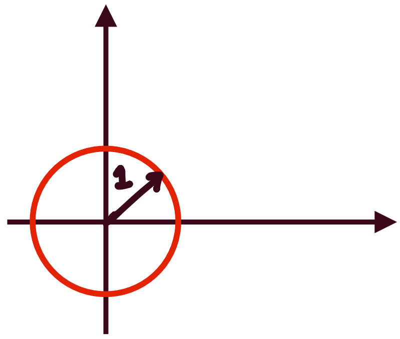
Chapter 4: Complex Line Integral



Parametrization

Consider, $x^2 + y^2 = 1$

Question: What is the range of values of x? $\longrightarrow [-1, 1]$

Question: Is there a way to generate the values of x in a functional manner?

$$x = \cos t \quad \text{where } 0 \leq t \leq 2\pi$$

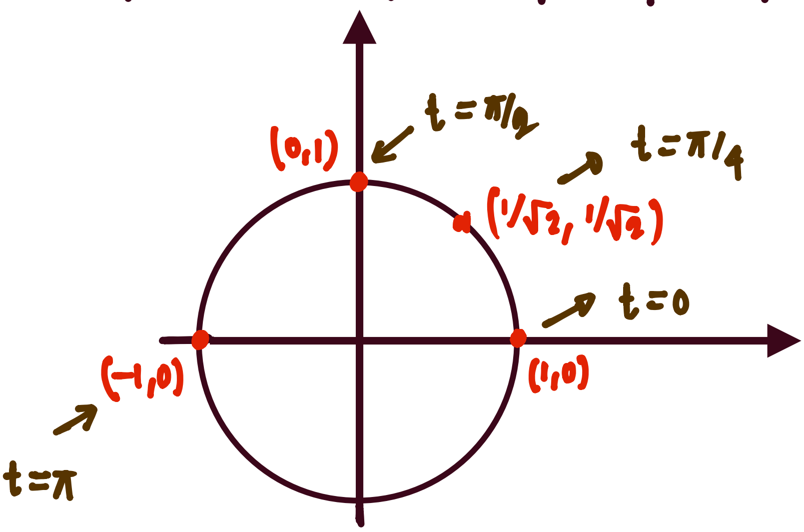
Question: What are the values of y then?

$$\cos^2 t + y^2 = 1$$
$$\Rightarrow y^2 = \sin^2 t$$
$$\Rightarrow y = \sin t \ (\text{or } -\sin t)$$

Now summarize,

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \text{where } 0 \leq t \leq 2\pi \quad \text{——— (i)}$$

t	0	$\pi/4$	.....	$\pi/2$	....	π	....	2π
(x,y)	$(1,0)$	$(1/\sqrt{2}, 1/\sqrt{2})$	.....	$(0,1)$	....	$(-1,0)$	....	$(0,0)$



(i) Describes the curve C. These are called parametric equations for C & t is the parameter.

Parametric curve (Defn): A curve in 2-D plane can be described by,

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \text{ where } a \le t \le b$$

These are called parametric equations for that curve & t is the parameter.

Equivalently,

$z = f(t) + i g(t)$ where $a \le t \le b$ → Describes the same curve as C

Complex form of parametric equations

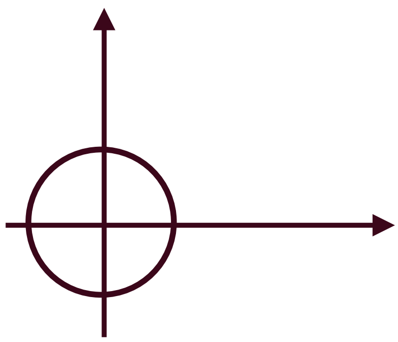
Remarks:

(i) $(f(a), g(a))$ → initial point of the curve

(ii) $(f(b), g(b))$ → terminal point of the curve

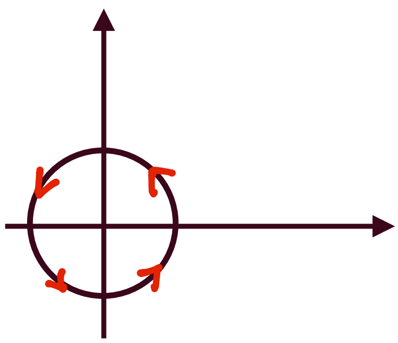
Difference between general and parametric curve:

$x^2 + y^2 = 1$



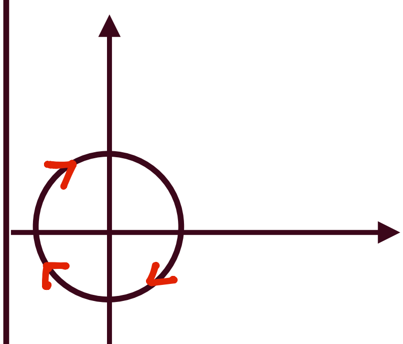
(No Direction)

$x = \cos t$
$y = \sin t$
$0 \le t \le 2\pi$



Direction of increasing parameter
↓
Orientation

$x = \cos t$
$y = -\sin t$
$0 \le t \le 2\pi$



Orientation

Problem

Plot the parametric curves defined by,

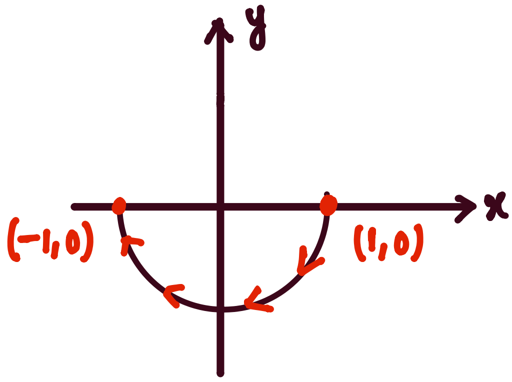
(a) $x = \cos t, y = -\sin t \; ; 0 \leq t \leq \pi$

(b) $x = t, y = 1 \; ; 0 \leq t \leq 4$

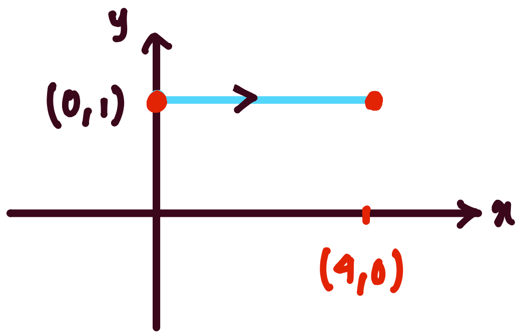
(a) $z = 1 + it \; ; 0 \leq t \leq 1$

Solution:

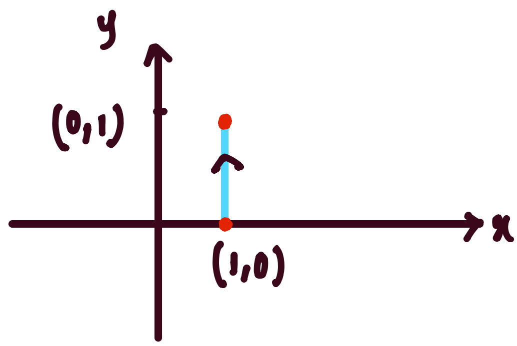
(a) $\left.\begin{aligned} x &= \cos t \\ y &= -\sin t \end{aligned}\right\} \; 0 \leq t \leq \pi$



(b) $\left.\begin{aligned} x &= t \\ y &= 1 \end{aligned}\right\} \; 0 \leq t \leq 4$

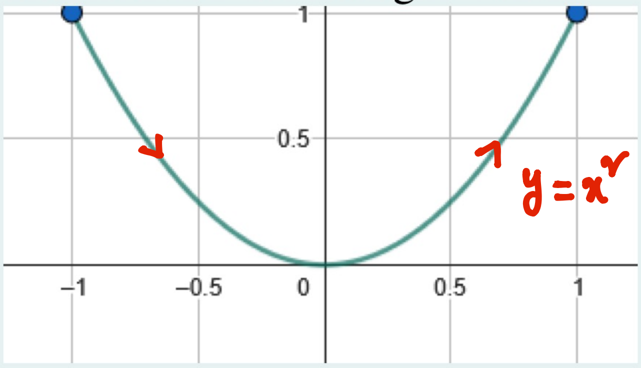


(c) $\left.\begin{aligned} x &= 1 \\ y &= t \end{aligned}\right\} \; 0 \leq t \leq 1$



Problem

Parametrize the following curves:

(a) 

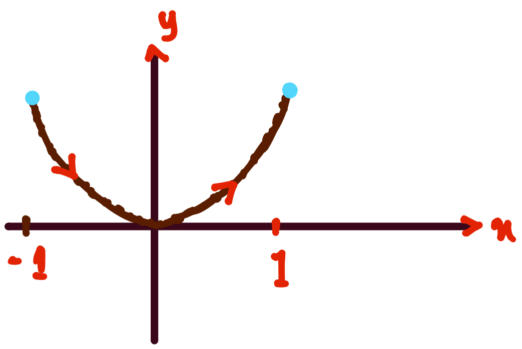
(b) Straight line from $(0,3)$ to $(2,3)$.

