Fourier

$f(x)$ = constant + cosine term + sine term

$f(x)$ is periodic if $f(x+p) = f(x)$ for all x

least value of $p>0$ is called period.

$f(x)$ is repeating after every p value.

$a \sin^b (cx+d) + f$

a = amplitude

d = phaseshift

f = takes the function up or down

} not related to period

new period = $\dfrac{\text{Original period}}{c}$

b — even period $= \pi$; odd period $= 2\pi$ } sin/cos

for tan nothing changes

$f(x) = \sin 6x - \sin 8x$ what is the period?

$\frac{2\pi}{6}$ $\frac{2\pi}{8}$

$$LCM\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \frac{LCM(\pi, \pi)}{gcd(3, 4)} = \frac{\pi}{1} = \pi$$

<u>Even function</u>

must be defined on $[-L, L]$ and $f(-x) = f(x)$

[symmetric about y axis] has same value on y axis.

Ex: $\cos x$

<u>Odd function</u>

must be defined on $[-L, L]$ and $f(-x) = -f(x)$

[symmetric about origin] has opposite values on y axis.

Ex: $\sin x$

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{d}{dx}u\int v\,dx\right)dx$$

faster method

$$\int (x^3+2x)\cos 3x\,dx$$

$$= (x^3+2x)\frac{1}{3}\sin 3x - (3x^2+2)\left(-\frac{1}{9}\right)\cos 3x + 6x\left(-\frac{1}{27}\sin 3x\right) - \frac{6}{81}\cos 3x$$

	$\frac{d}{dx}$ u	$\int dx$ v
$+$	x^3+2x	$\cos 3x$
$-$	$3x^2+2$	$\frac{1}{3}\sin 3x$
$+$	$6x$	$\frac{-1}{9}\cos 3x$
$-$	6	$\frac{-1}{27}\sin 3x$
$+$	0	$\frac{1}{81}\cos 3x$

$$\int_{-L}^{L} \underset{\text{odd}}{f(x)}\,dx = 0$$

$$\int_{-L}^{L} \underset{\text{even}}{f(x)}\,dx = 2\int_0^L f(x)\,dx$$

fourier expansion series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

interval $(-L, L)$, period $2L$/interval

→ can be different $(0, \pi), (\pi, 6\pi), \ldots$

$a_0 = a_n(0)$

fourier coefficient

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$n = 0, 1, 2, \ldots$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

$n = 1, 2, 3, \ldots$

If period is given we will find L from there. If not given we take the interval, as period.

*) $f(x) = \begin{cases} -2 & 0 \le x < \pi \\ 2 & \pi \le x < 2\pi \end{cases}$

$2L = 2\pi$

$\Rightarrow L = \pi$

$\therefore f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\dfrac{n\pi x}{L}\right) + b_n \sin\left(\dfrac{n\pi x}{L}\right) \right)$

$= \dfrac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$

$\therefore a_n = \dfrac{1}{\pi} \int_0^{2\pi} f(x) \cos \dfrac{n\pi x}{L} \, dx$

$\therefore b_n = \dfrac{1}{\pi} \int_0^{2\pi} f(x) \sin \dfrac{n\pi x}{L} \, dx$

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) \quad dx$$

$$= \frac{1}{\pi}\int_0^{\pi} -2 \quad dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2 \quad dx$$

$$= -\frac{2}{\pi}\Big[\quad x\Big]_0^{\pi} + \frac{2}{\pi}\Big[\quad x\Big]_{\pi}^{2\pi}$$

$$= 0$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos(nx)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} -2 \cos(nx)\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2\cos(nx)\, dx$$

$$= -\frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_{\pi}^{2\pi}$$

$$= 0$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} -2\sin(nx)\,dx + \frac{1}{\pi}\int_\pi^{2\pi} 2\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\cos(nx)}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\cos(nx)}{n}\right]_\pi^{2\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n} - \frac{(-1)^{2n}}{n} + \frac{(-1)^n}{n}\right]$$

$$= \frac{2}{\pi}\left[\frac{2(-1)^n}{n} - \frac{1}{n} - \frac{1}{n}\right]$$

$$= \frac{4(-1)^n - 4}{n\pi}$$

*)

$$f(x) = \begin{cases} 1 & -2 < x < 0 \\ 0 & 0 < x < 2 \end{cases}$$

$$L = \frac{4}{2} = 2$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right)$$

