

REVISITING MIXED GEOMETRY

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ABSTRACT. We construct a new theory which associates to each Artin stack of finite type \mathcal{Y} over $\overline{\mathbb{F}}_q$ a symmetric monoidal DG-category $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ of constructible graded sheaves on \mathcal{Y} along with the six-functor formalism, a perverse t -structure, and a weight (or co- t -)structure in the sense of Bondarko and Pauksztello. This category sits in between the category of constructible mixed ℓ -adic sheaves $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ in the sense of Beilinson–Bernstein–Deligne–Gabber for any \mathbb{F}_{q^n} -form \mathcal{Y}_n of \mathcal{Y} and the category of constructible ℓ -adic sheaves $\mathrm{Shv}_c(\mathcal{Y})$, compatible with the six-functor formalism. Moreover, the t -structure (resp. weight structure) is compatible with the perverse t -structures on $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ and $\mathrm{Shv}_c(\mathcal{Y})$ (resp. with Deligne’s notion of weight on $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$). Our theory has a natural interpretation in terms of mixed geometry in the sense of Beilinson–Ginzburg–Soergel, giving a uniform construction thereof.

As an application, we show that for any reductive group G with a fixed pair $T \subset B$ of a maximal torus and a Borel subgroup, we have an equivalence of monoidal DG weight categories $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B) \simeq \mathrm{Ch}^b(\mathrm{SBim}_W)$, where $\mathrm{Ch}^b(\mathrm{SBim}_W)$ is the monoidal DG-category of bounded chain complexes of Soergel bimodules and W is the Weyl group of G .

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1. INTRODUCTION

1.1. Motivation. Frobenius weights in the theory of mixed ℓ -adic sheaves developed by Deligne and Beilinson–Bernstein–Deligne–Gabber [De180, BBDG18] have long played an important role in geometric representation theory and categorification. This went back to the works of Beilinson, Ginzburg, and Soergel [BG86, BGS96] to more recent developments by Achar, Bezrukavnikov, Lusztig, Riche, Rider, Yun, and others [AR13, BY13, Rid13, LY20], to name a few. Here, Frobenius weights provide an extra grading which already appears in the most basic objects in the theory, the Hecke categories in their Soergel bimodule incarnations. In [Kho07], Khovanov used Soergel bimodules to construct a triply-graded link homology (also known as HOMFLY-PT homology), categorifying HOMFLY-PT knot polynomials. Frobenius weights thus made their way into the theory of categorified knot invariants as well via the works of, for example, Webster–Williamson, Shende–Tremann–Zaslow, and Trinh [WW17, STZ17, Tri21].

In all of these cases, to match geometric and combinatorially defined categories, a “mixed version” of the category of sheaves (on the geometric side) has to be constructed via some artificial “cooking” (in the words of Romanov–Williamson, see [RW21]), necessary to kill non-trivial extensions between pure objects of the same weights. Due to Frobenius’ non-semisimplicity, a fundamental problem already over a point, the mixed version is not simply the category of mixed sheaves. Roughly speaking, the construction of a mixed version, which is usually subtle and technical, involves picking a collection $\mathrm{Pur}(\mathcal{Y})$ of pure/semi-simple complexes on a stack \mathcal{Y} of interest and show that the homotopy category of chain complexes $K^b(\mathrm{Pur}(\mathcal{Y}))$ recovers a known object previously combinatorially defined.¹ However, to relate it back to the derived category of sheaves, known techniques only work for very special classes of schemes/stacks that have nice stratifications by affine spaces. Moreover, the construction $K^b(\mathrm{Pur}(-))$ loses most of the functoriality that makes the geometric language powerful: we no longer know how to pull (resp. push) along non-smooth (resp. non-proper) morphisms. This problem is not easy to fix: preservation of semi-simplicity is part of the Tate conjecture, which is still wide open.

