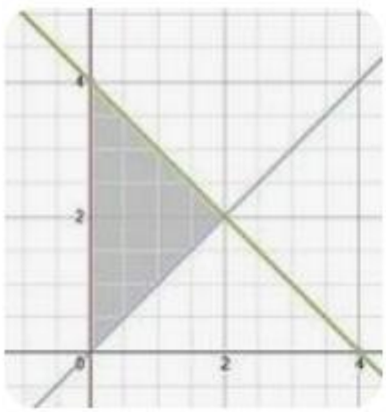
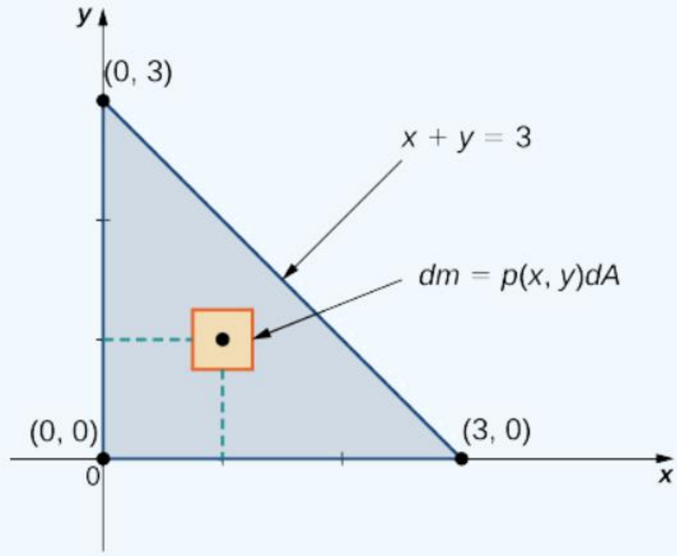
In mathematics, a planar lamina (or plane lamina) is **a figure representing a thin, usually uniform, flat layer of the solid**. ... Planar laminas can be used to determine moments of inertia, or center of mass of flat figures, as well as an aid in corresponding calculations for 3D bodies.



Consider a triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy\,kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.



Using the expression developed for mass, we see that

$$m = \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3}\left[x\frac{y^2}{2}\bigg|_{y=0}^{y=3}\right]dx = \int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right]\bigg|_{x=0}^{x=3} = \frac{27}{8}.$$

The computation is straightforward, giving the answer $m = \frac{27}{8}\,kg.$

Consider the same triangular lamina R with vertices $(0,0)$, $(0,3)$, $(3,0)$ and with density $\rho(x,y) = xy$. Find the moments M_x and M_y.

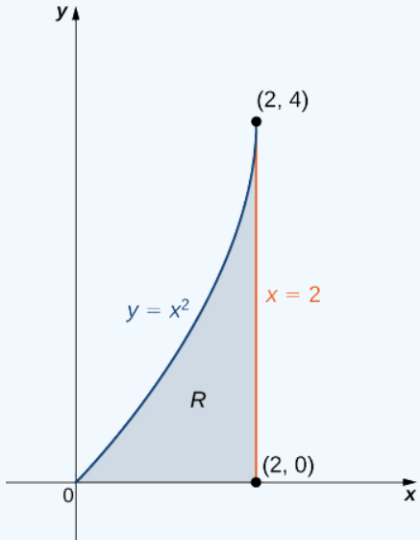
Solution

Use double integrals for each moment and compute their values:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx = \frac{81}{20},$$

$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx = \frac{81}{20},$$

Find the mass, moments, and the center of mass of the lamina of density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the following figure).



Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$m = \iint_R dm = \iint_R \rho(x, y)\, dA = \int_{x=0} x = 2 \int_{y=0}^{y=x^2} (x + y) dy\, dx = \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=x^2} \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right] \Big|_{x=0}^{x=2} = \frac{36}{5}.$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x, y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x + y)\, dy\, dx = \frac{80}{7},$$

$$M_y = \iint_R x\rho(x, y)\, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x + y)\, dy\, dx = \frac{176}{15}.$$

Finally, evaluate the center of mass,

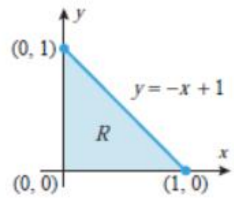
$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)\, dA}{\iint_R \rho(x, y)\, dA} = \frac{176/15}{36/5} = \frac{44}{27},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y)\, dA}{\iint_R \rho(x, y)\, dA} = \frac{80/7}{36/5} = \frac{100}{63}.$$

Hence the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{44}{27}, \frac{100}{63} \right)$.

