18CHNTST12

Himadri Mandal

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§1 Solution

Solution. I claim the only solutions are $k = p^{1009} \cdot C_1 \ \forall$ prime $p \geq 3$ or $2^{1009} \cdot C_2$ with $C_2 \neq 1$

Construction: For the first solution: we can take $a = p^{1008}(p-1), k = p^{1009} \cdot C_1$. This works because

$$a + nk = p^{1008}((p-1) + npC_1)$$

where $(p-1) + npC_1$ is never a square, as $p-1$ is a NQR mod $p \geq 3$.
For the second solution: we can take $a = C_2 - 1$ if $C_2 \neq 2$ and $a = 3$ if $C_2 = 2$. This works as

$$(C_2 - 1) + 2nC_2 \text{ and } (3 + 4n)$$

can never be a square.

Necessity:
Let $g = \gcd(a, b)$

$$a + nk = g(a' + nk') = C \cdot p_i^{1009 \cdot \alpha_i - 1} p_j^{2\alpha_j - 1}$$

Note that by dirichlet $a' + nk'$ is a prime infinitely often, and is unbounded, this means $g = \beta \cdot p^{1009 \cdot \alpha_i - 1}$ with $\gcd(\beta, p) = 1$
Note that

$$a' + nk' \not\equiv 0 \pmod{p} \implies k' \equiv 0 \pmod{p}$$

Take $k'' = k'/p$. So we get $k = p^{1009\alpha_i}\beta k'' = p^{1009}F$ as $\beta k'' p^{1009(\alpha_i - 1)}$ can take any value.

To finish we would have to prove $k = 2^{1009}$ doesnt work. We already have $a = 2^{1008}\beta$

Claim 1.1 — $\exists n$ such that $2^{1008}(\beta + 2n)$ is a square.

Proof. This is trivial, just take $n = \frac{d^2 - \beta}{2}$ with large enough odd d. □

This finishes the proof. □