$$\therefore a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2} \, dx$$

$$\therefore b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} \, dx$$

$$\therefore a_0 = \frac{1}{2} \int_{-2}^{2} f(x) \, dx$$

$$= \frac{1}{2} \int_{-2}^{0} 1 \, dx = 1$$

$$\therefore a_n = \frac{1}{2}\int_{-2}^{2} f(x) \cos\frac{n\pi x}{2}\, dx$$

$$= \frac{1}{2}\int_{-2}^{0} \cos\frac{n\pi x}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_{-2}^{0}$$

$$= \frac{1}{n\pi} \cdot 0$$

$$= 0$$

$$\therefore b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n x\pi}{2}\, dx$$

$$= \frac{1}{2}\int_{-2}^{0} \sin\frac{n\pi x}{2}\, dx$$

$$= -\frac{1}{n\pi}\left[\cos\frac{n\pi x}{2}\right]_{-2}^{0} = -\frac{1}{n\pi}\left(1-(-1)^n\right)$$

$$= \frac{-1+(-1)^n}{n\pi}$$

*) $f(x) = x \quad -1 \leq x \leq 5$; period $= 6$

$\therefore L = 3$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{3} + b_n \sin\frac{n\pi x}{3} \right)$$

$$\therefore a_n = \frac{1}{3} \int_{-1}^{5} f(x) \cos\frac{n\pi x}{3} \, dx$$

$$\therefore b_n = \frac{1}{3} \int_{-1}^{5} f(x) \sin\frac{n\pi x}{3} \, dx$$

$$\therefore a_0 = \frac{1}{3} \int_{-1}^{5} x \, dx$$

$$= \frac{1}{6} \left[x^2 \right]_{-1}^{5} = 4$$

$$\therefore a_n = \frac{1}{3}\int_{-1}^{5} x \cos\frac{n\pi x}{3}\, dx$$

$$= \frac{1}{3}\left[\frac{3}{n\pi} x \sin\frac{n\pi x}{3} + \frac{9}{n^2\pi^2}\cos\frac{n\pi x}{3}\right]_{-1}^{5}$$

$+$	x	$\cos\frac{n\pi x}{3}$
$-$	1	$\dfrac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}$
$+$	0	$\dfrac{-\cos\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}$

$$= \frac{1}{3}\left[\frac{3\cdot 5}{n\pi}\sin\frac{5}{3}n\pi + \frac{9}{n^2\pi^2}\cos\frac{5}{3}n\pi + \frac{3}{n\pi}(-1)\sin\frac{1}{3}n\pi - \frac{9}{n^2\pi^2}\cos\frac{1}{3}n\pi\right]$$

$$= \frac{5n\pi\sin\left(\frac{5n\pi}{3}\right) + 3\cos\frac{5n\pi}{3} - n\pi\sin\left(\frac{n\pi}{3}\right) - 3\cos\frac{3n\pi}{3}}{n^2\pi^2}$$

$$b_n = \frac{1}{3}\int_{-1}^{5} x \sin\frac{n\pi x}{3}\,dx$$

$+$	x	$\sin\frac{n\pi x}{3}$
$-$	1	$\frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}$
$+$	0	$\frac{-\sin\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}$

$$= \frac{1}{3}\left[\frac{-3}{n\pi} x \cos\frac{n\pi x}{3} + \frac{9}{n^2\pi^2}\sin\frac{n\pi x}{3}\right]_{-1}^{5}$$

$$= \frac{1}{3}\left[-\frac{3\cdot 5}{n\pi}\cos\frac{5n\pi}{3} + \frac{9}{n^2\pi^2}\sin\frac{5n\pi}{3} - \frac{3}{n\pi}\cos\frac{n\pi}{3} + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}\right]$$

$$= \frac{-5n\pi\cos\frac{5n\pi}{3} + 3\sin\frac{5n\pi}{3} - n\pi\cos\frac{n\pi}{3} + 3\sin\frac{n\pi}{3}}{n^2\pi^2}$$

*)

$$f(x) = \begin{cases} 2-x & 0 < x < 4 \\ x-6 & 4 < x < 8 \end{cases}$$

period 8

$\therefore L = 4$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{4} + b_n \sin \frac{n\pi x}{4} \right)$$

$$\therefore a_n = \frac{1}{4} \int_0^8 f(x) \cos \frac{n\pi x}{4} \, dx$$

$$\therefore b_n = \frac{1}{4} \int_0^8 f(x) \sin \frac{n\pi x}{4} \, dx$$

$$\therefore a_0 = \frac{1}{4} \int_0^8 f(x) \, dx$$

$$= \frac{1}{4} \int_0^4 (2-x) \, dx + \frac{1}{4} \int_4^8 (x-6) \, dx$$

$$= \frac{1}{4} \left[2x - \frac{x^2}{2} \right]_0^4 + \frac{1}{4} \left[\frac{x^2}{2} - 6x \right]_4^8$$

