# Multiple Integral "Jacobians" 

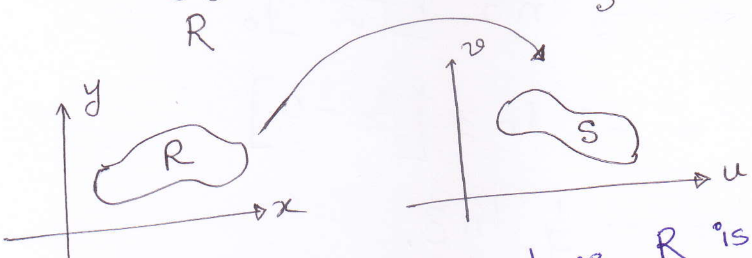
**Week 6**

If $T$ is a transformation from the $uv$-plane to the $xy$-plane defined by the equations $x = x(u,v)$, $y = (u,v)$, then the Jacobian of $T$ is denoted by $J(u,v)$ or by $\dfrac{\partial(x,y)}{\partial(u,v)}$ and is denoted by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}$$

And hence $\displaystyle\iint_R f(x,y)\, dA_{xy} = \iint_S f(u,v)\, J \, dA_{uv}$




**Examples:** ① Evaluate $\displaystyle\iint_R \frac{x-y}{x+y}\, dA$ where $R$ is the region enclosed by the lines $x-y=0$, $x-y=1$, $x+y=1$, $x+y=3$ using transformation.

Let $u = x+y$, $v = x-y$

$\therefore u = 1, 3$ $\quad$ $v = 1, 0$

$\therefore x+y=1, x+y=3$ $\quad$ $\therefore x-y=1, x-y=0$

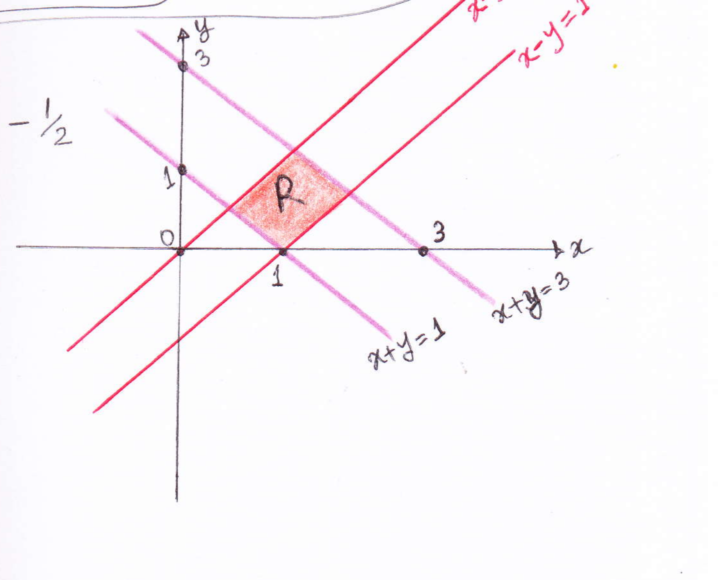
$$\begin{array}{l} u = x+y \\ v = x-y \\ \hline u+v = 2x \quad \text{Add} \end{array}$$

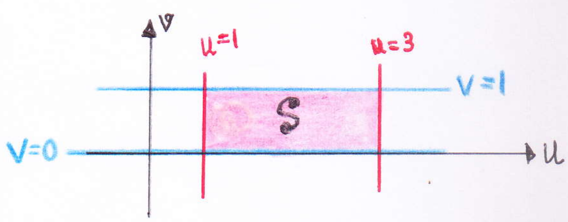
$\therefore x = \frac{1}{2}(u+v)$

$$\begin{array}{l} u = x+y \\ \underset{(-)}{v} = \underset{(-)}{x} - \underset{(+)}{y} \\ \hline u-v = 2y \end{array} \quad \text{subtract}$$

