

## MAT 120

### INTRODUCTION to HIGHER ORDER DE

#### Inhomogeneous case

Non-homogeneous or Inhomogeneous DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x) \quad \text{--- (1)}$$

$g(x) \neq 0$ , otherwise eqn (1) will become Homogeneous DE

Solve the complementary function:  $y_c = 0$

$$\Rightarrow \underbrace{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y}_{y_c} = 0 \quad \rightarrow \text{homogeneous DE with constant coefficient}$$

Then solve the particular solution:  $y_p = g(x)$

Solution of  $y = y_c + y_p$

Trial Particular Solutions

$g(x)$	Form of $y_p$
1 (or any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
$\sin 4x$	$A \cos 4x + B \sin 4x$
$\cos 4x$	$A \cos 4x + B \sin 4x$
$e^{5x}$	$Ae^{5x}$
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
$x^2 e^{bx}$	$(Ax^2 + Bx + C)e^{bx}$
$e^{3x} \sin 4x$	$e^{3x} (A \cos 4x + B \sin 4x)$
$5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
$x e^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + D)e^{3x} \sin 4x$

This table shows the pattern of forming  $y_p$

Exercise: Solve the following inhomogeneous DE

①  $y'' + 3y' + 2y = 6$

Solution: For complementary functions:

$$y'' + 3y' + 2y = 0 \quad \left[ \begin{array}{l} \text{let } y = e^{mx} \\ y' = me^{mx} ; y'' = m^2 e^{mx} \end{array} \right]$$
$$m^2 e^{mx} + 3me^{mx} + 2e^{mx} = 0$$
$$m^2 + 3m + 2 = 0 \quad ; \div \text{ by } e^{mx} \text{ \& } e^{mx} \neq 0$$

$$m^2 + 2m + m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m_1 = -1, m_2 = -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

For particular

$$y_p = A \quad \because g(x) = 6$$

$$\because y_p' = 0 \text{ \& } y_p'' = 0$$

$$\text{Given } y'' + 3y' + 2y = 6$$

$$\text{Consider } y_p'' + 3y_p' + 2y_p = 6$$

$$0 + 3(0) + 2A = 6$$

$$A = 3 \quad \because y_p = A = 3$$

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{-2x} + 3$$

$$(2) y'' - 10y' + 25y = 30x + 3$$

Solution:

For complimentary function:

$$y'' - 10y' + 25 = 0$$

$$\text{let } y = e^{mx}$$

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0$$

$$(m-5)(m-5) = 0$$

$$m_1 = m_2 = 5$$

$$y_c = c_1 e^{5x} + c_2 x e^{5x}$$

For particular solution:

$$y_p = Ax + B \quad \because g(x) = 30x + 3$$

$$y_p' = A \quad ; \quad y_p'' = 0$$

$$\text{Given: } y'' - 10y' + 25y = 30x + 3$$

$$0 - 10(A) + 25(Ax + B) = 30x + 3$$

$$-10A + 25Ax + 25B = 30x + 3$$

Equating coefficients of like terms:

$$\boxed{x} \quad 25A = 30$$

$$A = \frac{6}{5}$$

Constant

$$-10A + 25B = 3$$

$$-10\left(\frac{6}{5}\right) + 25B = 3$$

$$-12 + 25B = 3$$

$$25B = 15 \Rightarrow B = \frac{3}{5}$$

$$\therefore y_p = Ax + B = \frac{6}{5}x + \frac{3}{5}$$

$$y = y_c + y_p = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}$$

$$\textcircled{3} \quad \frac{1}{4}y'' + y' + y = x^2 - 2x$$

For complimentary function:

$$\frac{1}{4}y'' + y' + y = 0 ; \text{ let } y = e^{mx}$$

$$\Rightarrow \frac{1}{4}m^2 + m + 1 = 0 \quad \leftarrow$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

$$m_1 = m_2 = -2$$

$$y_c = C_1 e^{-2x} + C_2 x e^{-2x}$$

for particular solution:

$$y_p = Ax^2 + Bx + C \quad \because g(x) = x^2 - 2x$$

$$y_p' = 2Ax + B ; y_p'' = 2A$$

$$\text{Given } \frac{1}{4}y'' + y' + y = x^2 - 2x$$

$$\frac{1}{4}(2A) + (2Ax + B) + (Ax^2 + Bx + C) = x^2 - 2x$$

$$\frac{1}{2}A + 2Ax + B + Ax^2 + Bx + C = x^2 - 2x$$

Equating coefficients of like terms:

$$\boxed{x^2}$$

$$A = 1$$

$$\boxed{x}$$

$$2A + B = -2$$

$$2(1) + B = -2$$

$$B = -4$$

$$\boxed{\text{constant}}$$

$$\frac{1}{2}A + B + C = 0$$

$$\frac{1}{2}(1) + (-4) + C = 0$$

$$C = 4 - \frac{1}{2} = \frac{7}{2}$$

$$y_p = Ax^2 + Bx + C$$

$$\therefore y_p = x^2 - 4x + \frac{7}{2}$$

$$y = y_c + y_p = C_1 e^{-2x} + C_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

$$(4) y'' + 3y = -48x^2 e^{3x}$$

For complementary function:

