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Sec: 06

Ans. to the Q. No: 1

a) $(x, y, z) + (x', y', z') = (x+x', y+y', z+z')$

$k(x, y, z) = (0, 0, 0)$

M_5: Let, $\vec{u} = (1, 2, 3)$

$k = 1$

$\therefore 1(1, 2, 3) = (0, 0, 0) \neq \vec{u}$

so unit of scalar multiplication doesn't hold here.

b) $(x,y)+(x',y') = (x+x', y+y')$; $x \geq 0$

$k(x,y) = (kx+ky)$

Let, $\vec{u} = (x_1,y_1)$, $\vec{v} = (x_2,y_2)$, $\vec{w} = (x_3,y_3)$

A1: $\vec{u}+\vec{v} = (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$

$\therefore \vec{u}+\vec{v} \in V$

A2: $\vec{u}+\vec{v} = (x_1+x_2, y_1+y_2)$

$\therefore \vec{v}+\vec{u} = (x_2,y_2)+(x_1,y_1) = (x_2+x_1, y_2+y_1)$

$\therefore \vec{u}+\vec{v} = \vec{v}+\vec{u}$

A3: $\vec{u}+(\vec{v}+\vec{w}) = (x_1,y_1) + \{(x_2,y_2)+(x_3,y_3)\}$

$= (x_1,y_1) + (x_2+x_3, y_2+y_3)$

$= (x_1+x_2+x_3, y_1+y_2+y_3)$

$(\vec{u}+\vec{v})+\vec{w} = \{(x_1,y_1)+(x_2,y_2)\}+(x_3,y_3)$

$= (x_1+x_2, y_1+y_2)+(x_3,y_3)$

$= (x_1+x_2+x_3,\ y_1+y_2+y_3)$

$\therefore \vec{u} + (\vec{v}+\vec{w}) = (\vec{u}+\vec{v}) + \vec{w}$

A4: $\vec{u} + \vec{0} = \vec{u}$

$\Rightarrow (x_1, y_1) + (0,0) = (x_1, y_1)$

here, $\vec{0} \in \mathbb{R}^2$ & $x \geq 0$

A5: $\vec{u} + \vec{u'} = \vec{0}$

$\Rightarrow (x_1, y_1) + (-x_1, y_1) = (0,0)$

here in $\vec{u'}$,

$x < 0$

$\therefore$ Additive inverse doesn't hold here.

c) $(x,y) + (x',y') = (x+x'+1, y+y'+1)$

$k(x,y) = (kx, ky)$

Let, $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$, $\vec{w} = (x_3, y_3)$

A_1: $\vec{u}+\vec{v} = (x_1+x_2+1, y_1+y_2+1) \in V$

A_2: $\vec{v}+\vec{u} = (x_2+x_1+1, y_2+y_1+1) = \vec{u}+\vec{v}$

A_3: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$

$\Rightarrow (x_1,y_1) + ((x_2,y_2)+(x_3,y_3)) = ((x_1,y_1)+(x_2,y_2)) + (x_3,y_3)$

$\Rightarrow (x_1,y_1) + (x_2+x_3+1, y_2+y_3+1) = (x_1+x_2+1, y_1+y_2+1) + (x_3,y_3)$

$\Rightarrow (x_1+x_2+x_3+2, y_1+y_2+y_3+2) = (x_1+x_2+x_3+2, y_1+y_2+2+y_3)$

A_4: $\vec{u}+\vec{0} = (x_1,y_1) + (-1,-1)$

$= (x_1+1-1, y_1+1-1)$

$= (x_1,y_1) = \vec{u}$

A5:

$$\vec{u}+\vec{u'} = (x_1, y_1)+(-x_1-2, -y_1-2)$$

$$= (x_1-x_1-2+1, \; y_1-y_1-2+1)$$

$$= (-1,-1)$$

$$= \vec{0}$$

Let, $\alpha \in \mathbb{R}$

M_1: $\alpha\vec{u} = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1) \in \mathbb{R}^2$

