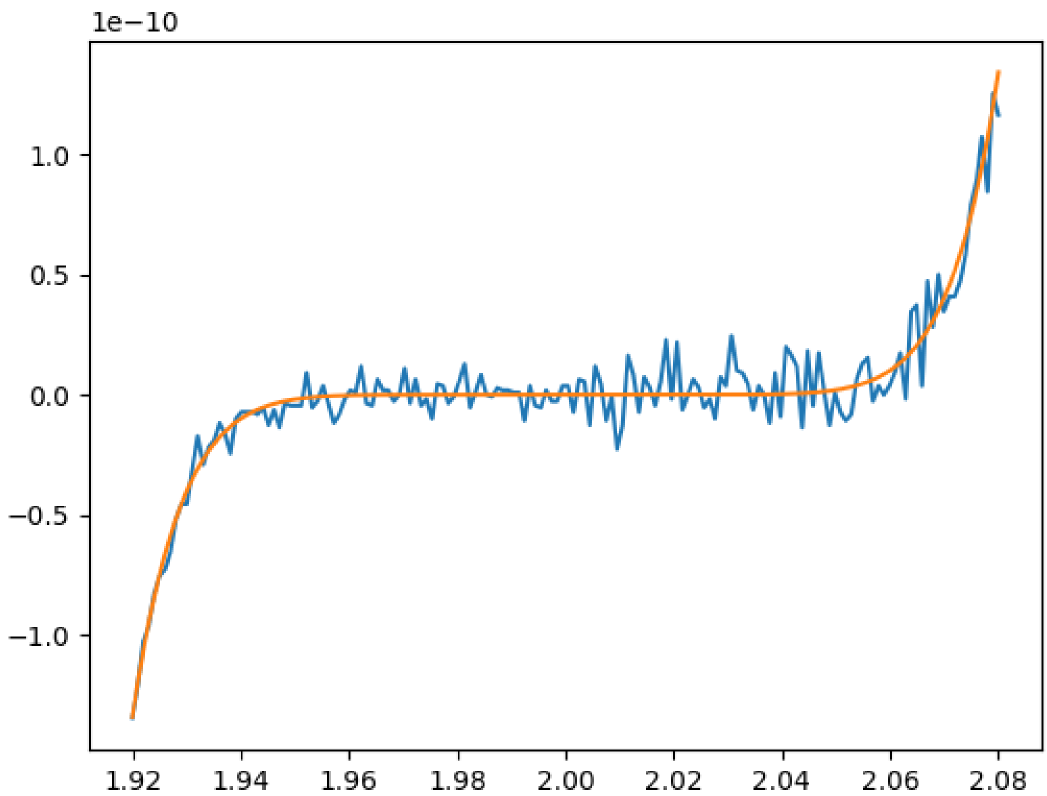
Homework 1

APPM 4600 Numerical Methods and Analysis

Quinn Lew

Question 1

This question asked us to plot (x-2)^9 using its factored and expanded form on the interval from 1.92 to 2.08 with .001 width intervals. I did this using python, specifically np.linspace to get the right number of intervals. In the figure below the blue line is the expanded form, and the orange line is the factored form.



1.iii.

I would guess that the differences in accuracy are from the operations and expansions that have use subtracting, resulting in the loss of accuracy in some data points.

Question 2.

This question is asking how we would approach the following if we wanted to keep all accuracy consistent, meaning not lose any digits or decimal places with the operation of subtraction.

2.iQuestion 2 was more simple problems asking us to manipulate equations to eliminate the operation of subtraction so the formula wouldn't lose accuracy.

Quinn Lew

Homework 1 Due 1/26/24

Question 2i

$\sqrt{x+1}-1$ for $x \approx 0$

$$\sqrt{x+1}-1 \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{(-\sqrt{x+1})^2-1}{\sqrt{x+1}+1} = \frac{x+1-1}{\sqrt{x+1}+1}$$

$$= \boxed{\frac{x}{\sqrt{x+1}+1}}$$ New algorithm without losing accuracy

2ii $\sin(x)-\sin(y) = \sin x - \sin y \times \frac{\sin x + \sin y}{\sin x + \sin y}$

$$\frac{\sin^2 x - \sin^2 y}{\sin x + \sin y} = \underbrace{\frac{\sin(x+y)\sin(x-y)}{\sin x + \sin y}}$$ for $x \approx y$

This works assuming $x = y + h$ meaning $h \approx 0$ with full precision

2iii $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$

$$\frac{1-\cos x}{\sin x} \times \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{\sin(1+\cos x)} = \frac{\sin^2 x}{\sin x(1+\cos x)}$$

$$= \boxed{\frac{\sin x}{1+\cos x}}$$ New algorithm without losing accuracy via subtraction

Qoinn Lew

Homework 1 continued Due 1/26/24

3. Find second degree polynomial $P_2(x)$ for $f(x) = (1+x+x^3)\cos(x)$ around $x_0 = 0$

a. Use $P_2(0.5)$ to approximate $f(0.5)$

Find an upper bound $|f(0.5) - P_2(0.5)|$ using an error formula and compare it to the actual error.