$$y'' + 3y = 0$$

$$\text{let } y = e^{mx}$$

$$m^2 + 3 = 0$$

$$m = \pm \sqrt{-3} = \pm i\sqrt{3} = 0 \pm i\sqrt{3}$$

$$y_c = e^{0 \cdot x} (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

$$y_c = C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x$$

For particular solution:

$$y_p = (Ax^2 + Bx + C)e^{3x} \quad \because g(x) = -48x^2 e^{3x}$$

$$y_p' = (Ax^2 + Bx + C)3e^{3x} + (2Ax + B)e^{3x}$$

$$= e^{3x}(3Ax^2 + 3Bx + 3C + 2Ax + B)$$

$$y_p'' = 3e^{3x}(3Ax^2 + 3Bx + 3C + 2Ax + B)$$

$$+ e^{3x}(6Ax + 3B + 2A)$$

$$= e^{3x}(9Ax^2 + 9Bx + 9C + 6Ax + 3B + 6Ax + 3B + 2A)$$

$$= e^{3x}(9Ax^2 + 9Bx + 12Ax + 2A + 6B + 9C)$$

$$\text{Given: } y'' + 3y = -48x^2 e^{3x}$$

$$e^{3x}(9Ax^2 + 12Ax + 9Bx + 2A + 6B + 9C) + 3e^{3x}(Ax^2 + Bx + C) = -48x^2 e^{3x}$$

$$9Ax^2 + 12Ax + 9Bx + 2A + 6B + 9C + 3Ax^2 + 3Bx + 3C = -48x^2$$

$$[ \div \text{ by } e^{3x} ]$$

$$12Ax^2 + 12Ax + 12Bx + 2A + 6B + 12C = -48x^2 + 0x + 0$$

Equating Coefficients of like terms:

$x^2$

$$12A = -48$$

$$A = -4$$

$x$

$$12A + 12B = 0$$

$$A + B = 0$$

$$(-4) + B = 0$$

$$B = 4$$

Constant

$$2A + 6B + 12C = 0$$

$$A + 3B + 6C = 0$$

$$(-4) + 3(4) + 6C = 0$$

$$6C = -8$$

$$C = -\frac{4}{3}$$

$$y_p = (Ax^2 + Bx + C)e^{3x}$$

$$\therefore y_p = \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$$

$$y = y_c + y_p = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x + \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$$

⑤  $y'' - 2y' + 5y = e^x \cos 2x$

For complementary function

$$y'' - 2y' + 5y = 0$$

let  $y = e^{mx}$

$$\rightarrow m^2 - 2m + 5 = 0$$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$m = 1 \pm 2i$$

$$y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$$

For particular solution:

$$y_p = e^x (A \cos 2x + B \sin 2x) \because g(x) = e^x \cos 2x$$

but  $y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$

In that case  $y_p$  is the duplicate of  $y_c$

$$\therefore y_p = x e^x (A \cos 2x + B \sin 2x)$$

⑥

$$y_p = x e^x (A \cos 2x + B \sin 2x)$$

$$y_p' = x e^x [-2A \sin 2x + 2B \cos 2x] + (x e^x + e^x) [A \cos 2x + B \sin 2x]$$

$$= x e^x [(-2A+B) \sin 2x + (2B+A) \cos 2x] + e^x [A \cos 2x + B \sin 2x]$$

$$y_p'' = x e^x [(-3A+4B) \cos 2x + (-3B-4A) \sin 2x] + e^x [(-4A+2B) \sin 2x + (4B+2A) \cos 2x]$$

$$\text{Given } y'' - 2y' + 5y = e^x \cos 2x$$

$$\Rightarrow x e^x (-3A+4B) \cos 2x + x e^x (-3B-4A) \sin 2x + e^x (-4A+2B) \sin 2x + e^x (4B+2A) \cos 2x - 2x e^x (-2A+B) \sin 2x - 2x e^x (2B+A) \cos 2x - 2e^x A \cos 2x - 2e^x B \sin 2x + 5x e^x A \cos 2x + 5x e^x B \sin 2x = e^x \cos 2x$$

$$\Rightarrow x (-3A+4B) \cos 2x + x (-3A-4B) \sin 2x + (-4A+2B) \sin 2x + (4B+2A) \cos 2x - 2x (-2A+B) \sin 2x - 2x (2B+A) \cos 2x - 2A \cos 2x - 2B \sin 2x + 5x A \cos 2x + 5x B \sin 2x = \cos 2x \quad \left[ \div \text{ by } e^x \right]$$

Equating coefficients of like terms:

$$\boxed{x \cos 2x}$$

$$-3A+4B-2(2B+A)+5A=0$$

$$-3A+4B-4B-2A+5A=0$$

$$0=0$$

$$\boxed{x \sin 2x}$$

$$-3A-4B-2(-2A+B)+5B=0$$

$$-3A-4B+4A-2B+5B=0$$

$$A-B=0$$

$$\boxed{\cos 2x}$$

$$4B+2A-2A=1$$

$$4B=1$$

$$\boxed{B = \frac{1}{4}}$$

$$\boxed{\sin 2x}$$

$$-4A+2B-2B=0$$

$$-4A=0$$

$$\boxed{A=0}$$

$$y_p = x e^x (A \cos 2x + B \sin 2x)$$

$$= x e^x (0 + \frac{1}{4} \sin 2x)$$

$$\therefore y_p = \frac{1}{4} x e^x \sin 2x$$

$$y = y_c + y_p$$

$$= e^x (c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{4} x e^x \sin 2x$$