MODELING (DE)

Chapter 3.1 Linear Models

Text: Differential Equations with Boundary-Value Problems 7th Ed. Dennis G. Zill, Michael R. Cullen

→ Growth and Decay

→ Newton's Law of Cooling/Warming

MAT 120

Ch 3.1 LINEAR EQN of GROWTH & DECAY

Linear DE: $\frac{dy}{dx} + P(x)y = f(x)$

Linear DE of population function w.r.t. time

$$\frac{dP}{dt} + P(t)P = f(t)$$

$P(t_0) = P_0 \rightarrow$ initial population at initial time

The population of a community is known to increase/decrease at a rate proportional to the number of people present at time t.

$$\frac{dP}{dt} \propto P$$

$\Rightarrow \frac{dP}{dt} = kP$ → k is a constant of proportionality serves as a model for diverse phenomena involving either growth or decay.

$\Rightarrow \frac{dP}{dt} - kP = 0$

I.F. $= e^{\int P(t)dt}$
$= e^{\int -k\,dt}$
$= e^{-kt}$

$\Rightarrow e^{-kt}\left(\frac{dP}{dt} - kP\right) = e^{-kt}(0)$

(multiply by I.F.)

$\Rightarrow \int \left[e^{-kt}\left(\frac{dP}{dt} - kP\right)\right]dt = \int 0\,dt$

$\Rightarrow Pe^{-kt} = C$ Refer to Ch 2.3 general sol. of Linear DE

$\therefore P = Ce^{kt}$

$\underbrace{\boxed{P(t) = Ce^{kt}}}_{(a)}$ → standard Linear eqn of growth & decay

Initially $t_0 = 0$, P_0 = initial Population

$P(t_0) = P(0) = Ce^{k(0)}$

$\Rightarrow P_0 = C$

→ Now Substitute $C = P_0$ into (a)

$\boxed{P(t) = P_0 e^{kt}}$

↳ initial Population function.

Growth & Decay

Examples

[1] The population of a community is known to increase at a rate $\frac{dp}{dt}$ proportional to the number of people present at time "t". If the population has doubled in 5 years, how long will it take to triple? to $\underbrace{\text{quadruple}}_{\text{4 times larger}}$?

Eqn of growth:

$P(t) = Ce^{kt}$ —— (i)

initially $t=0$ [$\because$ The countdown of time starts at '0']

$\therefore$ substitute into (i) we have

$P(0) = Ce^{k(0)} \Rightarrow P_0 = Ce^0 \Rightarrow C = \underbrace{P_0}_{\text{initial population}}$

Substitute C into (i)

$P(t) = P_0 e^{kt}$ —— (ii)

Given: Population doubled in 5 years (given) (The population has doubled in 5years)

$\underbrace{t=5, \quad P(5) = 2P_0}_{\text{Substitute in (ii)}}$

$P(5) = P_0 e^{k5}$

$\Rightarrow 2\cancel{P_0} = \cancel{P_0} e^{5k}$

$\Rightarrow e^{5k} = 2$

$\Rightarrow \ln e^{5k} = \ln 2$

$\boxed{e^{\ln x} = x}$

$\Rightarrow 5K = \ln 2$

$\Rightarrow \boxed{K = \frac{\ln 2}{5}}$

{How long will it take to triple?

$\Rightarrow t = ?$ when $P(t) = 3P_0$

substitute $P(t)$ into eqn (ii)

$P(t) = P_0 e^{kt}$ —— (ii)

$3\cancel{P_0} = \cancel{P_0} e^{\frac{\ln 2}{5}t}$ $\quad \because K = \frac{\ln 2}{5}$

$e^{\frac{\ln 2}{5}t} = 3 \Rightarrow \ln e^{\frac{\ln 2}{5}t} = \ln 3$

$\Rightarrow \frac{\ln 2}{5}t = \ln 3$

$\Rightarrow t = \frac{5\ln 3}{\ln 2}$

$= 7.92481$

≈ 8 years

{How long will it take to quadruple?

