi}{6}+2n\pi\right)} = x+iy$$

$$\Rightarrow x+iy = \cos\left(\frac{\pi}{6}+2n\pi\right) + i\sin\left(\frac{\pi}{6}+2n\pi\right)$$

$$\therefore x+iy = \frac{\sqrt{3}}{2} + i\,\frac{1}{2}$$

$$\left.\begin{aligned} \therefore x &= \frac{\sqrt{3}}{2} \\ \therefore y &= \frac{1}{2} \end{aligned}\right\} \underline{\text{Ans.}}$$

3)

a) $(-1+i)^{\frac{1}{3}} = (\sqrt{2})^{\frac{1}{3}} \left(\cos\left(\pi - \frac{\pi}{4} + 2n\pi\right) + i\sin\left(\pi - \frac{\pi}{4} + 2n\pi\right)\right)^{\frac{1}{3}}$

$= 2^{\frac{1}{6}} \left(\cos \frac{3\pi + 8n\pi}{12} + i \sin \frac{3\pi + 8n\pi}{12}\right)$

$n=0$,

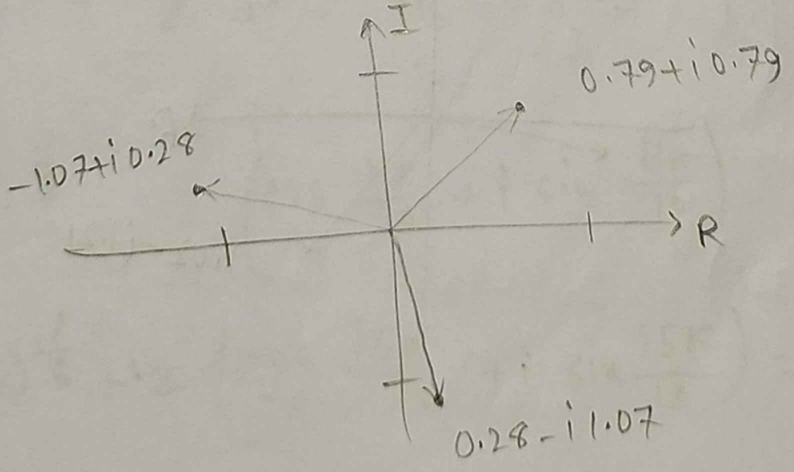
$(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right) = 0.79 + i0.79$

$n=1$,

$(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}} (-0.96 + i\,0.25) = -1.07 + i0.28$

$n=2$,

$(-1+i)^{\frac{1}{3}} = 2^{\frac{1}{6}} (0.25 - i0.96) = 0.28 - i1.07$



b) $(-2\sqrt{3}-2i)^{\frac{1}{3}} = 4^{\frac{1}{3}}\left(\cos\left(-\pi+\frac{\pi}{6}+2n\pi\right)+i\sin\left(-\pi+\frac{\pi}{6}+2n\pi\right)\right)$

$= 4^{\frac{1}{3}}\left(\cos\left(\frac{-5\pi+12n\pi}{6\cdot 3}\right)+i\sin\left(\frac{-5\pi+12n\pi}{6\cdot 3}\right)\right)$

when $n=0$,

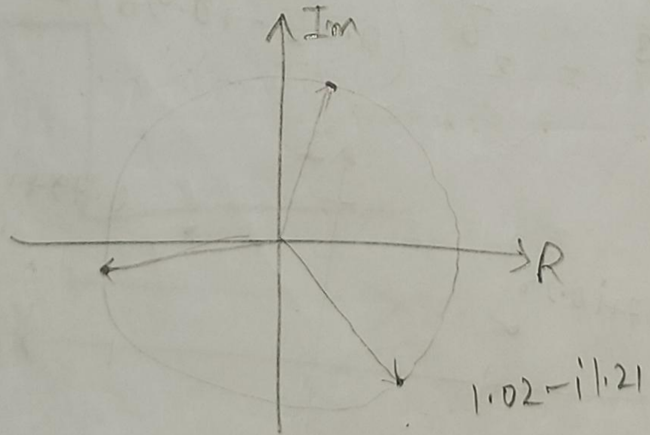
$(-2\sqrt{3}-2i)^{\frac{1}{3}} = 4^{\frac{1}{3}}\left(\cos\frac{-5\pi}{18}+i\sin\frac{-5\pi}{18}\right) \doteq 1.02-i1.21$

$n=1$,

$(-2\sqrt{3}-2i)^{\frac{1}{3}} = 4^{\frac{1}{3}}\left(\cos\frac{7\pi}{18}+i\sin\frac{7\pi}{18}\right)$

$n=2$,

$(-2\sqrt{3}-2i)^{\frac{1}{3}} = 4^{\frac{1}{3}}\left(\cos\frac{19\pi}{18}+i\sin\frac{19\pi}{18}\right)$



c) $\left(-32\sqrt{-1}\right)^{\frac{1}{5}} = \left(-32i\right)^{\frac{1}{5}}$

$= (32)^{\frac{1}{5}}\left(\cos\left(-\frac{\pi}{2}+2n\pi\right)+i\sin\left(-\frac{\pi}{2}+2n\pi\right)\right)^{\frac{1}{5}}$

$= 2\left(\cos\frac{-\pi+4n\pi}{2\cdot5}+i\sin\frac{-\pi+4n\pi}{2\cdot5}\right)$

$n=0,$

$(-32i)^{\frac{1}{5}} = 2\left(\cos\frac{-\pi}{10}+i\sin\frac{-\pi}{10}\right) = 1.9-i0.62$

$n=1,$

$(-32i)^{\frac{1}{5}} = 2\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)$