(c) Straight line from $(2,3)$ to $(2,4)$.

(d) Straight line from $(0,3)$ to $(2,4)$.

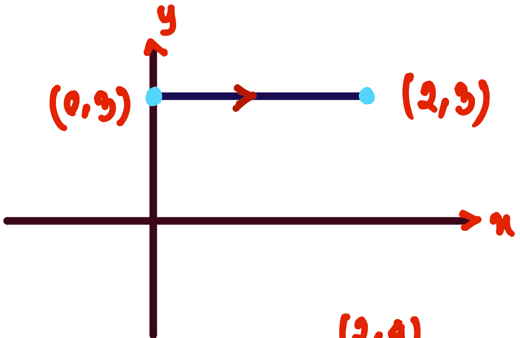
Solution:

(a)



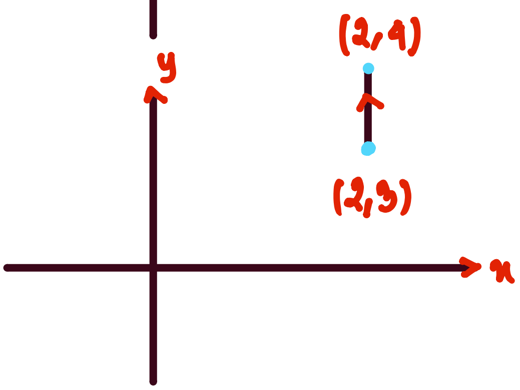
$$\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} \quad -1 \le t \le 1$$

(b)



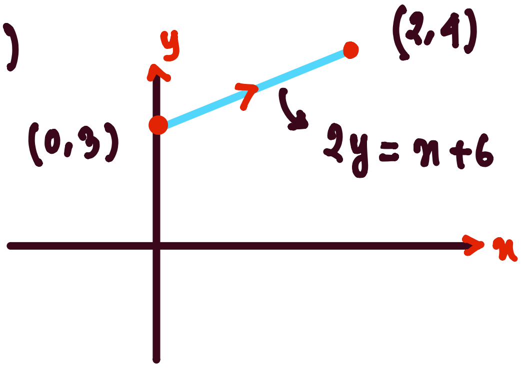
$$\left.\begin{array}{l} x = t \\ y = 3 \end{array}\right\} \quad 0 \le t \le 2$$

(c)



$$\left.\begin{array}{l} x = 2 \\ y = t \end{array}\right\} \quad 3 \le t \le 4$$

(d)



Straight line joining (x_1,y_1) & (x_2,y_2)

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

$$\boxed{\begin{aligned} x &= t \\ y &= t/2 + 3 \end{aligned} \quad \text{where } 0 \le t \le 2}$$

One parametrization

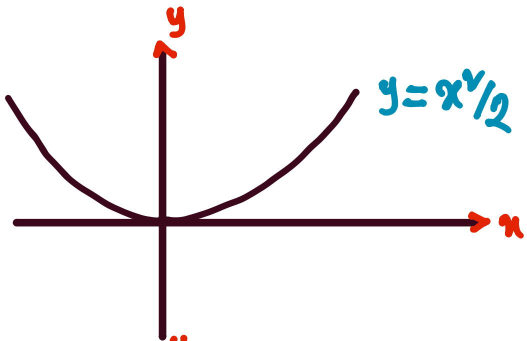
$$\boxed{\begin{aligned} x &= 2t - 6 \\ y &= t \end{aligned} \quad \text{where } 3 \le t \le 4}$$

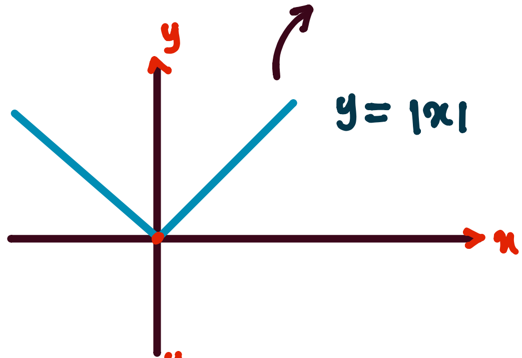
Another parametrization

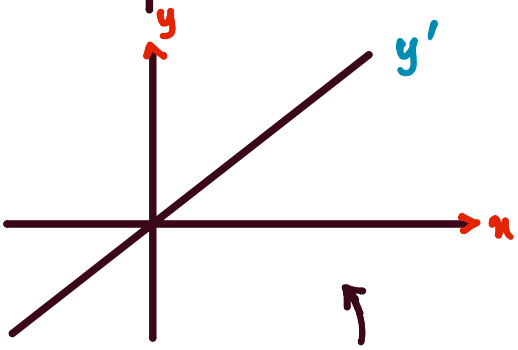
Smooth Curve:

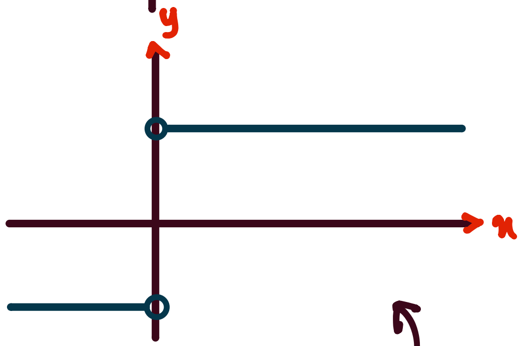
Made up of finitely many smooth segments

Piecewise Smooth









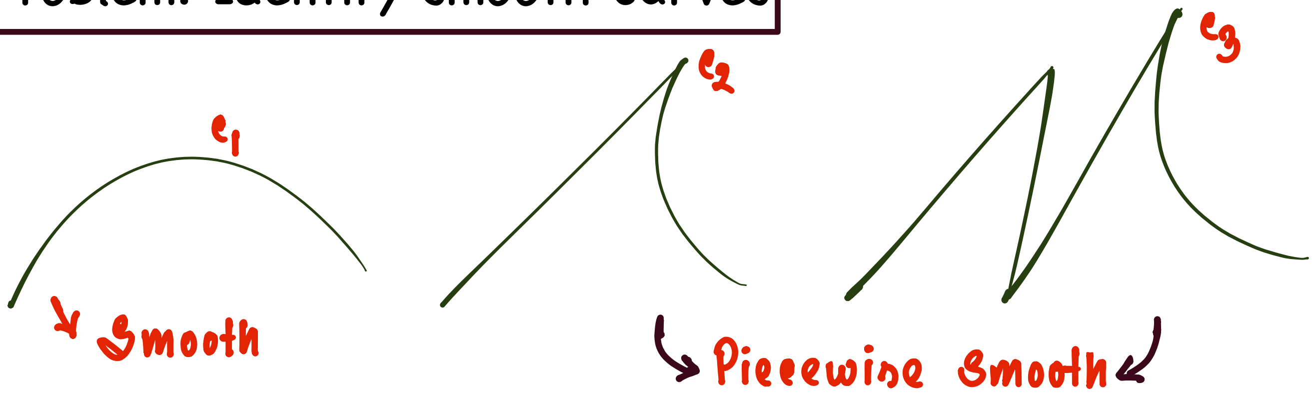
Derivative exists at all points and is continuous

Smooth curve

Derivative is not continuous

Not a smooth curve

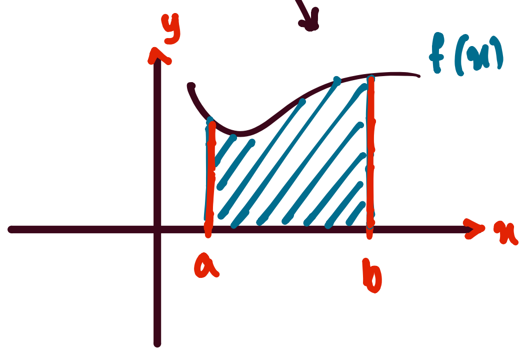
Problem: Identify smooth curves



Path in complex integration

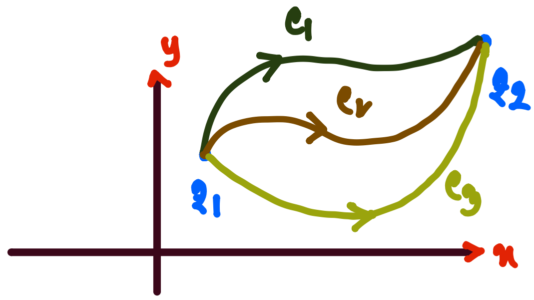
$f(x) \rightarrow$ real valued function

$\int_a^b f(x)\,dx$



$f(z) \rightarrow$ complex function

$\int_{z_1}^{z_2} f(z)\,dz \rightarrow$ meaningful ? (×)

