19SLA5

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§1 Solution

Solution. We will prove the version where x_i are complex numbers. Let the expression be P/V. The denominator is the **Vandermonde Determinant**

$$V = \prod_{i>j} (x_i - x_j)$$

Call $Q_i = V \cdot \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j}$

Claim — $P(x_1, x_2, \cdots, x_n)$ is divisible by the denominator V.

Proof. So we would be done if we show $P = 0$ if $x_i = x_j$ for $i \neq j$.

Assume $x_1 = x_k$, clearly $Q_2 = \cdots = Q_{k-1} = Q_{k+1} \cdots = Q_n = 0$,

$$Q_1 = (-1)^{n-1}(1 - x_1x_2)(1 - x_1x_3)\cdots(1 - x_1x_n) \cdot \frac{V}{\prod_{j>1}(x_j - x_1)}$$

$$Q_k = (-1)^{n-k} \frac{\prod_j (1 - x_k x_j)}{1 - x_j^2} \cdot \frac{V}{\prod_{j>k}(x_j - x_k) \prod_{j<k}(x_k - x_j)}$$

(There are no divide by 0 issues because we first divide and then equate the variables.) We get, $Q_k = (-1)^{k-1+k-2} \cdot Q_1$, which proves the claim. Other cases follow similarly. □

Claim — $P(x_1, x_2, \cdots, x_n) = C \cdot \prod_{i>j}(x_i - x_j)$, where $C \in \mathbb{R}$ and is constant for all variable x_i's.

Proof. Assume P is nonzero. We already know

$$P = V \cdot R$$

where V is the denominator and $R \in \mathbb{R}[X_1, X_2, \cdots, X_n]$. It is well known that $\deg_{x_i}(V) = n - 1$. Thus, $\deg_{x_i}(V \cdot R) \geq n - 1$ while $\deg_{x_i}(P) \leq n - 1$, because we can choose the terms in lexicographic order of degrees. Equality holds only when $\deg_{x_i}(P) = n - 1$ and $\deg_{x_i}(R) = 0$ □

To finish we have to show this holds true for 1 set of values, take $x_{n+1} = 1$,

$$\sum_{i,n+1} \prod_{i \neq j} \frac{1 - x_i x_j}{x_i - x_j} = (-1)^n \sum_{i,n} \prod_{i \neq j} \frac{1 - x_i x_j}{x_i - x_j} + 1 = (-1)^n \cdot \frac{1 - (-1)^n}{2} + 1 = \frac{1 + (-1)^n}{2}$$

As the base case is trivial, we are done by induction. $\square$