Ch 7.5 Integrating Rational Function by Partial Fraction

Anton's Calculus 10th Ed.

Decompositions:

$$\frac{3x-1}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}$$

$$\frac{5}{x(x^2-4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$\frac{2x-3}{x^3-x^2} = \frac{2x-3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

$$\frac{1-x^2}{x^3(x^2+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+2}$$

$$\frac{3x}{(x-1)(x^2+6)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+6}$$

$$\frac{4x^3-x}{(x^2+5)^2} = \frac{Ax+B}{x^2+5} + \frac{Dx+E}{(x^2+5)^2}$$

$$\frac{1-3x^4}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

Improper ~~Integrals~~ fraction

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$$

$$
\begin{array}{r l}
 & \quad 3x^2 \qquad\quad +1 \\
x^2+x-2 \,\big) & 3x^4+3x^3-5x^2+x-1 \\
 & \underline{\overset{(-)}{3x^4}\ \overset{(-)}{+3x^3}\ \overset{(+)}{-6x^2}} \\
 & \qquad\qquad\quad x^2+x-1 \\
 & \qquad\qquad\quad \underline{\overset{(-)}{x^2}\ \overset{(-)}{+x}\ \overset{(+)}{-2}} \\
 & \qquad\qquad\qquad\quad 1
\end{array}
$$

$x^2 \times \underline{\overset{3x^2}{?}} = 3x^4$

$x^2 \times \underline{\overset{1}{?}} = x^2$

$$x^2+x-2 = x^2-2x+x-2 = (x-2)(x+1)$$

$$\int \left[3x^2+1+\frac{1}{x^2+x-2}\right] dx$$

$$= \frac{3x^3}{3}+x+\int\frac{1}{x^2+x-2}\,d$$

$$= x^3+x+\int\left(\frac{A}{x+1}+\frac{B}{x-2}\right)dx$$

$$= x^3+x+\frac{1}{3}\int\left(\frac{1}{x+1}-\frac{1}{x-2}\right)dx$$

$$= x^3+x+\frac{1}{3}\Big[\ln|x+1|-\ln|x-2|\Big]+C$$

$$= x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$$

$$\frac{1}{x^2+x-2} = \frac{1}{(x+1)(x-2)}$$

$$\frac{1}{(x+1)(x-2)} = \frac{A}{(x+1)}+\frac{B}{x-2}$$

$$1 = A(x-2)+B(x+1)$$

$$1 = Ax-2A+Bx+B$$

$\boxed{x}$ $\qquad$ $\boxed{\text{Constant}}$

$0 = A+B \qquad 1 = -2A+B$

$A = \frac{1}{3}, \quad B = -\frac{1}{3}$

Integration by substitution

$$\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$$

let

$\sqrt{x} = z$

$\frac{1}{2\sqrt{x}}\,dx = dz$

$\frac{1}{\sqrt{x}}\,dx = 2dz$

$= 2\int \sec^2 z\,dz$

$= 2\tan z + c$

$= 2\tan\sqrt{x} + c$

$$\int x\sqrt{1-x}\,dx$$

let

$1 - x = z$

$-dx = dz$

$dx = -dz$

$x = 1 - z$

$= \int (1-z)\sqrt{z}(-dz)$

$= -\int (\sqrt{z} - z\sqrt{z})\,dz$

$= -\int (z^{1/2} - z^{3/2})\,dz$

$= -\frac{z^{3/2}}{\frac{3}{2}} + \frac{z^{5/2}}{\frac{5}{2}} + C$

$= -\frac{2}{3}z^{3/2} + \frac{2}{5}z^{5/2} + C$

$= -\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C$

$$\int x^2 e^{-2x^3}\,dx$$

$2x^3 = z$

$6x^2\,dx = dz$

$x^2\,dx = \frac{1}{6}\,dz$

$= \int \frac{1}{6} e^{-z}\,dz$

$= \frac{1}{6}\left[\frac{e^{-z}}{-1}\right] + c$

$= -\frac{1}{6} e^{-2x^3} + C$

Ch 7.2 Reduction Formula

$$\int \cos^5 x\,dx = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\int \cos^3 x\,dx$$

$$= \frac{1}{5}\cos^4 \sin x + \frac{4}{5}\left[\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\,dx\right]$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + c$$

$\int \sin^3 2x \cos^2 2x$ [redacted] dx

$$= \int \sin^2 2x \cos^2 2x \sin 2x\,dx$$

$$= \int (1-\cos^2 2x)\cos^2 2x \sin 2x\,dx$$

$$= \int (\cos^2 2x - \cos^4 2x)\sin 2x\,dx$$

$\longrightarrow$ Let

$\cos 2x = u$

$-\sin 2x(2)dx = du$

$-2\sin 2x\,dx = du$

$\sin 2x\,dx = -\frac{1}{2}du$

$$= -\int (u^2 - u^4)\frac{1}{2}du$$

$$= -\frac{1}{2}\int (u^2 - u^4)du$$

$$= -\frac{1}{2}\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + c$$

$$= -\frac{1}{2}\left[\frac{\cos^3 2x}{3} - \frac{\cos^5 2x}{5}\right] + c$$

$$\int \tan^5 x \sec^4 x \, dx$$

$$= \int \tan^5 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^5 x \left(1+\tan^2 x\right) \sec^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^5 (1+u^2) \, du$$

$$= \int (u^5 + u^7) \, du$$

$$= \frac{u^6}{6} + \frac{u^8}{8} + c$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$$