M_2: $\alpha(\vec{u}+\vec{v}) = \alpha(x_1+x_2+1, y_1+y_2+1)$

$$= (\alpha x_1+\alpha x_2+\alpha, \alpha y_1+\alpha y_2+\alpha)$$

$\alpha\vec{u}+\alpha\vec{v} = (\alpha x_1, \alpha y_1)+(\alpha x_2+\alpha y_2)$

$$= (\alpha x_1+\alpha x_2+1, \; \alpha y_1+\alpha y_2+1)$$

$\therefore$ Distributive law of vectors doesn't hold here.

d)

$$(1, y) + (1, y') = (1, y + y')$$

$$k(1, y) = (1, ky)$$

Let, $\vec{u} = (1, x_1)$, $\vec{v} = (1, x_2)$, $\vec{w} = (1, x_3)$

A1: $\vec{u} + \vec{v} = (1, x_1) + (1, x_2) = (1, x_1 + x_2) \in V$

A2: $\vec{v} + \vec{u} = (1, x_2) + (1, x_1) = (1, x_2 + x_1)$

$$= \vec{u} + \vec{v}$$

A3: $\vec{u} + (\vec{v} + \vec{w}) = (1, x_1) + ((1, x_2) + (1, x_3))$

$$= (1, x_1) + (1, x_2 + x_3)$$

$$= (1, x_1 + x_2 + x_3)$$

$$(\vec{u} + \vec{v}) + \vec{w} = (1, x_1 + x_2) + (1, x_3)$$

$$= (1, x_1 + x_2 + x_3)$$

A4: $\vec{u} + \vec{0} = (1, x_1) + (1, 0)$

$$= (1, x_1)$$

$$= \vec{u}$$

A5: $\vec{u}+\vec{u} = (1, x_1) + (1, -x_1)$

$= (1, 0)$

$= \vec{0}$

Let, $\alpha \in \mathbb{R}$

M1: $\alpha \vec{u} = \alpha(1, x_1) = (1, \alpha x_1) \in V$

M2: $\alpha(\vec{u}+\vec{v}) = \alpha(1, x_1+x_2)$

$= (1, \alpha x_1 + \alpha x_2)$

$\cancel{\alpha x_1 + \alpha}$

$\alpha\vec{u} + \alpha\vec{v} = \alpha(1, x_1) + \alpha(1, x_2)$

$= (1, \alpha x_1) + (1, \alpha x_2)$

$= (1, \alpha x_1 + \alpha x_2)$

M3: $\beta \in \mathbb{R}$

$(\alpha+\beta)\vec{u} = (\alpha+\beta)(1, x_1)$

$= (1, \alpha x_1 + \beta x_1)$

$$\alpha\vec{u} + \alpha\beta\vec{u} = \alpha(1, x_1) + \alpha\beta(1, x_1)$$

$$= (1, \alpha x_1) + (1, \beta x_1)$$

$$= (1, \alpha x_1 + \beta x_1)$$

M4:

$$(\alpha\beta)\vec{u} = \alpha\beta(1, x_1) = (1, \alpha\beta x_1)$$

$$\alpha(\beta\vec{u}) = \alpha(\beta(1, x_1))$$

$$= \alpha(1, \beta x_1) = (1, \alpha\beta x_1)$$

M5:

$$1\vec{u} = 1(1, x_1)$$

$$= (1, x_1)$$

$$= \vec{u}$$

$\therefore$ The set is vector space.

e)

$$x+y=xy$$

$$kx=x^k$$

M4: Let, $\vec{u}=x_1$

$\alpha, \beta \in \mathbb{R}$

$\therefore (\alpha\beta)\vec{u} = x_1^{\alpha\beta}$

$\therefore \alpha(\beta\vec{u}) = \alpha(x_1^{\beta})$

$= x_1^{\beta^{\alpha}}$

$\therefore$ Associativity of scalar multiplication doesn't hold.

