13SLN3

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§1 Solution

Solution. Let $f(n) = n^2 + n + 1$ and $P(n) = \max$ prime of n.
Note that $f(n^2) = n^4 + n^2 + 1 = f(n)f(n-1), f((n+1)^2) = f(n+1)f(n)$.
We have

$$f(n) | \gcd(f(n^2), f((n+1)^2))$$

So we would be done if we prove that

Claim 1.1 — There exist infinitely many n such that

$$P(f(n-1)) \leq P(f(n)) \geq P(f(n+1))$$

Proof. Assume for the sake of contradiction that there are only finitely many n that satisfy this.
So, $\exists N$ such that $\{P(f(n))\}_{n \geq N}$ is monotonic.
Now clearly if this were decreasing we would get a contradiction, so, assume $\{P(f(n))\}_{n \geq N}$ is strictly increasing.
But we already know,

$$P(f(n^2)) = \max(P(f(n)), P(f(n-1)))$$

so it cannot be strictly increasing and we are done. □

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