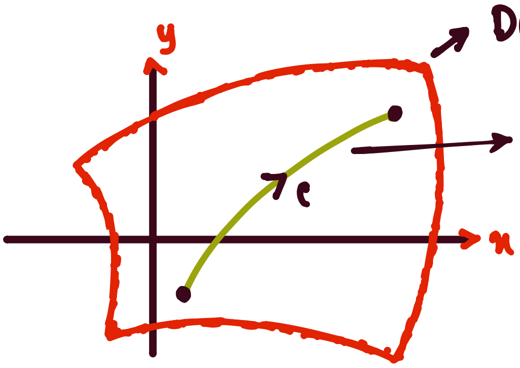


integration will be along a path c

$\int_c f(z)\,dz \rightarrow$ integral of f along c

Integral of a complex function along a smooth curve (Complex line integral)

$f(z) \rightarrow$ complex function



Dom(f)

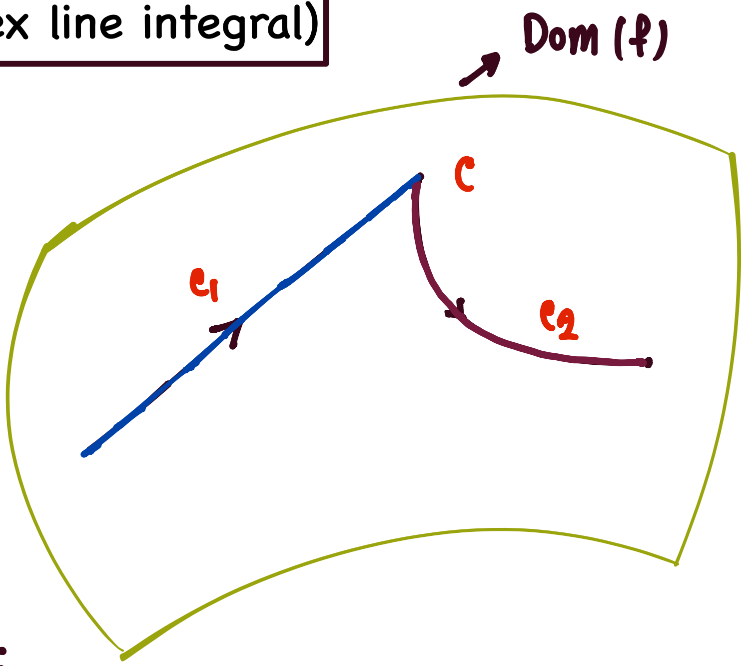
Smooth curve lying in the Domain of f

$c : z = x(t) + i y(t)\ ;\ a \leq t \leq b$
$= \gamma(t)$

Then line integral (or complex line integral) of f along C is defined as

$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Integral of a complex function along a Piecewise smooth curve (Complex line integral)



$f(z) \to$ complex function

$C \to$ piecewise smooth curve lying in the Dom(f)

C_1: $z = h_1(t) + i g_1(t) = \gamma_1(t)$; $a \le t \le b$

C_2: $z = h_2(t) + i g_2(t) = \gamma_2(t)$; $c \le t \le d$

Then line integral along C of f is defined as,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \int_a^b f(\gamma_1(t))\,\gamma_1'(t)\,dt + \int_c^d f(\gamma_2(t))\,\gamma_2'(t)\,dt$$

Solving complex integrals

Procedure for solving integral of the form $\int_C f(z)\,dz$

(i) Find parametric equations for C in the complex form

$$C : z = \gamma(t) \; ; \; a \leq t \leq b$$

(ii) Apply the definition

$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Problem

Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $z = t^2 + it$.

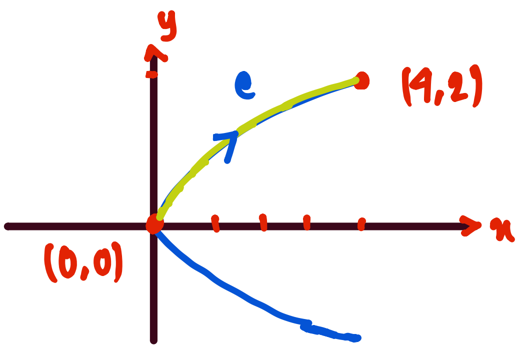
Solution: Step-01: Finding parametric equations for C (in the complex form)

$C : z = t^2 + it$

initial point : $z = 0 \Rightarrow t^2 + it = 0 \Rightarrow t = 0$

terminal point: $z = 4 + 2i \Rightarrow t^2 + it = 4 + 2i \Rightarrow t = 2$

$\therefore \; C : z = t^2 + it \; ; \; 0 \leq t \leq 2$



Step-02: Apply the definition

$$\int_C \bar{z}\,dz = \int_0^2 \overline{t^2 + it}\,(2t + i)\,dt$$

$$= \int_0^2 (t^2 - it)(2t + i)\,dt$$

$$= \int_0^2 (2t^3 - it^2 + t)\,dt$$

$$= 10 - 8/3\, i \quad \text{(Ans:)}$$

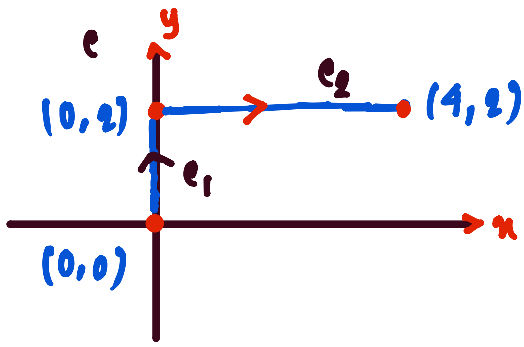
Problem

Evaluate $\int_c \bar{z}\,dz$ along the line from $z = 0$ to $z = 2i$ and then from $z = 2i$ to $z = 4 + 2i$.

Sol^n: step-01: Find parametric equations for c in complex form

$c_1 : z = 0 + it \; ; \; 0 \le t \le 2$

$c_2 : z = t + 2i \; ; \; 0 \le t \le 4$



Step-02: Apply the definition

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$$

$$= \int_{c_1} \bar{z}\,dz + \int_{c_2} \bar{z}\,dz$$

$$= \int_0^2 (-it)\, i\,dt + \int_0^4 \overline{t + 2i} \cdot 1\,dt$$

$$= \int_0^2 t\,dt + \int_0^4 (t - 2i)\,dt$$

$$= 10 - 8i \quad \text{(Ans:)}$$

Real line integral:

$$c: \begin{array}{l} x = f(t) \\ y = g(t) \end{array} \quad \text{where } a \leq t \leq b \longrightarrow \text{Smooth curve}$$

Then the real line integral of $P(x,y)\,dx + Q(x,y)\,dy$ along a curve c is defined as,

$$\int_c P(x,y)\,dx + Q(x,y)\,dy$$

$$= \int_a^b P(f(t), g(t))\, f'(t)\,dt + Q(f(t), g(t))\, g'(t)\,dt$$

$$= \int_a^b \{P(f(t), g(t))\, f'(t) + Q(f(t), g(t))\, g'(t)\}\,dt$$

Procedure for solving integral of the form

$$\int_c P(x,y)\,dx + Q(x,y)\,dy$$

(i) Find parametric equations for c

$$c: \left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\} \quad a \leq t \leq b$$

(ii) Apply the definition

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(f(t), g(t))\, f'(t) + Q(f(t), g(t))\, g'(t)\,dt$$

Problem

Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$ along

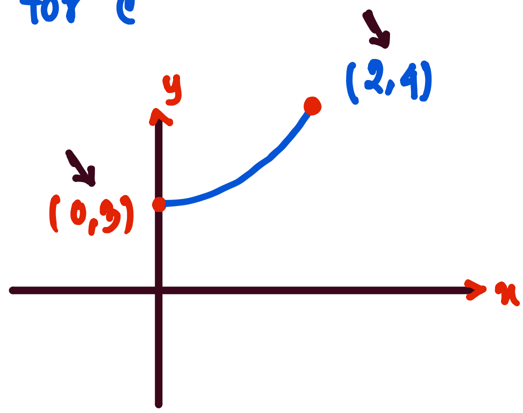
(a) the parabola $x = 2t, y = t^2 + 3$.

(b) the straight lines from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2,4)$.

(c) the straight line from $(0,3)$ to $(2,4)$.