The goal of this paper is to construct a uniform theory which associates to each scheme/stack \mathcal{Y} over $\overline{\mathbb{F}}_q$ a triangulated (in fact, DG-)category $\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}$ (resp. $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$) of *graded sheaves* (resp. *constructible graded sheaves*) on \mathcal{Y} , along with the six-functor formalism, a perverse t -structure, and a weight (or co- t -)structure in the sense of Pauksztello and Bondarko [Pau08, Bon10]. As part of the definition of a weight structure, pure objects are semi-simple. In particular, we have a decomposition theorem similar to, but in some sense, more convenient than [BBDG18]. Moreover, the weight complex functor of [Bon10, Sos19, Aok20] provides a comparison $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \rightarrow \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w})$, where $\mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w})$ is the DG-category of bounded chain complexes in the weight heart $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$, which plays the role of

¹For example, when $\mathcal{Y} = B\backslash G/B$, we obtain the homotopy category of Soergel bimodules.

$\mathrm{Pur}(\mathcal{Y})$. In special cases, such as the case of $B \backslash G/B$, this functor is an equivalence of categories. But in general, we get a genuinely new category.

We do this by circumventing the Frobenius non-semisimplicity problem: instead of picking semi-simple objects in the category, we systematically semi-simplify the Frobenius actions at the categorical level. Consequently, our theory interacts seamlessly with the theory of ℓ -adic sheaves in terms of both six-functor formalism and the perverse t -structure. More precisely, for any stack \mathcal{Y}_n over \mathbb{F}_{q^n} whose base change to $\overline{\mathbb{F}_q}$ is \mathcal{Y} , $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ fits into the following diagram

$$\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n) \xrightarrow{\mathrm{gr}_{\mathcal{Y}_n}} \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \xrightarrow{\mathrm{oblv}_{\mathrm{gr}}} \mathrm{Shv}_c(\mathcal{Y})$$

where $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y})$ is the category of mixed complexes. Here, $\mathrm{gr}_{\mathcal{Y}_n}$ and $\mathrm{oblv}_{\mathrm{gr}}$ are compatible with all of the six-functor formalism and are exact with respect to the perverse t -structures. Moreover, Frobenius weights translate directly to our new notion of weights in $\mathrm{Shv}_{\mathrm{gr},c}$. Thus, known results in the theory of mixed ℓ -adic sheaves have direct translations in our setting.

As mentioned above, in writing this paper, we are motivated by questions from geometric representation theory and categorified knot invariants. To demonstrate this, we will conclude the paper with a streamlined proof of the fact that $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)$ is equivalent to the DG-category of bounded chain complexes of Soergel bimodules $\mathrm{Ch}^b(\mathrm{SBim}_W)$, matching the geometrically and combinatorially defined versions of Hecke categories. More generally, we expect that the theory developed in this paper will have a much wider applicability. In particular, in forthcoming papers, we will utilize the robust theory of graded sheaves developed here

- (i) to compute the categorical trace and Drinfeld center of Hecke categories, using the techniques of Ben-Zvi, Nadler, and Preygel [BN09, BNP17];
- (ii) to upgrade Bezrukavnikov's theorem on the two geometric realizations of affine Hecke algebras [Bez16] to the graded setting; and
- (iii) to categorify quantum groups using geometry.

The first two items, due to their proximity to the last section §6 of the current paper, will be elaborated in more details in §2.

Remark 1.1.1. Recently, using the theory of motives, Soergel, Virk, and Wendt [SW18, SVW18] constructed a streamlined theory that shares many nice features with the theory developed in this paper. At a technical level, the main difference between the two approaches is that the results of [SW18, SVW18] are formulated using the theory of motives whereas ours make use of mixed ℓ -adic sheaves. On the one hand, their results are much more sophisticated and can be applied to other cohomology theories, such as K -theory, [Ebe19, ES21a]. On the other hand, our theory works for Artin stacks and admits a perverse t -structure in general. The general existence of such a t -structure on the triangulated categories of mixed motives depends on the standard conjectures of algebraic cycles. From the user's perspective, we believe that our results are easier to use and understand as they interface seamlessly with known results in the theory of mixed ℓ -adic sheaves.

