Week 4 Fun Assignment

Navier Stokes Equation

The Navier-Stokes equation, in fluid mechanics, is a set of partial differential equations that describe the motion of incompressible viscous fluids. The equation is a generalization of the equation devised by Swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and non-viscous fluids. In 1821, French engineer Claude-Louis Navier introduced the element of viscosity for the more realistic and vastly more difficult problem of viscous fluids. Throughout the middle of the 19th century, British physicist and mathematician, Sir George Gabriel Stokes improved on his work, though complete solutions were obtained only for the case of simple two-dimensional flows. The complex vortices and turbulence or chaos that occurred in three dimensional fluid flows as velocity increase have proven intractable to any but approximate numerical analysis methods.
The simulation of complex, dynamic processes that appear in nature or in industrial applications poses a lot of challenging mathematical problems, opening a long road from the basic problem, to the mathematical modelling, the numerical simulation, and finally to the interpretation of results. In order to achieve these goals, interdisciplinarity between applied mathematicians and experts in other fields becomes of increasing importance, since mathematical knowledge alone does not suffice in order to obtain a solution, but understanding the physics of the process is required as well. Computational Fluid Dynamics is one such state of the art problem.
Where do I fit in? Your job is to develop the Navier Stokes equation in numerical format and write a python program which can make these non-trivial computations for you.
Following links will guide you through the step by step development of the equation and code.
https://github.com/barbagroup/CFDPython/blob/master/lessons/15_Step_12.ipynb
https://www.sciencedirect.com/science/article/abs/pii/0167797787900116

Heat Equation

We have played a lot with the heat equation this week, here's the last problem statement on solving the heat equation. Assume that the entire plate was at a temperature of 0 degree Celsius. The temperature at the four edges remains constant at 0 degree celsius. At time t=0, a point heat source of 100 degree Celsius is placed at the mid point of the plate. Simulate the time evolution of the Heat Equation in 2D. (Note that this is different from Problem 3 in the mandatory assignment. Here we are simulating the actual time evolution, instead of assuming a steady state at each iteration)
Following link will guide you through the step by step development of Python code for Diffusion Equation.
https://scipython.com/book/chapter-7-matplotlib/examples/the-two-dimensional-diffusion-equation/

Acoustic Wave Equation

The acoustic wave equation governs the propagation of acoustic waves through a material medium. The form of the equation is a second order partial differential equation. The equation describes the evolution of acoustic pressure ‘p’ or particle velocity ‘u’ as a function of position ‘x’ and time ‘t’.

Simulate the solution of acoustic wave equation for the following scenario: Given that there are 10,000 steps in x direction, assume that the source is at the midpoint of this path and the receiver is stationary at 100 steps away from the source. Analyse the pressure as a function of position for 1000 time steps. You might want to plot position vs time with pressure amplitude contours. Make the necessary assumptions wherever needed.

The following explanation will be handy when simulating the solution:

Acoustic Wave Equation

The acoustic wave equation in 1D with constant density is:

$$\frac{\partial^2 p(x,t)}{\partial t^2} = c(x)^2 \frac{\partial^2 p(x,t)}{\partial x^2} + s(x,t)$$

where p denotes the pressure, c is the acoustic velocity, and s is the source term. This equation contains two second derivatives that can be approximated with the *Finite Difference Method* as follows:

$$\frac{\partial^2 p(x,t)}{\partial t^2} \approx \frac{p(x,t+dt) - 2p(x,t) + p(x,t-dt)}{dt^2}$$

and equivalently for the space derivative. Injecting these approximations into the wave equation allows us to formulate the pressure p(x) for the time step $t + dt$ (the future) as a function of the pressure at time t (now) and $t - dt$ (the past). We replace the time-dependent (upper index time, lower indices space) part by

$$\frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{\mathrm{d}t^2} = c^2(\partial_x^2 p) + s_i^n$$

solving for p_i^{n+1}. The extrapolation scheme is

$$p_i^{n+1} = c_i^2 \mathrm{d}t^2 \left[\frac{\partial^2 p(x,t)}{\partial x^2} p\right] + 2p_i^n - p_i^{n-1} + \mathrm{d}t^2 s_i^n$$

The space derivatives are determined by:

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{\mathrm{d}x^2}$$