$$= 0$$

$$\therefore a_n = \frac{1}{4}\int_0^8 f(x) \cos\frac{n\pi x}{4}\, dx$$

$$= \frac{1}{4}\int_0^4 (2-x) \cos\frac{n\pi x}{4}\, dx + \frac{1}{4}\int_4^8 (x-6) \cos\frac{n\pi x}{4}\, dx$$

$$= \frac{1}{4}\left[(2-x)\frac{4}{n\pi}\sin\frac{n\pi x}{4} - \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4$$

$+$	$2-x$	$\cos\frac{n\pi x}{4}$
$-$	-1	$\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}$
$+$	0	$\frac{-\cos\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}$

$$+ \frac{1}{4}\left[(x-6)\frac{4}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_4^8$$

$+$	$x-6$	$\cos\frac{n\pi x}{4}$
$-$	1	$\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}$
$+$	0	$\frac{-\cos\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}$

$$= \frac{1}{4}\left[0 - \frac{16}{n^2\pi^2}(-1)^n - 0 + \frac{16}{n^2\pi^2}\cdot 1\right]$$

$$+ \frac{1}{4}\left[0 + \frac{16}{n^2\pi^2}(-1)^{2n} - 0 - \frac{16}{n^2\pi^2}(-1)^n\right]$$

$x-6$ $+$	$2-x$	$\sin\frac{n\pi x}{4}$
1 $-$	-1	$\dfrac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}$
0 $+$	0	$\dfrac{-\sin\frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}$

$$= \frac{8-8(-1)^n}{n^2\pi^2}$$

$$\therefore b_n = \frac{1}{4}\int_0^8 f(x)\sin\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{4}\int_0^4 (2-x)\sin\frac{n\pi x}{4}\,dx + \frac{1}{4}\int_4^8 (x-6)\sin\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{4}\left[\left[-(2-x)\frac{4}{n\pi}\cos\frac{n\pi x}{4} - \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_0^4 + \left[-(x-6)\frac{4}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{\pi^2 n^2}\sin\frac{n\pi x}{4}\right]_4^8\right]$$

$$= \frac{1}{4}\left[\frac{2\cdot 4}{n\pi}(-1)^n + \frac{2\cdot 4}{n\pi} - \frac{2\cdot 4}{n\pi}(-1)^{2n} - \frac{2\cdot 4}{n\pi}(-1)^n\right]$$

$$= 0$$

Even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = 2 \frac{1}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Odd

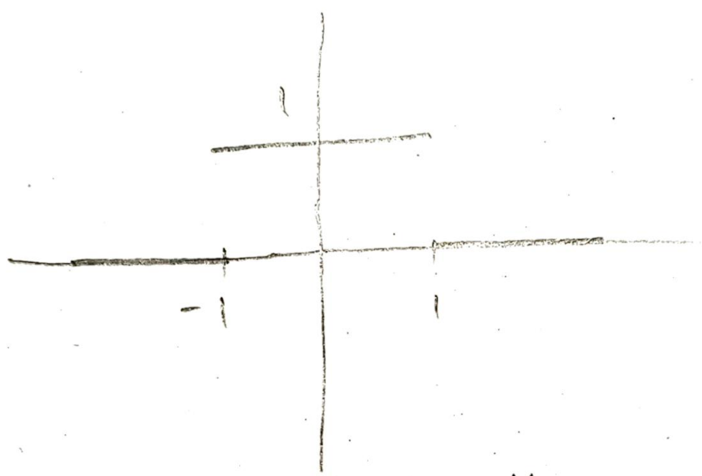
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = 2 \frac{1}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

①

*) $f(x) = \begin{cases} 1 & |x|<1 \\ 0 & \text{else} \end{cases}$

$2L = 2$
$\Rightarrow L = 1$



$f(x)$ is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{1}\cdot 2\int_0^1 f(x)\cos(n\pi x)$$

