Demystifying Robust Optimization: a Practical Guide

Qun Chen

Abstract- Despite a number of criticisms, the Markowitz Mean-Variance Optimization (MVO) model remains the norm for portfolio optimization. A major issue is sensitivity of the model's solution to its input parameters. One approach employed by portfolio optimization practitioners to overcome this problem is using robust optimization (RO). RO is an intuitive and efficient way to model parameter uncertainty, but is not widely used by practitioners in the financial community. A major reason is that the approach is relatively new and is considered too technical. In this document I will give a step by step guide to use RO without a need to specify any mysterious parameters.

I. Introduction

Modern portfolio theory establishes portfolio construction as an optimization problem, where the goal is to maximize return for a given risk level, the so called Mean-Variance Optimization (MVO). MVO is far from perfect due to the negative effect that estimation error has on Mean-Variance optimal portfolios. This does not necessarily imply that MVO is flawed, but MVO must be modified when used in practice to achieve robust results. There are different ways to address the estimation error. On the estimation side, one can use estimators that are less sensitive to outliers, such as Bayesian and shrinkage estimators. On the modeling side, one can constrain portfolio weights or apply robust optimization techniques to specify ranges of values for estimated parameters, thus incorporating uncertainty into the optimization process itself.

Within the context of robust estimator, the Black-Litterman (BL) model (Black F and Litterman R 1992) is the best known Bayesian estimator. The BL expected return is calculated as a weighted average of the market equilibrium (or any prior benchmark returns) and the expert's views. The resulting expected return (posterior distribution) is then used as input to the optimization model. Portfolio weights computed in this fashion tend to be more intuitive and less sensitive to small changes in the prior inputs.

Another robust estimator, shrinkage estimator (Ledoit O., and Wolf M., 2003), is based on the trade-off between variance and bias. The variance is an error from sensitivity to small fluctuations in the data set, while the bias is an error from erroneous assumptions in the parameter estimation model. The variance of the parameter estimates can be reduced by increasing the bias in the estimation model. The shrinkage estimator typically consists of three components: (1) an estimator with little or no structure (e.g. sample covariance matrix); (2) an estimator with a rigid structure (e.g. constant covariance matrix); and (3) the shrinkage intensity.

Inclusion of constraints (Jagannathan, R., and MA T., 2003) in an optimization model is the most straightforward way to avoid unintuitive asset weights. Care must be taken when imposing constraints for the purpose of stability. If the constraints used are too tight, they will completely determine the portfolio weights instead of the forecasts from data.

Robust optimization (RO) (Goldfarb, D., Iyengar, G., Robust 2003, Tütüncü R.H and Koenig M 2004, Ceria, S., and Stubbs, R. A., 2006, Yin, C., Perchet, R. and Soupé, F. 2021) takes uncertainty in the inputs into consideration directly in the optimization model. RO does not limit parameters to point estimates. Instead it specifies the parameter in an uncertainty set, and requires the optimization to remain feasible for any values of the uncertain parameters within the pre-specified sets. Since there are many ways to specify the uncertainty set, we may view RO as an optimization framework.

Despite the fame of the above mentioned methods, they remain mysterious to many portfolio optimization practitioners, who find it is difficult to apply robust techniques in practice, including what the parameters should be set. In this document, I propose to use RO and give a practical guide to use RO in case that there are uncertainty in both return and covariance estimations. The reasons to choose RO over other methods are:

1) All the above mentioned methods are deeply connected
2) RO is more intuitive to use, and under some assumptions one can remove all mysterious parameters.
3) RO can be extended to mean-VaR and mean-CVaR portfolio optimization (Quaranta, A.G., Zaffaroni, A., 2008)

II. PROBLEM WITH STANDARD MVO

The standard MVO have three equivalent formulations and we consider the risk-aversion version:

$$\max_{w} r'w - \frac{\gamma}{2} w'\Sigma w \qquad (1)$$

Where

w is portfolio weight, the decision variable

r is asset expected return vector

Σ is asset covariance matrix

γ is risk aversion parameter

The optimal solution to (1) is:

$$w^{MVO} = \frac{1}{\gamma}\Sigma^{-1}r \qquad (2)$$

In order to better understand the effect of estimation error in expected returns on optimal portfolios, consider the following example. Suppose we want to build a five-asset portfolio with monthly rebalance. The objective is to maximize expected return subject to a constraint that limits the total risk to be no more than 10% annually. The sample mean, volatility and correlation are given in table 1. In order to use MVO model (1), we can turn volatility budget 10% (σ_0) to risk aversion parameter using the following formula. The proof is given in appendix A.

