19INDTST2

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§1 Solution

Solution. With $\pmod 5$ we get $a_1 \equiv \cdots \equiv a_{2018} \equiv \{0, -1\} \pmod 5$.
So firstly, assume $a_1 \equiv 0 \pmod 5$. Let i_0 be the index such that $\nu_5(a_{i_0}) = \max_{i \in \mathbb{Z}} \nu_5(a_i)$. Now notice that,

$$a_{i_0}^{2018} + a_{i_0+1} = 5^{k_{i_0}},$$

we know $2018 \cdot \nu_5(a_{i_0}) > \nu_5(a_{i_0+1})$, thus $k_{i_0} = \nu_5(a_{i_0+1})$, but because of size reasons this is a clear contradiction.
Otherwise, $a_1 \equiv \cdots \equiv a_{2018} \equiv -1 \pmod 5$. Perform a change of variables $5b_i - 1 = a_i$.

$$\begin{aligned}
&\implies (5b_j - 1)^{2018} + (5b_{j+1} - 1) \\
&= (5b_j - 1)^{2018} - 1 + 5b_{j+1} \\
&= \left((5b_j)^{2018} + \left(\sum_{i=1}^{2016} \binom{2018}{i} (-1)^i (5b_j)^{2018-i} \right) - 2018 \cdot 5b_j \right) + 5b_{j+1}
\end{aligned}$$

If there exists any j such that $\nu_5(5b_j) > \nu_5(5b_{j+1})$ then we are done because of size reasons. So, $\nu_5(5b_1) \le \nu_5(5b_2) \le \cdots \le \nu_5(5b_{2018}) \le \nu_5(5b_1)$, which means all of them are equal, making another change of variables $5b_i = 5^\alpha \cdot c_i, \gcd(5, c_i) = 1$
We get,

$$\begin{aligned}
&\implies (5b_i - 1)^{2018} + (5b_{i+1} - 1) \\
&= \left((5^\alpha c_i)^{2018} + \left(\sum_{j=1}^{2016} \binom{2018}{j} (-1)^j (5^\alpha c_i)^{2018-j} \right) - 2018 \cdot 5^\alpha c_j \right) + 5^\alpha c_{j+1} \\
&\equiv -2018 \cdot 5^\alpha c_j + 5^\alpha c_{j+1} \pmod{5^{\alpha+1}} \\
&\implies -2018 \cdot c_j + c_{j+1} \equiv 2c_j + c_{j+1} \equiv 0 \pmod 5
\end{aligned}$$

as otherwise we are done because of size reasons. Thus,

$$c_2 \equiv -2c_1, c_3 \equiv 4c_1, \cdots, c_{2018} \equiv (-2)^{2017} c_1 \implies c_1 \equiv 2^{2018} c_1 \pmod 5.$$

From which we get $5 | c_1$, a contradiction.

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