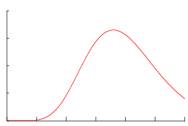
The Optimal Transport and its application in Color Transfer and Machine Learning

Abstract

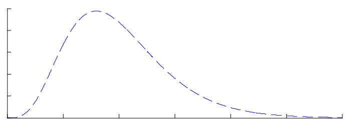
This report summarizes the development of optimal transport, with its application in color transfer and machine learning. The transport problem has been relaxed and regularized to reduce the artifacts caused by classic transport map. Other methods are added before or after color transfer process to improve the output results. Additional constraints are also considered to satisfy some specific requirements. Matlab codes are available to practice programming.

Introduction

1. Optimal transport
   a) Background: Transporting ore from mines to factories [10]. Presume that we have a collection of iron mines, producing w tonnes of iron ore every day, and a collection of factories that happen to consume w tonnes of iron ore every day. And the cost function $c(x,y)$ means the cost of transporting per tonne of ore in a mine at x to a factory at y. We want to find the optimal match between the offered mines and the received factories in order to get a minimum transportation cost.



Mines



Factories

   b) Monge problem: The history of Optimal Transport dates back to 1781[11] when this problem was first proposed by the French mathematician Gaspard Monge. Suppose that $f, g \geq 0$ are two distributions of mass on R^d, with $\int f(x)dx = \int g(y)dy$ =1. A transport plan $T: R^d \to R^d$ pushing forward the first one to the other, i.e. $\int_{T^{-1}(A)} f(x)dx = \int_A g(y)dy$ on any Borel subset $A \subset R^d$. We hope to find the optimal T that realizes the minimum

$$\int_{R^d} |T(x)-x| f(x)dx.$$

Rewrite the problem with a more general cost function, i.e.

$$\min\left\{\int c(x,T(x))f(x)dx\right\}.$$

   c) Kantorovich formulation: The Monge problem has stayed unsolved until a significant breakthrough made in 1940s. In 1942, the Russian mathematician Kantorovich relaxed the Monge problem by removing the one to one transfer (no splitting) constraint. He also introduced some useful mathematical tools which were later used by other mathematicians to propose the solution.

Creating a mathematical model, presume that μ and ν are two Borel probability measures defined on space X and space Y separately. The cost function $c: X \times Y \to [0,+\infty]$ is supposed to be continuous or semi-continuous. $c(x,y)$ can be interpreted as the amount of mass transferred from x to y. And $\Pi(\mu,\nu)$ is the set of all Borel probability measures γ on $X \times Y$ with fixed marginals, i.e. $\int_X \gamma dx = \mu, \int_Y \gamma dy = \nu$. $\Pi(\mu,\nu)$ is the so-called transport plan. A central problem is to find, among all plans γ, one which attain the minimum $\min\left\{\int_{X\times Y} c(x,y)d\gamma, \gamma \in \Pi(\mu,\nu)\right\}$.

This formulation differs with the Monge one in that instead of transferring T(x) of mass for all the mass originally located at x, it uses pair (x,y) to represent the number of mass of each x going to y. it turns the one to one problem into a one to N problem, which allows splitting, for each x can move to different y, with a broader solution set of the transport plan.

d) Brenier Theorem: Because of the contribution of Kantorovich and some other mathematicians, the solution of the original Monge problem has already been proposed. What`s more, in some cases, especially if $c(x,y) = |x-y|^2$, it has been proved to exist a unique transport plan induced by a map, i.e. no splitting of mass with the optimal γ. This means that the mass at x only move to the destination T(x), where T is the optimal map, thus providing solution to the Monge problem. This is what is done by Brenier in 1987. Generally, he proves these feature among some convex function and a special form for the optimal map: the optimal T is of the form $T(x) = \nabla u(x)$, for a convex function u. This means that T is a non-decreasing map.

e) Wasserstein distance: We can give a basic definition of Wasserstein distance based on the information above.

$$W_p(\mu,\nu) := \min\left\{\int_{X\times Y} |x-y|^p \, d\gamma, \gamma \in \Pi(\mu,\nu)\right\}^{\frac{1}{p}}$$

2. Color transfer

Color transfer aims to transform the colors of one source image or video to the colors of another target image or video. Color transfer is useful and receives a lot of attention in the computer vision and image processing fields recently, both for its academic research and application. For example, when clipping the videos, we can make sure all the different videos have the similar color mood and color grading using color transfer methods.

Related Literature

1. Beginning

Based on the statistical properties of a target images, Reinhard et al. proposed an approach to match the color distribution of a source image with the target image in $l\alpha\beta$ color space.

The method of histogram equalization in one-dimension (grayscale images) [4] happens to be related to the one-dimensional optimal transport. It`s reasonable for us to further extend the solution into three-dimension.