$n=2,$

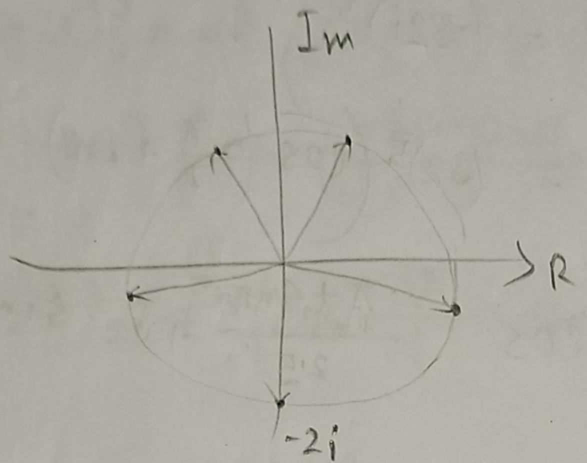
$(-32i)^{\frac{1}{5}} = 2\left(\cos\frac{7\pi}{10}+i\sin\frac{7\pi}{10}\right)$

$n=3,$

$(-32i)^{\frac{1}{5}} = 2\left(\cos\frac{11\pi}{10}+i\sin\frac{11\pi}{10}\right)$

$n=4,$

$(-32i)^{\frac{1}{5}} = 2\left(\cos\frac{15\pi}{10}+i\sin\frac{15\pi}{10}\right) = -2i$



d)

$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 64^{\frac{1}{6}}\left(\cos\left(\frac{-\pi}{6}+\pi+2n\pi\right)+i\sin\left(\frac{-\pi}{6}+\pi+2n\pi\right)\right)^{\frac{1}{6}}$$

$$= 2\left(\cos\frac{-\pi+6\pi+12n\pi}{6\cdot 6}+i\sin\frac{-\pi+6\pi+12n\pi}{6\cdot 6}\right)$$

$$= 2\left(\cos\frac{5\pi+12n\pi}{36}+i\sin\frac{5\pi+12n\pi}{36}\right)$$

$n=0,$

$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 2\left(\cos\frac{5\pi}{36}+i\sin\frac{5\pi}{36}\right)$$

$$= 1.81 + i\,0.85$$

$n=1,$

$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 2\left(\cos\frac{17\pi}{36} + i\sin\frac{17\pi}{36}\right)$$

$n=2,$

$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 2\left(\cos\frac{29\pi}{36} + i\sin\frac{29\pi}{36}\right)$$

$n=3,$

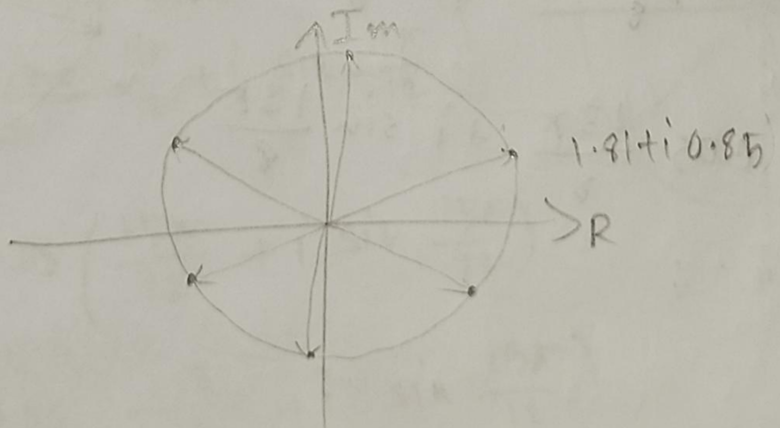
$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 2\left(\cos\frac{41\pi}{36} + i\sin\frac{41\pi}{36}\right)$$

$n=4,$

$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 2\left(\cos\frac{53\pi}{36} + i\sin\frac{53\pi}{36}\right)$$

$n=5,$

$$\left(-32\sqrt{3}+32i\right)^{\frac{1}{6}} = 2\left(\cos\frac{65\pi}{36} + i\sin\frac{65\pi}{36}\right)$$



4

a) $z^4 = i$

$$\Rightarrow z = (i)^{\frac{1}{4}}$$

$$= \left(\cos\left(\frac{\pi}{2} + 2n\pi\right) + i \sin\left(\frac{\pi}{2} + 2n\pi\right)\right)^{\frac{1}{4}}$$

$$= \left(\cos \frac{\pi + 4n\pi}{2 \cdot 4} + i \sin \frac{\pi + 4n\pi}{2 \cdot 4}\right)$$

$n = 0$,

$$z = \cos\frac{\pi}{8} + i \sin\frac{\pi}{8} = 0.92 + i\, 0.38$$

$n = 1$,

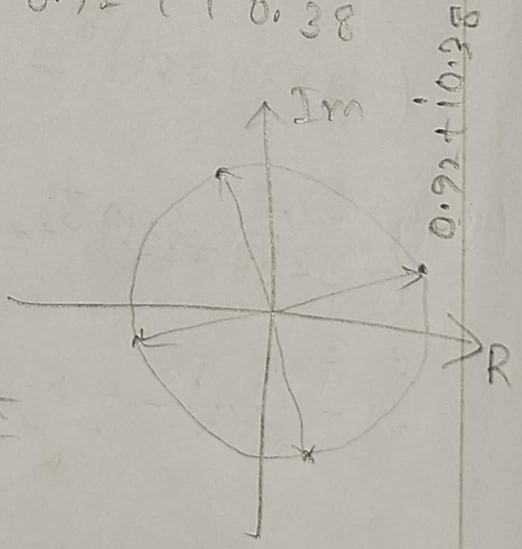
$$z = \cos\frac{5\pi}{8} + i \sin\frac{5\pi}{8}$$

$n = 2$,

$$z = \cos\frac{9\pi}{8} + i \sin\frac{9\pi}{8}$$

$n = 3$,

$$z = \cos\frac{13\pi}{8} + i \sin\frac{13\pi}{8}$$



b) $z^6 = -64i$

$\Rightarrow z = (-64i)^{\frac{1}{6}}$

$= 64^{\frac{1}{6}} \left(\cos\left(-\frac{\pi}{2} + 2n\pi\right) + i \sin\left(-\frac{\pi}{2} + 2n\pi\right)\right)^{\frac{1}{6}}$

$= 2 \left(\cos \frac{-\pi + 4n\pi}{2 \cdot 6} + i \sin \frac{-\pi + 4n\pi}{2 \cdot 6}\right)$

