Integrating Factor & 1st Order Differential Eqn

Consider a differential eqn:

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

coefficient of $\frac{dy}{dx}$ should be 1

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}\,y = \frac{g(x)}{a_1(x)} \qquad \div \text{ by } a_1(x)$$

Rename $\frac{a_0(x)}{a_1(x)} = P(x)$; $\frac{g(x)}{a_1(x)} = Q(x)$

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

→ Standard form of Differential Eqn.

Calculate integrating factor (IF) denoted by $e^{\int P(x)dx}$

$IF = e^{\int P(x)dx}$: An integrating factor (IF) is a function by which an ordinary differential eqn can be multiplied in order to make it integrable.

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$e^{\int P(x)dx}\left[\frac{dy}{dx} + P(x)\,y\right] = e^{\int P(x)dx}\,Q(x)$$

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)\,y = e^{\int P(x)dx}\,Q(x)$$

$$\int\Big[\underbrace{e^{\int P(x)dx}}_{u}\,\underbrace{\frac{dy}{dx}}_{v'} + \underbrace{e^{\int P(x)dx}P(x)}_{u'}\,\underbrace{y}_{v}\Big]dx = \int e^{\int P(x)dx}\,Q(x)\,dx$$

$v = y$

$v' = \frac{dy}{dx}$

$u = e^{\int P(x)dx}$

$u' = e^{\int P(x)dx}P(x)$

$$\frac{d}{dx}(uv) = uv' + u'v$$

$$\Rightarrow \int\left[\frac{d}{dx}\left(e^{\int P(x)dx}\,y\right)\right]dx = \int e^{\int P(x)dx}\,Q(x)\,dx$$

$$\Rightarrow y\,e^{\int P(x)dx} = \int e^{\int P(x)dx}\,Q(x)\,dx$$

Now we solve for "y"

Examples:

$\boxed{1}$ $y' + 3x^2 y = x^2$

$$\frac{dy}{dx} + 3x^2 y = x^2 \; ; \quad (1) \qquad P(x) = 3x^2 \qquad Q(x) = x^2$$

$$e^{\int P(x)dx} = e^{\int 3x^2 dx} = e^{3\frac{x^3}{3}} = e^{x^3}$$

multiply eqn (1) by $e^{\int P(x)dx}$

$$e^{x^3}\left[\frac{dy}{dx} + 3x^2 y\right] = x^2 e^{x^3}$$

$$ye^{x^3} = \int x^2 \cdot e^{x^3}\, dx$$

Let $x^3 = z$

$3x^2 dx = dz$

$x^2 dx = \frac{1}{3} dz$

$$= \frac{1}{3}\int e^z\, dz$$

$$= \frac{1}{3} e^z + C$$

$$ye^{x^3} = \frac{1}{3} e^{x^3} + C$$

$$y = \frac{1}{3} + Ce^{-x^3}.$$

$\boxed{2}$ $y' = 2y + x^2 + 5$

$$\frac{dy}{dx} - 2y = x^2 + 5 \;\; -(1) \; ; \quad P(x) = -2 \; ; \; Q(x) = x^2 + 5$$

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$$

multiply eqn (1) by $e^{\int P(x)dx}$

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx}\, Q(x)dx$$

$$ye^{-2x} = \int e^{-2x}(x^2+5)\,dx$$

$$ye^{-2x} = \int \underbrace{x^2}_{u}\,\underbrace{e^{-2x}}_{v}\,dx + 5\int e^{-2x}\,dx$$

$$= x^2\left[\frac{e^{-2x}}{-2}\right] - \int 2x\left[\frac{e^{-2x}}{-2}\right]dx + 5\left[\frac{e^{-2x}}{-2}\right]$$

$\int(uv)dx=u\int v\,dx-\int\{u'(\int v dx)\}\,dx$

$$= \frac{-x^2e^{-2x}}{2} + \int \underbrace{x}_{u}\,\underbrace{e^{-2x}}_{v}\,dx - \frac{5}{2}e^{-2x}$$

$$= \frac{-x^2e^{-2x}}{2} + x\left[\frac{e^{-2x}}{-2}\right] - \int 1\left[\frac{e^{-2x}}{-2}\right]dx - \frac{5}{2}e^{-2x}$$

$$= \frac{-x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx - \frac{5}{2}e^{-2x}$$

$$= \frac{-x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right] - \frac{5}{2}e^{-2x} + c$$

$$ye^{-2x} = \frac{-x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} - \frac{5e^{-2x}}{2} + c$$

$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} - \frac{5}{2} + \frac{c}{e^{-2x}}$$

$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{11}{4} + ce^{2x}$$

$\boxed{3}$ $(1+x)\dfrac{dy}{dx} - xy = x + x^2$

$$\frac{dy}{dx} - \frac{x}{1+x}y = \frac{x+x^2}{1+x}$$

$$\frac{dy}{dx} - \frac{x}{1+x}\cdot y = \frac{x(1+x)}{(1+x)} = x \quad \text{—— } (1)$$

$$P(x) = -\frac{x}{1+x}, \quad Q(x) = x$$

$$e^{\int P(x)dx} = e^{-\int \frac{x}{1+x}dx} = e^{-\int \frac{1+x-1}{1+x}dx}$$

$$= e^{-\int\left(1-\frac{1}{1+x}\right)dx}$$

$$= e^{-x+\ln(1+x)}$$

$$= e^{-x}e^{\ln(1+x)}$$

$$= e^{-x}(1+x) = (1+x)e^{-x}$$

multiply eqn (1) by I.F.

