1 Open Discussion to Our Work

Here we provide additional discussions and clarifications based on valuable feedback from the reviewers. These elaborations aim to enhance the practical relevance of our theoretical work, justify the assumptions made in our theorems, and illustrate the connection between our theoretical framework and real-world applications.

1.1 Practical Impact of Our Theoretical Results

To bridge the gap between theory and application, we expand on how our theoretical findings can guide prompt design and analysis in real-world scenarios.

Guidance from Theorems 5 and 8

Theorems 5 and 8 highlight that rank deficiency in the model's parameter matrices can lead to information loss when using graph prompts. In practice, if a prompt does not perform as expected, this insight suggests that the pre-trained model might exhibit a non-full-rank condition. Recognizing this issue allows practitioners to:

- **Diagnose Prompt Performance Issues**: By examining the rank properties of the model's weight matrices, practitioners can identify whether rank deficiency is causing suboptimal prompt performance.

- **Adjust Model Architecture or Training**: If rank deficiency is detected, steps can be taken to modify the model architecture or retrain the model to ensure full-rank weight matrices, thereby retaining essential information from prompts.

Implications of Theorem 6

Theorem 6 provides a theoretical upper bound on the performance of prompts with limited complexity for datasets of a certain scale. This result has practical implications:

- **Balancing Prompt Complexity and Dataset Size**: Practitioners can use this theorem to determine the necessary complexity level of prompts relative to the size of the dataset. For larger datasets, increasing prompt complexity may be necessary to achieve better results.

- **Optimizing Prompt Design**: By understanding the limitations imposed by prompt complexity, prompt designers can make informed decisions to enhance model performance without unnecessary complexity.

Application of Theorem 7

Theorem 7 offers a method to estimate the required size of a prompt through the parameter ϵ. Practical applications include:

- **Estimating Prompt Sufficiency**: Calculating ϵ allows practitioners to infer whether the current prompt size is sufficient for the dataset. An underperforming prompt with a high ϵ value may indicate the need for a larger prompt.

- **Guiding Further Training**: If the prompt size is deemed sufficient but performance is still lacking, this may suggest that the optimal parameters have not yet been found, indicating a need for further training or parameter tuning.

By applying these theoretical insights, practitioners can make informed decisions about model adjustments and prompt complexity, directly impacting the effectiveness of graph prompting in practical applications.

1.2 Validity of the Full-Rank Assumption in Practical Applications

We provide further justification for the full-rank assumption used in our theoretical framework, demonstrating its validity in real-world scenarios.

Empirical Observations

In practice, well-trained models often maintain full-rank weight matrices due to:

- **Initialization Techniques**: Standard initialization methods, such as orthogonal initialization and He initialization, start with full-rank weight matrices.

- **Training Dynamics**: Gradient-based optimization tends to preserve the full-rank property of weight matrices, as full-rank matrices offer greater expressive power, which the training process exploits to improve model performance.

Theoretical Justification

From a mathematical standpoint:

- **Measure Zero of Non-Full-Rank Matrices**: The set of non-full-rank matrices has measure zero in the space of all matrices. This is because a matrix is non-full-rank only when its determinant is exactly zero, an event with negligible probability under continuous distributions.

Addressing Non-Full-Rank Cases

We acknowledge that non-full-rank weight matrices may occur in certain situations. To ensure robustness:

- **Analyses Without Full-Rank Assumption**: Sections 3 and 5 of the main paper provide theoretical analyses that do not rely on the full-rank assumption, covering cases where weight matrices may be rank-deficient.

- **Comprehensive Understanding**: By including these analyses, our theoretical framework remains applicable even when the full-rank assumption does not hold, offering a comprehensive understanding of model behavior under different conditions.

1.3 Linking Theoretical Framework to Real-World Applications

We emphasize the practical implications of our theoretical work and its connection to real-world applications.

Graph Prompting Beyond Tuning

We advocate for viewing graph prompting not merely as a tuning technique but as a powerful tool capable of learning and implementing graph data manipulation strategies. This perspective enables:

- **Advanced Data Operations**: Utilizing graph prompts to perform complex data operations, such as integrating multi-source graph databases and reasoning over heterogeneous graph structures.

- **Enhanced Model Capabilities**: Leveraging prompts to extend the functionality of pre-trained models without retraining, facilitating adaptability and scalability.

Potential Applications

Our theoretical framework lays the groundwork for several practical applications:

- **Data-Operation Intensive Applications**: Designing advanced graph prompts for tasks like data fusion, network analysis, and knowledge graph completion.

- **Cross-Domain Transfer**: Enabling models trained on one domain to be effectively applied to different domains, improving efficiency in areas such as social networks, bioinformatics, and recommendation systems.

- **Enhanced Prompt Design**: Developing sophisticated prompts that capture complex patterns and relationships within graph data, benefiting applications like fraud detection and protein interaction networks.

Real-World Examples

In the Introduction, we have listed several real-world applications where graph prompts are utilized:

- **Social Network Analysis**: Understanding user behavior and community structures through graph prompts that manipulate social graph data.

- **Biological Networks**: Analyzing molecular interactions and gene regulatory networks using prompts that capture biological graph properties.

- **Graph Transfer Learning**: Transferring knowledge across different graph domains to improve performance on tasks with limited data.

By grounding our theoretical findings in these practical contexts, we aim to bridge the gap between theory and application, demonstrating the immediate and future benefits of our work.

1.4 Conclusion

These additional discussions enhance the practical relevance of our paper by:

- **Providing Actionable Insights**: Offering guidance derived from our theoretical results that can be directly applied in real-world settings.

- **Justifying Theoretical Assumptions**: Reinforcing the validity of our assumptions with empirical observations and mathematical justification.

- **Paving the Way for Future Research**: Highlighting potential applications and encouraging further exploration in the field of graph prompting.

We believe that these elaborations address the reviewers' concerns and contribute to a better understanding of the importance and applicability of our research. We are committed to further linking our theoretical findings with practical applications and encouraging future researchers to build upon this foundation.