19CHN1

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§1 Solution

Solution. The answer is $-512 \le S \le 288$. 288 is achieved at $a = b = 4$ and $c = d = e = -1$ and the -512 is achieved at $a = 9$ and $b = c = d = e = -1$. Lets prove the inequalities now,
Let $x_1 = a + b$, $x_2 = c + d$, $x_3 = e + a$, $x_4 = b + c$, $x_5 = d + e$. Then the conditions become

$$\sum_i x_i = 10 \quad \text{and} \quad x_i + x_{i+1} \le 6 \ \forall i$$

We also have $-2 \le x_i \le 8$ for each i, and $S = x_1 x_2 x_3 x_4 x_5$. Let $f(i) = \begin{cases} 0 & x_i \ge 0 \\ 1 & \text{otherwise} \end{cases}$.
$T = \sum_{\text{cyc}} f(i)$
Lets do some casework.

- If $T = 0$ then $S \ge 0$ and $S \le 2^5 = 32$ by AM-GM.
- If $T = 1$ $S \le 0$, and $|S| \le 2 \cdot 3^2 \cdot 3^2 = 162$ by AM-GM (since $x_1 x_2 \le 9$, $x_3 x_4 \le 9$).
- If $T = 2$, so $S \ge 0$. Two of the nonnegative x_i's must be adjacent, say x_1 and x_2, thus $x_1 x_2 \le 9$. So $|S| \le 2^2 \cdot 9 \cdot 8 = 288$.

  Equality is achieved at $(3, 3, -2, 8, -2)$.
- If $T = 3$, so $S \le 0$. In that case, $|S| \le 2^3 \cdot 8^2 = 512$.

  Equality is achieved at $(-2, -2, -2, 8, 8)$.
- If $T = 4$, so $S \ge 0$. Then $|S| \le 2^4 \cdot 8 = 128$.
- $T = 5$ is not possible since $\sum_i x_i = 10$.

Therefore, we deduce that

$$-512 \le S \le 288$$

as desired. $\square$