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Ans. to the Q. No.-2

$W = (x,0,0)$, clearly $W \subset \mathbb{R}^3$

a) Let, $\vec{u} = (x_1, 0, 0)$

A4: $\vec{u} + \vec{0} = (x_1, 0, 0) + (0,0,0)$

$= (x,0,0) = \vec{u}$

$\vec{0} \in W$

A1: Let, $\vec{v} = (x_2, 0, 0)$

$\therefore \vec{u} + \vec{v} = (x_1, 0, 0) + (x_2, 0, 0)$

$= (x_1 + x_2, 0, 0) \in W$

M1: $\alpha \in \mathbb{R}$

$\alpha \vec{u} = \alpha (x_1, 0, 0) = (\alpha x_1, 0, 0) \in W$

$\therefore$ all vectors ~~space~~ of the form $(a,0,0)$ are subspace of V.

b) $W = \{(a, b, c) : c = a - b\}$

clearly $W \subset \mathbb{R}^3$

Let, $\vec{u} = (x_1, y_1, z_1)$, $\vec{v} = (x_2, y_2, z_2)$, $\alpha \in \mathbb{R}$

$\therefore z_1 = x_1 - y_1$ $\qquad \therefore z_2 = x_2 - y_2$

A4: $\vec{u} + \vec{0} = (x_1, y_1, z_1) + (0, 0, 0)$

$= (x_1, y_1, z_1) = \vec{u}$

for $\vec{0}$,

$0 = 0 - 0$

$\therefore \vec{0} \in W$

A1: $\vec{u} + \vec{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$z_1 + z_2 = x_1 + x_2 - y_1 - y_2$

$= x_1 - y_1 + x_2 - y_2$

$= z_1 - z_2$

M_1: $\alpha\vec{u} = \alpha(x_1, y_1, z_1)$

$= (\alpha x_1, \alpha y_1, \alpha z_1)$

$\therefore \alpha z_1 = \alpha x_1 - \alpha y_1$

$\Rightarrow z_1 = x_1 - y_1$

⊗.

~~$\therefore$ w is a st~~

all w vectors are subspace of v.

c) $W = \{(a,b,c) : c = a+b+3\}$

Clearly $W \subset \mathbb{R}^3$

Let, $\vec{u} = (x_1, y_1, z_1)$

A4: $\vec{u} + \vec{0} = (x_1, y_1, z_1) + (0,0,0)$

$= (x_1, y_1, z_1) = \vec{u}$

for $\vec{0}$,

L.H.S. $= 0$, R.H.S. $= 0+0+3 = 3$

$\therefore$ L.H.S. $\neq$ R.H.S.

$\therefore$ all w vectors are not subspace of V.

d) $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+c+d=0 \right\}$

clearly, $W \subset M_{2\times2}$

$\Rightarrow$

Let, $\vec{u} = \begin{pmatrix} xa_1 & xb_1 \\ xc_1 & xd_1 \end{pmatrix}$; $\vec{v} = \begin{pmatrix} xa_2 & xb_2 \\ xc_2 & xd_2 \end{pmatrix}$, $x \in \mathbb{R}$

$\therefore xa_1 + xb_1 + xc_1 + xd_1 = 0$ $\qquad$ $\therefore xa_2 + xb_2 + xc + xd_2 = 0$

A4: $\vec{u} + \vec{0} = \begin{pmatrix} xa_1 & xb_1 \\ xc_1 & xd_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} xa_1 & xb_1 \\ xc_1 & xd_1 \end{pmatrix}$

$= \vec{u}$

for $\vec{0}$,

L.H.S. $= 0+0+0+0 = 0 =$ R.H.S.

$\therefore \vec{0} \in W$

A1: $\vec{u} + \vec{v} = \begin{pmatrix} xa_1 & xb_1 \\ xc_1 & xd_1 \end{pmatrix} + \begin{pmatrix} xa_2 & xb_2 \\ xc_2 & xd_2 \end{pmatrix}$

$= \begin{pmatrix} xa_1 + xa_2 & xb_1 + xb_2 \\ xc_1 + xc_2 & xd_1 + xd_2 \end{pmatrix}$

L.H.S. $= xa_1 + xa_2 + xb_1 + xb_2 + xc_1 + xc_2 + xd_1 + xd_2$
$= 0$

~~$\therefore$ L.H.S. $\neq$ R.H.S. $\therefore$ all w vectors are not subspace of V.~~

M_1:

$$\alpha\vec{u} = \alpha\begin{pmatrix} xa_1 & xb_1 \\ xc_1 & xd_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha xa_1 & \alpha xb_1 \\ \alpha xc_1 & \alpha xd_1 \end{pmatrix}$$

$\therefore$ L.H.S. $= \alpha(xa_1 + xb_1 + xc_1 + xd_1)$

$= \alpha \cdot 0$

$= 0$

$=$ R.H.S.

