18IMO5

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§1 Solution

Solution.
Call the thing $s(n)$ then $\triangle\, s(n) = \dfrac{a_{n+1} - a_n}{a_1} + \dfrac{a_n}{a_{n+1}}$

Claim — Define $\operatorname{rad}(n) = \prod_{p \text{ prime},\ p|n} p$. Then $\operatorname{rad}(a_1 a_2 \cdots)$ is finite.

Proof. We have,

$$a_1 a_{n+1} \triangle\, s(n) = a_{n+1}(a_{n+1} - a_n) + a_1 a_n$$

Assume for the sake of contradiction the expression is infinite. Then, we can choose a prime p such that $p|a_{n+1}, p \nmid a_n, p \nmid a_1$. Taking $\pmod p$, we have $a_1 a_n \equiv 0 \pmod p$ which is clearly impossible. □

Proposition
$\{\nu_p(a_n)\}_{n \geq N}$ is eventually constant.

Proof. Note that

$$\nu_p\left(\frac{a_{n+1} - a_n}{a_1} + \frac{a_n}{a_{n+1}}\right) \geq \min\left(\nu_p\left(\frac{a_n}{a_{n+1}}\right), \nu_p\left(\frac{a_{n+1} - a_n}{a_1}\right)\right) \tag{$\spadesuit$}$$

If there exists an n such that $\nu_p(a_n) < \nu_p(a_{n+1})$ then

$$\nu_p(a_n) - \nu_p(a_{n+1}) = \nu_p(a_n) - \nu_p(a_1) \implies \nu_p(a_{n+1}) = \nu_p(a_1),$$

because if one of the expressions is negative then both the expressions have to be equal as only then the complete expression "can" become non-negative. Now choose the next n' such that

$$\nu_p(a_1) = \nu_p(a_{n'}) > \nu_p(a_{n'+1})$$

the ν_p expression evaluates to

$$\nu_p(a_{n'+1}) - \nu_p(a_{n'}) < 0,$$

but this is not possible. Therefore, if there exists such n then the sequence becomes constant.
So, assume such n doesnt exist, this means that $\nu_p(a_n) \geq \nu_p(a_{n+1})\ \forall\ n \geq N$ but this is also not possible because a weakly decreasing whole number sequence ends up being constant. □

Because the ν_p sequence is eventually constant and there are only finitely many primes dividing $\operatorname{rad}(a_1 a_2 \cdots)$ thus we could just select $n = \max_{p \mid \operatorname{rad}(a_1 a_2 \cdots)} n_p$ where n_p denotes the index where the sequence ν_p becomes constant. $\square$