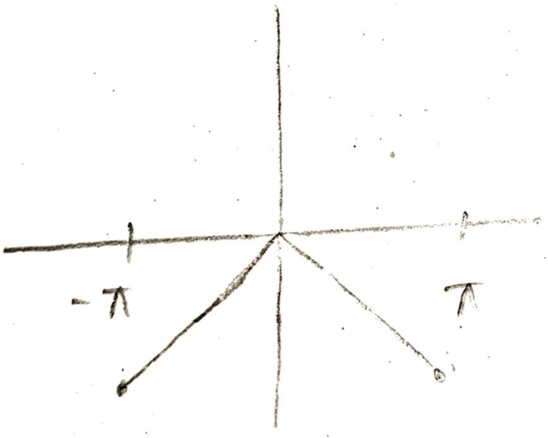
$$\therefore a_0 = 2\int_0^1 1\, dx = 2$$

$\therefore a_n = 2\int_0^1 \cos(n\pi x)$

$= \frac{2}{n\pi}\left[\sin(n\pi x)\right]_0^1$

$= 0$

*) $f(x) = \begin{cases} x & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases}$ $\quad 2L = 2\pi \Rightarrow L = \pi$



$f(x)$ is an even function

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$

$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} -x \, dx$

$= \frac{1}{\pi} \left[x^2 \right]_{\pi}^{0}$

$= -\pi$

$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} -x \cos nx \, dx$

$= -\frac{2}{\pi} \left[x \sin(nx) \frac{1}{n} + \cos(nx) \frac{1}{n^2} \right]_0^{\pi}$

$= -\frac{2}{\pi} \left(\frac{1}{n^2} (-1)^n - \frac{1}{n^2} \right)$

$= \frac{2 - 2(-1)^n}{n^2 \pi}$

*) $f(x) = x^2$ interval $(-3, 3)$

$\therefore L = 3$

$f(-x) = x^2 = f(x)$

$\therefore f(x)$ is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}$$

$$\therefore a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{n\pi x}{3}\,dx$$

$$\therefore a_0 = \frac{2}{3}\int_0^3 x^2\,dx$$

$$= \frac{2}{9}\left[x^3\right]_0^3$$

$$= 6$$

$\therefore a_n = \frac{2}{3}\int_0^3 f(x) \cos\frac{n\pi x}{3}\,dx$

$= \frac{2}{3}\int_0^3 x^2 \cos\frac{n\pi x}{3}\,dx$

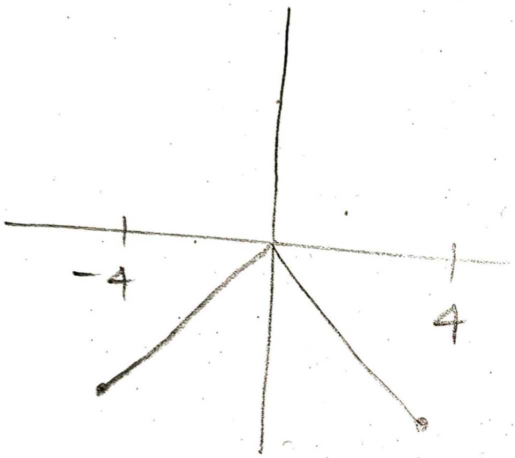
$+$	x^2	$\cos\frac{n\pi x}{3}$
$-$	$2x$	$\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}$
$+$	2	$\frac{-\cos\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}$
$-$	0	$\frac{-\sin\frac{n\pi x}{3}}{\frac{n^3\pi^3}{27}}$

$= \frac{2}{3}\left[x^2\frac{3}{n\pi}\sin\frac{n\pi x}{3} + 2x\frac{9}{n^2\pi^2}\cos\frac{n\pi x}{3} - \frac{2.27}{n^3\pi^3}\sin\frac{n\pi x}{3}\right]_0^3$

$= \frac{2}{3}\left[\frac{2.3.9}{n^2\pi^2}(-1)^n\right]$

$= \frac{36}{n^2\pi^2}(-1)^n$

*) $f(x) = \begin{cases} x & -4 < x < 0 \\ -x & 0 < x < 4 \end{cases}$ $\therefore L = 4$



$f(x)$ is an even function

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{4}$$

$$\therefore a_n = \frac{1}{2} \int_0^4 f(x) \cos \frac{n\pi x}{4} \, dx$$

$$\therefore a_0 = \frac{1}{2} \int_0^4 -x \, dx$$

$$= \frac{-1}{4} \left[x^2 \right]_0^4$$

$$= -4$$

$$\therefore a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{4}\,dx$$

$$= \frac{-1}{2}\int_0^4 x\cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{1}{2}\left[x\frac{4}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4$$

$$= -\frac{1}{2}\left(\frac{16}{n^2\pi^2}(-1)^n - \frac{16}{n^2\pi^2}\right)$$

$$= \frac{8 - 8(-1)^n}{n^2\pi^2}$$

Half Range

Even $\longrightarrow$ cosine

odd $\longrightarrow$ sine

normally $2L =$ interval length

For,
half range/ sine series/ cosine series
$L =$ Domain/ interval length / give half range

