14TWNQ3J5

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§1 Solution

Solution. One side is trivial, note that

$$\sum_{i \pmod n} \left(\frac{x_{i-1}}{x_i} + \frac{x_i}{x_{i-1}} \right) \geq 2n$$

by AM-GM. Now to prove the other side

Claim 1.1 — If i is the index such that x_i is the largest element in the sequence, then $x_i = x_{i-1} + x_{i+1}$ or it's a constant sequence.

Proof. Assume $k_i = \frac{x_{i-1}+x_{i+1}}{x_i} \geq 2$. Thus, $2x_i \leq x_{i-1} + x_{i+1} = k_i x_i$, but as x_i is the maximum element, we have $x_{i-1} \leq x_i, x_{i+1} \leq x_i$. Then we have $x_{i-1} = x_i = x_{i+1}$ but then $x_{i-2} + x_i = k_{i-1}x_{i-1} \implies x_{i-2} = x_i$ as $x_{i-2} \in \mathbb{Z}^+$, continuing we get that the sequence is constant. □

Claim 1.2 — If we remove the largest element(any one if there are multiple largest elements) in a non constant *good* sequence the resulting sequence is also *good*.

Proof. Let i be the index of the largest element. So we have,

$$\frac{x_{i-2} + x_i}{x_{i-1}} \in \mathbb{Z}$$

$$\frac{x_{i+2} + x_i}{x_{i+1}} \in \mathbb{Z}$$

We would be done if we show these hold:

$$\frac{x_{i-2} + x_{i+1}}{x_{i-1}} \in \mathbb{Z}, \frac{x_{i+2} + x_{i-1}}{x_{i+1}} \in \mathbb{Z}$$

But it is easy to see that this is true using $x_i = x_{i-1} + x_{i+1}$. □

To finish notice that we eventually either get a constant sequence or a *good* sequence with 3 terms both of which work. So, we are done. □