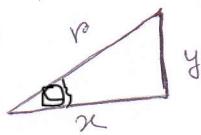


## Ch 7.4 Integration by Trigonometric substitution



$$x^2 + y^2 = r^2$$

$$\tan \theta = \frac{y}{x} \Rightarrow \cot \theta = \frac{x}{y}$$

$$\sin \theta = \frac{y}{r} \Rightarrow \cosec \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \Rightarrow \sec \theta = \frac{r}{x}$$

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \Rightarrow 1 - \cos^2 \theta = \sin^2 \theta \\ 1 + \tan^2 \theta &= \sec^2 \theta \quad \left\{ 1 - \sin^2 \theta = \cos^2 \theta \right. \end{aligned}$$

$$[3] \int \frac{x^2}{\sqrt{16-x^2}} dx$$

$$= \int \frac{x^2}{4\sqrt{1-\frac{1}{16}x^2}} dx$$

$$= \frac{1}{4} \int \frac{x^2}{\sqrt{1-(\frac{x}{4})^2}} dx$$

$$= \int \frac{16 \sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta$$

$$= \int \frac{16 \sin^2 \theta \cos \theta}{\cos \theta} d\theta$$

$$= 16 \int \sin^2 \theta d\theta$$

$$= 16 \int \frac{1}{2}(1-\cos 2\theta) d\theta$$

$$= 8 \left[ \theta + \frac{\sin 2\theta}{2} \right] + C$$

$$= 8 \left[ \sin^{-1} \frac{x}{4} + \frac{2 \sin \theta \cos \theta}{2} \right] + C$$

$$= 8 \left[ \sin^{-1} \frac{x}{4} + \sin \theta \cos \theta \right] + C$$

$$= 8 \left[ \sin^{-1} \frac{x}{4} + \frac{x}{4} \cdot \frac{\sqrt{16-x^2}}{4} \right] + C$$

Let

$$\frac{x}{4} = \sin \theta \quad \therefore \theta = \sin^{-1} \left( \frac{x}{4} \right)$$

$$\frac{x^2}{16} = \sin^2 \theta \quad \therefore x^2 = 16 \sin^2 \theta$$

$$\frac{1}{4} dx = \cos \theta d\theta$$

$$\begin{aligned} \because \sin \theta &= \frac{x}{4} \quad \therefore \cos \theta = \frac{x}{\sqrt{16-x^2}} \\ &\quad \text{y} \quad \text{r} \quad \text{y} = x \quad \text{r} = \sqrt{16-x^2} \\ &\quad \text{4} \quad \text{4} \quad \downarrow \quad \text{4} \\ &\quad \text{y} = \frac{x}{4} \end{aligned}$$

$$\begin{aligned} x &= \sqrt{r^2 - y^2} \\ &= \sqrt{16-x^2} \end{aligned}$$

$$= 8 \sin^{-1} \frac{x}{4} + \frac{x \sqrt{16-x^2}}{2} + C$$

$$⑤ \int \frac{dx}{(4x^2)^2}$$

$$1+ \tan^2 \theta = \sec^2 \theta$$

$$\begin{aligned} &= \int \frac{dx}{[4(1+x^2)]^2} \\ &= \frac{1}{16} \int \frac{dx}{(1+x^2)^2} \\ &= \frac{1}{16} \int \frac{\sec^2 \theta \, d\theta}{(1+\tan^2 \theta)^2} \end{aligned}$$

$$= \frac{1}{16} \int \frac{\sec^2 \theta}{\sec^4 \theta} \, d\theta$$

$$= \frac{1}{16} \int \frac{1}{\sec^2 \theta} \, d\theta$$

$$= \frac{1}{16} \int \frac{1}{\cos^2 \theta} \, d\theta$$

$$= \frac{1}{16} \int \cos^2 \theta \, d\theta$$

$$= \frac{1}{16} \int \frac{1}{2} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{32} \left[ \theta + \frac{\sin 2\theta}{2} \right] + C$$

$$= \frac{1}{32} \left[ \theta + \frac{2\sin \theta \cos \theta}{2} \right] + C$$

$$= \frac{1}{32} \left[ \theta + \sin \theta \cos \theta \right] + C$$

$$= \frac{1}{32} \left[ \tan^{-1} x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right] + C$$