$$\gamma = \sqrt{\frac{r'(\Sigma^{-1})r}{\sigma_0^2}} \qquad (3)$$

Table 1: source BBG from 1/1/2000-5/31/2021, annualized mean/std/SR, monthly time series sample covariance matrix

	Mean (%)	Std (%)	SR	Covar	EQUITY	TSY	INF	EQUITY_SMALL	IG
EQUITY	7.6	15.2	0.5		1.00	-0.35	0.06	0.84	0.27
TSY	4.6	4.4	1.1		-0.35	1.00	0.68	-0.36	0.53
INF	5.9	5.7	1.0		0.06	0.68	1.00	0.01	0.71
EQUITY_SMALL	11.0	20.3	0.5		0.84	-0.36	0.01	1.00	0.23
IG	6.1	5.6	1.1		0.27	0.53	0.71	0.23	1.00

Table 2: MVO portfolio and MVO portfolio with perturbation

	MVO			MVO with INF return changed from 5.9% to 7.4%	
	Weight (%)	Risk Contrib (%)		Weight (%)	Risk Contrib (%)
EQUITY	15.75	0.82		10.63	0.53
TSY	218.55	6.97		155.32	4.76
INF	-0.48	-0.02		82.37	4.00
EQUITY_SMALL	24.45	1.80		24.01	1.69
IG	10.42	0.44		-24.36	-0.98
Total	268.69	10.00		247.98	10.00

MVO portfolio weights and risk contribution are presented in table 2. MVO generates a very counter-intuitive portfolio with large long-positions in TSY and short position in INF even though these two assets have similar SR and highly correlated. If we change the expected return of INF from sample mean 5.9% to 7.4%, the new portfolio looks very different from the original portfolio with weights of INF and

IG flipped sign. In appendix B, we identify the cause of problematic portfolios created by standard MVO using principal component analysis.

III. Robust Optimization

Robust optimization (Ben-Tal A., Nemirovski 1998) is a subfield of optimization that deals with uncertainty in the parameters of optimization problems. Under this framework, the objective or constraint functions are assumed to belong to certain sets in function space (the so-called uncertainty sets). The goal is to make a decision that is feasible no matter what the constraints turn out to be, and optimal for the worst-case objective function.

Consider the MVO model defined in (1). The expected return r and covariance matrix Σ can be viewed as uncertain parameters in the optimization problem. As a result the objective is to maximize random variables instead of deterministic. In robust optimization, one makes the problem well-defined by assuming that the uncertain parameters, for example expected return, vary in a particular set defined by one's knowledge about its distribution. Then the worst-case (max-min) approach is taken: find portfolio weights such that the portfolio return is maximized even when the vector of realizations for the asset returns takes its worst value over the uncertainty set. (Kim, W., Kim, J., Mulvey, J., Fabozzi, F.) empirically examined equity market and demonstrate the importance of focusing on worst cases for achieving portfolio robustness.

In this section we first discuss the robust versions of MVO when uncertainty is assumed to be present only in the expected return estimates. We show a couple ways of modeling the uncertainty, box and quadratic. We then make a proposal to run RO with only data-driven parameters. We conclude this section by studying a RO model with uncertainty in the asset return covariance matrix

A. *Box Uncertainty Set For Expected Return*

The simplest possible choice for the uncertainty set for expected return r is a box around the estimated expected return $\overline{r_i}$.

$$U_k = \{\, r \mid |r_i - \overline{r_i}| \leq k_i,\ i = 1,2,\ldots \} \qquad (4)$$

The $k_i$could be specified by assuming some confidence interval around the estimated expected return For example, if expected returns are estimated using sample mean and returns are assumed to follow a normal distribution, the 95% confidence interval for $k_i$can be obtained by setting $k_i = 1.96\sigma_i/\sqrt{T}$, where $\sigma_i$is sample volatility and T is sample size.

