Olympiad questions

Pranjal

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This pdf contains a list of 50 questions, most of which have difficulty similar to the hardest question in any given year's INMO. Good Luck!

Some INMO+ questions

1. Some cells of a rectangular table with n rows and m columns ($n < m$) are filled with stones such that there is at least 1 stone in any column. Prove that there is a stone such that the number stones in the same row is at least $\frac{m}{n}$ times the number of stones in the same column

2. The incircle w of a triangle ABC touches the sides AC and BA at E and F respectively. N is the midpoint of arc BAC and P is the foot of altitude from the midpoint of BC onto EF. Prove that the line NP passes through the centre of w

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(1) = 1$ and
$$f(n) = n - f(f(n-1)), \quad \forall n \geq 2.$$
Prove that $f(n + f(n)) = n$ for each positive integer n.

4. One hundred and one of the squares of an $n \times n$ table are colored blue. It is known that there exists a unique way to cut the table to rectangles along boundaries of its squares with the following property: every rectangle contains exactly one blue square. Find the smallest possible value of n.

5. Let D be an arbitrary point on side BC of triangle ABC. Let I_1 and I_2 be the incenters of triangles ABD and ACD respectively. Let O_1 and O_2 be the circumcenters of triangles AI_1D and AI_2D respectively. Prove that the lines I_1O_2 and I_2O_1 meet on the perpendicular from D to BC.

6. Find all natural numbers m, n such that $m^2 + 3 = n^3$

7. Let P be a polynomial of degree $2n$, all of whose coefficients belong to the closed interval $[2020, 2021]$. Given that P has a real root, what is the smallest possible value of n

8. We call a natural number n honourable, if when a single corner cell is removed from an $n \times n$ grid, there are an odd number of ways of tiling the remaining cells using L-trominoes. Prove that a number is honourable if and only if it is a power of 2
Note: An L-tromino consists of three unit cells connected edge to edge, such that the two corner squares make an angle of $90°$ at the center square

9. Let a, b and c be positive real numbers. Prove that

$$\frac{\sqrt{a^2+3bc}}{a} + \frac{\sqrt{b^2+3ca}}{b} + \frac{\sqrt{c^2+3ab}}{c} \geq 6.$$

10. $ABCD$ is a convex quadrilateral. Points I_1, I_2, J_1, J_2 are respectively the incenter, incentre, A-excenter, A-excentre of ABC, ADC, ABC, ADC. Prove that I_1J_2 intersects I_2J_1 on the angle bisector of $\angle BCD$

11. Find all positive integers n such that $n^3 - 5n^2 + 9n - 6$ is a perfect square.

12. Let BE and CF be altitudes in a scalene triangle ABC. Let M be the midpoint of BC and let N be the midpoint of EF. Show that the circumcenter of AMN lies on the line through A parallel to BC.

13. We call a number n perfect if the sum of its positive integer divisors(including 1 and n) equals $2n$. Determine all perfect numbers n for which $n-1$ and $n+1$ are twin primes.

14. There are n^2 lights of some n colours (not necessarily n of each colour). Prove that they can be arranged on n christmas trees with n lights per christmas tree, such that no tree has lights of three or more different types.

15. Find all real numbers $x_1, \ldots, x_{2016}$ that satisfy the following equation for each $1 \leq i \leq 2016$. (Here $x_{2017} = x_1$.)

$$x_i^2 + x_i - 1 = x_{i+1}$$

16. Let ABC be an acute-angled triangle with $AB < AC$, and let H be its orthocenter. The circumference with diameter AH meets the circumscribed circumference of ABC at $P \neq A$. The tangent to the circumscribed circumference of ABC through P intersects line BC at Q. Show that $QP = QH$.

17. Let $a_1, a_2, \cdots, a_n$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \cdots, n$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

18. The roof of Eisler Memorial Hospital is a polygon with integer co-ordinate vertices. Dr Tenma decides to tile the roof with 1×4 and 2×2 rectangular tiles (which may be rotated). He had a set of tiles which would allow him to complete this task with no remaining tiles. Unfortunately overnight, in an attempt to manipulate Tenma, Johan stole one of the tiles, and replaced it with a tile of the opposite kind. Is it possible that Dr Tenma is still able to successfully tile his roof?