$n = 0,$
$z = 2\left(\cos \frac{-\pi}{12} + i \sin \frac{-\pi}{12}\right) = 1.93 - i\,0.51$

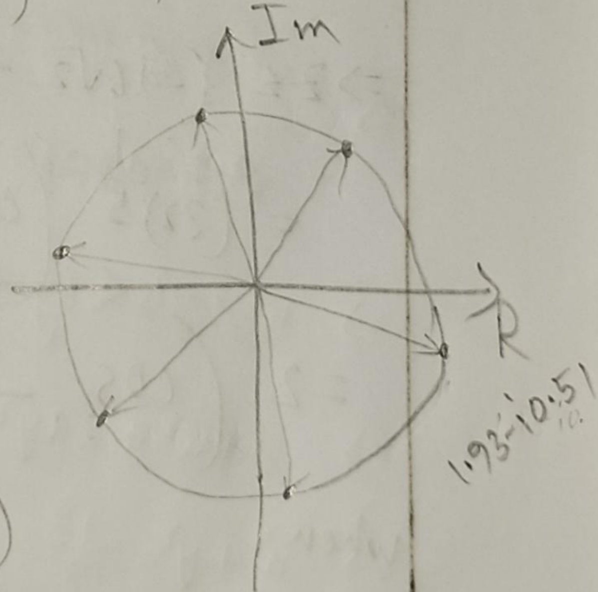
$n = 1,$
$z = 2\left(\cos \frac{3\pi}{12} + i \sin \frac{3\pi}{12}\right)$

$n = 2,$
$z = 2\left(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12}\right)$

$n = 3,$
$z = 2\left(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12}\right)$

$n = 4,$
$z = 2 \cos\left(\frac{15\pi}{12} + i \sin \frac{15\pi}{12}\right)$

$n = 5,$
$z = 2\left(\cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12}\right)$



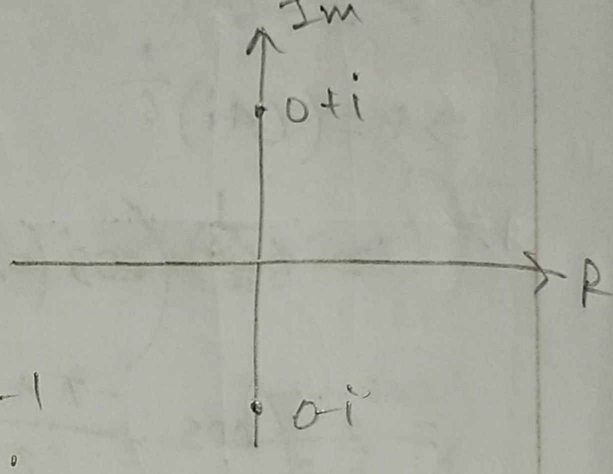
c) $z^3 + z^2 + z + 1 = 0$

$\Rightarrow z^2(z+1) + z + 1 = 0$

$\Rightarrow (z+1)(z^2+1) = 0$

$\therefore z = -1$ Or, $z^2 = -1$

$\Rightarrow z = \pm i$



d) $z^5 = -16\sqrt{2} + 16\sqrt{-2}$

$\Rightarrow z = (-16\sqrt{2} + 16\sqrt{2}\,i)^{\frac{1}{5}}$

$= (32)^{\frac{1}{5}} \left(\cos\left(\pi - \frac{\pi}{4} + 2n\pi\right) + i \sin\left(\pi - \frac{\pi}{4} + 2n\pi\right)\right)^{\frac{1}{5}}$

$= 2 \left(\cos \frac{3\pi + 8n\pi}{4.5} + i \sin \frac{3\pi + 8n\pi}{4.5} \right)$

when

$n = 0$,

$z = 2\left(\cos\frac{3\pi}{20} + i \sin\frac{3\pi}{20}\right) = 1.78 + i\, 0.9$

$n=1,$

$$z = 2\left(\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right)$$

$n=2,$

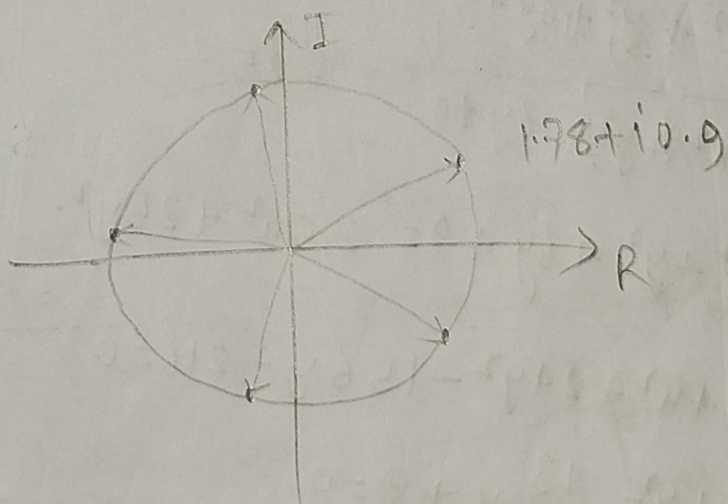
$$z = 2\left(\cos\frac{19\pi}{20} + i\sin\frac{19\pi}{20}\right)$$

