21STEMSB2

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§1 Solution

Solution. Let $\{p_1, \cdots, p_k\}$ be the set of first k odd primes with $p_k < 10^{100} < p_{k+1}$. If $P(0) \neq 0$, choose a big n_0, $P(2^{n_0}) = 2^{\nu_2(P(0))} \prod_i p_i^{\alpha_i}$ Note that $p_1^{\alpha_1} \mid\mid P(2^{n_0})$, now $p_1^{\alpha_1} \mid\mid P(2^{n_0+\varphi(p_1^{\alpha_1+1})})$ because

$$a_k(2^{n_0+\varphi(p_1^{\alpha_1+1})})^k \equiv a_k(2^{n_0})^k \cdot (2^{\varphi(p_1^{\alpha_1+1})})^k \equiv a_k(2^{n_0})^k \pmod{p_1^{\alpha_1+1}}$$

and $\varphi(p_1^{\alpha_1})|\varphi(p_1^{\alpha_1+1})$. Therefore,

$$P(2^{n_0}) = P(2^{n_0+\lambda\varphi(\prod_i p_i^{\alpha_i+1})}).$$

This is not possible so $P(0) = 0$. Now, $Q(x) \stackrel{\text{def}}{=} \frac{P(x)}{x}$, if $Q(0) \neq 0$ we could derive the same contradiction, continuing this way we see that $P(x) = x^n$ is the only solution. $\square$