$\hookrightarrow t = ?$ when $\underbrace{P(t) = 4P_0}_{\text{substitute into (ii)}}$

$$P(t) = P_0 e^{kt} \quad \text{— (ii)}$$

$$4\cancel{P_0} = P_0 e^{kt} = \cancel{P_0} e^{\frac{\ln 2}{5} t}$$

$$e^{\frac{\ln 2}{5} t} = 4 \Rightarrow \ln e^{\frac{\ln 2}{5} t} = \ln 4$$

$$\Rightarrow \frac{\ln 2}{5} t = \ln 4 \Rightarrow t = \frac{5 \ln 4}{\ln 2} \approx 10 \text{ years}$$

$\boxed{2}$ Suppose it is known that the population of the community in Problem $\boxed{1}$ is 10,000 after 3 years. What was the initial population? What will be the population in 10 years?

Given $P(3) = 10,000$

Find $P_0 = ?$, $P(10) = ?$

Population eqn of Growth: $P(t) = P_0 e^{kt}$ from Problem $\boxed{1}$ — (i)

$$\therefore P(3) = P_0 e^{k3}$$

$$\Rightarrow 10,000 = P_0 e^{3\left[\frac{\ln 2}{5}\right]}$$

$$= P_0 e^{\frac{3}{5}\ln 2}$$

$$= P_0 e^{\ln 2^{3/5}}$$

$$= P_0 2^{3/5}$$

$$P_0 = \frac{10,000}{2^{3/5}} = 6597.54$$

Substitute $t = 10$ into (i):

$$P(10) = P_0 e^{k(10)}$$

$$= 6597.54 e^{\left[\frac{\ln 2}{5}\right](10)}$$

$$= 6597.54 e^{\frac{10}{5}\ln 2} = 6597.54 e^{\ln 2^2} = 6597.54 e^{\ln 4}$$

$$= 6597.54(4)$$

$$= 26390.2$$

$$\approx 26390$$

3 The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years?

$$\frac{dP}{dt} = KP \; ; \quad P_0 = 500$$

$$P(10) = 500 + \frac{15}{100} \cdot (500) = 575$$

$\because$ Population is $\uparrow$

Find: $P(30) = ?$

Solving $\frac{dP}{dt} - KP = 0$

we get $P(t) = Ce^{Kt}$ — (i)

initially $t = 0$, $P(0) = P_0 \Rightarrow C = P_0$.

Substitute C into (i)

$P(t) = P_0 e^{Kt}$

$P(t) = 500 e^{Kt}$ (ii) $\because P_0 = 500$

$\therefore P(10) = 500 e^{K10}$

$575 = 500 e^{10K}$

$e^{10K} = \frac{575}{500} = \frac{23}{20}$

$\ln e^{10K} = \ln \frac{23}{20}$

$10K = \ln \frac{23}{20}$

$$\boxed{K = \frac{1}{10} \ln \frac{23}{20}}$$

Again in eqn (ii)

$P(t) = 500 e^{Kt}$

if $t = 30$ then

$$P(30) = 500 e^{\left[\frac{1}{10}\ln\left(\frac{23}{20}\right)\right]30}$$

$$= 500 e^{\frac{30}{10}\ln\frac{23}{20}}$$

$$= 500 e^{3\ln\frac{23}{20}}$$

$$= 500 e^{\ln\left(\frac{23}{20}\right)^3}$$

$$= 500 \left(\frac{23}{20}\right)^3$$

$$= 760.438$$

$$\approx 760$$

4 a A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated.

b Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

a Let $A(t)$ denote the amount of plutonium remaining at time t.

As The ~~linear~~ function of Growth/Decay, the solution of the initial value problem:

$$\frac{dA}{dt} = kA \; ; \; A(0) = A_0 \longrightarrow t=0 \text{ initial time}$$

is $A(t) = A_0 e^{kt}$ —— (i)

$A(15) =$ 0.043% of the initial amount A_0 of plutonium has disintegrated

which is $(100 - 0.043)\% = 99.957\%$ of A_0 remains

$\therefore A(15) = 0.99957 A_0$

$\cancel{A_0} e^{k15} = 0.99957 \cancel{A_0}$ $\left[(i) \Rightarrow A(t) = A_0 e^{kt} \Rightarrow A(15) = A_0 e^{k15} \right]$