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx}\, Q(x)\, dx$$

$$y(1+x)e^{-x} = \int (1+x)e^{-x}(x)\, dx$$

∫(uv)dx=u∫v dx-∫{u' (∫vdx)} dx

$$= \int xe^{-x}dx + \int \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v}dx$$

$$= \int xe^{-x}dx + x^2\left[\frac{e^{-x}}{-1}\right] - \int\left(2x\left[\frac{e^{-x}}{-1}\right]\right)dx$$

$$= \int xe^{-x}dx - x^2e^{-x} + \int 2xe^{-x}dx$$

$$= \int 3xe^{-x}dx - x^2e^{-x}$$

$$= 3\int \underbrace{x}_{u}\underbrace{e^{-x}}_{v}\,dx - x^2e^{-x}$$

$$= 3\left[x\left[\frac{e^{-x}}{-1}\right] - \int 1\left[\frac{e^{-x}}{-1}\right]dx\right] - x^2e^{-x}$$

$$= 3\left[-xe^{-x} + \int e^{-x}dx\right] - x^2e^{-x}$$

$$y(1+x)e^{-x} = 3\left[-xe^{-x} + \left(\frac{e^{-x}}{-1}\right)\right] - x^2e^{-x} + c$$

$$y(1+x) = 3\left[-x - 1\right] - x^2 + ce^{-x}$$

$$= -3x - 3 - x^2 + ce^{-x}$$

$$y = -\frac{1}{1+x}\left(x^2 + 3x + 3 - ce^{-x}\right)$$

$\boxed{4}$ $y\,dx - 4(x + y^6)\,dy = 0$ $\quad\because \frac{dy}{dx} + P(x)y = Q(x)$

Similarly

$$\frac{dx}{dy} + P(y)x = Q(y)$$

$$\frac{dx}{dy} - 4\frac{(x+y^6)}{y} = 0 \quad (\div\, y\,dy)$$

$$\frac{dx}{dy} - \frac{4}{y}\cdot x - 4y^5 = 0$$

$$\frac{dx}{dy} - \frac{4}{y}\cdot x = 4y^5 \quad \text{— (1)}$$

$P(y) = -\frac{4}{y}$, $Q(y) = 4y^5$

$$\text{I.F.} = e^{\int P(y)dy} = e^{\int -\frac{4}{y}dy} = e^{-4\ln y} = e^{-\ln y^4} = e^{\ln\left(\frac{1}{y^4}\right)} = \frac{1}{y^4}$$

multiply eqn (1) by I.F.

$$xe^{\int P(y)dy} = \int e^{\int P(y)dy}\, Q(y)\,dy$$

$$x\cdot\frac{1}{y^4} = \int \frac{1}{y^4}\cdot 4y^5dy = \int 4y\,dy$$

$$x \cdot \frac{1}{y^4} = 4\frac{y^2}{2} + c$$

$$= 2y^2 + c$$

$$\frac{x}{y^4} = 2y^2 + c$$

$$x = 2y^2 \cdot y^4 + cy^4$$

$$x = 2y^6 + cy^4$$

$\boxed{5}$ $\cos x \frac{dy}{dx} + (\sin x) y = 1$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x}$$

$$\frac{dy}{dx} + \tan x \cdot y = \sec x \quad \text{—} (1)$$

$P(x) = \tan x$; $Q(x) = \sec x$

$$e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$$

multiply eqn (1) by I.F.

$$y e^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x)dx$$

$$y \sec x = \int \sec x \cdot \sec x \, dx$$

$$= \int \sec^2 x \, dx$$

$$y \sec x = \tan x + c$$

$$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}$$

$$y = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} + C \cdot \frac{1}{\sec x}$$

$$= \frac{\sin x}{\cos x} \times \frac{\cos x}{1} + c \cdot \cos x$$

$$y = \sin x + c \cos x$$

$\boxed{6}$ Solve the initial value problem

$$y' + \tan x \cdot y = \cos^2 x \quad ; \quad \underbrace{y(0) = -1}_{x=0,\ y=-1}$$

$$\frac{dy}{dx} + \tan x \cdot y = \cos^2 x \quad — (1) \qquad P(x) = \tan x, \ Q(x) = \cos^2 x$$

$$IF = e^{\int P(x)dx} = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x$$

multiply eqn (1) by I.F.

$$y e^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x) \, dx$$

$$y \sec x = \int \sec x \cos^2 x \, dx$$

$$= \int \frac{1}{\cos x} \cdot \cos^2 x \, dx$$

$$= \int \cos x \, dx$$

$$y \sec x = \sin x + c \quad — (2)$$

Substitute $x=0$, $y=-1$ into (2)

$$(-1)\sec(0) = \sin(0) + C$$

$$-1 = 0 + C$$

$$C = -1$$

substitute $C=-1$ into (2)

$$y\sec x = \sin x - 1$$

$$y = \frac{\sin x}{\sec x} - \frac{1}{\sec x}$$

$$= \frac{\sin x}{\frac{1}{\cos x}} - \cos x$$

$$= \sin x\cos x - \cos x$$

$$y = \cos x(\sin x - 1)$$ ■