Optimal Transport and Its Application in Color Transfer

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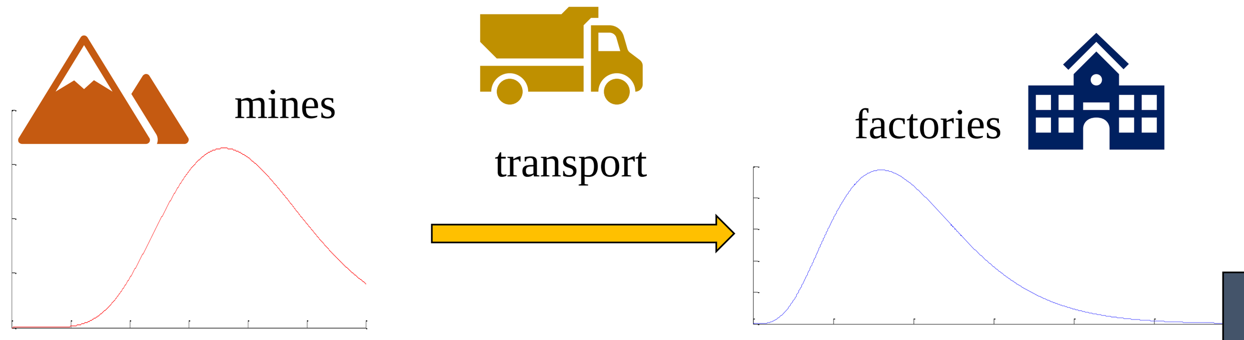
Optimal Transport

Background

Monge-Kantorovich problem : Transporting ore from mines to factories.

Imagine we have a distribution of iron mines, producing a total of 1000 tonnes of iron ore weekly, and a distribution of factories that consume a total of 1000 tonnes of iron ore weekly.

Knowing the cost c(x, y) per tonne of ore transported from a mine at x to a factory at y, the problem is to decide which mines should supply which factories so as to minimize the total transportation cost.



Introduction

The history of Optimal Transport started long-time ago and the problem was formalized by the French mathematician Gaspard Monge in 1781.

Given two densities of mass $f, g \geq 0$ on R^d, with $\int f(x)dx = \int g(y)dy = 1$, find a map $T: R^d \to R^d$ pushing the first one onto the other, i.e. such that $\int_{T^{-1}(A)} f(x)dx = \int_A g(y)dy$ for any Borel subset $A \subset R^d$, and minimizing the quantity

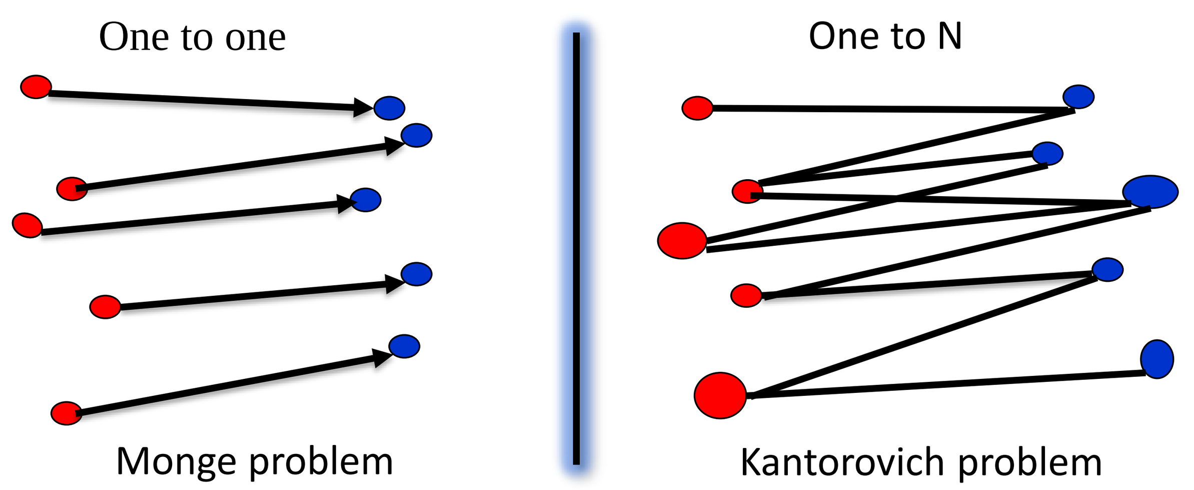
$$M(T) := \int_{R^d} |T(x) - x| \, f(x)dx.$$

More generally, compute the problem

$$\min \left\{ M(T) := \int c(x, T(x)) f(x)dx \right\}$$

for a more general cost $c: X \times Y \to R.$

A significant breakthrough was made by Kantorovich (1942) and he relaxed the optimization problem (the Monge problem) by dropping the requirement that all the ore from a given mine goes to a single factory.



