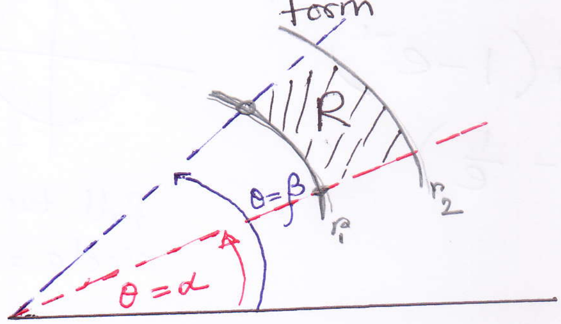
Reference Book
Anton's Calculus
10^{th} Ed.

Chapter 14.3
DOUBLE INTEGRAL
(Polar form)

$$\text{Volume} = \iint_R f(x,y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

where "R" is the region $= \{(r,\theta) \mid \theta_1 \le \theta \le \theta_2;\ r_1(\theta) \le r \le r_2(\theta)\}$

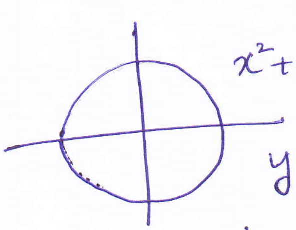
$$dA = \underbrace{dx\,dy}_{\text{Cartesian form}} = \underbrace{r\,dr\,d\theta}_{\text{polar form}}$$



If R is a simple polar region whose boundaries are the rays $\theta=\alpha$, $\theta=\beta$ and curves $r=r_1(\theta)$, $r=r_2(\theta)$ and $f(r,\theta)$ is continuous on R, then $\iint_R f(r,\theta)\, dA$

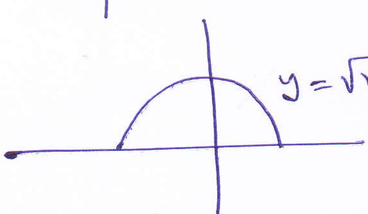
$$= \int_{\theta=\alpha}^{\beta} \int_{r=r_1}^{r_2} f(r,\theta)\, r\, dr\, d\theta$$

Note



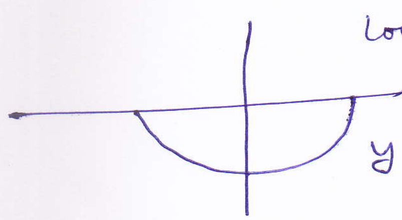
$x^2+y^2=r^2$

$y = \pm\sqrt{r^2-x^2}$



$y=\sqrt{r^2-x^2}$

upper semi circle



Lower semi circle

$y=-\sqrt{r^2-x^2}$

Examples:

① $\iint_R e^{-(x^2+y^2)}\, dA$, where R is the region bounded by the circle $x^2+y^2=1$.

$$\iint_R e^{-(x^2+y^2)}\, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 e^{-z}\,\tfrac{1}{2}\, dz\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{e^{-z}}{-1}\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(-e^{-1}+1\right) d\theta$$

$$= \frac{1}{2}\, 2\pi\left(1-e^{-1}\right)$$

$$= \pi\left(1-\frac{1}{e}\right)$$

$x^2+y^2=r^2$

$\downarrow$

$x^2+y^2=1$

$\Rightarrow x^2+y^2=1^2$

Eqn of a circle

$\therefore\ \theta\in[0,2\pi]$

$r\in[0,1]$

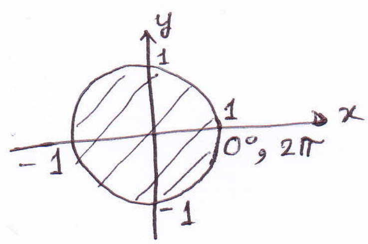
let $r^2=z$

$2r\,dr=dz$

$r\,dr=\frac{1}{2}dz$

$r=0\rightarrow z=0$

$r=1\rightarrow z=1$



full turn of a circle $=2\pi$

Exercise

1 d $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{\sqrt{x^2+y^2}}\, dy\, dx$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} e^{r}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \left(r e^{r}\right) dr\, d\theta$$

