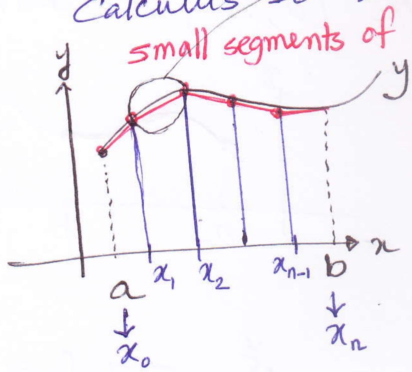
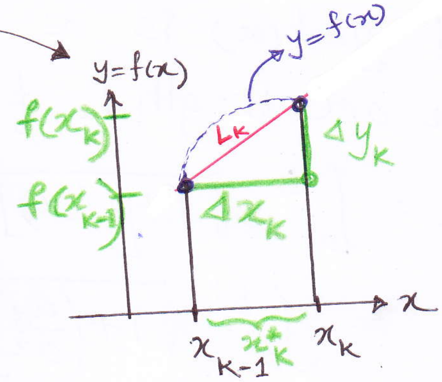
MAT 120 Week 4

Ch 6.4 LENGTH of PLANE CURVE

Anton's Calculus 10^{th} Ed.

small segments of straight lines





$x_k^* \in [x_{k-1}, x_k]$

$L_k = \sqrt{\Delta x_k^2 + (\Delta y_k)^2}$ by Pythagorean Theorem

single segment of the curve $\rightarrow$ $= \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$

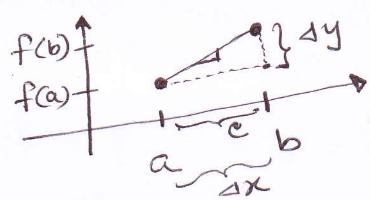
$$\therefore \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{\left[(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2\right]} \quad \text{(1)}$$

n many segment of the curve

The arc length is the measure of the distance along the curved line making up the arc. It is longer than the straight line distance between its end points.

Recall Mean Value Theorem

There exist c between a & b such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



$c \in [a, b]$

There exist a number x_k^* between x_{k-1} and x_k such that $\dfrac{f(x_k) - f(x_{k-1})}{\underbrace{x_k - x_{k-1}}_{\Delta x_k}} = f'(x_k^*)$

$$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k \quad \text{(2)}$$

Substitute (2) into (1)

$$L = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_k^*)]^2 (\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 \left(1 + [f'(x_k^*)]^2\right)}$$

$$= \sum_{k=1}^{n} \Delta x_k \sqrt{1 + [f'(x_k^*)]^2}$$

Riemann Sum to Riemann Integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

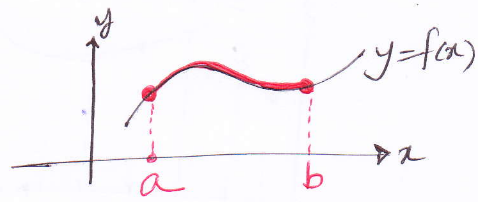
A smooth function is a function that has continuous derivatives up to some desired order over some domain.

Definition

If $y=f(x)$ is a smooth curve on the interval $[a,b]$, the arc length L of this curve $[a,b]$ is defined as

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

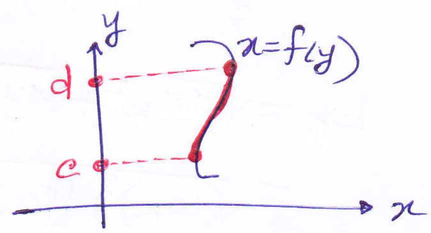
$$=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



$x=f(y)$
f is continuous on $[c,d]$.

$$L=\int_c^d \sqrt{1+[f'(y)]^2}\,dy$$

$$=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$



Exercise:

Find the exact arc length of the curve over the stated interval:

