19TSTST7

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§1 Solution

Solution. Call $\lambda = \operatorname{lcm}(1, 2, \ldots, 10^{100})$. Let $S = \{1, 2, \ldots, \lambda\}$.

Claim — $f(x)$ is periodic.

Proof. Set $y = x + \lambda$, we get

$$\gcd(f(x), f(y)) = \gcd(f(x), \lambda) = f(x) \implies f(x)|f(x+\lambda)\forall x \in \mathbb{Z},$$

but now setting $x = y + \lambda$, we get

$$\gcd(f(y+\lambda), f(y)) = \gcd(f(y+\lambda), \lambda) = f(y+\lambda) \implies f(y+\lambda)|f(y)\forall y \in \mathbb{Z}$$

Therefore, $f(x) = f(x+\lambda)\forall x \in \mathbb{Z}$, which means it is periodic. $\square$

Now to finish, fix a prime p, choose $x = x_p$ such that $\nu_p(f(x_p)) = \max_{x \in S} \nu_p(f(x))$
Therefore,

$$\nu_p(f(y)) = \nu_p(\gcd(f(x_p), f(y))) = \nu_p(\gcd(f(x_p), x_p - y))$$

Notice that y's of the form

$$y \equiv x_p \pmod{p^{\nu_p(f(x_p))}}$$

clearly have maximal ν_p, so performing CRT on

$$y \equiv x_p \pmod{p^{\nu_p(f(x_p))}}$$

over all primes $p| \operatorname{lcm}(f(1), \ldots, f(\lambda))$. Let α be a number that satisfies all those congruences.
Note that $f(x) = \gcd(f(\alpha), x - \alpha)$. Now, taking $m = -\alpha, n = f(\alpha)$. We are done. $\square$