$$= \int_0^{\pi/2} \left[r \int e^{r} dr - \int \left(\frac{d}{dr} r \int e^{r} dr \right) dr \right]_0^2 d\theta$$

$$= \int_0^{\pi/2} \left[r e^{r} - \int \left[1 (e^{r}) \right] dr \right]_0^2 d\theta$$

$$= \int_0^{\pi/2} \left[r e^{r} - e^{r} \right]_0^2 d\theta$$

$$= \int_0^{\pi/2} \left[2e^{2} - e^{2} - 0e^{0} + e^{0} \right] d\theta$$

$$= \int_0^{\pi/2} \left(e^{2} + 1 \right) d\theta$$

$$= \left(e^{2} + 1 \right) \left[\theta \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} \left(e^{2} + 1 \right)$$

Note

$x^2 + y^2 = r^2$

$\Rightarrow r = \sqrt{x^2+y^2}$

$\to y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

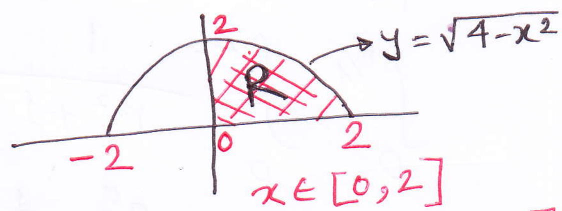
given the upper limit

$y = \sqrt{4-x^2}$

$= \sqrt{2^2 - x^2}$

$= \sqrt{r^2 - x^2}$

$\therefore r = 2$



$x \in [0, 2]$

$y \in [0, \sqrt{4-x^2}]$

$\therefore r \in [0, 2]$

$\theta \in [0, \frac{\pi}{2}]$

$\therefore$ R $\in$ Quarter of a circle

Exercise

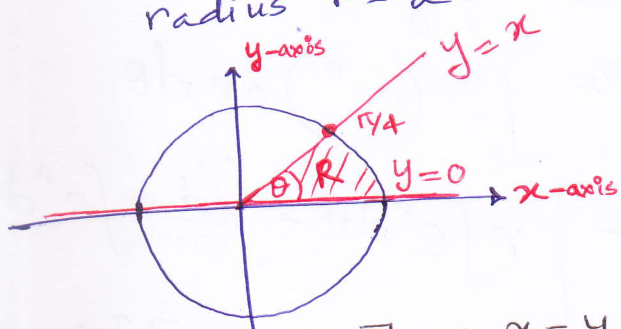
2 (b) $\iint_R \frac{1}{x^2+y^2+1}\, dA$, where R is in the 1^{st} quadrant bounded by $y=0, y=x, \underbrace{x^2+y^2=4}$

Eqn of a circle with radius $r = 2$

$x^2+y^2 = r^2$

Given $x^2+y^2 = 4 = 2^2$

$r = 2$



$$\iint_R \frac{1}{x^2+y^2+1}\, dA$$

$\theta \in [0, 45°] \because x=y$

$$\int_0^{\pi/4}\int_0^2 \frac{1}{r^2+1}\, r\,dr\,d\theta$$

$r \in [0, 2]$

let $r^2+1 = z$

$2r\,dr = dz$

$r\,dr = \frac{1}{2}dz$

$r=0 \rightarrow z=1$

$r=2 \rightarrow z=5$

$$= \int_0^{\pi/4}\int_1^5 \frac{\frac{1}{2}dz}{z}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[\ln z\right]_1^5 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[\ln 5 - \ln 1\right] d\theta$$

$$= \frac{\ln 5}{2}\int_0^{\pi/4} d\theta = \frac{\ln 5}{2}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi \ln 5}{8}$$