3) $y=3x^{3/2}-1$ from $x=0$ to $x=1$

$$y=f(x)=3x^{3/2}-1$$

$$f'(x)=3\cdot\frac{3}{2}x^{\frac{3}{2}-1}-0=\frac{9}{2}x^{1/2}$$

$$L=\int_0^1 \sqrt{1+\left(\frac{9}{2}x^{1/2}\right)^2}\,dx=\int_0^1 \sqrt{1+\frac{81}{4}x}\,dx$$

$$=\frac{4}{81}\int_1^{85/4}\sqrt{z}\,dz$$

$$=\frac{4}{81}\left[\frac{z^{3/2}}{3/2}\right]_1^{85/4}$$

$$=\frac{4}{81}\cdot\frac{2}{3}\left[\left(\frac{85}{4}\right)^{3/2}-(1)^{3/2}\right]$$

$$=3.19$$

Let

$1+\frac{81}{4}x=z$

$\frac{81}{4}dx=dz$

$dx=\frac{4}{81}dz$

limits

$x=0\rightarrow z=1$

$x=1\rightarrow z=\frac{85}{4}$

$\boxed{5}$ $f(x) = y = x^{2/3}$; from $x = 1$ to $x = 8$

$f'(x) = \frac{2}{3}x^{-1/3}$

$[f'(x)]^2 = \frac{4}{9}x^{-2/3}$

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

$L = \int_1^8 \sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx$

$= \int_1^8 \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\,dx$

$= \frac{1}{3}\int_1^8 \frac{\sqrt{9x^{2/3}+4}}{x^{1/3}}\,dx$

$= \frac{1}{3}\int_{13}^{40} \sqrt{z}\,\frac{1}{6}\,dz$

⋮

continue.

Let

$9x^{2/3} + 4 = z$

$9 \cdot \frac{2}{3}x^{-1/3}\,dx = dz$

$\frac{6}{x^{1/3}}\,dx = dz$

$\frac{1}{x^{1/3}}\,dx = \frac{1}{6}\,dz$

limits:

$x = 1 \rightarrow z = 13$

$x = 8 \rightarrow z = 40$

(7) $24xy = y^4 + 48$; $y = 2$ to $y = 4$

$x = \frac{y^4 + 48}{24y} \rightarrow f(y)$

$\frac{dx}{dy} = \frac{24y\,(4y^3) - (y^4 + 48)\,24}{(24y)^2}$

$= \frac{24\,[4y^4 - y^4 - 48]}{24^2y^2} = \frac{3y^4 - 48}{24y^2} = \frac{y^4 - 16}{8y^2}$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\left(\frac{y^4-16}{8y^2}\right)^2}$$

$$= \sqrt{1+\frac{(y^4-16)^2}{64y^4}}$$

$$= \sqrt{\frac{64y^4+y^8-32y^4+256}{64y^4}}$$

$$= \sqrt{\frac{32y^4+y^8+256}{64y^4}}$$

$$= \frac{\sqrt{(y^4)^2+2y^4.16+16^2}}{8y^2}$$

$$= \frac{\sqrt{(y^4+16)^2}}{8y^2} = \frac{y^4+16}{8y^2}$$

$$L = \int_2^4 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_2^4 \frac{y^4+16}{8y^2}\, dy$$

$$= \int_2^4 \left(\frac{y^2}{8}+\frac{2}{y^2}\right) dy$$

$$= \left[\frac{y^3}{24}+\frac{2y^{-2+1}}{-2+1}\right]_2^4$$

$$= \frac{64-8}{24} - 2\left[(4)^{-1}-(2)^{-1}\right]$$

$$= \frac{56}{24} - 2\left[\frac{1}{4}-\frac{1}{2}\right] = \frac{7}{3} - 2\left[-\frac{1}{4}\right]$$

$$= \frac{7}{3}+\frac{1}{2} = \frac{14+3}{6} = \frac{17}{6}.$$

Parametric Eqn

Ex

$x = x(t), \quad y = y(t)$

Consider

① $x = 2t$, $y = t^2$ } • pair of these together called parametric eqns

• t = parameter

② $x = \sin\theta + 2$, $y = \cos\theta - 3$ } → parametric eqn

θ = parameter.