(a) Step-01: Parametric equations for C

$$C: \begin{cases} x = 2t \\ y = t^2 + 3 \end{cases}$$



initial point: $0 = 2t \Rightarrow t = 0$

terminal point: $2 = 2t \Rightarrow t = 1$

$\therefore$ C: $x = 2t,\ y = t^2 + 3;\ 0 \leq t \leq 1$

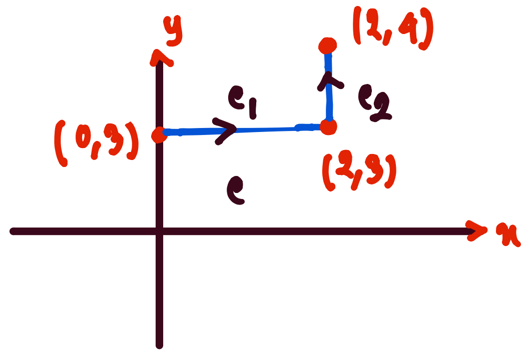
Step-02: Apply the definition

$$\int_C (2y + x^2)\,dx + (3x - y)\,dy$$

$$= \int_0^1 \left\{ (2t^2 + 6 + 4t^2)\cdot 2 + (6t - t^2 - 3)\,2t \right\} dt$$

$$= \frac{33}{2} \quad \text{(Ans:)}$$

(b) Step-01: Parametric equations for C



$$C_1: \begin{cases} x = t \\ y = 3 \end{cases} \quad \text{where } 0 \leq t \leq 2$$

$$C_2: \begin{cases} x = 2 \\ y = t \end{cases} \quad \text{where } 3 \leq t \leq 4$$

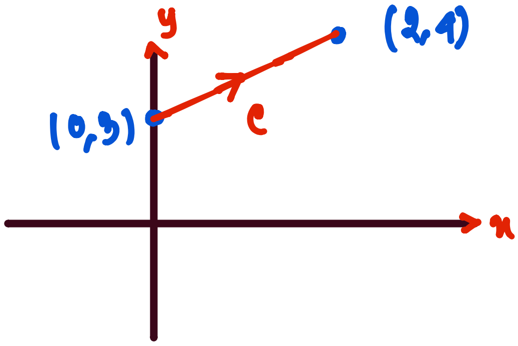
Step-02: Apply the definition

$$\int_C (2y + x^2)\,dx + (3x - y)\,dy$$

$$= \int_{C_1} (2y + x^2)\,dx + (3x - y)\,dy + \int_{C_2} (2y + x^2)\,dx + (3x - y)\,dy$$

$$= \int_0^2 (6 + t^2)\cdot 1\,dt + \int_3^4 (6 - t)\cdot 1\,dt = \frac{44}{3} + \frac{5}{2} \quad \text{(Ans:)}$$

(c) Step-01: Parametric equations



$$\boxed{\begin{aligned} x &= 2t-6 \\ y &= t \end{aligned} \qquad \text{where } 3 \le t \le 4}$$

Step-02: Apply the definition

$$\int_C (2y+x^2)\,dx + (3x-y)\,dy$$

$$= \int_3^4 \{2t+(2t-6)^2\}\cdot 2 + \{3(2t-6)-t\}\cdot 1 \, dt$$

$$= \frac{97}{6} \quad \text{(Ans:)}$$

Simply Connected region: A region with no holes

Multiply connected region: A region with one or more holes

Simply connected

Multiply connected

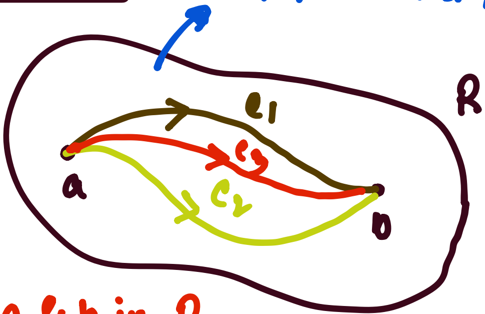
A Result for simply connected region:

R: Simply connected region

$f(z) \rightarrow$ Analytic in R

$\int_a^b f(z)\,dz \rightarrow$ independent of the path in R joining points a & b in R

f(z) is analytic here



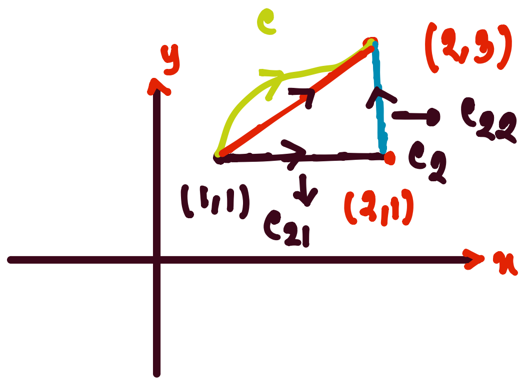
Problem

If C is the curve defined by $y = x^3 - 3x^2 + 4x - 1$ joining $(1,1)$ to $(2,3)$, then find $\int_C (12z^2 - 4iz)\,dz$

Solution: $\boxed{\text{Step-01 : Reformulation}}$

$f(z) = 12z^2 - 4iz$ is analytic everywhere in the complex plane.

& Entire complex plane is simply connected



The integral is independent of the path joining (1,1) to (2,3)

let,

c_2: line segment from (1,1) to (2,1) & then from (2,1) to (2,3)

$$\int_c (12z^2 - 4iz)\,dz = \int_{c_2} (12z^2 - 4iz)\,dz$$

Step-02: $\int_{c_2} (12z^2 - 4iz)\,dz$ computation

(2.1) c_2 Parametric form:

c_{21} : $z = t + i$; $1 \leq t \leq 2$

c_{22} : $z = 2 + it$; $1 \leq t \leq 3$

(2.2) Apply the definition

$$\int_{c_2} (12z^2 - 4iz)\,dz = \int_{c_{21}} (12z^2 - 4iz)\,dz + \int_{c_{22}} (12z^2 - 4iz)\,dz$$

$$= \int_1^2 \{12(t+i)^2 - 4i(t+i)\} \cdot 1\,dt$$

$$+ \int_1^3 \{12(2+it)^2 - 4i(2+it)\} \cdot i\,dt$$

$$= -156 + 38i$$

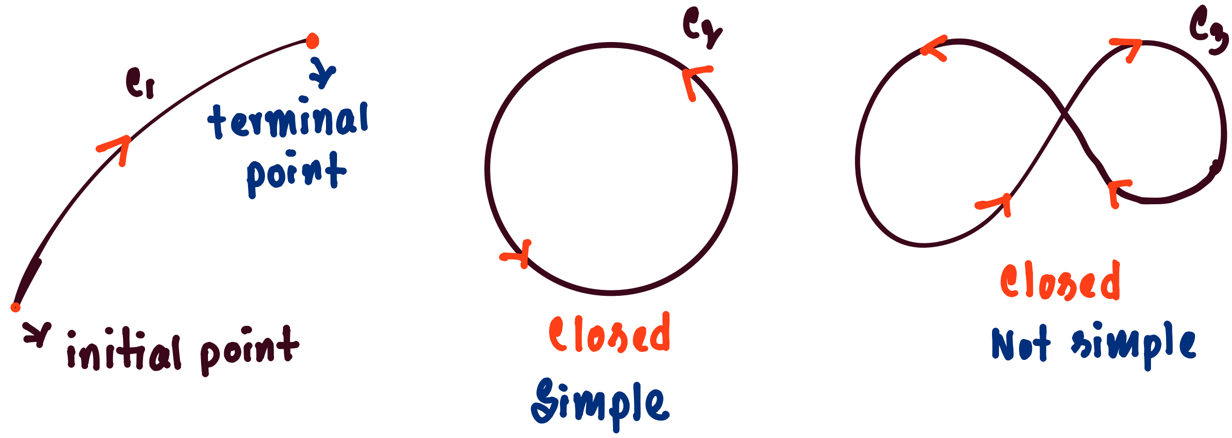
$\int_{c_2} (12z^2 - 4iz)\,dz = -156 + 38i$

$\Rightarrow \int_{c} (12z^2 - 4iz)\,dz = -156 + 38i$ (Ans:)

Closed curve and simple closed curve

(No self intersection)

(initial point = terminal point)

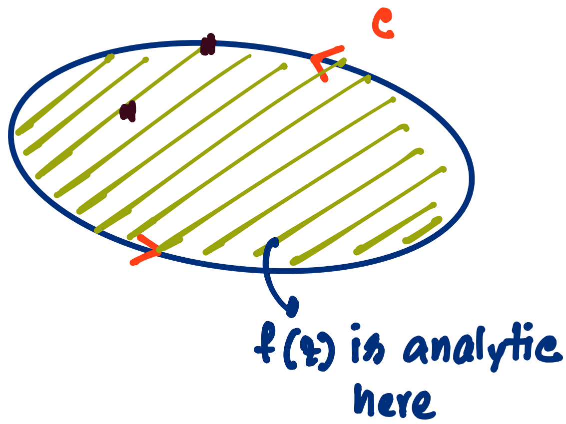


Cauchy's theorem (Cauchy Goursat theorem)

c : closed curve

$f(z) \rightarrow$ analytic in the region enclosed by c (inside & on c)

Then, $\int_c f(z)\,dz = 0$



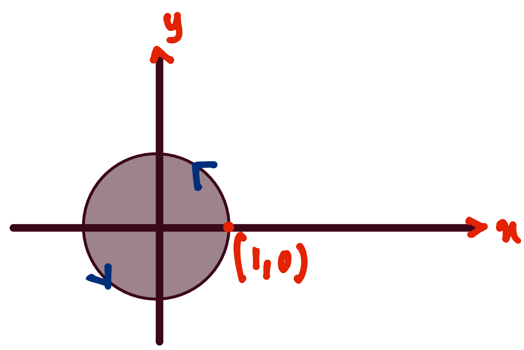
Example: $f(z) = 12z^2 - 9z + 4i$

c : a circle with centre $(0,0)$, radius 1 (counter-clockwise)