$\therefore$ all ω vectors are subspace of V.

e) $W = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$

clearly $W \subset M_{2\times2}$

let, $\vec{u} = \begin{pmatrix} a_1 & a_2 \\ -a_3 & -a_4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} a_1' & a_2' \\ -a_3' & -a_4' \end{pmatrix}$, $\alpha \in \mathbb{R}$

A4: $\vec{u} + \vec{o} = \begin{pmatrix} a_1 & a_2 \\ -a_3 & -a_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} a_1 & a_2 \\ -a_3 & -a_4 \end{pmatrix}$

$\vec{o} \in W$

A1: $\vec{u} + \vec{v} = \begin{pmatrix} a_1 & a_2 \\ -a_3 & -a_4 \end{pmatrix} + \begin{pmatrix} a_1' & a_2' \\ -a_3' & -a_4' \end{pmatrix}$

$= \begin{pmatrix} a_1 + a_1' & a_2 + a_2' \\ -(a_3 + a_3') & -(a_4 + a_4') \end{pmatrix} \in W$

M1: $\alpha \vec{u} = \alpha \begin{pmatrix} a_1 & a_2 \\ -a_3 & -a_4 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ -\alpha a_3 & -\alpha a_4 \end{pmatrix} \in W$

$\therefore$ all w vectors are subspace of V.

Ans. to the Q. No.- 3

a) $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{pmatrix}$ $\quad R_2' = R_2 - 2R_1$

$\quad R_3' = R_3 - 2R_1$

$= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad R_3' = 3R_3 - 2R_2$

$\therefore$ Basis of col space $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \right\}$

$\therefore$ b is in the col space of A.

b) Let,

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad \vec{v_3} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ 2\alpha_1 \\ 2\alpha_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3\alpha_2 \\ 2\alpha_2 \end{pmatrix} + \begin{pmatrix} 3\alpha_3 \\ 2\alpha_3 \\ 3\alpha_3 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \quad + 3\alpha_3 \\ 2\alpha_1 + 3\alpha_2 + 2\alpha_3 \\ 2\alpha_1 + 2\alpha_2 + 3\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 2 & 3 & 2 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \quad R_3' = 3R_3 - 2R_2$$

∴ column vectors of A are linearly independent.

Let, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$ Null space

~~from~~ $\therefore A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$

from above,

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\therefore \begin{aligned} x \qquad + 3z &= 0 \\ 3y - 4z &= 0 \\ -z &= 0 \end{aligned}$$

$\therefore z = 0$

$\therefore y = 0$

$\therefore x = 0$

$$\text{Null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Ans. to the Q. No.-4

a) $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $v_4 = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}$

$\therefore \alpha_1 v_1 + \alpha_3 v_3 + \alpha_4 v_4 = \vec{0}$

$$\Rightarrow \begin{pmatrix} \alpha_1 + 4\alpha_3 + 6\alpha_4 \\ 2\alpha_1 + 5\alpha_3 + 7\alpha_4 \\ 3\alpha_1 + 6\alpha_3 + 8\alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 2 & 5 & 7 & 0 \\ 3 & 6 & 8 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & -6 & -10 & 0 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3' = R_3 - 2R_2$$

$\therefore v_1, v_3, v_4$ are not linearly independent.