The robust formulation of MVO under the box uncertainty set is :

$$\max_w \bar{r}'w - k'|w| - \frac{\gamma}{2} w'\Sigma w \qquad (5)$$

This formulation is obvious. If the weight of asset i in the portfolio is negative, the worst-case expected return for asset i is $r_i + k_i$. If the weight of asset i in the portfolio is positive, then the worst case expected return for asset i is $r_i - k_i$. Therefore, the mathematical expression in the objective agrees with our intuition: it tries to maximize the worst-case expected portfolio return.

Table 3: RO portfolios BOX uncertainty with different confidence interval

	Sample Vol	MVO	RO BOX 90%	RO BOX 95%	RO BOX 99%
EQUITY	15.2	15.75	0.00	0.00	0.00
TSY	4.4	218.55	172.59	157.20	113.68
INF	5.7	-0.48	7.57	8.73	10.45
EQUITY_SMALL	20.3	24.45	19.36	13.96	0.00
IG	5.6	10.42	57.04	71.08	104.16
Total		268.69	256.56	250.97	228.29

We run the same MVO asset allocation problem with expected return in the confidence intervals 90%, 95% and 99%. There are a few observations from results reported in table 3:

- There are less extreme long-short weights
- Assets whose mean return estimates are less accurate (larger sample volatility) are penalized in the objective function, and tend to have smaller weights in the optimal portfolio allocation. For two volatile assets EQUITY and EQUITY_SMALL allocations using the largest uncertainty set 99% are zero.

B. Quadratic Uncertainty Set For Expected Return

Another frequently used uncertainty sets for the expected returns vector are quadratic or ellipsoidal uncertainty set. The uncertainty $r - \overline{r}$ is assumed to follow a multivariate normal distribution with mean zero and covariance matrix Ω. The uncertainty set around the estimated mean return can be written as follows (k^2represents the level of uncertainty):

$$U_k = \{ r \mid (r - \overline{r})' \, \Omega^{-1} (r - \overline{r}) \leq k^2 \} \qquad (6)$$

The intuition behind this uncertainty set is as follows. The random realizations of return r in the optimization horizon are more likely to deviate from $\overline{r}$ if their volatility is higher, so deviations are scaled by the inverse of the estimation uncertainty covariance matrix. The parameter k^2 corresponds to the overall amount of scaled deviations of the realized returns from the forecasts against which the investor would like to be protected. Note that estimation uncertainty covariance matrix Ω is different from return covariance matrix Σ.

RO allocates the portfolio for the worst possible estimates of the expected return. Mathematically, this can be expressed as:

$$\max_{w} \min_{r \in U_k} r'w - \frac{\gamma}{2} w'\Sigma w \qquad (7)$$

Problem (7) is called the robust counterpart of standard MVO problem (1). It is a max-min model, and is not in the form that can be input into a standard optimization solver. It can be reformulated as below (proof is given in appendix C).

$$\max_{w} r'w - \frac{\gamma}{2} w'\Sigma w - k\sqrt{w'\Omega w} \qquad (8)$$

Problem (8) is convex, but it is not in the form understood by an optimization solver as a convex model. Let y = $\sqrt{w'\Omega w}$, problem (8) becomes:

$$\max_{w} r'w - \frac{\gamma}{2} w'\Sigma w \text{ - ky}$$

$$\text{s.t. } w'\Omega w = y^2 \; and \; y \geq 0 \qquad (9)$$

It is still not convex since there is a nonlinear equity constraint. It is fine to convert the equality to inequality because at the optimum it will become equity.

$$\max_{w} r'w - \frac{\gamma}{2} w'\Sigma w \text{ - ky} \qquad (10)$$

$$\text{s.t. } w'\Omega w \leq y^2 \; and \; y \geq 0$$

Problem (10) is a second order cone programming (SOCP) model, and can be solved by a number of optimization solvers (CPLEX, GUROBI, MOSEK).