$\int_c f(z)\,dz = ?$

$f(z)$ is analytic everywhere & so is on the region enclosed by c

$\therefore \int_c f(z)\,dz = 0$ (Using Cauchy's theorem)

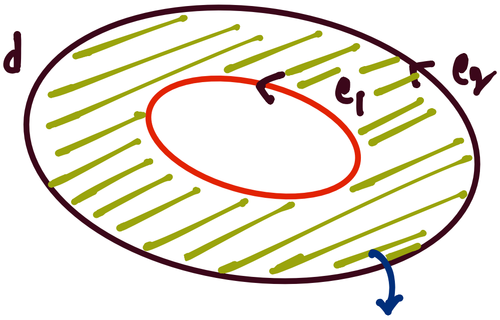


A Result for region bounded by two simple closed curves

$C_1, C_2 \rightarrow$ simple closed curves (with same direction)

$f(z) \rightarrow$ analytic in the region enclosed between C_1 & C_2 $\Rightarrow$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



$f(z)$ is analytic here

Parametrizing a circle

$|z| = r$ → Mod (z) is fixed
→ We don't know about the argument
→ Argument can be anything

$\therefore \quad z = re^{i\theta}\ ; \quad 0 \le \theta \le 2\pi$

- circle with centre at z_0, radius r

  $|z - z_0| = r$

  $\Rightarrow \quad z - z_0 = re^{i\theta}\ ; \quad 0 \le \theta \le 2\pi$

  $\Rightarrow \quad z = z_0 + re^{i\theta}\ ; \quad 0 \le \theta \le 2\pi$

→ Parametric equation of a circle

Convention:

(i) $\int_C f(z)\,dz \longrightarrow \oint_C f(z)\,dz$ if C is closed

(ii) Assume counter-clockwise direction if not mentioned explicitly.

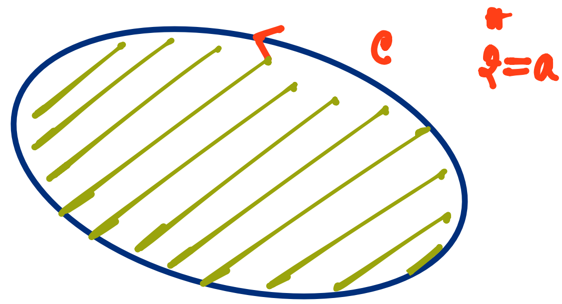
Problem

Evaluate $\oint_C \dfrac{dz}{(z-a)}$ where C is any simple closed curve and

(a) $z=a$ is outside C.

(b) $z=a$ is inside C.

Solution: (a) $f(z) = \dfrac{1}{z-a}$



$f(z)$ is analytic on $\mathbb{C} \setminus \{a\}$

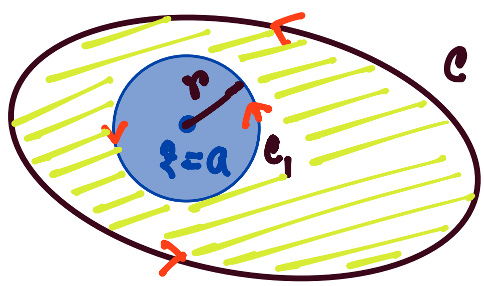
$\because$ $z=a$ lies outside of C

$\Rightarrow$ $f(z)$ is analytic in the region enclosed by C

$\therefore$ $\oint_C f(z)\,dz = 0$ (Using Cauchy's theorem)

(b) $z=a$ is inside C:

Step-01: Reformulation



Construct a circle of radius r, centred at $z=a$ which lies within C (& denote it by C_1)

$$\therefore \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

step-02: $\oint_{C_1} f(z)\,dz$ computation

C_1: $z = a + re^{i\theta}$; $0 \le \theta \le 2\pi$

$$\therefore \oint_{C_1} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{re^{i\theta}}\, re^{i\theta}.i\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i$$

$$\therefore \oint_{C_1} \frac{1}{z-a}\,dz = 2\pi i$$

$$\Rightarrow \oint_{C} \frac{1}{z-a}\,dz = 2\pi i \quad \text{(Ans:)}$$

H.W.

Problem

Evaluate $\oint_C \frac{dz}{(z-a)^n}$ where

(a) $z = a$ is outside C.

(b) $z = a$ is inside C.

for any simple closed curve C and $n \geq 2$

Problem

Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0),(1,0),(1,1),(0,1)$.

Solution:

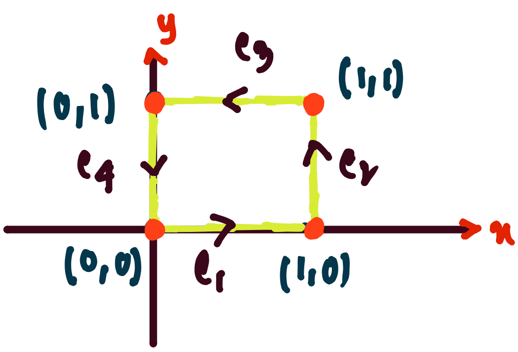
$\boxed{\text{step-01: Find parametric equations}}$

$C_1: z = t + i0\ ;\ 0 \le t \le 1$

$C_2: z = 1 + it\ ;\ 0 \le t \le 1$

$C_3: z = (1-t) + i\ ;\ 0 \le t \le 1$

$C_4: z = (1-t)i\ ;\ 0 \le t \le 1$



step-02: Apply the definition

$$\oint_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz$$

$$= \int_0^1 |t|^2 \cdot 1 \, dt + \int_0^1 |1+it|^2 \, i \, dt + \int_0^1 |(1-t)+i|^2 (-1) \, dt$$

$$+ \int_0^1 |(1-t)i|^2 (-i) \, dt$$

$$= \int_0^1 t^2 dt + \int_0^1 (1+t^2) \, i \, dt + \int_0^1 \{(1-t)^2 + 1\} (-1) \, dt$$

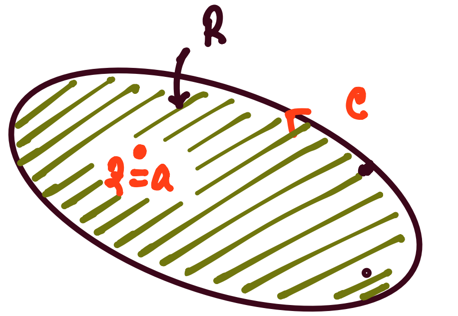
$$+ \int_0^1 (1-t)^2 (-i) \, dt$$

$$= \tfrac{1}{3} + i\tfrac{4}{3} - \tfrac{4}{3} - \tfrac{i}{3} = i - 1 \quad \text{(Ans:)}$$

Chapter 5: Cauchy's Integral Formula :

C : Simple closed curve (anti-clockwise)

$f(z) \to$ analytic in the region enclosed by C
(inside & on C)



$z = a \to$ a point inside C. Then

(i) $$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

(ii) $$\oint_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i\, f^{(n-1)}(a)}{(n-1)!} \quad \forall\, n \geq 2$$

Problem

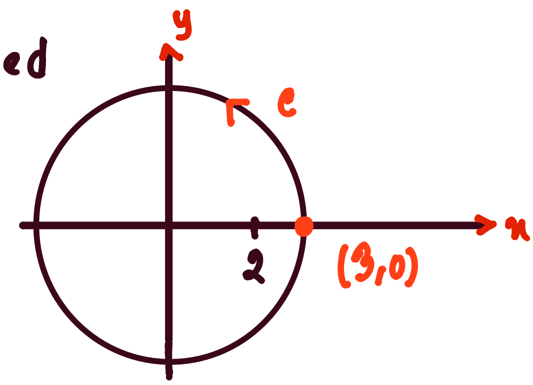
Evaluate $\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{e^z}{z-2}\,dz$ if C is:

(a) the circle $|z| = 3$, (b) the circle $|z| = 1$.

Solution: (a) $f(z) = e^z$ is analytic everywhere

$\therefore$ $f(z)$ is analytic in the region enclosed by C

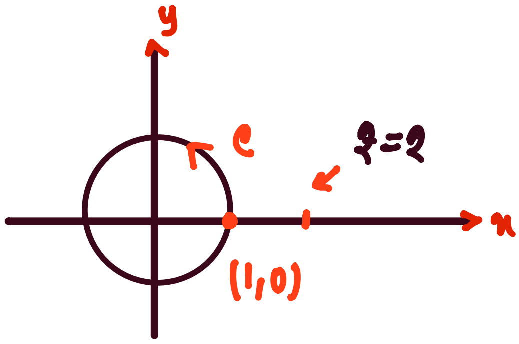
& $z = 2$ lies inside C



$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i e^2$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = e^2 \quad \text{(Ans:)}$$

(b) $\dfrac{e^z}{z-2}$ is analytic on $\mathbb{C}\setminus\{2\}$



$\Rightarrow \dfrac{e^z}{z-2}$ is analytic in the region enclosed by C

$$\Rightarrow \oint_C \frac{e^z}{z-2}\,dz = 0 \quad \text{(Using Cauchy's Theorem)}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0 \quad \text{(Ans:)}$$

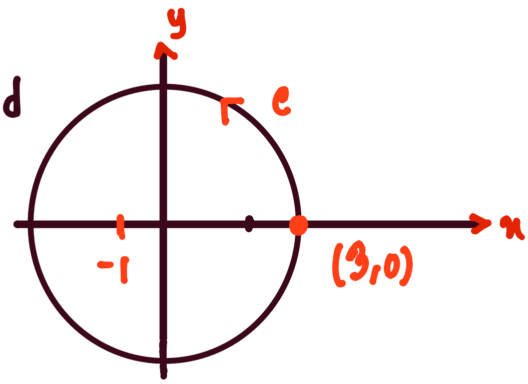
Problem

Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}dz$ where C is the circle $|z| = 3$.