*) $f(x) = \cos x$; $0 < x < \pi$; sine series

$L = \pi$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin(nx)\, dx \qquad [n \neq 1]$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos x \sin(nx)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left\{ \sin(1+n)x + \sin(n-1)x \right\} dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{1+n}}{1+n} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{1+n} + \frac{1}{n-1} \right]$$

$$= \frac{1}{\pi}\left(\frac{2n}{n^2-1} + \frac{2n(-1)^n}{n^2-1} \right) = \frac{2n+2n(-1)^n}{(n^2-1)\pi}$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin 2x \quad dx$$

$$= \frac{-1}{\pi} \Big[\cos 2x\Big]_0^{\pi}$$

$$= 0$$

*) $f(x) = 3\sin x$; $0 < x < \pi$; cosine expansion

$L = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

$$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} 3 \sin x \, dx$$

$$= -\frac{6}{\pi} \left[\cos x\right]_0^{\pi}$$

$$= \frac{12}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} 3\sin x \cos(nx)\, dx \qquad [n \neq 1]$$

$$= \frac{3}{\pi}\int_0^{\pi}\left(\sin(n+1)x + \sin(1-n)x\right) dx$$

$$= \frac{3}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{3}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{1-n}}{1-n} + \frac{1}{n+1} + \frac{1}{1-n}\right]$$

$$= \frac{3}{\pi}\left[\frac{-(-1)^{n+1} + n(-1)^{n+1} - n(-1)^{1-n} - (-1)^{n-1}}{1-n^2} + \frac{1-n+1+n}{1-n^2}\right]$$

$$= \frac{3}{\pi}\left[\frac{-(-1)^n(-1) + n(-1)^n(-1) - n(-1)(-1)^n - (-1)^n(-1)}{1-n^2} + \frac{2}{1-n^2}\right]$$

$$= \frac{3}{\pi}\left(\frac{2(-1)^n}{1-n^2} + \frac{2}{1-n^2}\right)$$

$$= \frac{6\left((-1)^n+1\right)}{(1-n^2)\pi}$$

$$\therefore a_1 = \frac{2}{\pi}\int_0^{\pi} 3 \sin x \cos x \, dx$$

$$= \frac{3}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{-3}{2\pi}\left[\cos 2x\right]_0^{\pi}$$

$$= 0$$

Convergence of fourier series

*) $f(x) = x^2$ for $-\pi < x < \pi$, show that,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$$

$f(x)$ is even. after solving it,

take $x = 0, 1, \pi, 2\pi$, or something else

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

$$\therefore f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n$$

$$\Rightarrow \frac{(-\pi)^2 + \pi^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

as we are taking the boundary

$$\Rightarrow \frac{2\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} \quad \text{(Showed)}$$

Parseval's Identity

period = 2L

interval = (-L, L)

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

diff for odd/even

Ex) $f(x) = x$; $0 < x < 2$; cosine

use Parseval's identity to show that,

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}, \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

$L = 2$

(do all the process....)

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\cos(nx)$$

$a_0 = 2$

using Parseval's identity,

$$2 \cdot \frac{1}{2}\int_{0}^{2} x^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\Rightarrow \int_0^2 x^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\right]^2$$

$$\Rightarrow \frac{1}{3}\left[x^3\right]_0^2 = 2 + \sum_{n=1}^{\infty} \frac{16}{n^4\pi^4}\left((-1)^n - 1\right)^2$$

$$\Rightarrow \frac{2}{3} = \frac{16}{1^4\pi^4}(-2)^2 + 0 + \frac{16}{3^4\pi^4}(-2)^2 + 0 + \cdots$$

$n=1$ $(-2)^2$

$n=2$ 0^2

$$\Rightarrow \frac{2}{3} = \frac{16(-2)^2}{\pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$$

$n=3$ $(-2)^2$

$n=4$ 0^2

$$\Rightarrow 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$ (Showed)