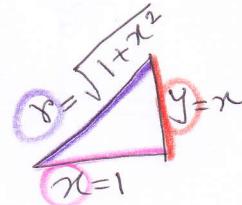
$$= \frac{1}{32} \left[ \tan^{-1} x + \frac{x}{1+x^2} \right] + C$$

$$\begin{aligned} x &= \tan \theta \\ dx &= \sec^2 \theta \, d\theta \end{aligned}$$

$$\begin{aligned} x &= \tan \theta \\ \theta &= \tan^{-1} x \end{aligned}$$

$$\therefore \tan \theta = \frac{y}{x}$$

$$\text{We have } \tan \theta = \frac{x}{1}$$



$$r^2 = \sqrt{x^2 + y^2}$$

$$= \sqrt{1+x^2}$$

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r}$$

$$= \frac{x}{\sqrt{1+x^2}} \quad = \frac{1}{\sqrt{1+x^2}}$$

$$\textcircled{7} \quad \int \frac{\sqrt{x^2 - 9}}{x} dx$$

$$= \int \frac{\sqrt{9(\frac{x^2}{9} - 1)}}{x} dx$$

$$= 3 \int \frac{\sqrt{(\frac{x}{3})^2 - 1}}{x} dx$$

$$= 3 \int \frac{\sqrt{\sec^2 \theta - 1}}{3 \sec \theta} (3 \sec \theta \tan \theta) d\theta$$

$$= +3 \int \frac{\tan \theta (\tan \theta)}{\cos \theta} d\theta$$

$$= 3 \int \tan^2 \theta d\theta$$

$$= 3 \int (\sec^2 \theta - 1) d\theta$$

$$= 3 [\tan \theta - \theta] + C$$

$$= 3 \left[ \frac{\sqrt{x^2 - 9}}{3} - \sec^{-1} \left( \frac{x}{3} \right) \right] + C$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

$$\frac{x}{3} = \sec \theta \quad (\because (\frac{x}{3})^2 = \sec^2 \theta)$$

$$\frac{1}{3} dx = \sec \theta \tan \theta d\theta \quad | \begin{array}{l} x = 3 \sec \theta \\ \therefore \theta = \sec^{-1} \left( \frac{x}{3} \right) \\ \therefore dx = 3 \sec \theta \tan \theta d\theta \end{array}$$

$$\sec \theta = \frac{x}{3} = \frac{r}{z}$$

$y = \sqrt{r^2 - z^2}$   
 $= \sqrt{r^2 - 9}$   
 $\therefore \tan \theta = \frac{y}{z} = \frac{\sqrt{x^2 - 9}}{3}$

$$(23) \int_{\sqrt{2}}^2 \frac{dx}{x^2 \sqrt{x^2 - 1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec \tan \theta d\theta}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\tan \theta d\theta}{\sec \theta \sqrt{\tan^2 \theta}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec \theta} d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cos \theta d\theta$$

$$= \left[ + \sin \theta \right]_{\pi/4}^{\pi/3}$$

$$= + \left[ \sin \frac{\pi}{4} - \sin \frac{\pi}{3} \right]$$

$$= + \left[ \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \right]$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}$$

$$\sec^2 \theta - 1 = \tan^2 \theta$$

let

$$x = \sec \theta \rightarrow \theta = \sec^{-1} x$$

$$x^2 = \sec^2 \theta$$

$$\Rightarrow dx = \sec \tan \theta d\theta$$

$$\left\{ \begin{array}{l} x = \sqrt{2} \Rightarrow \theta = \sec^{-1} \sqrt{2} = \frac{\pi}{4} \\ x = 2 \Rightarrow \theta = \sec^{-1} 2 = \frac{\pi}{3} \end{array} \right.$$

$$\sec \theta = \frac{x}{1} \quad \sec \theta = \frac{r}{x}$$

$$\begin{array}{c} r=x \\ y=\sqrt{r^2-x^2} \\ x=1 \\ y=\sqrt{x^2-1} \end{array}$$

