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§1 Solution

Solution.

$$Q(x) \stackrel{\text{def}}{=} P(x)P(x+1) - P(x^2+x+1)$$

Q is a polynomial and is $= 0$ for all reals, this means $Q(x) \equiv 0$. So,
$P(x)P(x+1) = P(x^2+x+1)\ \forall\ x \in \mathbb{C}$
Let α be a root of P, then clearly $\alpha^2+\alpha+1, \alpha^2-\alpha+1$ are also roots.

Claim — Only possible root is $\pm i$.

Proof. Assume FTSOC, there exists a root which is not $= \pm i$ We know
$|x^2-x+1| = |-x^2+x-1|$, so using triangle inequality,

$$|x^2+x+1| + |-x^2+x-1| \geq 2|x|$$

Unless $|x^2+x+1| = |x^2-x+1| = |x|$, we run into problems. It is easy to see that this happens only for $x = \pm i$. So, the only possible roots are $\pm i$. $\square$

Let $P = (x-i)^a(x+i)^b$ which always works. So,
$\boxed{P = \{\{(x-i)^a(x+i)^b : a, b \geq 0\}, 0, 1\}}$ are the only solutions.

$\square$