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of distinct prime factors of n, with $P(1) = 1$. Show that for any $a_0 \in \mathbb{N}$, if we define a sequence $a_{k+1} = a_k + P(a_k)$ for $k \geq 0$, there exists some $k \in \mathbb{N}$ with $a_k / P(a_k) = 2015$.

20. Let I be the incentre of the triangle ABC. Suppose the incircle is tangent to the sides AB, BC, CA at F, D, E. Let the line through I perpendicular to MI meets DE, DF at U, V respectively, where M is the midpoint of BC and. Prove that $MU = MV$.

21. Is it true that for integer $n \geq 2$, and given any non-negative reals ℓ_{ij}, $1 \leq i < j \leq n$, we can find a sequence $0 \leq a_1, a_2, \ldots, a_n$ such that for all $1 \leq i < j \leq n$ to have $|a_i - a_j| \geq \ell_{ij}$, yet still $\sum_{i=1}^{n} a_i \leq \sum_{1 \leq i < j \leq n} \ell_{ij}$

22. H is the orthocentre and N is the nine-point centre of acute scalene triangle ABC. Let P and Q be points on BC such that $PH \perp AN$ and $QN \perp AN$. If the perpendicular from H onto AN, meets the tangent from A to the circumcircle of ABC, at K, prove that AK is tangent to the circumcircle of KPQ.

23. Let $a_1, a_2, \cdots, a_n$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \cdots, n$. Prove that
$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$
for each positive integer n.

24. We tile a convex polygon with 2021 parallelograms. Prove that we could have tiled the same convex polygon with 2020 parallelograms

25. Some congruent paper squares of k distinct colors are placed on a rectangular table, with sides parallel to the sides of the table. Suppose that for any k squares of distinct colors, some two of them can be nailed on the table with only one nail. Prove that there is a color such that all squares of that color can be nailed with $2k - 2$ nails.

26. Each of three lines cuts chords of equal lengths in two given circles. The points of intersection of these lines form a triangle. Prove that the circumcircle of this triangle passes through the midpoint of the segment joining the centres of the circles.

27. For a positive integer n define $S_n = 1! + 2! + \ldots + n!$. Prove that there exists an integer n such that S_n has a prime divisor greater than 10^{2021}.

28. There are 50 students in a classroom, some pairs of which are friends. Everyone has at least 25 friends. Is it always possible to divide the students into 25 teams of 2 such that every pair is a friend?

29. Find all polynomials P with integer coefficients such that $P(0) \neq 0$ and
$$P^n(m) \cdot P^m(n)$$
is a square of an integer for all nonnegative integers n, m.

    Remark: For a nonnegative integer k and an integer n, $P^k(n)$ is defined as follows: $P^k(n) = n$ if $k = 0$ and $P^k(n) = P(P(^{k-1}(n))$ if $k > 0$

30. The following configuration is extremely rich: Here is a really hard geometry problem bisected into many parts.

    Let ABC be an acute angled triangle with orthocenter H. w_1 and w_2 denote the incircle of ABH and ACH respectively. Prove that one of the common tangents of w_1 and w_2 passes through the midpoint of BC

31. Prove that one of the common tangents of w_1 and w_2 is parallel to BC

32. Conclude that the last common tangent of w_1 and w_2 is also tangent to (BC)

33. There are k piles of 2020 stones each. In a turn, Petya chooses any two non-empty piles of stones, and Vasya takes one stone from one of the two chosen piles and puts it into the other pile. Petya wins if she can eventually make an empty pile. Find the smallest k for which Petya can guarantee victory?