$y = \frac{1}{2}(u-v)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$



$$\iint_R \frac{x-y}{x+y}\, dA_{xy} = \iint_S \frac{v}{u} |J| \, dA_{uv}$$

$$= \int_{v=0}^{1} \int_{u=1}^{3} \frac{v}{u} \left|-\frac{1}{2}\right| du\, dv$$

$$= +\frac{1}{2} \int_{v=0}^{1} v \int_{u=1}^{3} \frac{1}{u}\, du\, dv$$

$$= +\frac{1}{2} \int_{v=0}^{1} v \left[ \ln u \right]_1^3 dv$$

$$= +\frac{1}{2} \int_0^1 v \left[ \ln 3 - \ln 1 \right] dv$$

$$= +\frac{1}{2} \ln 3 \left[ \frac{v^2}{2} \right]_0^1$$

$$= +\frac{1}{2} \ln 3 \left[ \frac{1}{2} - 0 \right]$$

$$= +\frac{1}{4} \ln 3.$$

Elample : 2 Evaluate $\int_{y=0}^{4} \int_{x=y/2}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx dy$ by applying transformation where $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating over an appropriate region in $uv$-plane.

$x = \frac{y}{2}$

$x = \frac{y}{2}+1$

$y = 0$

$y = 4$

$$u+v = \frac{2x-y}{2} + \frac{y}{2} = \frac{2x-y+y}{2} = \frac{2x}{2}$$

$$u+v = x$$

$$\therefore \boxed{x = u+v}$$

$$v = \frac{y}{2} \text{ (given)}$$

$$\boxed{y = 2v}$$

$y=0 \Rightarrow 2v=0 \Rightarrow \boxed{v=0}$

$y=4 \Rightarrow 2v=4 \Rightarrow \boxed{v=2}$

$x = \frac{y}{2} \Rightarrow x = \frac{2v}{2} \Rightarrow x = v$

$\therefore u+v = v \quad \because x = u+v$

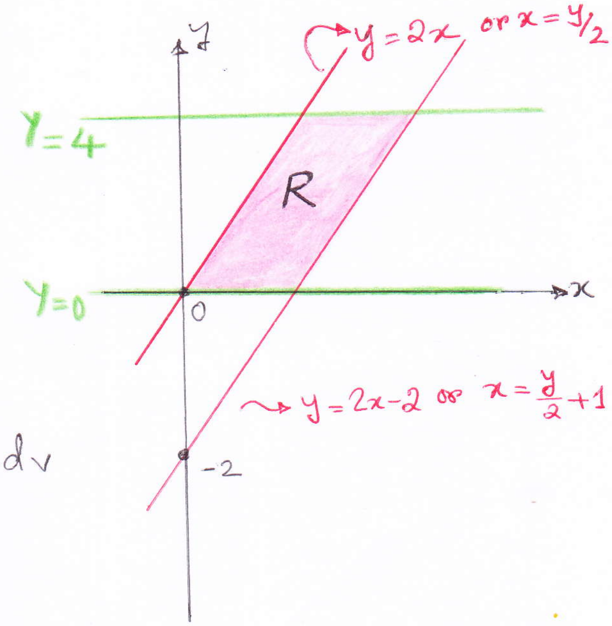
$\Rightarrow u = 0$

$x = \frac{y}{2}+1 \Rightarrow x = \frac{2v}{2}+1 \Rightarrow x = v+1$

$\Rightarrow u+v = v+1$

$\Rightarrow u = 1$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$




$$\int_{y=0}^{4} \int_{x=y/2}^{y/2+1} \frac{2x-y}{2}\, dx dy$$

$$= \int_{v=0}^{2} \int_{u=0}^{1} \frac{2(u+v)-(2v)}{2} |J|\, du dv$$

$$= \int_{v=0}^{2} \int_{u=0}^{1} u|2|\, du dv$$

$$= 2\int_{0}^{2} \left[\frac{u^2}{2}\right]_0^1 dv$$

$$= \int_{0}^{2} [1^2 - 0^2]\, dv$$

$$= [v]_0^2 = 2$$