$n=3,$

$$z = 2\left(\cos\frac{27\pi}{20} + i\sin\frac{27\pi}{20}\right)$$

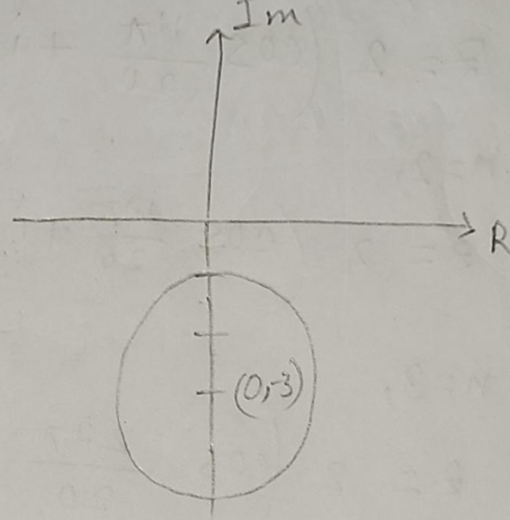
$n=4,$

$$z = 2\left(\cos\frac{35\pi}{20} + i\sin\frac{35\pi}{20}\right)$$



5)

a) $|z+3i| = 2$



$\Rightarrow |x + i(y+3)| = 2$

$\Rightarrow x^2 + (y+3)^2 = 2^2$

b) $\left|\frac{z+3i}{z-3i}\right| = 5$

$\Rightarrow \frac{x^2+y^2+6y+9}{x^2+y^2-6y+9} = 25$

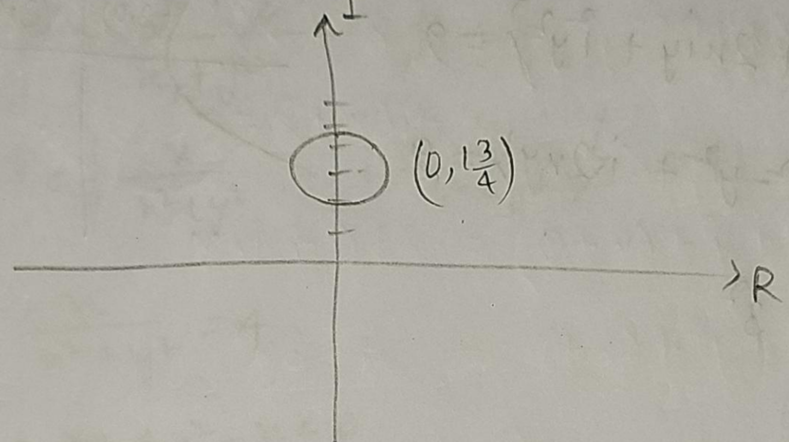
$\Rightarrow x^2+y^2+6y+9 = 25x^2+25y^2-150y+225$

$\Rightarrow 24x^2+24y^2-156y+216=0$

$\Rightarrow x^2+y^2-\frac{13}{2}y+9=0$

$\Rightarrow x^2+y^2-2\cdot y\cdot\frac{13}{4}+9+\left(\frac{13}{4}\right)^2 = \left(\frac{13}{4}\right)^2$

$$\Rightarrow x^2+\left(y-\frac{13}{4}\right)^2=\left(\frac{5}{4}\right)^2$$



c) $|z+4i|+|z-4i|=10$

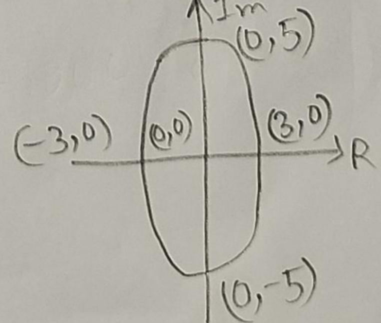
$$\Rightarrow \cancel{x^2}+y^2+8y+\cancel{16}=100-20\sqrt{x^2+y^2-8y+16}$$
$$+\cancel{x^2}+y^2-8y+\cancel{16}$$

$$\Rightarrow 10\sqrt{x^2+y^2-8y+16}=50-8y$$

$$\Rightarrow 100\cdot(x^2+y^2-8y+16)=2500-800y+64y^2$$

$$\Rightarrow 100x^2+36y^2=900$$

$$\Rightarrow \frac{x^2}{3^2}+\frac{y^2}{5^2}=1$$

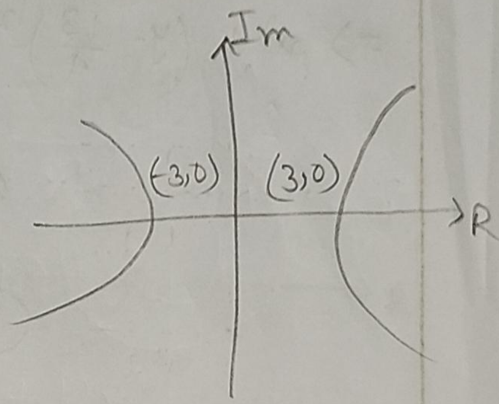


d) $Re\{z^2\} = 9$

$\Rightarrow Re\{x^2 + 2xiy + i^2y^2\} = 9$

$\Rightarrow Re\{x^2 - y^2 + i\,2xy\} = 9$

$\therefore x^2 - y^2 = 3^2$

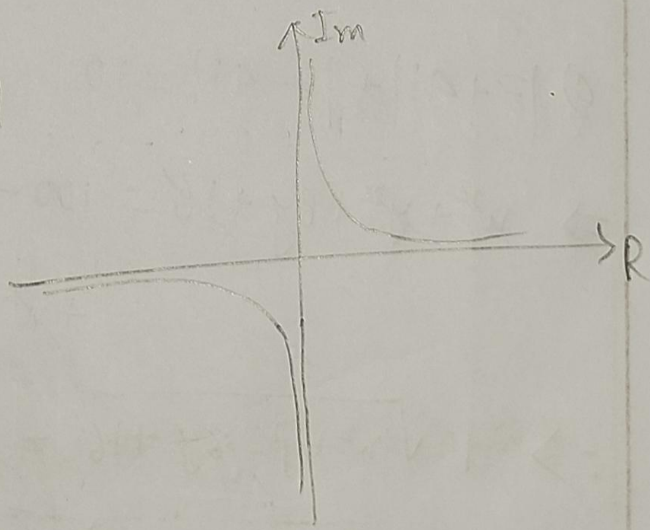


e) $Im\{z^2\} = 1$

$\Rightarrow Im\{x^2 - y^2 + i2xy\} = 1$

$\therefore 2xy = 1$

$\Rightarrow y = \frac{1}{2x}$



f) $Re\left\{\frac{1}{z}\right\} = 4$

$\Rightarrow Re\left\{\frac{x - iy}{x^2+y^2}\right\} = 4$

$\Rightarrow Re\left\{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right\} = 4$