The remaining question is how to choose uncertainty matrix Ω. In the RO literatures, a few types of uncertainty matrix are proposed. (Yin and Romain 2019) compared them and conclude that $\Omega = \text{diag}(\Sigma)$is the best candidate. Replacing Ω with $\text{diag}(\Sigma)$, we have the final format of robust MVO with quadratic uncertainty set:

$$\max_{w} r'w - \frac{\gamma}{2} w'\Sigma w \text{ - ky} \qquad (11)$$

$$\text{s.t. } w' \text{diag}(\Sigma) w \leq y^2 \; and \; y \geq 0$$

We run the same MVO asset allocation using RO model (11) with k equal to half of the assets average SR, as suggested by (Yin and Romain 2019). The allocation and marginal risk contribution for standard MVO, RO with box uncertainty set and RO with quadratic uncertainty set are summarized in table 4. The results demonstrate that RO Quadratic has more balance allocations than standard MVO and RO Box. Risk contribution also demonstrates the same pattern. Quadratic uncertainty set ensures that the total budget of tolerance to uncertainty is distributed evenly among expected return estimates that have different variability, while box uncertainty set assumes that all assets will achieve their worst-case return at the same time.

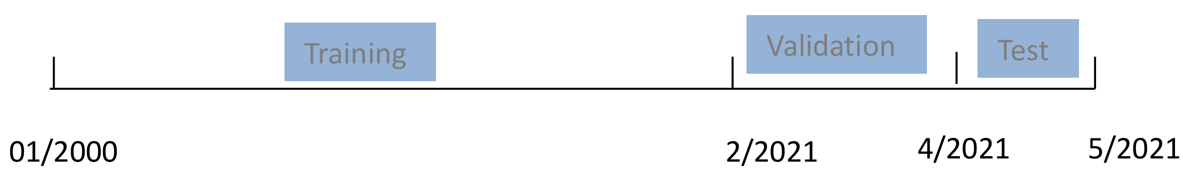
Table 4: Comparison of different allocations

	Standard MVO		RO Box k=95% ci		RO Quadratic k=half of average SR	
	Weight (%)	Risk Contrib (%)	Weight (%)	Risk Contrib (%)	Weight (%)	Risk Contrib (%)
EQUITY	15.75	0.82	0.00	0.00	13.69	0.73
TSY	218.55	6.97	157.20	5.76	128.31	3.90
INF	-0.48	-0.02	8.73	0.40	43.12	2.06
EQUITY_SMALL	24.45	1.80	13.96	0.36	14.81	1.01
IG	10.42	0.44	71.08	3.48	47.26	2.31
Total	268.69	10.00	250.97	10.00	247.18	10.00

C. *Making Parameter k^2 Data Driven*

By far, the parameter for level of uncertainty (k^2) still remains mysterious. In this section, we borrow the parameter tuning idea from the machine learn community. In machine learning, the input data are usually divided into three subsets: training, validation and test. A model is initially fit on a training dataset. Next the validation dataset is evaluated against the fitted model to tune model parameters. Finally the test dataset is used to evaluate the out of sample performance.

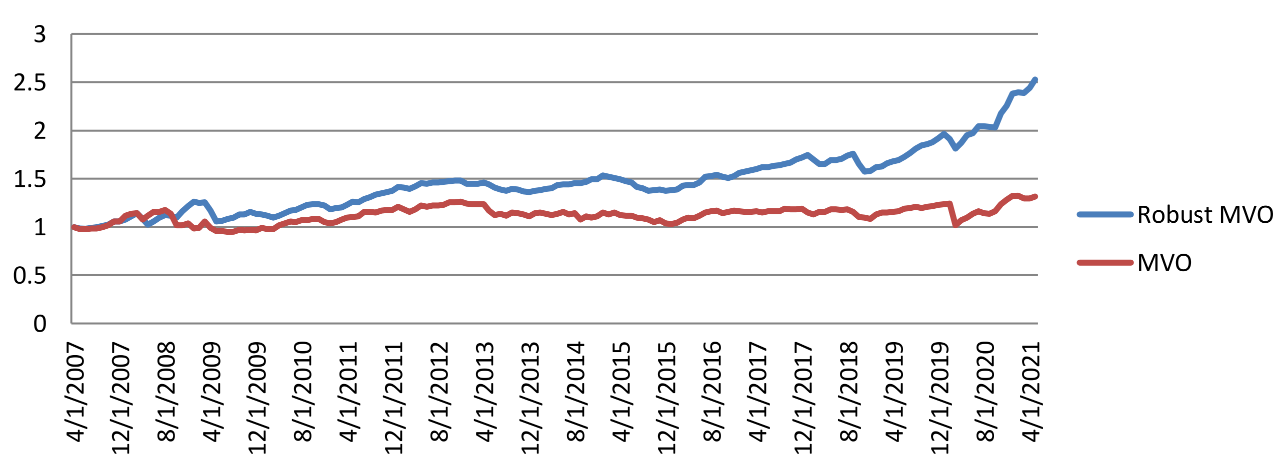
For robust portfolio optimization, the split of return time series is illustrated as below. Suppose we run asset allocation for May 2021. May 2021 is the test dataset. Three months before May 2021, i.e. February-April 2021, are the validation dataset. All the months before February 2021 belong to the training dataset.