Solution: $f(z) = e^{2z}$ is analytic everwhere

$\therefore$ $f(z)$ is analytic in the region enclosed by C

& $z = -1$ lies inside C



$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i\, f'''(-1)}{3!}$$

$$= \frac{2\pi i \cdot 8}{6e^2}$$

$$= \frac{8\pi i}{3e^2} \quad \text{(Ans:)}$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-1) = \dfrac{8}{e^2}$

Problem

Evaluate $\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz$ where C is the circle $|z|=3$.

Solution: step-01: Partial fraction decomposition.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

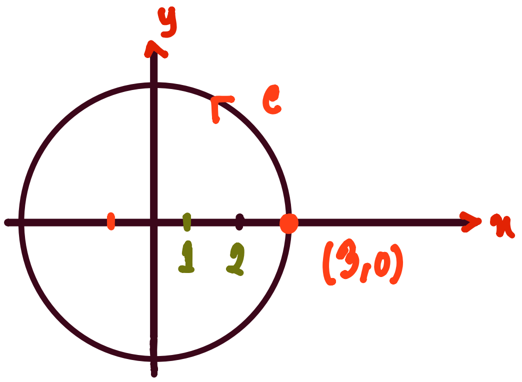
(i) $z=1 \Rightarrow 1 = A(-1) \Rightarrow A=-1$

(ii) $z=2 \Rightarrow 1 = B(1) \Rightarrow B=1$

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\Rightarrow \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)} = \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2} - \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}$$

$$\Rightarrow \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz = \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}dz$$



$$- \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}dz$$

$$= 2\pi i\{\sin(4\pi)+\cos(4\pi)\}$$

$$- 2\pi i\{\sin(\pi)+\cos(\pi)\}$$

$$= 2\pi i\{\cos(4\pi) - \cos(\pi)\}$$

$$= 2\pi i\{1-(-1)\} = 4\pi i \quad \text{(Ans:)}$$

Singular point (of $f(z)$): A point at which $f(z)$ fails to be analytic

Pole: $z = z_0 \rightarrow$ singular point of $f(z)$

$$\lim_{z \to z_0} (z - z_0)^n f(z) = \underset{\substack{\downarrow \\ \text{(Some defined value)}}}{A} \neq 0 \quad \text{for some } n \in \mathbb{Z}^+$$

$\Bigg\}$ $z = z_0$ is a pole of order n

(*) if $n = 1 \rightarrow z_0 \rightarrow$ simple pole

Example: (i) $f(z) = \dfrac{1}{(z-2)^3} \rightarrow z = 2$

$\therefore$ $z = 2$ is a pole of order 3

$$\because \lim_{z \to 2} (z-2)^3 f(z) = 1$$

(ii) $f(z) = \dfrac{3z - 2}{(z-1)^2 (z+1)(z-4)^5}$, $z = 1, -1, 4 \rightarrow$ Pole of order 5

$z = 1 \rightarrow$ Pole of order 2

$z = -1 \rightarrow$ Simple pole

(iii) $z = a$ is a pole of order n of $f(z)$

$$\therefore f(z) = \frac{g(z)}{(z-a)^n}$$

lemma: $f(z) \rightarrow$ complex function

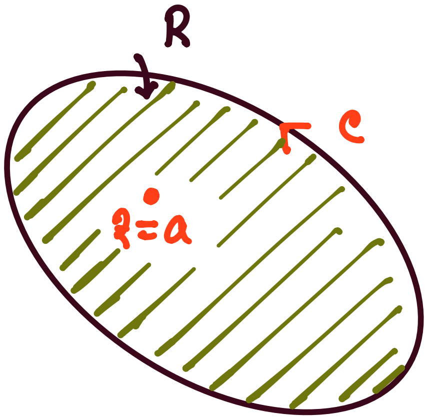
f is analytic in a region R

$\Rightarrow$ all its higher derivatives also exist in R

Question: (i) Why → (Hint: Cauchy's Integral Formula)

(ii) Is it true for real valued function?

Problem-04:



c: closed curve (anti-clockwise)

R: Region enclosed by c

$F(z) \to$ Analytic on $R \setminus \{a\}$

$z = a \to$ pole of order m inside c of $F(z)$

$$\frac{1}{2\pi i} \oint_c F(z)\, dz = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}$$

Solution: $z = a \to$ pole of order m of $F(z)$

$$F(z) = \frac{f(z)}{(z-a)^m} \text{ on } R, \quad f(z) \text{ is analytic on } R$$

$$\frac{1}{2\pi i} \oint_c F(z)\, dz = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^m}\, dz$$

$$= \frac{1}{2\pi i} \frac{2\pi i\, f^{(m-1)}(a)}{(m-1)!}$$

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (f(z)) \Big|_{z=a}$$

why?

$$= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (f(z))$$

$$= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left(F(z)(z-a)^m \right)$$

$$\therefore \boxed{\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left(F(z)(z-a)^m\right)}$$

$F(z)$ has a single pole inside C

C : Closed curve

R : Region enclosed by C

$z=a$: a pole of order m inside C of $F(z)$

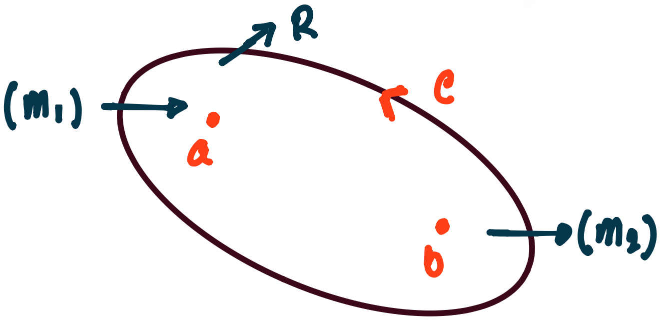
$F(z)$: analytic on $R\backslash\{a\}$

Then,

$$\frac{1}{2\pi i}\oint_C F(z)dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

Problem

Modify the result of previous problem in case of multiple poles.



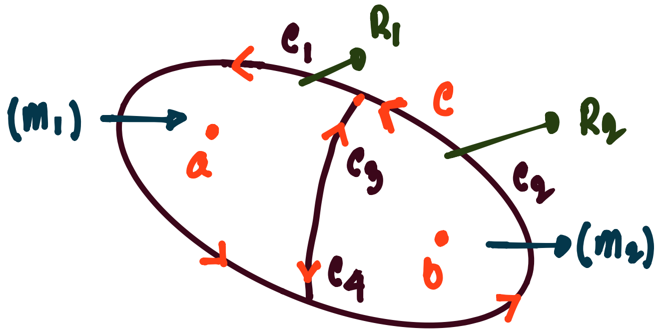
C: closed curve (anti-clockwise)

R: region enclosed by C

$F(z) \to$ Analytic on $R\backslash\{a,b\}$

$z=a,b \to$ pole of order m_1, m_2 inside of $F(z)$

We need: $\frac{1}{2\pi i}\oint_C F(z)dz$



$F(z)$ is analytic on $R_1 \setminus \{a\}$

$$\frac{1}{2\pi i} \oint_{C_1 \cup C_3} F(z)\,dz = \underbrace{\lim_{z \to a} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \left\{F(z)(z-a)^{m_1}\right\}}_{M_1} \quad \text{—(i)}$$

$F(z)$ is analytic on $R_2 \setminus \{b\}$

$$\frac{1}{2\pi i} \oint_{C_2 \cup C_4} F(z)\,dz = \underbrace{\lim_{z \to b} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \left\{F(z)(z-b)^{m_2}\right\}}_{M_2} \quad \text{—(ii)}$$

(i) + (ii) $\Rightarrow$

$$\frac{1}{2\pi i}\left(\oint_{C_1 \cup C_3} F(z)\,dz + \oint_{C_2 \cup C_4} F(z)\,dz\right) = M_1 + M_2$$

$$\Rightarrow \frac{1}{2\pi i}\left(\int_{C_1} F(z)\,dz + \int_{C_3} F(z)\,dz + \int_{C_2} F(z)\,dz + \int_{C_4} F(z)\,dz\right)$$

$$= M_1 + M_2$$

$$\Rightarrow \frac{1}{2\pi i}\left(\int_{C_1} F(z)\,dz + \int_{C_2} F(z)\,dz\right) = M_1 + M_2$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{C_1 \cup C_2} F(z)\,dz = M_1 + M_2$$

$$\Rightarrow \boxed{\begin{aligned} \frac{1}{2\pi i} \oint_C F(z)\,dz = {} & \lim_{z \to a} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \left\{F(z)(z-a)^{m_1}\right\} \\ & + \lim_{z \to b} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \left\{F(z)(z-b)^{m_2}\right\} \end{aligned}}$$