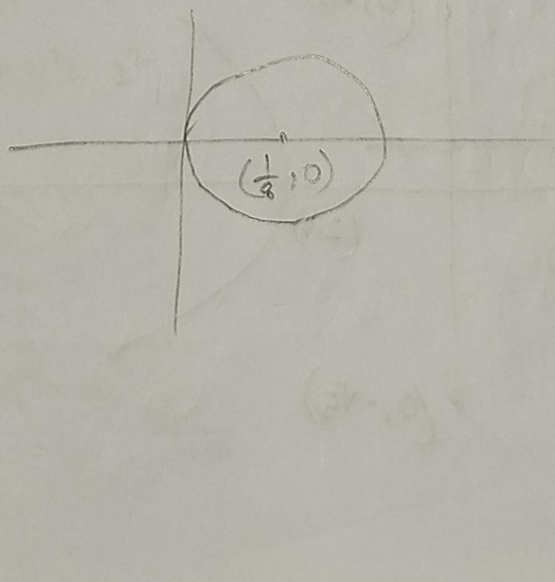
$\therefore \frac{x}{x^2+y^2} = 4$

$\Rightarrow 4x^2 + 4y^2 - x = 0$

$\Rightarrow x^2 - \frac{x}{4} + y^2 = 0$

$\Rightarrow x^2 - 2x\frac{1}{8} + \left(\frac{1}{8}\right)^2 + y^2 = \left(\frac{1}{8}\right)^2$

$\Rightarrow \left(x - \frac{1}{8}\right)^2 + y^2 = \left(\frac{1}{8}\right)^2$



9) $|z-3| - |z+3| = 4$

$\Rightarrow \cancel{x^2} - 6x + \cancel{9} + \cancel{y^2} = 16 + 8\sqrt{x^2+6x+9+y^2} + \cancel{x^2}$
$+ 6x + \cancel{9} + \cancel{y^2}$

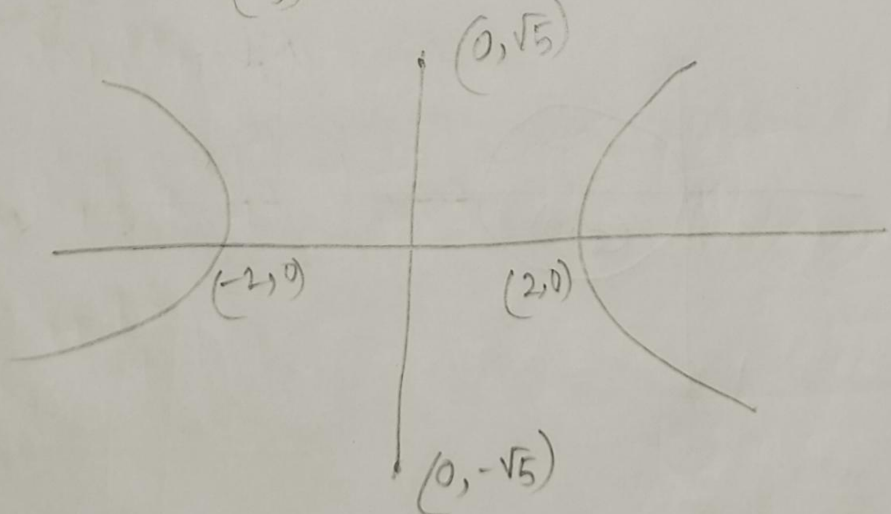
$\Rightarrow 8\sqrt{x^2+y^2+6x+9} = 16 + 12x$

$\Rightarrow 2\sqrt{x^2+y^2+6x+9} = 4 + 3x$

$\Rightarrow 4x^2 + 4y^2 + \cancel{24x} + 36 = 16 + \cancel{24x} + 9x^2$

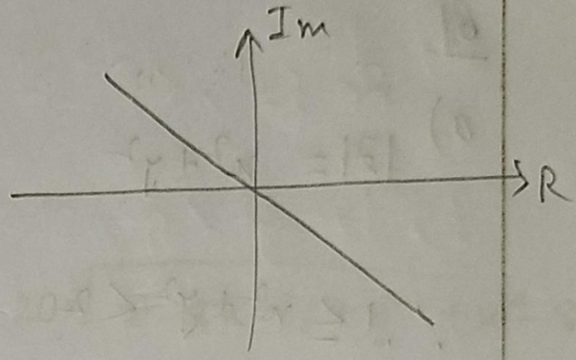
$\Rightarrow -5x^2 + 4y^2 = -20$

$\Rightarrow \dfrac{x^2}{2^2} - \dfrac{y^2}{(\sqrt{5})^2} = 1$

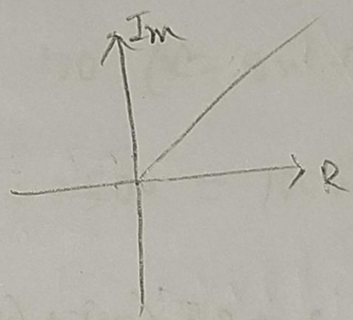


h) $Re\{z\} + Im\{z\} = 0$

$\Rightarrow x + y = 0$



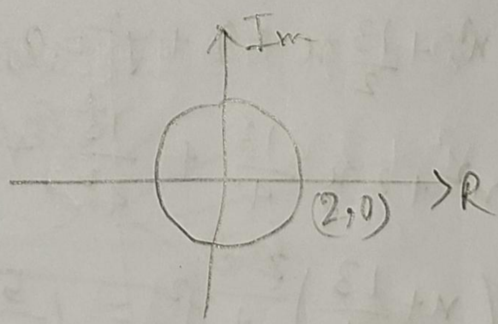
i) $Arg\{z\} = \frac{\pi}{4}$



j) $z \cdot \bar{z} = 4$

$\Rightarrow |z|^2 = 4$

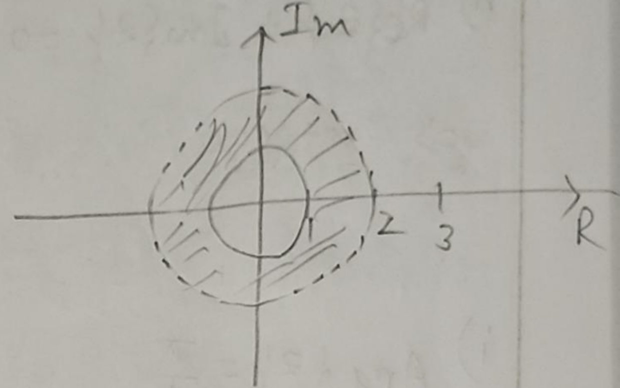
$\Rightarrow x^2 + y^2 = 2^2$



6)