1.2. Conventions. Throughout the paper, we fix $k_1 = \mathbb{F}_q$ where q is a fixed power of a fixed prime p . More generally, for any positive integer n , we use $k_n = \mathbb{F}_{q^n}$ to denote the degree n field extension of k_1 . We also use $k = \overline{\mathbb{F}_q}$ to denote the algebraic closure of \mathbb{F}_q . A scheme/stack defined over k_n will be written as X_n . We will use $X_{n'}$ to denote its base change to $k_{n'}$ and X its base change to k .

All schemes that appear in the paper are separated and of finite type. All stacks are Artin, of finite type, and have affine stabilizers. For any field \mathbb{F} , we use $\mathrm{Sch}_{\mathbb{F}}$ (resp. $\mathrm{Stk}_{\mathbb{F}}$) to denote the category of schemes (resp. stacks) over \mathbb{F} , satisfying the conditions above. Unless otherwise specified, $\mathbb{F} \in \{k_n, k\}_{n \in \mathbb{Z}} = \{\mathbb{F}_{q^n}, \overline{\mathbb{F}_q}\}_{n \in \mathbb{Z}}$ in this paper. We also use $\mathrm{pt}_{\mathbb{F}} \in \mathrm{Sch}_{\mathbb{F}}$ to denote the final object. Following the convention written in the above paragraph, we also use pt_n to denote $\mathrm{Spec} k_n = \mathrm{Spec} \mathbb{F}_{q^n}$ and $\mathrm{pt} = \mathrm{Spec} k = \mathrm{Spec} \overline{\mathbb{F}_q}$.

Unless otherwise specified, by a category, we always mean an ∞ -category (or more precisely, an $(\infty, 1)$ -category). Given a triangulated/stable ∞ -category \mathcal{C} equipped with a t -structure, the heart of the t -structure is denoted by \mathcal{C}^{\heartsuit} . Similarly, the heart of a weight structure is denoted by $\mathcal{C}^{\heartsuit_w}$. We will

also use \mathcal{C}^{\heartsuit} to denote the heart of a t -structure on \mathcal{C} to emphasize the t -structure, especially when a weight structure is also present. Lastly, categories and functors are derived by default.

1.3. The main results.

1.3.1. Abstract theory. For each stack $\mathcal{Y} \in \text{Stk}_k$, we construct a symmetric monoidal DG-category $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ of *constructible graded sheaves* on \mathcal{Y} along with its “large” cousin, the category of renormalized graded sheaves $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}} \simeq \text{Ind}(\text{Shv}_{\text{gr},c}(\mathcal{Y}))$, where Ind denotes the process of taking ind-completion, i.e., formally adding all filtered colimits.

For any k_n -form $\mathcal{Y}_n \in \text{Stk}_{k_n}$ of \mathcal{Y} , we have

$$\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}} \simeq \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}}.$$

Here, $\text{Vect}^{\text{gr}} := \text{Fun}(\mathbb{Z}, \text{Vect})$ denotes the category of graded chain complexes of vector spaces over $\overline{\mathbb{Q}}_\ell$ and $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} := \text{Ind}(\text{Shv}_{m,c}(\mathcal{Y}_n))$ where $\text{Shv}_{m,c}(\mathcal{Y}_n)$ is the (derived) category of constructible mixed ℓ -adic sheaves over \mathcal{Y}_n in the sense of [BBDG18, LO09]. Moreover, the tensor product is the one defined by Lurie, where $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ is tensored over $\text{Shv}_m(\text{pt}_n)$ via pulling back and the symmetric monoidal functor $\text{Shv}_m(\text{pt}_n) \rightarrow \text{Vect}^{\text{gr}}$ is obtained by turning Frobenius weights into a grading. In particular, $\text{Shv}_{\text{gr}}(\text{pt}) \simeq \text{Vect}^{\text{gr}}$, which is semi-simple.

In fact, we take this as the definition of $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$, see Definition 4.4.2, and prove that this definition is independent of the choice of a k_n -form, see Theorem 4.7.12. Moreover, the category $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is defined to be the full subcategory of $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$ spanned by compact objects.

Remark 1.3.2. While the small category of constructible sheaves is more explicit, dealing with its large cousin is unavoidable since, for example, the push-forward functor between stacks does not preserve constructibility most of the time.