In some cases, and in particular if $c(x, y) = |x - y|^2$, it is even possible to prove that particles at x are only sent to a unique destination T(x), where T is the optimal map, thus providing a solution to the original problem by Monge. This is what is done by Brenier (1987), where he also proves a very special form for the optimal map: the optimal T is of the form $T(x) = \nabla u(x)$, for a convex function u .

The Model[5]

The source and the target image are stored as a vector $X^0, Y^0 \in R^{N_0 \times d}$, *respectively*, where $d = 3$ is the number of channels (in RGB color space) and N_0 is the number of pixels. The color histogram of two images X^0, Y^0 can be represented using the distribution μ_{X^0}, μ_{Y^0}.

$$\mu_{X^0} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^0_i} \quad \mu_{Y^0} = \frac{1}{N} \sum_{j=1}^{N} \delta_{Y^0_j}$$

where δ_x is the Dirac measure at location $x \in R^d$, and where the positions of the supporting points are $X^0 = \left(X_i^0\right)_{i=1}^N$ and $Y^0 = \left(Y_j^0\right)_{j=1}^N$, where $X_i^0, Y_j^0 \in R^d$.

The goal of color transfer algorithms is to compute a transformation T^0 such that $\left(\tilde{X}^0\right)_i = T^0\left(X_i^0\right)_i$, where the new distribution $\mu_{\tilde{X}^0}$ is close (or equal) to μ_{Y^0}.

Σ is the color transfer matrix , $\Sigma_{i,j}$ = what percent of X_i^0 transferred to Y_j^0.

We introduce the cost matrix $C_{X^0,Y^0} \in R^{N \times N}$, $\forall (i,j) \in \{1,\ldots,N\}^2$, $(C_{X^0,Y^0})_{i,j} = c(X_i^0, Y_j^0)$.

We want to solve the relaxed optimization problem

$$\min_{\Sigma \in S_1} \left\langle C_{X^0,Y}, \Sigma \right\rangle = \sum_{i,j=1}^{N} c(X_i, Y_j)\Sigma_{i,j},$$

where S_1 is the set of permutation matrices

$$S_1 = \left\{ \Sigma \in R^{N \times N} \setminus \Sigma \mathrm{I} = \mathrm{I}, \Sigma^* \mathrm{I} = \mathrm{I}, \Sigma_{i,j} \in [0,1] \right\}.$$

We have defined $\mathrm{I} = (1, \ldots 1)^* \in R^N$ and denoted A^* as the adjoint of the matrix *A*, which for real matrices amounts to the transpose operation.

This is exactly (a discrete version of) Kantorovich's formulation of the optimal transport problem.

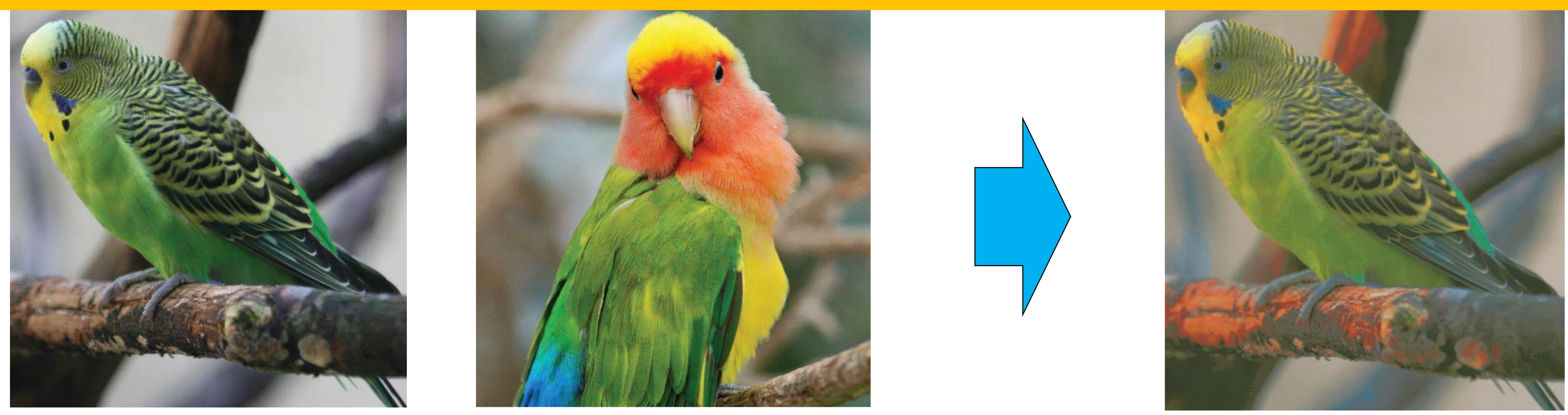
Algorithm:

1. **Histogram down-sample.** Before computing the transport, we define two smaller point clouds X and Y from X^0 and Y^0. These clouds are created such that their respective distributions μ_X and μ_Y are close to the two original distributions μ_{X^0} and μ_{Y^0}. Compute the smaller points sets X, Y from X^0, Y^0, respectively using K-means clustering.