~~remains~~

b) to find if $b_1, b_2 \in \text{Span}\{v_1, v_2, v_3, v_4\}$,

~~we have to find~~

Let,

$$\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \alpha_3\vec{v_3} + \alpha_4\vec{v_4} = \vec{b_1}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 \\ 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 7\alpha_4 \\ 3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 8\alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\therefore \left(\begin{array}{cccc|c} 1 & 3 & 4 & 6 & 0 \\ 2 & 4 & 5 & 7 & 1 \\ 3 & 5 & 6 & 8 & -2 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 4 & 6 & 0 \\ 0 & -2 & -3 & -5 & 1 \\ 0 & -4 & -6 & -10 & -2 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 4 & 6 & 0 \\ 0 & -2 & -3 & -5 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right) \quad R_4' = R_4 - 2R_2$$

$\therefore$ No solution

$\therefore b_1 \notin \text{Span}\{v_1, v_2, v_3, v_4\}$

Let, $\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \alpha_3\vec{v_3} + \alpha_4\vec{v_4} = \vec{b_2}$

$$\Rightarrow \begin{pmatrix} \alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 \\ 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 7\alpha_4 \\ 3\alpha_1 + 5\alpha_2 + 6\alpha_3 + 8\alpha_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{cccc|c} 1 & 3 & 4 & 6 & 11 \\ 2 & 4 & 5 & 7 & 7 \\ 3 & 5 & 6 & 8 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 4 & 6 & 11 \\ 0 & -2 & -3 & -5 & -15 \\ 0 & -4 & -6 & -10 & -33 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$= \left(\begin{array}{cccc|c} 1 & 3 & 4 & 6 & 11 \\ 0 & -2 & -3 & -5 & -15 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right) \quad R_4' = R_4 - 2R_2$$

$\therefore$ No solution

$\therefore b_2 \notin \text{Span}\{v_1, v_2, v_3, v_4\}$

Ans. to the Q. No.- 5

a) $A = \begin{bmatrix} 1 & 4 & -2 & 5 \\ 2 & 9 & -1 & 8 \\ 2 & 9 & -1 & 9 \\ -1 & -4 & 2 & -5 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4 & -2 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad R_2' = R_2 - 2R_1$, $R_3' = R_3 - 2R_1$, $R_4' = R_4 + R_1$

$= \begin{bmatrix} 1 & 4 & -2 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad R_3' = R_3 - R_2$

$\text{Row}(A) = \{ (1,4,2,5), (2,9,-1,8), (2,9,-1,9) \}$

$\text{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 9 \\ -4 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 9 \\ -5 \end{pmatrix} \right\}$

Let, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \text{Null}(A)$

$$\therefore \quad A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left[\begin{array}{cccc|c} 1 & 4 & -2 & 5 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

here, x_3 is free variable (let, $x_3 = t$)

$$\therefore \quad x_1 + 4x_2 - 2x_3 + 5x_4 = 0$$
$$x_2 + 3x_3 - 2x_4 = 0$$
$$x_4 = 0$$

$\therefore x_4 = 0, \; x_3 = t$

$\therefore x_2 = -3t$

$\therefore x_1 = 2t - 4(-3t) = 14t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t\begin{pmatrix} 14 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathrm{Null}(A) = \left\{ \begin{pmatrix} 14 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 4 & 9 & 9 & -4 \\ -2 & -1 & -1 & 2 \\ 5 & 8 & 9 & -5 \end{bmatrix}$$

Let, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathrm{Null}(A^T)$

$$\therefore \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 4 & 9 & 9 & -4 & 0 \\ -2 & -1 & -1 & 2 & 0 \\ 5 & 8 & 9 & -5 & 0 \end{array}\right]$$

$R_2' = R_2 - 4R_1$

$R_3' = R_3 + 2R_1$

$R_4' = R_4 - 5R_1$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_3' = R_3 - 3R_2 \\ R_4' = R_4 + 2R_2 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad R_1' = R_1 - 2R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad R_2' = R_2 - R_4$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$$x_1 - x_4 = 0$$
$$x_2 = 0$$
$$x_3 = 0$$

Let, x_4 is free variable $x_4 = t$

$\therefore x_1 = t, \quad x_2 = x_3 = 0, \quad x_4 = t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \text{Null}(A^T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