a) $|z| = x^2 + y^2$

$\therefore 1 \leq x^2 + y^2 < 2$



b) $\left|\frac{z-3}{z+3}\right| = 5$

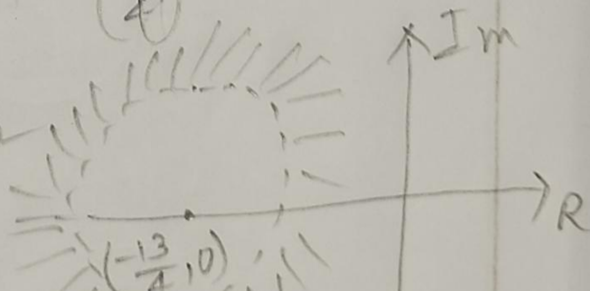
$\Rightarrow x^2 - 6x + 9 + y^2 = 25\,(x^2 + 6x + 9 + y^2)$

$\Rightarrow 24x^2 + 156x + 216 + 24y^2 = 0$

$\Rightarrow x^2 + \frac{13}{2}x + 9 + y^2 = 0$

$\Rightarrow x^2 + 2x\frac{13}{4} + \frac{13^2}{4^2} + y^2 = \left(\frac{5}{4}\right)^2$

$\Rightarrow \left(x + \frac{13}{4}\right)^2 + y^2 = \left(\frac{5}{4}\right)^2$



c) $|z+4| + |z-4| > 10$

$\therefore |z+4| + |z-4| = 10$

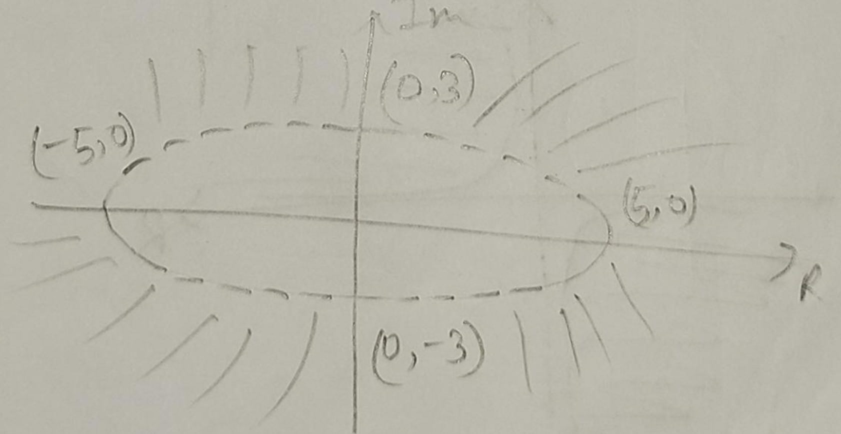
$\Rightarrow \cancel{x^2} + 8x + \cancel{16} + \cancel{y^2} = 100 - 20\sqrt{x^2 - 8x + 16 + y^2} + \cancel{x^2} - 8x + \cancel{16} + \cancel{y^2}$