2. **Compute mapping.** Compute the Σ by solving the optimization problem above and $T(X) = diag(\Sigma \mathrm{I})^{-1} \Sigma Y$.

3. **Obtain high resolution result.** The transformation *T* is extended to the whole space using a nearest neighbor interpolation.

Results [5]



X^0 $\quad$ Y^0 $\quad$ $\tilde{X}^0$

Future work[4]

Three important visual artifacts can be caused due to local irregularities of the transformation T^0.

Noise enhancement: this happens if the variance of the noise in X^0 increases after the application of T^0 to X^0.

Compression artifacts: these artifacts appear when the image X^0 is highly compressed and when pixels with a very similar color are mapped to different colors .

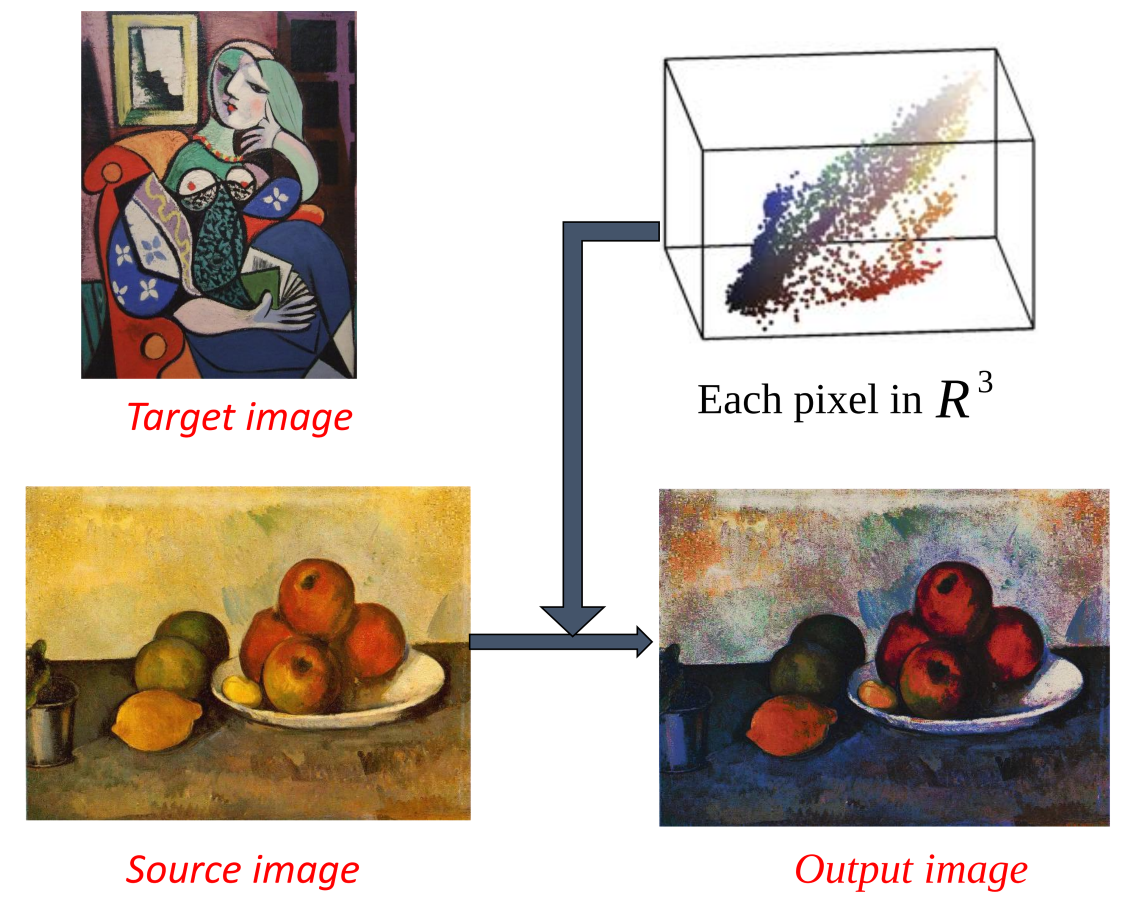
Detail loss: this results from a reduction of contrast between X^0 and $\tilde{X}^0$.

Need to further relax the problem and regularize the transportation T^0 to solve the problem.

Color Transfer

Introduction

Color transfer is a process that transforms the colors of one *source image* to the colors of another *target image*.



Two types of algorithms for color transfer

1. Employ the statistics of the colors of two images, such as histogram matching.
2. Give the pixel correspondence between the images, such as building joint-histogram.

We choose the first one.

Assessment

Less noise for the output image.

Adapt the different color saturation between target and source image.

Avoid visual artifacts.

Better without pre or post processing image regularization.

Keep the computational space and time complexity as low as possible.

Reference

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5. Ferradans, S., Papadakis, N., Peyré, G., and Aujol, J.-F. 2014. Regularized discrete optimal transport .