Again,

Let $S = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

$= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \cdots\right)$

$$= \frac{\pi^4}{96} + \left(\frac{1}{(2\cdot 1)^4} + \frac{1}{(2\cdot 2)^4} + \frac{1}{(3\cdot 2)^4} + \cdots \right)$$

$$= \frac{\pi^4}{96} + \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$\Rightarrow S = \frac{\pi^4}{96} + \frac{1}{2^4} S$$

$$\Rightarrow S = \frac{\pi^4}{96\left(1 - \frac{1}{2^4}\right)}$$

$$\therefore S = \frac{\pi^4}{90} \quad \underline{\text{(Shwd)}}$$

complex fourier Series

$e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ euler's formula

$e^{i\pi} = -1$

$\Rightarrow e^{i\pi} + 1 = 0$ euler's identity

$e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$e^{i\theta} - e^{-i\theta} = 2i \sin\theta \Rightarrow \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$f(x) \to$ period $2L$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}$$ plus

$$\therefore C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx$$ minus

*) Use euler's formula to prove that, complex form of fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$ and

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx$$

General fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \cos\frac{n\pi x}{L} = \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow \sin\frac{n\pi x}{L} = \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} + b_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{i\frac{n\pi x}{L}} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-i\frac{n\pi x}{L}} \right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-i\frac{n\pi x}{L}} + \frac{a_0}{2} e^{i\frac{0\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{i\frac{n\pi x}{L}}$$

$$\sum_{n=-\infty}^{-1} = \sum_{-n=1}^{\infty}$$

$$= \sum_{-n=1}^{\infty}\left(\frac{a_{-n}}{2} - \frac{b_{-n}}{2i}\right) e^{i\frac{n\pi x}{L}} + \frac{a_0}{2} e^{i\frac{0\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

n is less than 0
and we have a $(-)$ so it turns
out as positive

where,

$$C_n = \begin{cases} \dfrac{a_{-n}}{2} - \dfrac{b_{-n}}{2i} & n<0 \\[2em] \dfrac{a_0}{2} & n=0 \\[2em] \dfrac{a_n}{2} + \dfrac{b_n}{2i} & n>0 \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{n\pi x}{L}}$$

$$\Rightarrow \int_{-L}^{L} f(x) \, e^{-i\frac{m\pi x}{L}} dx = \int_{-L}^{L} \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{n\pi x}{L}} e^{-i\frac{m\pi x}{L}} dx$$

$$\Rightarrow \int_{-L}^{L} f(x) \, e^{-i\frac{m\pi x}{L}} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{i\frac{n\pi x}{L}} e^{-i\frac{m\pi x}{L}} dx \quad \text{———} (1)$$

for $n = m$,

$$\int_{-L}^{L} e^{i\frac{n\pi x}{L}} e^{-i\frac{m\pi x}{L}} dx$$

$$= \int_{-L}^{L} 1 \, dx$$

$$= 2L$$

for $n \neq m$,

$$\int_{-L}^{L} e^{i\frac{n\pi x}{L}} e^{-i\frac{m\pi x}{L}} dx$$

$$= \int_{-L}^{L} e^{i\frac{(n-m)\pi x}{L}} dx$$

$$= \left[\frac{e^{i\frac{(n-m)\pi x}{L}}}{i\frac{(n-m)\pi}{L}} \right]_{-L}^{L}$$

[use cos, sin here]

$= 0$

① $\Rightarrow$

$$\therefore \int_{-L}^{L} f(x) e^{-i\frac{m\pi x}{L}} dx = C_m \cdot 2L$$

$$\Rightarrow c_m = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{m\pi x}{L}} dx$$

[rest of the terms are 0. 2L when $n = m$]

$$\therefore C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{m\pi x}{L}}\, dx$$

(proved)

*) $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$ use complex form

$2L = 2\pi$
$\Rightarrow L = \pi$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{inx}$$

$$\therefore C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (-1)\, e^{-inx}\, dx + \frac{1}{2L}\int_{0}^{\pi} e^{-inx}\, dx$$

$$= \frac{-1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi}$$

$$\frac{1}{2\pi}\left(\frac{e^{0}}{in} - \frac{e^{-in\pi}}{in}\right) + \frac{1}{2\pi}\left(\frac{e^{-in\pi}}{-in} - \frac{e^{0}}{-in}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{in} - \frac{\cos(n\pi) - i\sin(n\pi)}{in}\right) + \frac{1}{2\pi}$$

$$\left(\frac{\cos(n\pi) - i\sin(n\pi)}{-in} + \frac{1}{in}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2}{in} - \frac{2(-1)^n}{in}\right)$$

$$= \frac{1-(-1)^n}{in}$$

General fourier transformation

to

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{iwx}\, dw$$

inverse transform

$$F(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

fourier transform

to

$f(x)$ is a continuous signal and the fourier transform is $F(w)$. x is time domain variable and w is frequency domain variable.