Example from Calculus 10th Ed. Anton, Chapter 14.8 (This book is available in buX)

Example 1 A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density function $\delta(x, y) = xy$. Find its total mass.



▲ Figure 14.8.4

Solution. Referring to (3) and Figure 14.8.4, the mass M of the lamina is

$$M = \iint_R \delta(x, y)\, dA = \iint_R xy\, dA = \int_0^1 \int_0^{-x+1} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left[\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right] dx = \frac{1}{24} \text{ (unit of mass)}$$ ◄

Alternative Formulas for Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R} \iint_R x\delta(x, y)\, dA$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R} \iint_R y\delta(x, y)\, dA$$

Example 2 Find the center of gravity of the triangular lamina with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ and density function $\delta(x, y) = xy$.

Solution. The lamina is shown in Figure 14.8.4. In Example 1 we found the mass of the lamina to be

$$M = \iint_R \delta(x, y)\, dA = \iint_R xy\, dA = \frac{1}{24}$$

The moment of the lamina about the y-axis is

$$\begin{aligned} M_y &= \iint_R x\delta(x, y)\, dA = \iint_R x^2 y\, dA = \int_0^1 \int_0^{-x+1} x^2 y\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{2}x^2 y^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right) dx = \frac{1}{60} \end{aligned}$$

and the moment about the x-axis is

$$\begin{aligned} M_x &= \iint_R y\delta(x, y)\, dA = \iint_R xy^2\, dA = \int_0^1 \int_0^{-x+1} xy^2\, dy\, dx \\ &= \int_0^1 \left[\frac{1}{3}xy^3\right]_{y=0}^{-x+1} dx = \int_0^1 \left(-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\right) dx = \frac{1}{60} \end{aligned}$$

From (11) and (12),

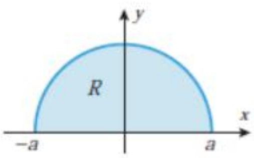
$$\bar{x} = \frac{M_y}{M} = \frac{1/60}{1/24} = \frac{2}{5}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/60}{1/24} = \frac{2}{5}$$

so the center of gravity is $\left(\frac{2}{5}, \frac{2}{5}\right)$. ◄

Centroid of a Region R

$$\bar{x} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} = \frac{1}{\text{area of } R} \iint\limits_R x\,dA \tag{13}$$

$$\bar{y} = \frac{\iint\limits_R y\,dA}{\iint\limits_R dA} = \frac{1}{\text{area of } R} \iint\limits_R y\,dA \tag{14}$$



▲ Figure 14.8.6

► **Example 3** Find the centroid of the semicircular region in Figure 14.8.6.

Solution. By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance. From (14),

$$\begin{aligned}
\bar{y} &= \frac{1}{\text{area of } R} \iint\limits_R y\,dA = \frac{1}{\frac{1}{2}\pi a^2} \iint\limits_R y\,dA \\
&= \frac{1}{\frac{1}{2}\pi a^2} \int_0^{\pi} \int_0^{a} (r \sin\theta) r\,dr\,d\theta \qquad \boxed{\text{Evaluating in polar coordinates}} \\
&= \frac{1}{\frac{1}{2}\pi a^2} \int_0^{\pi} \left[\frac{1}{3} r^3 \sin\theta\right]_{r=0}^{a} d\theta \\
&= \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{1}{3}a^3\right) \int_0^{\pi} \sin\theta\,d\theta = \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{2}{3}a^3\right) = \frac{4a}{3\pi}
\end{aligned}$$

so the centroid is $\left(0, \frac{4a}{3\pi}\right)$. ◄

Compare the calculation in Example 3 to that of Example 3 in Section 6.7.