Pitie et al. [2] give an overview of the existing techniques and proposes a method to find a one to one color mapping to transfer the color palette. They use an iterative, non-linear algorithm to transform the probability distribution, which can be extended from one-dimension (grayscale pictures) to N-dimension (color pictures). Furthermore, they [3] focus on the linear color transformation to find the best one and propose a novel transformation which is based on the Monge-Kantorovicth problem. The different techniques are compared to each other with same source images and target images in RGB space and. The linear Monge-Kantorovicth solution turns out to be optimal among the output results for it minimizes the color variation in the pictures and some good monotonicity property to preserve the relevance of color position and illuminance. It is also compared under various color space to analyze the influence of color space on output results. They made a contribution to the first-time combination of Monge-Kantorovicth problem and color transfer, leading to follow-up research to improve the transformation.

2. Possible Challenges

While the evaluation of color transfer is not that straightforward, for it has something to do with individual personal aesthetic feeling and preference. However, there are some artifacts are not regarded as expected results. A common drawback of color transfer is visual artifacts and the output image receives the unnatural and unpleasant results. Moreover, because of the local irregularities of the classic optimal transport mapping [6], noise enhancement and detail loss happen due to the changing distance between the neighbor color points. An improvement of transport map is needed to achieve a visually-pleasing and proper feel. Meanwhile, it is supposed to improve the computation accuracy and expedite computation speed and the computational space and time complexity should be as low as possible.

3. Development

a) Julien et al. [6] improve the method by regularizing the transportation map with non-local filters. This concept was introduced by Yaroslavsky. The regularized map combines the output image filtered by non-local mean operator and the regularized source image. The regularized map decreases the noise contained in the transfer process and also preserve the lost details of the classic transport map. In practice, iteration is needed to remove the drawbacks mentioned above and the map gradually converges when the iteration number $k \to \infty$.

b) Because of the highly irregular transport map and the unwanted artifacts and noise amplification, Sira et al. [5] propose a variational formalism to relax and regularize the discrete transport map. Several parameters are introduced and further constraints are imposed, such as the range of every possible transportation and the total transportation amount. The extended gradient operator, applied to the color points, based on the graph structure is added as a regularization term. It is considered to further regularize the transport map and make use of the color gradient and relative position of the color points, especially several nearest neighbor points.
In practice, before the transfer process, the K-means clustering method is applied to extract a subsampled point cloud from large scale images. And after the transfer, the nearest neighbor algorithm is used to extend the map to the whole space.
Rabin et al. [7] extend the work by adjusting the spatial distribution of the colors and trying to adapt the relaxation parameters automatically.

c) Oriel et al. [8] add the pre-processing step to match the illuminance between the source image and the target image, which improve adaptation of the color saturation. The post-processing algorithm [2] is also a solution to reduce the noise and artifacts. It helps restore the gradient information of the source image.

d) Additional visual or semantic constraints can be considered depending on the different input expectation and specific requirements. Oriel et al. [8] offers a choice to split the transportation into two problems and transfer the two parts of the image separately (e.g. the background and the foreground) based on some constraints as visual saliency and face detection.

What`s more, displacement interpolation can be used to observe how the input transformed into the output result and is used by Bonneel et al.[9] in computer graphics applications.

Matlab codes for Optimal Transport

1.Optimal Transport with Linear Programming[1]
2.Optimal Transport in 1-D[2]
3.Sliced Optimal Transport[3]
4. Automated color grading using color distribution transfer[4]
5. The Linear Monge-Kantorovitch Linear Color Mapping for Example-Based Color Transfer[5]
6. Regularized Discrete Optimal Transport[6]

Machine learning and Wasserstein distance

A Restricted Boltzmann machine (RBM) is a generative stochastic neural network. It consists of one layer of visible units, one layer of hidden units and a bias unit. RBMs use Kullback Leibler divergence to measure the difference between the original data and learned model. Grégoire et al. [12] propose a new method to train the RBMs which uses the known metric between the observations to help define the Wasserstein distance. It is proved to have practical potential on data completion and denoising.

Conclusion

This report has introduced the optimal transport and its application in color transfer and machine learning. Several methods are proposed to improve the output image from different aspects. The prominent problem may reside in the irregularity of the transport map which may be solved by regularization. Other problems in computer vision as visual saliency and face detection are also combined with color transfer to expand the scope of application. The programming tasks help to understand the process thoroughly.

Recommendation

1. Avoid the beginning pre-processing or post-processing by incorporating more image details in the improved transport map.

[1]See http://nbviewer.jupyter.org/github/gpeyre/numerical-tours/blob/master/matlab/optimaltransp_1_linprog.ipynb.
[2]See http://nbviewer.ipython.org/github/gpeyre/numerical-tours/blob/master/matlab/optimaltransp_3_matching_1d.ipynb.
[3]See http://nbviewer.ipython.org/github/gpeyre/numerical-tours/blob/master/matlab/optimaltransp_4_matching_sliced.ipynb.
[4] See https://github.com/frcs/colour-transfer.
[5] See https://github.com/frcs/colour-transfer.
[6] See https://github.com/siraferradans/ColorTransfer.

2. Take more specific requirements into consideration to satisfy different needs in color tansfer.
3. Expedite computation speed and reduce computation complexity almost without any loss in the output results.

Reference

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