20BXMO1

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November 20, 2021

§1 Solution

Solution. This is equivalent to finding d such that $\exists P \in \mathbb{Z}[X]$ with

$$S(P) = \#_{\mathbb{Z}}(P-1) + \#_{\mathbb{Z}}(P+1) \geq d+1$$

where $\#_{\mathbb{Z}}(P)$ denotes the number of distinct integral roots of P.

Claim — $S(P) \leq 4$ which also means $d \leq 3$.

Proof.

$$P(x) + 1 = Q(x)(x-\alpha_1)^{a_1}(x-\alpha_2)^{a_2}\cdots(x-\alpha_k)^{a_k}$$
$$P(x) - 1 = Q(x)(x-\alpha_1)^{a_1}(x-\alpha_2)^{a_2}\cdots(x-\alpha_k)^{a_k} - 2$$

If $P(x_0) - 1 = 0$ with $x_0 \in \mathbb{Z}$,

$$Q(x_0)(x_0-\alpha_1)^{a_1}(x_0-\alpha_2)^{a_2}\cdots(x_0-\alpha_k)^{a_k} = 2$$

So, either $Q(x_0) = \pm 2, k \leq 2$ or $Q(x_0) = \pm 1, k \leq 3$, translating we get,
Case 1:

$$\pm 2x^{a_1}(x+c)^{a_2}$$

$S(P) \leq 3$
Case 2:

$$\pm 1x^{a_1}(x+c_1)^{a_2}(x+c_2)^{a_3}$$

Sorting $x, x+c_1, x+c_2$ and translating accordingly such that $x > x+c_1 > x+c_2$, but then $x = 2$ or 1 which determines the root itself. Thus, $S(P) \leq 4$ □

Construction:

- $d = 1 : x$
- $d = 2 : 2x^2 - 1$
- $d = 3 : -x(x-1)(x-3) - 1$

□