$\Rightarrow (8x - 50)^2 = 100(x^2 - 8x + 16 + y^2)$

$\Rightarrow 64x^2 - 800x + 2500 = 100x^2 - 800x + 1600 + 100y^2$

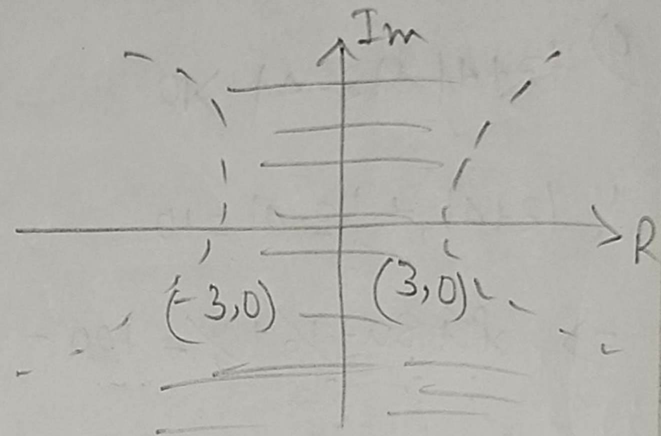
$\Rightarrow 36x^2 + 100y^2 = 900$

$\Rightarrow \frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$



d) $\operatorname{Re}\{z^2\} < 9$

$\Rightarrow x^2 - y^2 < 9$

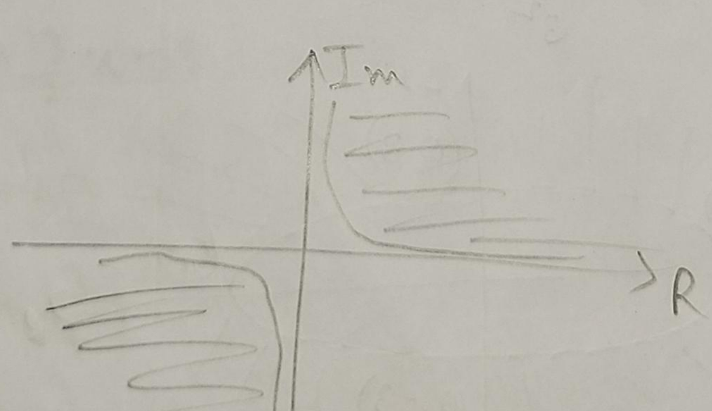


e) $\operatorname{Im}\{z^2\} \geq 1$

$\therefore \operatorname{Im}\{z^2\} = 1$

$\Rightarrow \operatorname{Im}\{x^2 - y^2 + i\,2xy\} = 1$

$\Rightarrow 2xy = 1 \Rightarrow y = \frac{1}{2x}$

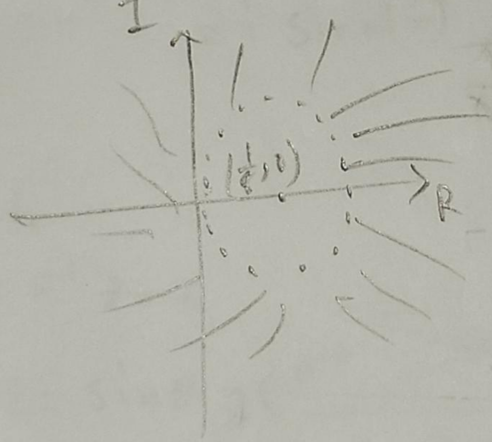


f) $\mathrm{Re}\left\{\frac{1}{z}\right\} < 4$

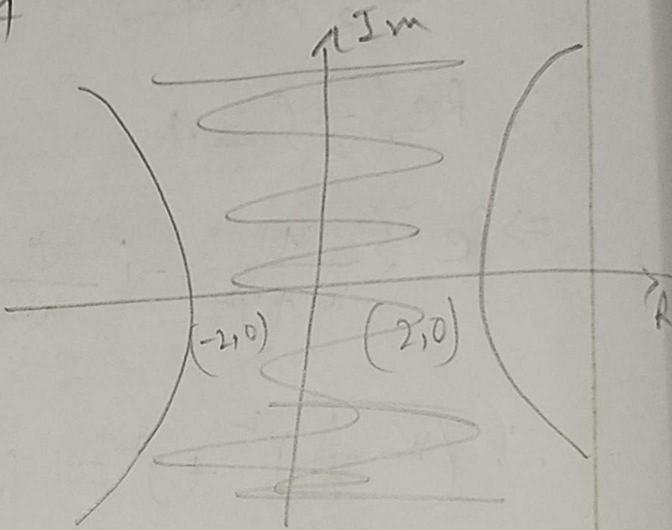
$\therefore \mathrm{Re}\left\{\frac{1}{z}\right\} = 4$

$\Rightarrow \mathrm{Re}\left\{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right\} = 4$

$\therefore \left(x - \frac{1}{8}\right)^2 + y^2 = \left(\frac{1}{8}\right)^2$



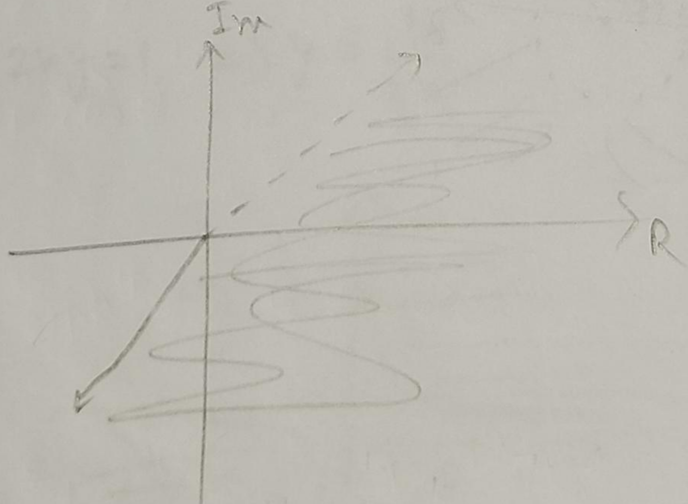
g) $|z-3| - |z+3| \le 4$



$\therefore |z-3| - |z+3| = 4$

$\Rightarrow \dfrac{x^2}{2^2} - \dfrac{y^2}{(\sqrt{5})^2} = 1$

h) $\dfrac{-2\pi}{3} \le Arg\{z\} < \dfrac{\pi}{4}$



8)

a) $\sin(iz) = \dfrac{e^{iiz} - e^{-iiz}}{2i} = \dfrac{e^{-z} - e^{z}}{2i}$

$= -\dfrac{(e^{z} - e^{-z})}{2i}$

$= i\left(\dfrac{e^{z} - e^{-z}}{2}\right)$

$= i\sinh(z)$ (proved)

b) Let,

$\sin^{-1} z = \theta$

$\Rightarrow z = \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\Rightarrow e^{i\theta} - e^{-i\theta} = 2iz$ —— (I)

$\therefore e^{i\theta} + e^{-i\theta} = \sqrt{(e^{i\theta} - e^{-i\theta})^2 + 4}$

$= \sqrt{-4z^2 + 4}$

$\Rightarrow e^{i\theta} + e^{-i\theta} = \pm 2\sqrt{1 - z^2}$ —— (II)

$(i) + (ii) \Rightarrow$

$$2e^{i\theta} = 2iz \pm 2\sqrt{1-z^2}$$

$$\Rightarrow i\theta = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow \sin^{-1} z = -i\ln\left(iz \pm \sqrt{1-z^2}\right)$$

(proved)

c) Let,

$$\cos^{-1} z = \theta$$

$$\Rightarrow z = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow e^{i\theta} + e^{-i\theta} = 2z \quad \text{--- (i)}$$

$$\therefore e^{i\theta} - e^{-i\theta} = \sqrt{(e^{i\theta} + e^{-i\theta})^2 - 4}$$

$$= \sqrt{4z^2 - 4}$$

$$\Rightarrow e^{i\theta} - e^{-i\theta} = 2\sqrt{z^2 - 1} \quad \text{--- (ii)}$$

