97IMO5

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§1 Solution

Solution. Firstly notice that $\text{rad}(a) = \text{rad}(b)$, this is quite easy to prove.

Claim — $b^3|a$

Proof. Assume $a < b$, this leads to a clear contradiction. So $a \geq b$. For $a = b$ we easily get $a = b = 1$. Now assume $a > b$,

$$a^{b^2} = b^a < a^a \implies a > b^2$$

$$a^{b^2} = b^a = (b^2)^{\frac{a}{2}} < a^{\frac{a}{2}} \implies a > 2b^2$$

$$\frac{\nu_p(a)}{\nu_p(b)} = \frac{a}{b^2} > 2$$

Thus, $b^2|a$, but using this in the previous equation again we would have

$$\frac{\nu_p(a)}{\nu_p(b)} = \frac{a}{b^2} \geq 3$$

So, $b^3|a$. □

Set $a = kb^3$. The original equation reduces to $k = b^{bk-3}$. To finish,

$$k = b^{bk-3} = b^{b^{bk-2}-3} = b^{b^{b^{bk-2}-2}-3}$$

If $bk - 2 \geq 3$ then the size of the RHS increases unboundedly. Thus, $bk = \{1, 2, 3, 4\}$ Checking these we get $(a, b) \in (1, 1), (16, 2), (27, 3)$ are the only solutions. □