The robust optimization is backtested in the following steps:

1. For a given k, obtain the optimal asset allocations using the training data and evaluate the SR in the validation period based on the allocations. SR in the validation period is computed in daily frequency.
2. Search for k to optimize SR in the validation period, denote k corresponding to the optimal SR as k^{opt}.
3. Rebuild the robust optimization model using the combined data in Training and Validation periods, and k^{opt} as the level of uncertainty. The allocations from this final model are evaluated against the test month to produce the out of sample portfolio return.
4. Move the training, validation and test dataset one month ahead, and repeat 1-3.

Figure 1 Backtest Robust vs Standard MVO



Portfolio values from the backtest (full and non-leverage investment constraint, i.e. sum of weight to 100%, is added) between 4/2007 and 5/2021 are shown in Figure 1. Robust allocation achieved 0.82 annual SR, while SR of stand MVO is 0.25. Note we restrict the scope of this document to compare optimization approaches (RO vs standard MVO). A good asset allocation requires more advanced expected return and covariance forecast techniques that we will explore in the future document. In addition, we cheated in the title of this section since length of validation period is still a parameter. However unlike the uncertainty level k^2, it is easier to choose and justify in practice.

D. Uncertainty in Covariance Matrix

MVO is less sensitive to inaccuracies in the estimate of the covariance matrix than it is to estimation errors in expected returns (Chopra, V and Ziemba, W.T.). Nonetheless, investors may still have concerns about big drawdowns due to correlation breakdown in the optimization horizon. Robust MVO with respect to covariance matrix uncertainty can be expressed similarly as to expected return:

$$\max_{w} \ (r'w - \frac{\gamma}{2} \max_{r \in U_\Sigma} w'\Sigma w) \qquad (12)$$

The worst-case portfolio variance is given by $\max_{r \in U_\Sigma} w'\Sigma w$. As described before, RO model in (12) seeks to optimize the expected return for the worst possible risk realization in the uncertainty set U_Σ.

Let's consider the box uncertainty set for the covariance matrix:

$$U_\Sigma = \{ \Sigma \mid \underline{\Sigma} \leq \Sigma \leq \overline{\Sigma}, and\ \Sigma\ is\ PSD \} \qquad (13)$$

Where $\underline{\Sigma}$ and $\overline{\Sigma}$ are elementwise lower and upper bounds of Σ.

For any fixed portfolio weights **w**, we can find the worst case risk by solving the optimization problem

$$\max_{\Sigma} w'\Sigma w$$

$$\text{s.t. } \underline{\Sigma} \leq \Sigma \leq \overline{\Sigma}\ and\ \Sigma\ is\ PSD \qquad (14)$$

The dual problem of (14) is given by (Lobo M.S. and Boyd S 2000).

$$\min_{\underline{\Lambda}, \overline{\Lambda}} Tr(\overline{\Sigma}\,\overline{\Lambda} - \underline{\Sigma}\,\underline{\Lambda})$$

