09AMO6

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§1 Solution

Solution. Notice that displacing all of the terms in both the sequences by k doesnt change the statement. Let the sequences be $\{s_1, s_2, ...\}, \{t_1, t_2, ...\}$ displacing by $-s_1$ and $-t_1$ we get that the sequences become $\{0, s_2 - s_1, ...\}, \{0, t_2 - t_1, ...\}$. Rename these to $\{0, s_1, ...\}, \{0, t_1, ...\}$.
Define $s_i = \frac{a_i}{b_i}, t_j = \frac{c_j}{d_j}$ where $a_i, b_i, c_j, d_j \in \mathbb{Z}$ and $\gcd(a_i, b_i) = \gcd(c_j, d_j) = 1$ So the condition becomes,

$$s_i t_j \in \mathbb{Z} \ \forall \ i, j \in \mathbb{Z}^+$$

and we have to show that $\exists \frac{m}{n} \in \mathbb{Q}$ such that

$$\frac{ma_i}{nb_i}, \frac{nc_j}{md_j} \in \mathbb{Z} \ \forall \ i, j \in \mathbb{Z}^+$$

Choose a large enough N, now using the previous conditions, we have the following conclusions $\forall \ i, j \in [N]$ and $\forall$ prime p.

- $\min_{i \le N}(\nu_p(a_i)) \ge \max_{i \le N}(\nu_p(d_i))$
- $\min_{i \le N}(\nu_p(c_i)) \ge \max_{i \le N}(\nu_p(b_i))$

Choose $m_p, n_p \in \mathbb{Z}_{\ge 0}$, such that

$$\nu_p(c_j) \ge \boxed{\min_{i \le N}(\nu_p(c_i)) \ge m_p \ge \max_{i \le N}(\nu_p(b_i))} \ge \nu_p(b_i)$$

$$\nu_p(a_i) \ge \boxed{\min_{i \le N}(\nu_p(a_i)) \ge n_p \ge \max_{i \le N}(\nu_p(d_i))} \ge \nu_p(d_j)$$

it is easy to see that such m_p, n_p exist. Repeat the same operation for all primes p. And choose $r = \prod_{\text{prime } p} p^{m_p - n_p}$. Now, taking $N \to \infty$, we are done. $\square$