14PTNMB4

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§1 Solution

Solution. Note that

$$P(x) = \sum_{k=0}^{n} \frac{(2^n \cdot x)^k}{2^{k^2}}$$

Obviously, $P(x)$ can be replaced with $P(\frac{x}{2^n})$.

Claim — Consider the sequence $a_i = 2i, 0 \le i \le n$. Then, $\operatorname{sgn}(P(-2^{a_i})) = (-1)^i$.

Proof. Notice that $|\pm 2^m| > |\sum_{k=0}^{m-1} \pm 2^k|$, thus

$$\operatorname{sgn}(P(-2^{a_i})) = \operatorname{sgn}\left(\sum_{k=0}^{n} (-1)^k \cdot 2^{k(a_i-k)}\right) = \operatorname{sgn}((-1)^k)$$

such that $k(a_i - k)$ is the biggest, this might not work when there are atleast two of the same terms but this cannot happen because a_i is even. To finish note that $k(a_i - k) = k(2i - k)$ achieves maximum at $k = i$, so $\operatorname{sgn}(P(-2^{a_i})) = (-1)^i$ $\square$

Now by IVT P has atleast n real roots, so all roots of P are real. $\square$