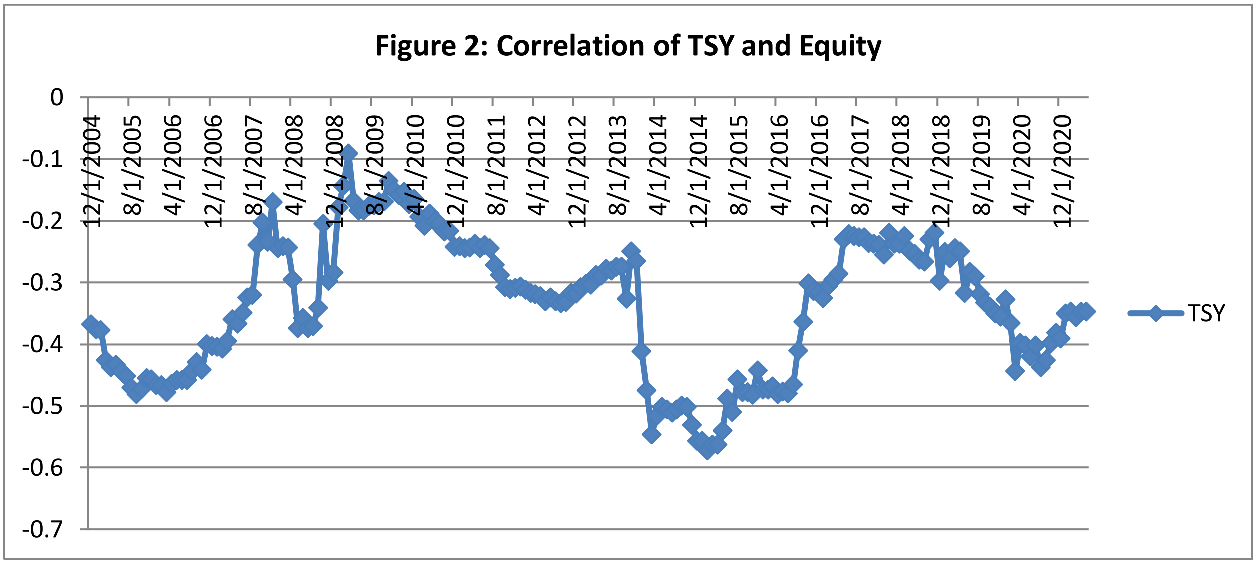
$$\text{s.t. } \underline{\Lambda} - \overline{\Lambda} - ww' \text{is PSD}, and\ \overline{\Lambda}_{ij} \geq 0, \underline{\Lambda}_{ij} \geq 0 \qquad (15)$$

Function Tr is the trace of a matrix. Dual decision variables $\overline{\Lambda}$ and $\underline{\Lambda}$ are corresponding to constraints $\Sigma \leq \overline{\Sigma}$ and $\underline{\Sigma} \leq \Sigma$. The optimal value of the dual problem will be at least as large as the optimal value of the primal problem. Therefore, we can use the expression $\min_{\underline{\Lambda}, \overline{\Lambda}} Tr(\overline{\Sigma}\,\overline{\Lambda} - \underline{\Sigma}\,\underline{\Lambda})$ to replace $\max_{r \in U_\Sigma} w'\Sigma w$ in (12) and arrive the following RO model with respect to U_Σ:

$$\max_{w, \overline{\Lambda}, \underline{\Lambda}} \ (r'w - \frac{\gamma}{2} Tr(\overline{\Sigma}\,\overline{\Lambda} - \underline{\Sigma}\,\underline{\Lambda})$$

$$\text{s.t. } \underline{\Lambda} - \overline{\Lambda} - ww' \text{is PSD}, and\ \overline{\Lambda}_{ij} \geq 0, \underline{\Lambda}_{ij} \geq 0 \qquad (16)$$

Optimization model (16) is a semidefinite program (SDP), which is convex, but harder to solve than a quadratic or SOCP model. The solvers that can handle SDP include commercial solver MOSEK and open source solver CVXOPT.



In the asset allocation example, we noticed the correlation dynamics among assets. For example sample correlation of TSY and Equity varies between -0.57 and -0.09 during 2004 to 2021 using 6 month rolling window. We build an RO model (16) assuming the correlations vary between the minimum and maximum realized values, and there is no uncertainty about volatility estimation, i.e.

$$\underline{\Sigma}_{ij} = \underline{\rho}_{ij}\sqrt{\Sigma_{ii}\,\Sigma_{jj}} \;\le\; \Sigma_{ij} \le \overline{\rho}_{ij}\sqrt{\Sigma_{ii}\,\Sigma_{jj}} = \overline{\Sigma}_{ij}$$

Where Σ_{ii} and $\Sigma_{jj}$are constant and $\underline{\rho}_{ij}$ is the minimum realized sample correlation and $\overline{\rho}_{ij}$ is the maximum realized sample correlation.