$F(z)$ has a two poles inside C

C : Closed curve

R : Region enclosed by C

$z = a, b$: poles order m_1, m_2 inside C of $F(z)$

$F(z)$: analytic on $R \setminus \{a, b\}$

Then,

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a}\frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\left\{(z-a)^{m_1}F(z)\right\} + \lim_{z\to b}\frac{1}{(m_2-1)!}\frac{d^{m_2-1}}{dz^{m_2-1}}\left\{(z-b)^{m_2}F(z)\right\}$$

$F(z)$ has n poles inside C

C : Closed curve

R : Region enclosed by C

$z = a_k$: poles of order m_k inside C of $F(z)$, where $k = 1, 2, \ldots, n$

$F(z)$: analytic on $R \setminus \{a_1, a_2, \ldots, a_n\}$

Then,

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \sum_{k=1}^{n}\lim_{z\to a_k}\frac{1}{(m_k-1)!}\frac{d^{m_k-1}}{dz^{m_k-1}}\left\{(z-a_k)^{m_k}F(z)\right\}.$$

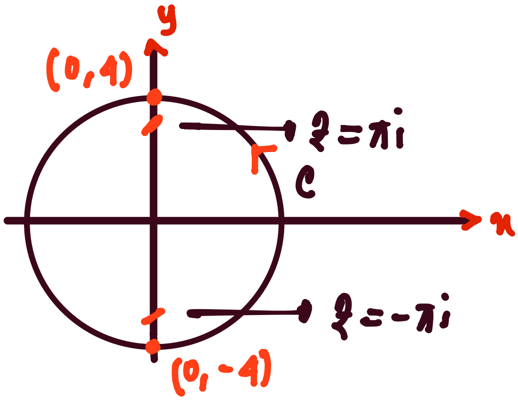
Problem

Evaluate $\displaystyle\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z| = 4$.

Solution: $$f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z^2 - i^2\pi^2)^2}$$

$$= \frac{e^z}{\{(z+\pi i)(z-\pi i)\}^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$$

$f(z)$ has poles at $z = \pm \pi i$ both of order 2 lying inside C



$$\frac{1}{2\pi i}\oint_C f(z)\,dz$$

$$= \lim_{z\to -\pi i} \frac{1}{1!}\frac{d}{dz}\left(\frac{e^z}{(z-\pi i)^2}\right) + \lim_{z\to \pi i} \frac{1}{1!}\frac{d}{dz}\left(\frac{e^z}{(z+\pi i)^2}\right)$$

$$= \lim_{z\to -\pi i} \frac{(z-\pi i)^2 e^z - e^z\, 2\,(z-\pi i)}{(z-\pi i)^4}$$

$$+ \lim_{z\to \pi i} \frac{(z+\pi i)^2 e^z - e^z\, 2\,(z+\pi i)}{(z+\pi i)^4}$$

$$= \lim_{z\to -\pi i} \frac{\{(z-\pi i)-2\}e^z}{(z-\pi i)^3} + \lim_{z\to \pi i} \frac{\{(z+\pi i)-2\}e^z}{(z+\pi i)^3}$$

$$= \frac{(-2\pi i-2)\,e^{-\pi i}}{(-2\pi i)^3} + \frac{(2\pi i-2)\,e^{\pi i}}{(2\pi i)^3} = \cdots\cdots = \frac{1}{2\pi^2}$$

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi^2}$$

$$\Rightarrow \oint_C f(z)\,dz = \frac{2\pi i}{2\pi^2} = \frac{i}{\pi} \quad \text{(Ans:)}$$

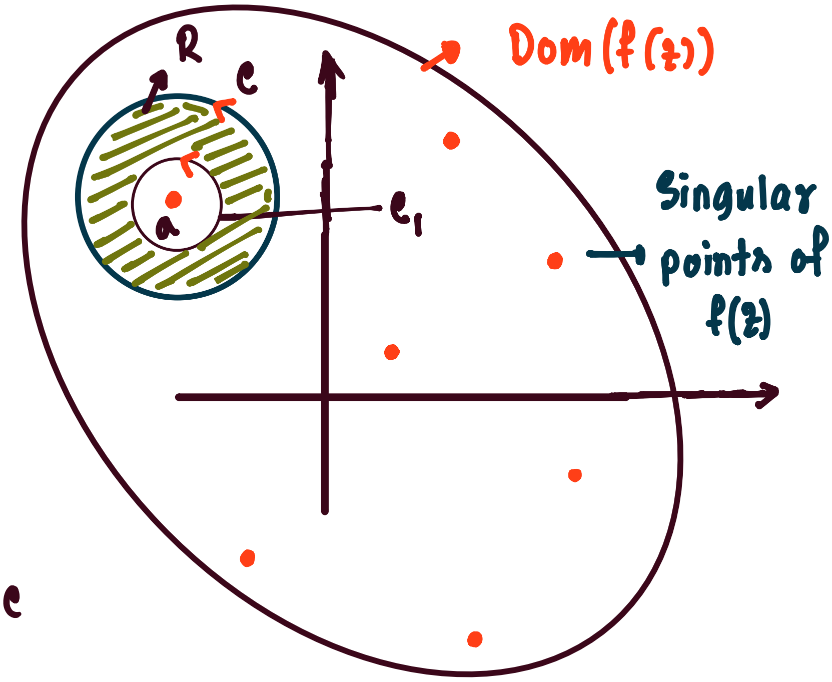
H.W.

Problem

Evaluate $\dfrac{1}{2\pi i}\displaystyle\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ when $t>0$ and C is the circle $|z|=3$.

Chapter 7:Cauchy's Residue Theorem

What is residue:



f(z) is analytic on R

$$\Rightarrow \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz$$

Also since f(z) is analytic on R $\Rightarrow$

$$\therefore f(z) = \cdots\cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots$$

on R (laurent series expansion about $z=a$)

$$\Rightarrow \oint_{C_1} f(z)\,dz = \cdots\cdots + a_{-2}\oint_{C_1}\frac{1}{(z-a)^2}dz + a_{-1}\oint_{C_1}\frac{1}{z-a}dz + \oint_{C_1}\{a_0 + a_1(z-a) + \cdots\cdots\}\,dz$$

$$= 0 + \cdots + 0 + a_{-1}(2\pi i) + 0$$

$$= a_{-1}(2\pi i)$$

$$\oint_{\Gamma} f(z)\,dz = a_{-1}(2\pi i)$$

$$\oint_{C} f(z)\,dz = a_{-1}(2\pi i) \Rightarrow a_{-1} = \frac{1}{2\pi i}\oint_{C} f(z)\,dz$$

This a_{-1} : Residue of $f(z)$ at $z=a \rightarrow \text{Res}(f,a)$

Calculation of residue:

Residue of $f(z)$ at $z=a$ (singular point)

$\downarrow$

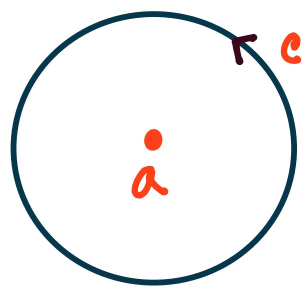
laurent series expansion about $z=a$

$\downarrow$

a_{-1} (co-efficient of $(z-a)^{-1}$)

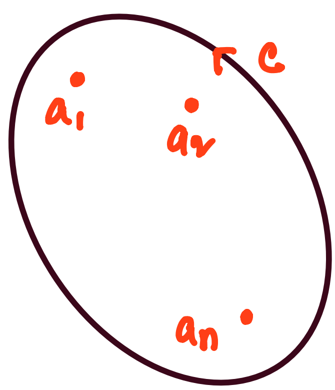
$$\boxed{\text{case : } z=a \text{ is a pole of order } m}$$

$$\text{Res}(f,a) = \frac{1}{2\pi i}\oint_{C} f(z)\,dz$$



$$\text{Res}(f,a) = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{f(z)(z-a)^m\right\}$$

$$\boxed{\text{Cauchy's Residue Theorem :}}$$



C: Simple closed curve (anti-clockwise)

R: region enclosed by C

$a_1, a_2, \ldots, a_n$: singular points inside C of $f(z)$

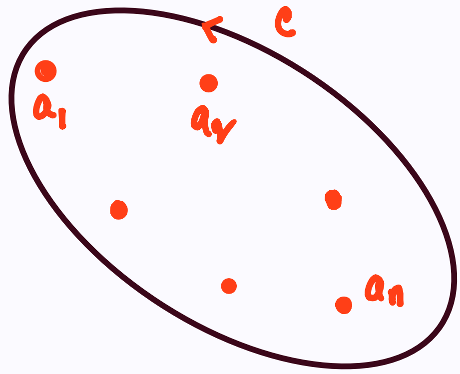
$f(z)$: analytic on $R \setminus \{a_1, a_2, \ldots, a_n\}$. Then,

$\oint_{C} f(z)\,dz = 2\pi i$ (Sum of the residues of singularities inside C)

$$\oint_{C} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f, a_i)$$

Cauchy's Residue Theorem

C : Closed curve (anti-clockwise)



R : Region enclosed by C

$a_1, a_2, \ldots, a_n$: singularities inside C of $F(z)$

$F(z)$: analytic on $R \backslash \{a_1, a_2, \ldots, a_n\}$

Then,

$$\oint_C F(z)\,dz = 2\pi i(\text{Sum of residues of singularities of } f(z) \text{ inside } C)$$
$$= 2\pi i \sum_{i=1}^{n} Res(f, a_i)$$

Residue of a pole of order m

If z=a is a pole of order m of a function f(z), then

$$Res(f,a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

Problem

Find residues of $f(z) = \dfrac{z}{(z-1)^2(z+1)}$.

