13TSTST8

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§1 Solution

Solution.

Claim 1.1 — 2 is a primitive root $\pmod{3^k}$

Proof. Assume $\text{ord}_{3^k}(2) = \alpha \leq 2 \cdot 3^{k-1}$. It is obvious that α has to be even.
Note that $v_3(2^\alpha - 1) = v_3(4^{\frac{\alpha}{2}} - 1) = 1 + v_3(\alpha/2) \geq k \implies \alpha \geq 2 \cdot 3^{k-1}$.

So, $\alpha = 2 \cdot 3^{k-1}$ □

Claim 1.2 — $f(n)$ is periodic $\pmod{3^k}$ with period 3^k

Proof. We will induct, base case is trivial, assume the proposition is true for $n \leq N$. Now notice that the interval

$$\begin{aligned} [a, a + 3^{N+1} - 1] &= [a, a + 3^N - 1] \cup [a + 3^N, 2 \cdot 3^N - 1] \cup [a + 2 \cdot 3^N, a + 3^{N+1} - 1] \\ &= I_1 \cup I_2 \cup I_3 \qquad \text{(say)} \end{aligned}$$

As, I_1 is a residual class mod 3^N.

$$\begin{aligned} f(a + 3^N) - f(a) = \sum_{i \in I_1} 2^{f(i)} &\equiv 2^1 + 2^3 \cdots + 2^{2 \cdot 3^N - 1} \pmod{3^{N+1}} \\ &\equiv \frac{2}{3} \cdot (4^{3^N} - 1) \pmod{3^{N+1}} \end{aligned}$$

by virtue of **Claim 1.1** and the fact that $f(i)$ is odd. Clearly this is analogous over I_2, I_3, and we get,

$$f(a + 3^{N+1}) - f(a) \equiv 2(4^{3^N} - 1) \equiv 0 \pmod{3^{N+1}}$$

thanks to LTE.
To finish, just see that $f(a + 2 \cdot 3^N) \not\equiv f(a) \not\equiv f(a + 3^N) \pmod{3^{N+1}}$.
Thus, $f(a) \equiv f(b) \pmod{3^{N+1}} \iff 3^{N+1} | a - b$. Which finishes the problem. □

□