Table 5: RO portfolios with correlation uncertainty set

	EQUITY	TSY	INF	EQUITY_SMALL	IG
Weight	0.00	176.72	0.00	22.10	0.00

Table 6: Minimum, maximum, sample and RO selected correlation matrices

CORR MIN	EQUITY	TSY	INF	EQUITY_SMALL	IG	CORR MAX	EQUITY	TSY	INF	EQUITY_SMALL	IG
EQUITY	1.00	-0.57	-0.34	0.67	-0.17	EQUITY	1.00	-0.09	0.34	0.96	0.55
TSY	-0.57	1.00	0.44	-0.61	0.27	TSY	-0.09	1.00	0.91	-0.11	0.93
INF	-0.34	0.44	1.00	-0.33	0.50	INF	0.34	0.91	1.00	0.27	0.86
EQUITY_SMALL	0.67	-0.61	-0.33	1.00	-0.16	EQUITY_SMALL	0.96	-0.11	0.27	1.00	0.45
IG	-0.17	0.27	0.50	-0.16	1.00	IG	0.55	0.93	0.86	0.45	1.00
CORR SAMPLE	EQUITY	TSY	INF	EQUITY_SMALL	IG	CORR RO	EQUITY	TSY	INF	EQUITY_SMALL	IG
EQUITY	1.00	-0.35	0.06	0.84	0.27	EQUITY	1.00	-0.27	-0.02	0.80	0.15
TSY	-0.35	1.00	0.68	-0.36	0.53	TSY	-0.27	1.00	0.66	-0.11	0.58
INF	0.06	0.68	1.00	0.01	0.71	INF	-0.02	0.66	1.00	0.00	0.66
EQUITY_SMALL	0.84	-0.36	0.01	1.00	0.23	EQUITY_SMALL	0.80	-0.11	0.00	1.00	0.16
IG	0.27	0.53	0.71	0.23	1.00	IG	0.15	0.58	0.66	0.16	1.00

The RO with correlation uncertainty allocations are shown in table 5. The confidence intervals we have chosen is too wide, as a result only two negatively correlated assets TSY and Small Equity are picked to make sure the worst case risk is minimized. RO picks the worst case correlations that lie between the minimum and maximum realized values and ensures this correlation matrix is PSD.

A. How to set risk aversion on optimization

The risk aversion parameter represents the trade-off between the portfolio's risk and expected return. An experience portfolio manager often translates risk to expected return based on a rule-of-thumb premium. For example S&P long term volatility is 20% and expected return is 6%. If the portfolio manager thinks taking 20% risk is equivalent to 3% return haircut, then the risk aversion parameter γ can be set to 0.015 since $\frac{1}{2} * 0.015 * 20^2 = 3$. Alternatively, portfolio managers may plot the efficient frontier to get an idea of their target risk and return trade-off. Often time, there is a risk budget to the portfolio construction process. This risk budget can be translated into risk aversion parameter using the formula described below. Consider the risk budget version MVO:

$$\max_{w} r'w$$

$$\text{s.t. } w'\Sigma w \le \sigma_0^2 \qquad (20)$$

Since the constraint always binds, we can rewrite it as equality constraint:

$$\max_{w} r'w$$

$$\text{s.t. } w'\Sigma w = \sigma_0^2 \qquad (21)$$

The Lagrangian of (21) is:

$$L(w,\lambda) = r'w - \lambda(w'\Sigma w - \sigma_0^2) \qquad (22)$$

Take partial derivative with respect to w and set it to zero, we have:

$$w = \frac{1}{\gamma}\Sigma^{-1} r \qquad (23)$$

Plug (23) into the constraint in (21) and solve for γ, we prove equation (3)

$$\gamma = \sqrt{\frac{r'(\Sigma^{-1})r}{\sigma_0^2}}$$

In case that there are other constraints in problem (20), we can let optimizer solve the problem and report shadow price of the constraint $w'\Sigma w \le \sigma_0^2$, which is corresponding to the risk aversion parameter γ.