$\ln e^{15k} = \ln 0.99957$

$15k = \ln 0.9957$

$k = \frac{1}{15} \ln 0.9957$

$= -0.00002867$

$\therefore A(t) = A_0 e^{kt}$

$\Rightarrow A(t) = A_0 e^{-0.00002867t}$

$\boxed{b}$ Now the half-life is the corresponding value of time at which $A(t) = \frac{1}{2} A_0$

From (a) we have:

$$A(t) = A_0 e^{-0.00002867 t}$$

$$\frac{1}{2} \cancel{A_0} = \cancel{A_0} e^{-0.00002867 t} \qquad \because A(t) = \frac{1}{2} A_0$$

$$\therefore e^{-0.00002867 t} = \frac{1}{2}$$

$$\frac{1}{e^{0.00002867 t}} = \frac{1}{2}$$

$$e^{0.00002867 t} = 2$$

$$\ln e^{0.00002867 t} = \ln 2$$

$$0.00002867 t = \ln 2$$

$$t = \frac{\ln}{0.00002867} = 24.180 \approx 24 \text{ years}$$

Newton's Law of Cooling / Warming

It is given by

$$\frac{dT}{dt} = k\,(T - T_m)$$

k – constant

$T(t)$ – temperature of the object with respect to time

time is non-negative

T_m – ambient temperature
(It is the air temperature of any environment where computers and related equipment are kept)
OR (Room temperature)

Example:

When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take for the cake to cool off to a room temperature of $\underbrace{70°F}_{T_m}$?

$T(0) = 300°F$

$T(3) = 200°F$ (3 minutes later)

$T_m = 70°F$

$$\frac{dT}{dt} = k(T-70)$$

$$\frac{dT}{T-70} = k\,dt$$

$$\int \frac{dT}{T-70} = \int k\,dt$$

$$\ln|T-70| = kt + c_1$$

$$\log_e|T-70| = kt + c_1$$

$$T-70 = e^{kt+c_1}$$

$$T = 70 + e^{kt}e^{c_1}$$

$$T = 70 + e^{kt}c_2 \quad \left[\text{Relabel constant } e^{c_1} = c_2\right]$$

$$T(t) = 70 + c_2e^{kt}$$

$$T(0) = 70 + c_2e^{0}$$

$$300 = 70 + c_2$$

$\therefore c_2 = 230$

$\therefore T(t) = 70 + 230e^{kt}$

Now substitute $t = 3$ $\because$ Given $T(3) = 200°F$

$$T(3) = 70 + 230e^{k(3)}$$

$$200 = 70 + 230e^{3k}$$

$$230e^{3k} = 130$$

$$e^{3k} = \frac{130}{230} = \frac{13}{23}$$

$$\ln e^{3k} = \ln\frac{13}{23}$$

$$3k = \ln\frac{13}{23}$$

$$k = \frac{1}{3}\ln\frac{13}{23} = -0.19018$$

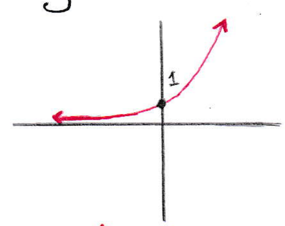
$t = ?$ when $T(t) = 70°$

$$T(t) = 70 + 230e^{-0.19018t}$$

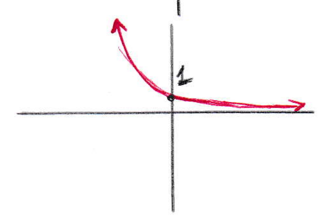
↳ There is no finite solution to $T(t) = 70$ $\because \lim_{t\to\infty} T(t) = 70$.

Yet infinitely we expect the cake to reach the room temperature which is 70°F after a reasonably long period of time $\therefore$ Consider $T(t) = 71$ $\because$ 71° is close to 70°. Note that the room temperature is always changing, hence it won't be always 70° (fixed).

e^{x}



e^{-x}



We have:

$$T(t) = 70 + 230e^{-0.19018t}$$

$$71 = 70 + 230e^{-0.19018t}$$

$$1 = 230e^{-0.19018t}$$

$$e^{-0.19018t} = \frac{1}{230}$$

$$\ln e^{-0.19018t} = \ln\left(\frac{1}{230}\right)$$

$$-0.19018t = -5.438079$$

$$t = 28.59 \approx 29 \text{ minutes}$$