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

$$\left\{ \begin{array}{l} \sec^{-1} \sqrt{2} = \theta \\ \sec \theta = \sqrt{2} = \frac{r}{x} = \frac{\sqrt{2}}{1} \end{array} \right.$$

$$\begin{array}{c} r=\sqrt{2} \\ x=1 \\ \therefore y=1 \\ \tan \theta = \frac{y}{x} = 1 \\ \therefore \theta = 45^\circ \end{array}$$

$$\left\{ \begin{array}{l} \sec^{-1} 2 = \theta \\ \sec \theta = 2 = \frac{r}{x} = \frac{2}{1} \end{array} \right.$$

$$\begin{array}{c} r=2 \\ x=1 \end{array}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \cos^{-1} \frac{1}{2}$$

$$= 60^\circ = \frac{\pi}{3}$$

## Examples

$$1 \int \frac{dx}{x^2 \sqrt{4-x^2}}$$

$$= \int \frac{dx}{x^2 \sqrt{4(1-\frac{x^2}{4})}}$$

$$= \frac{1}{2} \int \frac{dx}{x^2 \sqrt{1-(\frac{x}{2})^2}}$$

$$= \frac{1}{2} \int \frac{2 \cos \theta d\theta}{4 \sin^2 \theta \sqrt{1-\sin^2 \theta}}$$

$$= \int \frac{\cos \theta d\theta}{4 \sin^2 \theta \sqrt{\cos^2 \theta}}$$

$$= \frac{1}{4} \int \frac{1}{\sin^2 \theta} d\theta$$

$$= \frac{1}{4} \int \csc^2 \theta d\theta$$

$$= \frac{1}{4} (-\cot \theta) + C$$

$$= -\frac{1}{4} \cot \theta + C$$

Solve  $\int_1^{\sqrt{2}} \frac{dx}{x^2 \sqrt{4-x^2}}$

$$= \frac{1}{4} \int_{\pi/6}^{\pi/4} \csc^2 \theta d\theta$$

$$= -\frac{1}{4} [\cot \theta]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{4} \left[ \frac{1}{\tan \theta} \right]_{\pi/6}^{\pi/4}$$

$$\begin{aligned} \frac{x}{2} &= \sin \theta & \therefore x &= 2 \sin \theta \\ \frac{1}{2} dx &= \cos \theta d\theta & x^2 &= 4 \sin^2 \theta \\ dx &= 2 \cos \theta d\theta & \theta &= \sin^{-1} \left( \frac{x}{2} \right) \end{aligned}$$

$$\text{cosec} = \csc$$

$$x=1 \Rightarrow \sin^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{6} = 30^\circ$$

$$x=\sqrt{2} \Rightarrow \theta = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right)$$

$$= \sin^{-1} \left( \frac{1}{\sqrt{2}} \right) = \frac{\pi}{4} = 45^\circ$$

$$= -\frac{1}{4} \left[ \frac{1}{\tan \frac{\pi}{4}} - \frac{1}{\tan \frac{\pi}{6}} \right]$$

$$= -\frac{1}{4} \left[ \frac{1}{1} - \frac{1}{\frac{1}{\sqrt{3}}} \right]$$

$$= -\frac{1}{4} [1 - \sqrt{3}]$$

$$= \frac{\sqrt{3}-1}{4}$$

$$\begin{aligned}
 & \boxed{2} \quad \int \frac{x}{x^2 - 4x + 8} dx \\
 &= \int \frac{x}{x^2 - 2 \cdot x \cdot 2 + 2^2 + 4} dx \\
 &= \int \frac{x}{(x-2)^2 + 4} dx \quad \text{let } u = x-2 \\
 &\qquad\qquad\qquad du = dx \\
 &= \int \frac{u+2}{u^2 + 4} du \quad \therefore x = u+2 \\
 &= \int \frac{u}{u^2 + 4} du + 2 \int \frac{1}{u^2 + 4} du \\
 &= \int \frac{\frac{1}{2} dz}{z} + 2 \int \frac{du}{4(\frac{u^2}{4} + 1)} \quad u^2 + 4 = z \\
 &\qquad\qquad\qquad 2udu = dz \\
 &= \frac{1}{2} \int \frac{1}{z} dz + \frac{1}{2} \int \frac{du}{(\frac{u}{2})^2 + 1} \\
 &= \frac{1}{2} \ln z + \frac{1}{2} \tan^{-1} \left( \frac{u}{2} \right) + C \\
 &= \frac{1}{2} \ln(u^2 + 4) + \frac{1}{2} \tan^{-1} \left( \frac{u}{2} \right) + C \\
 &= \frac{1}{2} \ln[(x-2)^2 + 4] + \frac{1}{2} \tan^{-1} \left( \frac{x-2}{2} \right) + C
 \end{aligned}$$