Solution:

$$f(z) = \frac{z}{(z-1)^2(z+1)}$$

Singular point :

$z = 1 \longrightarrow$ pole of order 2

$z = -1 \longrightarrow$ simple pole

$$\operatorname{Res}(f,1) = \lim_{z\to 1} \frac{1}{(2-1)!} \frac{d}{dz}\left(\frac{z}{z+1}\right)$$

$$= \lim_{z\to 1} \frac{(z+1)\cdot 1 - z\cdot 1}{(z+1)^2}$$

$$= \lim_{z\to 1} \frac{1}{(z+1)^2} = \frac{1}{(1+1)^2} = \frac{1}{4} \quad \text{(Ans:)}$$

$$\operatorname{Res}(f,-1) = \lim_{z\to -1} \frac{1}{(1-1)!} \{f(z)\,(z+1)\}$$

$$= \lim_{z\to -1} \frac{1}{0!} \frac{z}{(z-1)^2}; \qquad 0! = 1$$

$$= \lim_{z\to -1} \frac{z}{(z-1)^2} = \frac{-1}{(-1-1)^2} = -\frac{1}{4} \quad \text{(Ans:)}$$

Problem

Using the previous result, find:

(i) $\oint_C \frac{z}{(z-1)^2(z+1)} dz$ where C: $|z| = 2$. $f(z)$

(ii) $\oint_C \frac{z}{(z-1)^2(z+1)} dz$ where C: $|z+1| = \frac{3}{2}$.

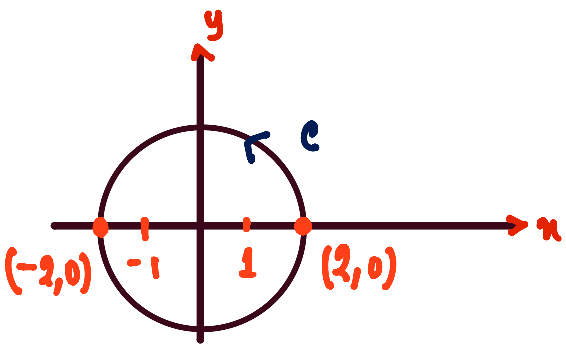
(iii) $\oint_C \frac{z}{(z-1)^2(z+1)} dz$ where C: $|z-1| = \frac{3}{2}$.

Solution:

(i) $\oint_C f(z)\,dz$

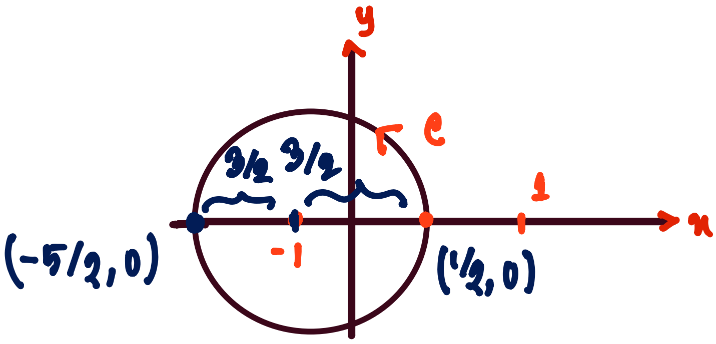
$= 2\pi i \{\operatorname{Res}(f,1) + \operatorname{Res}(f,-1)\}$

$= 2\pi i \left(1/4 - 1/4\right) = 0$ (Ans:)

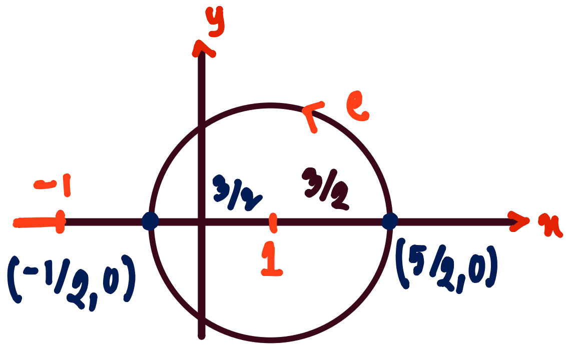


(ii)

$$\oint_C f(z)\,dz$$
$$= 2\pi i \cdot \text{Res}(f,-1)$$
$$= 2\pi i\left(-1/4\right) = -\pi i/2 \quad \text{(Ans:)}$$



(iii) $$\oint_C f(z)\,dz$$
$$= 2\pi i \cdot \text{Res}(f,1)$$
$$= 2\pi i\cdot(1/4) = \pi i/2 \quad \text{(Ans:)}$$



H.W.

Problem

$f(z) = \dfrac{1}{z^2+1}$. Find

(i) $\oint_C f(z)dz$ where C: $|z-i| = 1$.

(i) $\oint_C f(z)dz$ where C: $|z+i| = 1$.

(i) $\oint_C f(z)dz$ where C: any path enclosing $z = i$ and $z = -i$.

Problem

Evaluate $\oint_C \dfrac{e^{zt}}{z^2(z^2+2z+2)}dz$ around the circle C with equation $|z| = 3$.

Solution: $$f(z) = \frac{e^{zt}}{z^2(z^2+2z+2)}$$

singular point: $z=0 \rightarrow$ pole of order 2

$$z^2+2z+2=0 \Rightarrow z = \frac{-2\pm\sqrt{4-4\cdot 2}}{2} = \frac{-2\pm 2i}{2} = -1\pm i$$

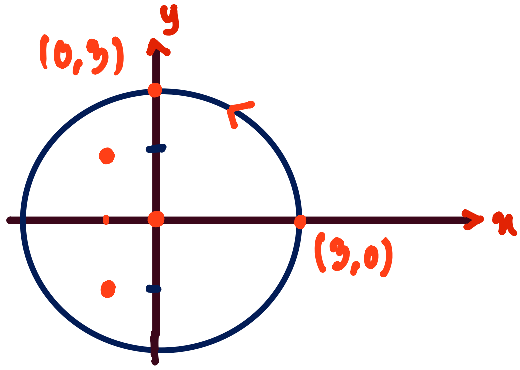
$$f(z) = \frac{e^{zt}}{z^2 \{z-(-1+i)\}\{z-(-1-i)\}}$$

$z = -1+i \longrightarrow$ simple pole

$z = -1-i \longrightarrow$ simple pole

$$\oint_C f(z)\,dz$$

$$= 2\pi i \{\text{Res}(f,0) + \text{Res}(f,-1+i) + \text{Res}(f,-1-i)\}$$



$$\text{Res}(f,0) = \lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left(\frac{e^{zt}}{z^2+2z+2}\right)$$

$$= \lim_{z\to 0} \frac{(z^2+2z+2)\,e^{zt}.t - e^{zt}\,(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{2t-2}{2^2} = \frac{t-1}{2}$$

$$\text{Res}(f,-1-i) = \lim_{z\to -1-i} \frac{1}{(1-1)!}\frac{e^{zt}}{z^2\{z-(-1+i)\}}$$

$$= \lim_{z\to -1-i} \frac{e^{zt}}{z^2\{z-(-1+i)\}}$$

$$= \frac{e^{(-1-i)t}}{(-1-i)^2\{-1-i-(-1+i)\}}$$

$$= \frac{e^{(-1-i)t}}{(1+2i-1)(-2i)} = \frac{e^{(-1-i)t}}{4}$$

$$\operatorname{Res}(f,-1+i) = \lim_{z\to -1+i} \frac{1}{(1-1)!} \frac{e^{zt}}{z^2\{z-(-1-i)\}}$$

$$= \lim_{z\to -1+i} \frac{e^{zt}}{z^2\{z-(-1-i)\}}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2\{(-1+i)-(-1-i)\}}$$

$$= \frac{e^{(-1+i)t}}{4}$$

$$\oint_C f(z)\,dz = 2\pi i\,\{\operatorname{Res}(f,0) + \operatorname{Res}(f,-1+i) + \operatorname{Res}(f,-1-i)\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{e^{-t}}{4}(e^{it}+e^{-it})\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{e^{-t}}{2}\left(\frac{e^{it}+e^{-it}}{2}\right)\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{e^{-t}}{2}\cos t\right\} \quad \text{(Ans:)}$$