16AMO2

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November 14, 2021

§1 Solution

Problem 1.1

Prove that for any positive integer k,

$$f(k) = \left(k^2\right)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Solution. We would be done if we show $\nu_p(f(k)) \geq 0 \ \forall \ k \in \mathbb{Z}$. Choose a prime power λ. Note that $k^2 + \sum_{j=0}^{k-1} j = \sum_{j=0}^{k-1}(j+k)$. It suffices to show that,

$$\left\lfloor \frac{k^2}{\lambda} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j!}{\lambda} \right\rfloor > \sum_{j=0}^{k-1} \left\lfloor \frac{(j+k)!}{\lambda} \right\rfloor - 1$$

$$\left\{ \frac{k^2}{\lambda} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j!}{\lambda} \right\} < \sum_{j=0}^{k-1} \left\{ \frac{(j+k)!}{\lambda} \right\} + 1$$

$$\left\{ \frac{k^2 \pmod{\lambda}}{\lambda} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j! \pmod{\lambda}}{\lambda} \right\} < \sum_{j=0}^{k-1} \left\{ \frac{(j+k)! \pmod{\lambda}}{\lambda} \right\} + 1$$

which is obvious.

$\square$