Ch 7.2 $\boxed{\text{Integration by Parts}}$

$\int \sec^2 x \, dx = \tan x$

$\int \tan x = -\ln|\cos x|$

$= \ln|\sec x|$

$$\int x \tan^2 x \, dx$$

$$= x \int \tan^2 x \, dx - \int \left(\frac{d}{dx} x \int \tan^2 x \, dx\right) dx$$

$$= x \int (\sec^2 x - 1) \, dx - \int \left(1 \int (\sec^2 x - 1) \, dx\right) dx$$

$$= x(\tan x - x) - \int (\tan x - x) \, dx$$

$$= x \tan x - x^2 - \left[\ln|\sec x| - \frac{x^2}{2}\right] + c$$

$$= x \tan x - x^2 - \ln|\sec x| + \frac{x^2}{2} + c$$

$$= x \tan x - x^2 - \ln|\sec x| + c$$

$$= x \tan x - \frac{x^2}{2}$$

$$\int uv = u \int v - \int \left(u' \int v\right)$$

Integration by Parts

$$\int \cos(\ln x)\, dx \qquad\qquad \ln x = z$$

$$\frac{1}{x}\, dx = dz$$

$$= \int e^z \cos z\, dz \qquad\qquad dx = x\, dz$$

$$= e^z \int \cos z\, dz - \int \left(e^z \int \cos z\right) dz \qquad = e^z\, dz \qquad \because \ln x = z,\ x = e^z$$

$$= e^z \sin z - \int e^z \sin z\, dz \qquad\qquad \int uv = u\int v - \int \left(u' \int v\right)$$

$$= e^z \sin z - \left[e^z \int \sin z\, dz - \int \frac{d}{dz} e^z \int \sin z\, dz \right]$$

$$= e^z \sin z - \left[-e^z \cos z - \int e^z (-\cos z)\, dz \right]$$

$$\int e^z \cos z\, dz = e^z \sin z + e^z \cos z - \int e^z \cos z\, dz$$

$$2\int e^z \cos z\, dz = e^z(\sin z + \cos z)$$

$$\int e^z \cos z\, dz = \frac{1}{2} e^z (\sin z + \cos z)$$

$$\int \cos(\ln x)\, dx = \frac{1}{2} e^{\ln x} \left[\sin(\ln x) + \cos(\ln x)\right]$$

$$= \frac{1}{2} x \left[\ ,,\ \right]$$

Gamma Function

See the Exercise Sheet in buX

$$\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma(n)$$

or $$\int_0^\infty e^{-t} t^{x-1}\, dt = \Gamma(x)$$

$$\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx$$

$x^2 = z$

$2x\,dx = dz$

$dx = \frac{1}{2x} dz$

$dx = \frac{1}{2\sqrt{z}} dz$

limits are unchanged

$$= \int_0^\infty z^{1/4} e^{-z} \frac{1}{2\sqrt{z}}\, dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{1/4 - 1/2}\, dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{-1/4}\, dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{3}{4}-1}\, dz = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

$z^{-\frac{1}{4}} = z^{n-1}$

$-\frac{1}{4} = n-1$

$n = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore n-1 = \frac{3}{4} - 1$

$$\int_0^1 \frac{1}{\sqrt{x \ln(\frac{1}{x})}}\, dx$$

$$= \int_\infty^0 \frac{-e^{-z} dz}{\sqrt{e^{-z} z}}$$

$$= -\int_\infty^0 \frac{e^{-z} dz}{e^{-z/2} z^{1/2}}$$

$$= -\int_\infty^0 e^{-z+z/2} z^{-1/2} dz$$

$$= \int_0^\infty e^{-z/2} z^{-1/2} dz$$

$$= \int_0^\infty e^{-y} (2y)^{-1/2} \, \cancel{dz}\, 2dy$$

$$= 2 \cdot 2^{-1/2} \int_0^\infty e^{-y} y^{-1/2} dy$$

$$= 2^{1/2} \int_0^\infty e^{-y} y^{1/2 - 1} dy$$

$$= \sqrt{2}\, \Gamma(\tfrac{1}{2}).$$

$\ln(\frac{1}{x}) = z$

$\ln 1 - \ln x = z$

$0 - \ln x = z$

$\ln x = -z$

$x = e^{-z}$

$dx = -e^{-z} dz$

$x = 0 \longrightarrow z = \infty$

$x = 1 \longrightarrow z = 0$

$\frac{z}{2} = y$

$z = 2y$

$dz = 2dy$

Limits

$z = 0 \rightarrow y = 0$

$z = \infty \rightarrow y = \infty$

$y^{-\frac{1}{2}} = y^{n-1}$

$n - 1 = -\frac{1}{2}$

$n = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore n - 1 = \frac{1}{2} - 1$