18SLC5

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§1 Solution

Never expected to solve this. Could'nt. Nice problem, however.

The required minimum is $k(4k^2 + k - 1)/2$.

Construction: We denote the players $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ in general. In the first $\binom{k}{2}$ days, the players $P_1, \ldots, P_k$ arrive in order, and when each player arrives they play everyone else. The last $\binom{k}{2}$ are done in a symmetric way. In the middle k^2 days we have two phases. First $Q_1, \ldots, Q_k$ arrive in order; when Q_i arrives they immediately play $P_1, \ldots, P_{k+1-i}$. Then $P_1, \ldots, P_k$ depart in order; before P_i departs they play the rest of their matches.

Minimality: Let us denote the arrival and departure dates by

$$1 = a_1 \geq a_2 \cdots \geq a_{2k} \text{ and } \binom{2k}{2} = b_1 \geq b_2 \geq \cdots \geq b_{2k}$$

Clearly, the total cost is

$$S = \sum_n (b_n - a_n + 1) .$$

Claim — $a_n - 1 \leq \binom{n-1}{2}$ and $\binom{2k}{2} - b_n \leq \binom{n-1}{2}$ for $n = 1, 2, \ldots, k$.

Proof. Quite obvious, just notice that before n arrives we have at most $n - 1$ players. 2nd bound can be derived similarly. □

Claim — For $n > k$ we have

$$(a_n - 1) + \left(\binom{2k}{2} - b_n\right) \leq 2\binom{n-1}{2} - \binom{2(n-1) - 2k}{2}$$

Proof. The idea of this proof is similar to the previous one.

However, when $n > k$ we double-count certain games. Specifically, there are $n - 1$ players that arrive before a_n and $n - 1$ players who depart after b_n. Therefore, there are at least $2(n - 1) - 2k$ players' games overcounted. This gives us the $\binom{2(n-1)-2k}{2}$ term above. □

To finish, note that

$$S = \sum_{n=1}^{2k}(b_n - a_n + 1) \geq \sum_{n=1}^{2k}\left(\binom{2k}{2}\right) - 2\binom{n-1}{2} + \sum_{n=k+2}^{2k}\binom{2(n-1)-2k}{2}$$

$$= \frac{1}{2}k(4k^2 + k - 1)$$