~~x, y, z~~

$$(x_1, x_2, x_3, x_4)\begin{bmatrix} 1 & 4 & -2 & 5 \\ 2 & 9 & -1 & 8 \\ 2 & 9 & -1 & 9 \\ -1 & -4 & 2 & -5 \end{bmatrix} = (0,0,0,0)$$

We know,

Null(AT) = left Null space of A

$\therefore$ lef null space of A = {(1, 0, 0, 1)}

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix}$ $R_2' = R_2 - 2R_1$, $R_3' = R_3 + R_1$

$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3' = R_3 + R_2$

$\text{Row}(A) = \{ (1,4,5,2), (2,1,3,0) \}$

$\text{col}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \right\}$

Let, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \text{Null}(A)$

$$\therefore A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

following the same steps

$$\therefore x_1 + 4x_2 + 5x_3 + 2x_4 = 0$$
$$-7x_2 - 7x_3 - 4x_4 = 0$$

here, x_3 and x_4 are free variable.

$$x_3 = t_1 \; , \; x_4 = t_2$$

$$\therefore x_2 = \frac{7t_1 + 4t_2}{-7} = -t_1 - \frac{4}{7}t_2$$

$$\therefore x_1 = -2t_2 - 5t_1 + 4t_1 + \frac{16}{7}t_2 = -t_1 + \frac{2}{7}t_2$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -t_1 + \frac{2}{7}t_2 \\ -t_1 - \frac{4}{7}t_2 \\ t_1 \\ t_2 \end{pmatrix}$$

$$= t_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \text{Null}(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \right.$$

1

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

Let, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \text{Null}(A^T)$

$$\therefore A^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \cancel{(0,0,0,0)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 4 & 1 & 3 & 0 \\ 5 & 3 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 4R_1 \\ R_3' = R_3 - 5R_1 \\ R_4' = R_4 - 2R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right) \quad \begin{array}{l} R_3' = R_3 - R_2 \\ \cancel{R_4' = R_4 - 2R_1} \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \qquad R_2' = \frac{R_2}{7}, \quad R_4' = \frac{R_4}{4}$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad R_1' = R_1 + 2R_2, \quad R_4' = R_4 - R_1$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad R_2' = -1 \cdot R_2$$

$\therefore$ $x_1 + x_3 = 0$

$x_2 - x_3 = 0$

Let, $x_3 = t$

$\therefore x_1 = -t, \; x_2 = t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore \text{Null}(A^T) = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Null (A^T) = left Null space of $A = \{(-1,1,1)\}$

c) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$ $\quad R_2' = R_2 - 5R_1$, $\quad R_3' = R_3 - 7R_1$

$= \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad R_3' = R_3 - R_2$

Row $(A) = \{(1,-1,3), (5,-4,-4)\}$

col $(A) = \left\{ \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix} \right\}$

Let, $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in$ Null (A)

$\therefore A \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

following same steps

$$\therefore x_1 - x_2 + 3x_3 = 0$$
$$x_2 - 19x_3 = 0$$

Let, $x_3 = t_1$

$$\therefore x_2 = 19t_1 \;,\; x_1 = 19t_1 - 3t_1 = 16t_1$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}$$

$$\therefore \text{Null}(A) = \left\{ \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -4 & -6 \\ 3 & -4 & 2 \end{bmatrix}$$

let, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in Null(A^T)$

$$\therefore \left(\begin{array}{ccc|c} 1 & 5 & 7 & 0 \\ -1 & -4 & -6 & 0 \\ 3 & -4 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 5 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -19 & -19 & 0 \end{array}\right) \qquad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 5 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad R_3' = R_3 + 19R_2$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad R_1' = R_1 - 5R_2$$

$\therefore x_1 + 2x_3 = 0$

$x_2 + x_3 = 0$

Let, $x_3 = t$ $\therefore x_2 = -t$ $\therefore x_1 = -2t$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \text{Null}(A^T) = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Null (A^T) = left null space of A

$\therefore$ leftnull space of $A = \{(-2, -1, 1)\}$