04RUS113

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§1 Solution

Solution. Let $\deg Q = m, \deg P = n$, and let ζ_k be a kth root of unity.

Claim — $m \mid n$

Proof. Let $R \in \mathbb{R}[x, y]$ such that

$$(Q(x) - Q(y)) \cdot R(x, y) = P(x) - P(y)$$

Clearly, $\deg(R) = n - m$. Setting $x = \zeta_m x, y = x$,

$$(Q(\zeta_m x) - Q(x))R(\zeta_m x, x) = P(\zeta_m x) - P(x)$$

As $\deg(Q(\zeta_m x) - Q(x)) \leq m - 1 \implies \deg(P(\zeta_m x) - P(x)) \leq n - 1$. Therefore, $x^n(\zeta_m^n - 1) = 0 \; \forall \; x \in \mathbb{C}$. So $m \mid n$.

□

To finish, WLOG assume leading coeffs of P, Q are 1, now we will induct on the degree, base case is clear.

$$Q(x) - Q(y) \mid (P(x) - Q(x)) - (P(y) - Q(y)),$$

which has degree $n - m$. By hypothesis, $P(x) - Q(x) = S_1(Q(x))$. Now, taking $S(x) = S_1(x) + x$, we would have $P(x) = S(Q(x))$. □