34. Acute scalene triangle ABC has a circumcircle Ω and incentre I. The circle with diameter AI intersects Ω at point P, and the tangent at P to Ω intersects the perpendicular bisector of BC at R. Prove that if RA intersects Ω again at Q, that

$$QB \cdot \tan \angle B = QC \cdot \tan \angle C$$

35. For his birthday, math prodigy Tom received a positive integer m and two increasing positive sequences $\{a_n\}$ and $\{b_n\}$. Tom claims that no element of the set $\left\{a_i b_j + m \mid (i,j) \in \mathbb{N}^2\right\}$ has a prime divisor larger than 2020!. Prove that young Tom's a liar.

36. Two circles Γ_1 and Γ_2 intersect at A and B. A line through A intersects Γ_1 and Γ_2 at C and D respectively. The tangent line to Γ_1 at C intersects AB at E. The circumcircle of CDE intersects Γ_1 and Γ_2 again at F and G respectively. Line EF intersects Γ_1 again at P and EG intersects CD at Q. Let X be the second intersection of PQ and Γ_1. Show that BX passes through the midpoint of CE.

37. Given a quadratic trinomial $f(x) = x^2 + ax + b$. Assume that the equation $f(f(x)) = 0$ has four different real solutions, and that the sum of two of these solutions is -1. Prove that $b \leq -\frac{1}{4}$.

38. Prove that for any 2 different triangulations of a cyclic $n-$gon into $n-2$ triangles, the sum of the inradii of the triangles used is the same

39. Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^{+}x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} \leq \sqrt{n}$$

40. The 2001 towns in a country are connected by some roads, at least one road from each town, such that no town is connected by a road to every other city. We call a set D of towns dominant if every town not in D is connected by a road to a town in D. Suppose that each dominant set consists of at least k towns. Prove that the country can be partitioned into $2001 - k$ republics in such a way that no two towns in the same republic are connected by a road.

41. Let I be the incenter of a circumscribed quadrilateral $ABCD$. The tangents to circle AIC at points A, C meet at point X. The tangents to circle BID at points B, D meet at point Y. Prove that X, I, Y are collinear.

42. Is there a natural number $n > 10^{1000}$ which is not divisible by 10 and which satisfies: in its decimal representation one can exchange two distinct non-zero digits such that the set of prime divisors does not change.

43. A number of people observed a cat for a period of t minutes. Each observed it for a period of one minute and saw the cat move 1m. There wasn't a single moment when the cat wasn't observed. What is the maximal and minimal length that the cat could have travelled in these t minutes?

44. Call the improvement of a positive number its replacement by a power of two, for which the increases, but by less than 2 times. Given 2^{100} positive numbers with a sum of 2^{100}, prove that you can erase some of the numbers, and improve each of the other numbers once, so that the sum of the resulting numbers is again 2^{100}.

45. Let M be midpoint of angle bisector AD of triangle ABC. Circle ω_1 with diameter AC meets segment BM at E, and circle ω_2 with diameter AB meets segment CM at F. Prove that B, E, F and C are concyclic.

46. Let $P(x)$ be a polynomial with integer coefficients. Can one guarantee the existence of an integer c so that $P(x) - c$ is irreducible?

47. Let p and q be (not necessarily distinct) primes. Prove that at most $\frac{p-1}{2}$ numbers n satisfy:

$$p! + n! + q = p(n^p)$$

48. Prove that for every positive integer n, there exists a $2n$-digit number $\overline{a_{2n}a_{2n-1}\cdots a_1}$ for which the following equality holds:

$$\overline{a_{2n}\cdots a_1} = (\overline{a_n \cdots a_1})^2$$

49. For a real number x, define $\lfloor x \rfloor$ to be the largest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$.

    a) Prove that there are infinitely many positive real numbers x that satisfy the inequality

$$\{x^2\} - \{x\} > \frac{2019}{2020}.$$

    b) Prove that there is no positive real number x less than 1000 that satisfies the inequality.

50. Can an infinite sequence of natural numbers $a_1, a_2, \ldots$ of natural numbers simultaneously satisfy that

    - $a_1 + a_2 + \cdots + a_n \leq n^2$
    - $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq 2020$

    for each n?