20IMO5

Himadri Mandal

December 10, 2021

§1 Solution

Solution. I claim all n work. Sort $\{a_i\}$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. WLOG assume $\gcd(a_1, a_2, \cdots, a_n) = 1$. It is clear that all of a_i have to be odd because $(x + \frac{1}{2})^k$ is never an integer for $x, k \in \mathbb{Z}^+$ Choose the smallest m such that $\operatorname{rad}(a_1) \nmid \operatorname{rad}(a_m)$

$$\frac{a_1 + a_m}{2} = \sqrt[k]{a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)}}$$

Claim — There is no such m.

Proof. Assume not, if $\exists\ i$ such that $\sigma(i) < m$ then we are done, otherwise

$$a_m^{2k} \geq (a_{\sigma(1)} \cdots a_{\sigma(k)})^2 \geq a_1^k a_m^k$$

$a_m \geq a_1$ meaning $a_m = a_1$, a contradiction. □

Therefore, $\operatorname{rad}(a_1) | a_t\ \forall\ t$ meaning $\operatorname{rad}(a_1) = 1 = a_1$, so $a_t = 1\ \forall\ t$. □