$$P_2(x) = f(x_0) + (x - x_0)\frac{f'(x_0)}{1!} + \frac{(x-x_0)^2 f''(x_0)}{2!}$$

if $x_0 = 0$ $\quad f(0) = (1+0+0)(\cos(0)) = 1$

$$f'(x) = \underbrace{(1+x+x^3)(-\sin(x))}_{0} + (1+3x^2)\cos(x)$$

$f'(0) = \quad 0 \quad 1 \times 1 = 1$

$$f''(x) = (1+3x^2)(-\sin x) + (1+x+x^3)(-\cos x) + 6x(\cos x) + (1+3x^2)(-\sin x)$$

$$f''(0) = -(1+0+0)(\cos(0)) = -1$$

$$P_2(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2 f''(0)}{2!}$$

$$= 1 + x - \frac{x^2}{2}$$

$$P_2(0.5) = 1 + 0.5 - \frac{(0.5)^2}{2}$$

$$1.5 - 0.125$$

$$= 1.375$$

Error formula for 2nd degree

$$f^3 = \frac{d}{dx}(f''(x))$$

$$= -2(1+3x^2)\cos(x) - 2(6x)\sin(x) + 6x(-\sin x) + 6\cos x - (1+3x^2)\cos x - (1+x+x^3)(-\sin x)$$

$$\approx (2 - 6x^2 - 1 - 3x^2 + 6)\cos x + (12x - 6x + 1 + x + x^3)\sin x$$

$$f^3 = (3 - 9x^2)\cos x + (x^3 - 17x + 1)\sin x$$

$$\text{Error for } x = 0.5 = \frac{|f^3(0.5)|\,|(0.5-0)|^2}{3!}$$

Question 3 was a lot more algebra and calculations to simplify equations and find error bounds and error itself. This question wanted a second degree polynomial to use as an approximation for a much more difficult formula. More work below.

Homework 1 cont, Due 1/26 Quinn Wu

$f^3(0.5) = (3-9(0.5)^2)\cos(0.5) + (0.5^3 - 17(0.5)+1)\sin(0.5)$
$= 0.658186921 - 4.015188886$
$= -3.357001965$

$\dfrac{|f^3(0.5)|\,x(0.5)^2}{6} = \underbrace{0.139875081}_{\text{Bounds the error for } P_2(0.5)}$

Actual error $|f(0.5) - P_2(0.5)|$
$|(1+0.5+0.5)^3(\cos(0.5)) - 1.375|$
$= 0.051071663 \rightarrow$ much less than the bound calculated above

3c. Instead of evaluating a trickier integral such as $\int_0^1 (1+x+x^3)\cos(x)dx$

Evaluating $\int_0^1 P_2(x)dx$

$\int_0^1 (1+x-\frac{x^2}{2})dx = (x+\frac{x^2}{2}-\frac{x^3}{6})\Big|_0^1$
$= 1+\frac{1}{2}-\frac{1}{6} = 1.\overline{33}$

d. Error in the integral

$|\int_0^1 f(x)dx - \int_0^1 P_2(x)dx|$

I used a calculator for this

$1.394982434 - 1.\overline{33}$
$= \boxed{0.0616491}$

3b. I was just unsure how to approach without given bounds

Question 4:

This question used the quadratic formula to find roots to a relatively simple equation. However there is a “bad” root because of the nature of subtraction to lose accuracy, so we formulate a new algorithm that does not lose accuracy with arbitrary values.

Quinn Lew

Homework 1 cont. Due 1/26

4. the quadratic $ax^2+bx+c=0$ $a=1$ $b=-56$ $c=1$

a. assume you can calculate $\sqrt{2}$ to 3 decimals using

$$r_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

calculate the roots

$$\frac{56\pm\sqrt{3136-4}}{2}=\frac{56\pm\sqrt{3132}}{2}=28\pm\sqrt{783}$$

$=28\pm 27.982=$ 55.982, $=0.018$ } accurate to 3 decimal places

relative error $\frac{x-\tilde{x}}{|x|}$

b. Manipulating $(x-r_1)(x-r_2)=0$ to relate r_1 & r_2 in terms of $a, b,$ and c for the "bad" root

$x^2-xr_1-xr_2+r_1r_2=0$

$x^2-(r_1+r_2)x+r_1r_2=0$ $a=1$

"Bad" root for $b^2-4ac<0$ $b=-(r_1+r_2)$

$c=r_1r_2$

$(r_1+r_2)^2-4r_1r_2<0$

$(r_1+r_2)^2<4r_1r_2$

$r_1+r_2<\pm 2\sqrt{r_1r_2}$

$r_1+r_2<2\sqrt{r_1r_2}$ $r_1+r_2<-2\sqrt{r_1r_2}$

$\underbrace{\frac{r_1+r_2}{\sqrt{r_1r_2}}<2}$ and $\frac{r_1+r_2}{\sqrt{r_1r_2}}<-2$

I would probably use this one as there are no negative signs present, so I'm guessing it is more reliable

Question 5

A much trickier question with the same idea of manipulating equations to minimalize error and maximize accuracy. Thanks to subtraction, all of these formulas have lost accuracy when carried out. I had a lot of difficulty with this question, regarding the arbitrary error and taylor series usage to minimalize subtraction.

Quinn Lew

Homework 1 continued Due 1/26

5. $y = x_1 - x_2$ $\quad \tilde{x}_1 = x_1 + \Delta x_1$

$\tilde{x}_2 = x_2 + \Delta x_2$

$\tilde{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$

a. Find upper bounds on absolute error (Δy) and relative error $\frac{|\Delta y|}{|y|}$

Relative error $\frac{|\tilde{y} - y|}{|y|}$

$\Delta y = \Delta x_1 - \Delta x_2$

$= \tilde{y} - y$

$|\Delta y| = |\tilde{y} - y|$

b. manipulate $\cos(x + \delta) - \cos(x)$ into an expression without subtraction

use taylor expansion because of sin & cos ± derivatives?

Wasn't really sure how to do this one