B. Why MVO Creates Problematic Portfolio:

The problematic portfolios from standard MVO can be explained using principal components of the correlation matrix. Define the principal component portfolios as:

$w^p = P'\sigma w$ where P is eigenvectors of the correlation matrix, $\sigma$is asset volatility

The expected return of the principal component portfolios is:

$r^p = P'\sigma^{-1}r$

The MVO optimal solution with respect to the principal component portfolios is:

$w^{p,MVO} = \frac{1}{\gamma}D^{-1}r^p$, where D is a diagonal matrix of the variance of principal component portfolios

Since the principal component portfolios are not correlated (D is diagonal), we can write the MVO optimal solution as:

$w_i^{p,MVO} = \frac{1}{\gamma} * \frac{r_i^p}{\sqrt{D_i}} * \frac{1}{\sqrt{D_i}}$ i = 1,2,… is the index for principal portfolios (24)

The least important principal component portfolio is the one with the lowest volatility (D_{min}), and corresponding to the smallest eigenvalue. Any error in the estimation of the covariance matrix is likely to lead to an underestimation of the volatility of the least important portfolio ($\frac{1}{\sqrt{D_i}}$). Furthermore, any estimation error for expected return will make the SR term $\frac{r_i^p}{\sqrt{D_i}}$ relatively large.

C. Solution for RO with Quadratic Uncertainty Set

To prove (8), we first look at the inner minimization problem for expected return r:

$\min_r r'w$

s.t. $(r - \overline{r})'\,\Omega^{-1}(r - \overline{r}) \leq k^2$ (25)

The Lagrangian of this problem takes the form:

$$L(r, \lambda) = (r'w) - \lambda((r - \overline{r})'\,\Omega^{-1}(r - \overline{r}) - k^2)$$

Differentiating this with respect to r, we have the first-order condition:

$$w - 2\,\lambda\;\Omega^{-1}(r - \overline{r}) = 0$$

Therefore, the optimal value of $r$is:

$$r = \overline{r} - \frac{1}{2\,\lambda}\Omega w$$

The optimal value of λcan be found by maximizing the Lagrangian after substituting r in $L(r, \lambda)$, and we get:

$$\lambda = \frac{1}{2k}\sqrt{w'\Omega \mathrm{w}}$$

Finally substitute λin the Lagrangian, we have the RO model with quadratic uncertainty set:

$$\max_w r'w - \frac{\gamma}{2}w'\Sigma w - k\sqrt{w'\Omega w}$$

REFERENCES

[1] Ben-Tal A., Nemirovski, A., Robust convex optimization, Mathematics of Operation research, Vol. 23 No. 4, 1998.

[2] Black F and Litterman R, Global portfolio optimization. Financial Analysts Journal, pages 28–43, September-October 1992.

[3] Ceria, S., and Stubbs, R. A., 2006. Incorporating estimation errors into portfolio selection: robust portfolio construction. Journal of Asset Management 7(2), 109–127.

[4] Chopra, V and Ziemba, W.T., The effects of errors in means, variances, and covariances on optimal portfolio choice. Journal of Portfolio Management, pages 6–11, 1993.

[5] Goldfarb, D., Iyengar, G., Robust portfolio selection problems, Mathematics of Operation research, Vol. 28, 2003.

[6] Jagannathan, R., and MA T., 2003, Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps, The Journal of Finance, vol. 58(4), pages 1651-1684, 08.

[7] Kim, W.; Kim, J.; Mulvey, J.; Fabozzi, F. Focusing on the worst state for robust investing. Int. Rev. Financ. Anal. 2015, 39, 19–31.

[8] Ledoit O., and Wolf M., 2003, Honey, I Shrunk the Sample Covariane Matrix, Journal of Portfolio Management, Volume 31, 219-255.

[9] Lobo M.S. and Boyd S., The worst-case risk of a portfolio. Unpublished paper, 2000. https://web.stanford.edu/~boyd/papers/pdf/risk_bnd.pdf

[10] Pedersen, HP, A Babu, and A Levine (2021). “Enhanced Portfolio Optimization” Financial Analysts Journal, pp. 124-151, doi: https://doi.org/10.1080/0015198X.2020.1854543

[11] Quaranta, A.G., Zaffaroni, A., 2008. Robust optimization of conditional value at risk and portfolio selection. Journal of Banking and Finance 32, 2046–2056

[12] Tütüncü R.H and Koenig M. (2004), Robust asset allocation, Annals of Operations Research, 132, pp. 132-157.

[13] Yin, C., Perchet, R. and Soupé, F., (2021), A practical guide to robust portfolio optimization, *Quantitative Finance*, **21**, issue 6, p. 911-928.