*) $f(x) = \begin{cases} \pi & |x| < 1 \\ 0 & |x| > 1 \end{cases}$ find the fourier transformation

hence, evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

$$\therefore F(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \int_{-1}^{1} \pi\, e^{-iwx}\, dx + 0$$

$$= \pi \left[\frac{e^{-iwx}}{-iw} \right]_{-1}^{1}$$

$$= \pi \left(\frac{e^{-iw}}{-iw} + \frac{e^{iw}}{iw} \right) = \pi\, \frac{e^{iw} - e^{-iw}}{iw}$$

$$= \pi \frac{2i \sin\omega}{i\omega}$$

$$= \frac{2\pi \sin\omega}{\omega}$$

I.F.T.,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\pi \sin\omega}{\omega}\, e^{i\omega x}\, dx$$

$x=0$,

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\pi \sin\omega}{\omega}\, e^{0}\, d\omega$$

$$\Rightarrow \pi = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \frac{\cancel{2\pi} \sin\omega}{\omega}\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

$\hookrightarrow$ even f.

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Ans.

*) Find the fourier transformation of

$$f(x) = \begin{cases} \frac{1}{m} & |x| < m \\ 0 & |x| > m \end{cases} \qquad \int_0^\infty \frac{\sin x}{x}\, dx = ?$$

F.T. $F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

$$= \int_{-m}^{m} \frac{1}{m} e^{-i\omega x}\, dx + 0$$

$$= \frac{1}{m} \frac{1}{-i\omega} \left[e^{-i\omega x} \right]_{-m}^{m}$$

$$= -\frac{1}{i\omega m} \left(e^{-i\omega m} - e^{i\omega m} \right)$$

$$= \frac{2i\sin(\omega m)}{i\omega m} = \frac{2\sin(\omega m)}{\omega m}$$

I.F.T.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{i\omega x}\, d\omega$$

$$\Rightarrow f(0) = \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{\cancel{2}\sin \omega m}{\omega m}\, e^{i\omega \cdot 0}\, d\omega$$

$$\Rightarrow \frac{1}{m} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \omega m}{\omega m}\, d\omega$$

$$\Rightarrow 1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin \omega m}{\omega}}_{\text{even}}\, d\omega$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin \omega m}{\omega}\, d\omega = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{\frac{x}{m}}\, \frac{dx}{m} = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2} \quad \text{Ag.}$$

$\omega m = x$

$\Rightarrow m = \frac{dx}{d\omega}$

*) $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$ $\int_0^\infty \frac{\sin(ax)\cos(ax)\,dx}{x} = ?$

$$\therefore f(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \int_{-a}^{a} e^{-iwx}\, dx + 0$$

$$= \left[\frac{e^{-iwx}}{-iw} \right]_{-a}^{a}$$

$$= \frac{-e^{-iwa} + e^{iwa}}{iw}$$

$$\therefore \text{IFT},\ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w)\, e^{iwx}\, dw$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-e^{-iwa} + e^{iwa}}{iw}\, e^{iwx}\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\cancel{i}\sin wa}{\cancel{i}w} e^{iwx} dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin wa}{w} (\cos wx + i\sin wx)\, dw$$

$$\Rightarrow f(x) + i \cdot 0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin wa}{w} \cos wx\, dw +$$

$$i\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin wa}{w} \sin wx\, dw$$

$$\therefore \frac{1}{\cancel{2}\pi} \int_{-\infty}^{\infty} \frac{\cancel{2}\sin wa}{w} \cos wx\, dw = f(x)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin wa}{w} \cos wa\, dw = f(a)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(aw)}{w} \cdot \cos(aw)\, dw = \frac{1+0}{2}$$

a is boundary

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(aw)\cos(aw)}{w}\, dw = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin(a\omega)\cos(a\omega)}{\omega}\,d\omega = \frac{\pi}{4}$$

even

$$\Rightarrow \int_0^\infty \frac{\sin(ax)\cos(ax)}{x}\,dx = \frac{\pi}{4}$$ As.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