Say, if $x = 2$

then $2 = \sin\theta + 2$

$0 = \sin\theta$

$\theta = \sin^{-1}(0) = 0, \pi, 2\pi, \ldots$

if $\theta = 0$ then $y = \cos(0) - 3 = -2$

and so on.

Evaluate Cartesian Form:

eqn with x, y and parameter eliminated

Case ① $x = 2t$ — (i), $y = t^2$ — (ii)

from (i) $t = \frac{x}{2}$

Substitute (i) into (ii) we get:

$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$ → This is the cartesian form

110 Coordinate Geometry

Case ② $x = \sin\theta + 2$ — (i)

$\sin\theta = x - 2$ — (i)

$\sin^2\theta = (x-2)^2$

$y = \cos\theta - 3$ — (ii)

$\cos\theta = y + 3$ — (ii)

$\cos^2\theta = (y+3)^2$

(i) + (ii) ⇒ $\sin^2\theta + \cos^2\theta = (x-2)^2 + (y+3)^2$

$1 = (x-2)^2 + (y+3)^2$ → θ is eliminated

Differentiation:

$x = x(t)$, $y = y(t)$

Consider $x = 2t$, $y = t^2$

$\frac{dx}{dt} = 2$; $\frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 2t \cdot \frac{1}{2} = t$$

110 differentiation

For Parametric curves: [redacted]

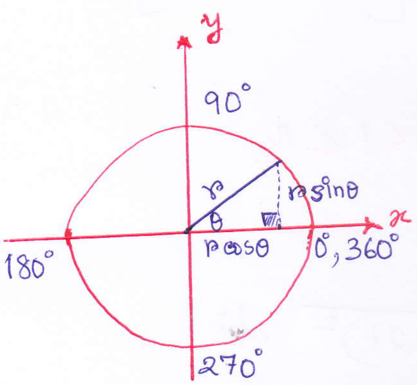
If the number of segments of the curve represented by the parametric eqn say $x = x(t)$, $y = y(t)$, $(a \leq t \leq b)$ is traced more than once as t increases from a to b, and if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous functions for $t \in [a, b]$, then the arc length L of the curve is given by $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ex $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$, $0 \leq \theta \leq 2\pi$

$$\frac{dx}{d\theta} = -\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta$$

$$L = \int_0^{2\pi} \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_0^{2\pi} d\theta = [\theta]_0^{2\pi} = 2\pi$$

arc length of full circle.



Find the arc length of the parametric curve

27) $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \le t \le 1$

$\frac{dx}{dt} = \frac{1}{3} \cdot 3t^2$, $\frac{dy}{dt} = \frac{1}{2} \cdot 2t$

$= t^2$ $\quad = t$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{t^4 + t^2}\, dt$$

$$= \int_0^1 \sqrt{t^2(t^2+1)}\, dt$$

$$= \int_0^1 t\sqrt{t^2+1}\, dt$$

$$= \int_1^2 \sqrt{z}\, \tfrac{1}{2}\, dz$$

$$= \frac{1}{2}\left[\frac{z^{3/2}}{3/2}\right]_1^2 = \frac{1}{2} \cdot \frac{2}{3}\left[(2)^{3/2} - (1)^{3/2}\right] = \frac{1}{3}(2\sqrt{2} - 1)$$

let $t^2 + 1 = z$

$2t\,dt = dz$

$t\,dt = \frac{1}{2}dz$

$t = 0 \rightarrow z = 1$

$t = 1 \rightarrow z = 2$

29) $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le \pi/2$

$\frac{dx}{dt} = -2\sin 2t$; $\frac{dy}{dt} = 2\cos 2t$

$$L = \int_0^{\pi/2} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{4\sin^2 2t + 4\cos^2 2t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{4(\sin^2 2t + \cos^2 2t)}\, dt$$

$$= 2\int_0^{\pi/2} \sqrt{1}\, dt = 2\left[t\right]_0^{\pi/2} = \pi.$$