A Non-Exact DE Made Exact

Considering $M(x,y)\,dx + N(x,y)\,dy = 0$ —— (1)

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for (1) is $e^{\int \frac{M_y - N_x}{N} dx}$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for (1) is $e^{\int \frac{N_x - M_y}{M} dy}$

Consider the following non-linear DE:

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0 \quad \text{—— (2)}$$

$$M = xy \qquad N = 2x^2 + 3y^2 - 20$$

$$M_y = \frac{\partial M}{\partial y} = x \qquad N_x = \frac{\partial N}{\partial x} = 4x$$

Eqn (2) is not an exact DE

__Consider:__

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20} \rightarrow \text{depends on } x \text{ and } y$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y} \rightarrow \text{depends only on } y$$

$\therefore$ The integrating factor: $e^{\int \frac{3}{y} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3.$

Multiply eqn (2) by integrating factor we have:

$$y^3\left(xy\,dx + (2x^2+3y^2-20)\,dy\right) = (0)\,y^3$$

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0 \quad \text{—— (3)}$$

$M = xy^4$ $\qquad$ $N = 2x^2y^3 + 3y^5 - 20y^3$

$\frac{\partial M}{\partial y} = 4xy^3$ $\qquad$ $\frac{\partial N}{\partial x} = 4xy^3$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence eqn (3) is an exact DE

$$M = xy^4$$

$$\frac{\partial f}{\partial x} = M = xy^4$$

$$\int \frac{\partial f}{\partial x} = \int xy^4$$

$$\int \partial f = \int xy^4\,\partial x$$

$$f = \frac{x^2y^4}{2} + \phi(y) \quad \text{—— (4)}$$

$$\frac{\partial f}{\partial y} = \frac{4x^2y^3}{2} + \phi'(y) \longrightarrow \text{derive w.r.t. } y$$

$$N = 2(x^2 3y^2) + \phi'(y)$$

$$2x^2y^3 + 3y^5 - 20y^3 = 6x^2y^2 + \phi'(y)$$

$$\int [2x^2y^3 + 3y^5 - 20y^3]\,dy = \int [6x^2y^2 + \phi'(y)]\,dy$$

$$\frac{2x^2y^4}{4} + \frac{3y^6}{6} - \frac{20y^4}{4} = \frac{6x^2y^3}{3} + \phi(y)$$

$$\phi(y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 - 2x^2y^3$$

$$= \frac{1}{2}x^2y^4 - 2x^2y^3 + \frac{1}{2}y^6 - 5y^4$$

Sub $\phi(y)$ into (4)

$$f(x,y) = \frac{x^2y^4}{2} + \left[\frac{1}{2}x^2y^4 - 2x^2y^3 + \frac{1}{2}y^6 - 5y^4\right] = C$$

$$\Rightarrow x^2y^4 - 2x^2y^3 + \frac{1}{2}y^6 - 5y^4 = C$$