Exercise

(3) Use polar coordinates to evaluate the double integral $\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{\frac{1}{2}} \, dy\, dx$

upper limit of y:

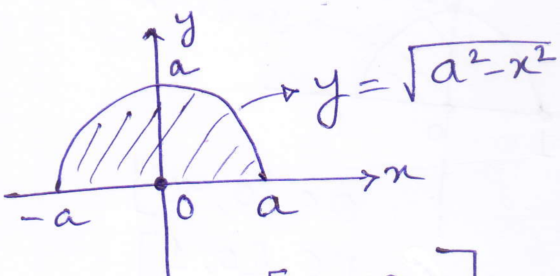
$y = \sqrt{a^2-x^2}$

$y^2 = a^2 - x^2$

$x^2+y^2 = a^2$

$\therefore r^2 = a^2$

$r = a$



$\theta \in [0, \pi]$

$$\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{1/2} \, dy\, dx$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a} (r^2)^{1/2} \, r\, dr\, d\theta$$

$$= \int_{0}^{\pi} \int_{0}^{a} (r)\, r\, dr\, d\theta$$

$$= \int_{0}^{\pi} \int_{0}^{a} r^2\, dr\, d\theta$$

$$= \int_{0}^{\pi} \left[\frac{r^3}{3}\right]_{0}^{a} d\theta$$

$$= \frac{a^3}{3} \left[\theta\right]_{0}^{\pi}$$

$$= \frac{\pi a^3}{3}.$$

Exercise

(5) Evaluate the $\underbrace{\text{iterated}}_{\text{function more than one variable}}$ integral by converting to polar coordinates:

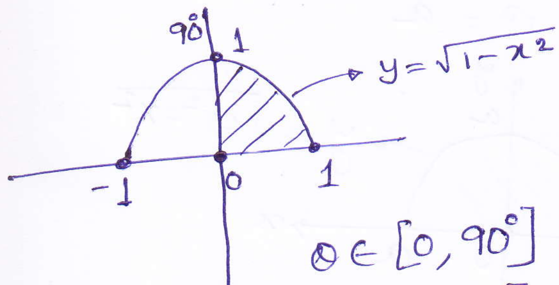
(a) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

$y = \sqrt{1-x^2}$

$y = \sqrt{1^2-x^2} \Rightarrow y^2 = 1^2 - x^2$

$\Rightarrow x^2+y^2 = 1^2$

$\therefore r = 1$



$\theta \in [0, 90°]$

$r \in [0, 1]$

$$= \int_0^{\pi/2}\int_0^1 r^2\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} \left[1^4 - 0^4\right] d\theta$$

$$= \frac{1}{4}\left[\theta\right]_0^{\pi/4}$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{8}$$

(b) $$\int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

upper limit of y:

$y=\sqrt{2x-x^2}$

$y^2=2x-x^2$

$x^2+y^2=2x$

$r^2=2r\cos\theta$

$(x=r\cos\theta)$

$r=2\cos\theta$

$\therefore r\in[0,2\cos\theta]$

Note that $x\in[0,2]$ which means the region 'R' is in the 1st quadrant

$\therefore \theta\in[0,\pi/2]$

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta}\sqrt{r^2}\;r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{2\cos\theta} r.r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$=\frac{1}{3}\int_0^{\pi/2} 8\cos^3\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{\pi/2}\cos^2\theta\cos\theta\,d\theta$$

Let

$\sin\theta=z$

$\cos\theta\,d\theta=dz$

$\theta=0\rightarrow z=0$

$\theta=\frac{\pi}{2}\rightarrow z=1$

$$=\frac{8}{3}\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{1}(1-z^2)\,dz$$

$$=\frac{8}{3}\left[z-\frac{z^3}{3}\right]_0^1=\frac{8}{3}\left[1-\frac{1}{3}\right]=\frac{8}{3}\cdot\frac{2}{3}=\frac{16}{9}$$

See the Examples of Chapter 14.3

Example '2' — page 1022

Example '4' — page 1023