(i) + (ii) =>

$$2e^{i\theta} = 2z + 2\sqrt{z^2-1}$$

$$\Rightarrow i\theta = \ln\left(z + 2\sqrt{z^2-1}\right)$$

$$\Rightarrow \cos^{-1} z = -i \ln\left(z + \sqrt{z^2-1}\right)$$ (proved)

d) Let,

$$\cot^{-1} z = \theta$$

$$\Rightarrow z = \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{i(e^{i\theta} + e^{-i\theta})}{(e^{i\theta} - e^{-i\theta})}$$

$$\Rightarrow \frac{z}{i} = \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} \Rightarrow \frac{z+i}{z-i} = \frac{2e^{i\theta}}{2e^{-i\theta}}$$

$$\Rightarrow e^{2i\theta} = \frac{z+i}{z-i} \Rightarrow 2i\theta = \ln\frac{z+i}{z-i}$$

$$\Rightarrow \cot^{-1} z = \frac{1}{2i} \ln\frac{z+i}{z-i}$$ (proved)

e) Let, $\sinh^{-1} z = \theta$

$$\Rightarrow z = \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\Rightarrow e^{\theta} - e^{-\theta} = 2z \quad \text{---} \quad (1)$$

$$e^{\theta} + e^{-\theta} = \sqrt{(e^{\theta} - e^{-\theta})^2 + 4}$$

$$= \sqrt{4z^2 + 4}$$

$$\Rightarrow e^{\theta} + e^{-\theta} = 2\sqrt{z^2+1} \quad \text{---} \quad (11)$$

(1), (11) $\Rightarrow$

$$2e^{\theta} = 2z + 2\sqrt{z^2+1}$$

$$\Rightarrow \sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right) \quad \text{(proved)}$$

9)

a) $$\lim_{z\to 0} \frac{z-\sin z}{z^3}$$

$$= \lim_{z\to 0} \frac{1-\cos z}{3z^2}$$

$$= \lim_{z\to 0} \frac{\sin z}{6z}$$

$$= \lim_{z\to 0} \frac{\cos z}{6}$$

$$= \frac{1}{6} \quad \underline{As.}$$

b) $$y = \lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$$

$$\Rightarrow \ln y = \lim_{z\to 0} \frac{1}{z^2} \ln\left(\frac{\sin z}{z}\right) = \lim_{z\to 0} \frac{\ln(\sin z) - \ln z}{z^2}$$

$$= \lim_{z\to 0} \frac{\frac{1}{\sin z}\cos z - \frac{1}{z}}{2z} = \lim_{z\to 0} \frac{z\cos z - \sin z}{2z \cdot z\sin z}$$

$$= \lim_{z\to 0} \frac{z\cos z - \sin z}{2z^2 \sin z}$$

$$= \lim_{z \to 0} \frac{-z\sin z + \cancel{\cos z} - \cancel{\cos z}}{2\left(z^2 \cos z + \sin z \cdot 2z\right)}$$

$$= \lim_{z \to 0} \frac{-\sin z}{2\left(z\cos z + 2\sin z\right)}$$

$$= \lim_{z \to 0} \frac{-\cos z}{2\left(-z\sin z + \cos z + 2\cos z\right)}$$

$$= -\frac{1}{6}$$

$$\Rightarrow y = e^{-\frac{1}{6}}$$ Ans.

c) $\lim_{z \to 0} \left(\frac{\tan z}{z}\right)^{\frac{1}{z^2}}$

$= \lim_{z \to 0} e^{\ln\left(\frac{\tan z}{z}\right)^{\frac{1}{z^2}}}$

$= e^{\lim_{z \to 0} \frac{\ln\left(\frac{\tan z}{z}\right)}{z^2}}$

$= e^{\lim_{z \to 0} \frac{z}{\tan z} \cdot \frac{z\sec^2 z - \tan z}{z^2} \cdot \frac{1}{2z}}$

$= e^{\lim_{z \to 0} \frac{z\sec^2 z - \tan z}{2z^2 \tan z}}$

$= e^{\lim_{z \to 0} \frac{z \cdot 2\sec z \cdot \sec z \tan z + \cancel{\sec^2 z} - \cancel{\sec^2 z}}{\cancel{z} \cdot (2z\tan z + z^2 \sec^2 z)}}$

$= e^{\lim_{z \to 0} \frac{\sec^2 z \tan z}{2\tan z + z\sec^2 z}}$

$$= e^{\lim_{z \to 0} \frac{\sec^2 z \cdot \sec^2 z + 2\sec^2 z \tan^2 z}{2\sec^2 z + \sec^2 z + z \cdot 2\sec^2 z \tan z}}$$

$$= e^{\frac{1}{3}}$$ <u>As.</u>

d) $\lim_{z \to 0} (\sec z)^{\frac{1}{z^2}}$

$$= e^{\lim_{z \to 0} \frac{\ln(\sec z)}{z^2}}$$

$$= e^{\lim_{z \to 0} \frac{\cancel{\sec z} \tan z}{2z \cdot \cancel{\sec z}}}$$

$$= e^{\lim_{z \to 0} \frac{\sec^2 z}{2}} = e^{\frac{1}{2}}$$ <u>As.</u>

10)

$$f(z) = \frac{2z-1}{3z+2}$$

$$\Rightarrow f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{2(z_0+h)-1}{3(z_0+h)+2} - \dfrac{2z_0-1}{3z_0+2}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(2z_0+2h-1)(3z_0+2) - (2z_0-1)(3z_0+3h+2)}{(3z_0+3h+2)(3z_0+2)}}{h}$$

$$= \lim_{h \to 0} \frac{6z_0^2 + 6hz - 3z_0 + 4z_0 + 4h - 2 - 6z_0^2 - 6hz - 4z_0 + 3z_0 + 3h + 2}{h(3z_0+3h+2)(3z_0+2)}$$

$$= \frac{7}{(3z_0+2)^2}$$

(proved)

iii)

$$f(z) = \frac{1-\cos(az)}{z^2}$$

$$\Rightarrow \lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{1-\cos(az)}{z^2}$$

$$= \lim_{z\to 0} \frac{a\sin(az)}{2z}$$

$$= \lim_{z\to 0} \frac{a^2\cos(az)}{2}$$

$$= \frac{a^2}{2}$$

for continuity,

$$\frac{a^2}{2} = f(0)$$

$$\Rightarrow \frac{a^2}{2} = 1$$

$$\Rightarrow a = \pm\sqrt{2} \quad \underline{\text{As.}}$$