*)

$$f(x) = \begin{cases} 1-|x| & |x| < 1 \\ 0 & |x| > 1 \end{cases} \quad \therefore \int_0^\infty \frac{\sin^2 x}{x^2}\, dx = ?$$

$$\therefore f(w) = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \int_{-1}^{1} (1-|x|)\, e^{-iwx}\, dx$$

$$= \int_0^1 (1-x)\, e^{-iwx}\, dx + \int_{-1}^{0} (1+x)\, e^{-iwx}\, dx$$

$$= \left[(1-x)\frac{e^{-iwx}}{-iw} + \frac{e^{-iwx}}{i^2 w^2} \right]_0^1 +$$

$$\left[(1+x)\frac{e^{-iwx}}{-iw} - \frac{e^{-iwx}}{i^2 w^2} \right]_{-1}^{0}$$

	$1+x$	e^{-iwx}
$+$		
$-$	1	$\frac{e^{-iwx}}{-iw}$
$+$	0	$\frac{e^{-iwx}}{i^2 w^2}$

$$= -\frac{e^{-iw}}{w^2} + \frac{1}{iw} + \frac{1}{w^2} - \frac{1}{iw} + \frac{1}{w^2} - \frac{e^{iw}}{w^2}$$

$$= \frac{2}{\omega^2} - \frac{e^{i\omega} + e^{-i\omega}}{\omega^2}$$

$$= \frac{2 - 2\cos\omega}{\omega^2}$$

$$= \frac{2(1-\cos\omega)}{\omega^2}$$

$$= \frac{4\sin^2\frac{\omega}{2}}{\omega^2}$$

IFT,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2\frac{\omega}{2}}{\omega^2}\, e^{i\omega x}\, d\omega$$

$$\Rightarrow f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\sin^2\frac{\omega}{2}}{\omega^2}\, d\omega$$

$$\Rightarrow 1 - |0| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\sin^2\frac{\omega}{2}}{\omega^2}\, d\omega$$

$$\Rightarrow 2\int_0^\infty \frac{\sin^2\frac{w}{2}}{w^2}\,dw = \frac{\pi}{2}$$

$\frac{w}{2} = x$

$\Rightarrow dw = 2\,dx$

$$\Rightarrow \int_0^\infty \frac{\sin^2 x}{4x^2}\,2\,dx = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

Fourier Cosine & Sine Transform

$$f(w) = \int_{-\infty}^{\infty} f(x)\, \underbrace{e^{-iwx}}\, dx$$

cos wx — even
sin wx — odd

$$= \int_{-\infty}^{\infty} f(x) \cos wx \, dx - i \int_{-\infty}^{\infty} f(x) \sin wx \, dx$$

even   odd

$$f(x) = 4 \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w)\, \underbrace{e^{iwx}}\, dw$$

cos wx — even
sin wx — odd

cosine

$$f_c(w) = \int_0^{\infty} f(x) \cos(wx)\, dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f_c(w) \cos(wx)\, dw$$

sine

$$f_s(w) = \int_0^{\infty} f(x) \sin(wx)\, dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} f_s(w) \sin(wx)\, dw$$

*) fourier cosine transformation of :

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

$$f_c(\omega) = \int_0^\infty f(x) \cos(\omega x)\, dx$$

$$= \int_0^1 \cos \omega x \, dx$$

$$= \frac{\sin \omega}{\omega}$$

*) find fourier sine transform, e^{-x}, $x \geq 0$

show,

$$\int_0^\infty \frac{x\sin(mx)}{x^2+1} dx = \frac{\pi}{2} e^{-m}, \quad m>0$$

$$f_s(w) = \int_0^\infty f(x) \sin(wx) dx$$

$$I = \int_0^\infty e^{-x} \sin(wx) dx$$

$+ \ e^{-x}$	$\sin wx$
$- \ -e^{-x}$	$-\frac{\cos wx}{w}$
$+ \ e^{-x}$	$-\frac{\sin wx}{w^2}$

$$= \left[e^{-x} \frac{-\cos(wx)}{w} - e^{-x} \frac{\sin(wx)}{w^2} \right]_0^\infty - \int_0^\infty e^{-x} \frac{\sin(wx)}{w^2} dx$$

$$= \left(-0-0-\frac{1}{w}-0\right) - \frac{1}{w^2} I$$

$$\Rightarrow I = \frac{1}{w} - \frac{1}{w^2} I$$

$$\Rightarrow w^2 I = w - I$$

$$\Rightarrow I = \frac{w}{1+w^2}$$

IFST,

$$f(x) = \frac{2}{\pi}\int_0^\infty f_s(\omega)\sin(\omega x)\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(\omega x)\,d\omega$$

$$\Rightarrow f(m) = \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(m\omega)\,d\omega$$

$$\Rightarrow \frac{\pi}{2}e^{-m} = \int_0^\infty \frac{\omega\sin(\omega m)}{1+\omega^2}\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{u\sin(mu)}{1+u^2}\,du = \frac{\pi}{2}e^{-m}$$

(Solved)

$$f(x) = \begin{cases} 2\sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

$2L = 2\pi$

$\therefore L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) \, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) \, dx$$