Remark 1.3.3. For any stack $\mathcal{Y} \in \text{Stk}_k$, there is also the usual category $\text{Shv}(\mathcal{Y})$ obtained, for instance, by using descent via a smooth atlas $S \rightarrow \mathcal{Y}$ where $S \in \text{Sch}_k$.² However, for many purposes, this category is not what we want: even in the simplest case $\mathcal{Y} = B\mathbb{G}_m$, the classifying stack of the multiplicative group \mathbb{G}_m , the constant sheaf on \mathcal{Y} is not compact in $\text{Shv}(\mathcal{Y})$. See Example 4.1.12. Note, however, that the distinction between renormalized and usual is only relevant for stacks.

The functoriality for the renormalized sheaf theory is similar to that of the usual sheaf theory and the two are closely related. To distinguish between the two, we will include the subscript *ren* in the notations of the various pull and push functors when working with renormalized sheaves. Note, however, that when applied to constructible sheaves, functors on the two sides are identified in a precise sense. This is reviewed in §4.1.8 and subsequent subsections.

By construction, $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$ is tensored over Vect^{gr} and moreover, this structure is compatible with usual operations of sheaves. We have the following result that compares graded sheaf theory and the theory of usual/mixed sheaves.

Theorem 1.3.4 (Proposition 4.5.5 and Lemma 4.7.16). *Let $\mathcal{Y} \in \text{Stk}_k$ and $\mathcal{Y}_n \in \text{Stk}_{k_n}$ a k_n -form of \mathcal{Y} . Then, we have a sequence of symmetric monoidal functors*

$$\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \xrightarrow{\text{gr}_{\mathcal{Y}_n}} \text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}} \xrightarrow{\text{oblv}_{\text{gr}}} \text{Shv}(\mathcal{Y})^{\text{ren}}$$

where oblv_{gr} is conservative. Moreover, these functors are compatible various pull and push functors along $f_n : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ and its base change to k , $f : \mathcal{Y} \rightarrow \mathcal{Z}$.

Even though $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is defined in an abstract way, one can understand it quite explicitly. Namely, its objects are direct summands of finite colimits of objects in the essential image of $\text{gr}_{\mathcal{Y}_n}$. Moreover, given $\mathcal{F}_n, \mathcal{G}_n \in \text{Shv}_{m,c}(\mathcal{Y}_n)$,

$$\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\text{gr}_{\mathcal{Y}_n}(\mathcal{F}_n), \text{gr}_{\mathcal{Y}_n}(\mathcal{G}_n)) \in \text{Vect}$$

²We ignore Frobenius action/mixedness since it is not relevant for current discussion.

is simply the Frobenius weight 0 part of the mixed- $\mathcal{H}\text{om}$ complex, see Proposition 4.6.2. This simple, but important, observation allows us to show that $\text{Shv}_{\text{gr},c}(\mathcal{Y}_n)$ has a transversal weight and t -structures in the sense of [Bon12]. See Theorem 5.4.3 for a quick review.

Theorem 1.3.5 (Theorem 5.4.8). *For any $\mathcal{Y} \in \text{Stk}_k$, $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is equipped with a weight and a t -structure such that the t -structure is transversal to the weight structure.*

This t -structure will be referred to as the perverse t -structure, whose heart is the category of graded perverse sheaves $\text{Perv}_{\text{gr},c}(\mathcal{Y}) := \text{Shv}_{\text{gr},c}(\mathcal{Y})^{\heartsuit}$.

Moreover, the weight and t -structures are compatible with various functors as expected. For instance, we have the following results. Many similar results are established in §5.5.

Proposition 1.3.6 (Proposition 5.5.1). *Let $\mathcal{Y}_n \in \text{Stk}_k$ and \mathcal{Y} its base change to k . Then, the functors*

$$\text{Shv}_{m,c}(\mathcal{Y}_n) \xrightarrow{\text{gr}} \text{Shv}_{\text{gr},c}(\mathcal{Y}) \xrightarrow{\text{oblv}_{\text{gr}}} \text{Shv}_c(\mathcal{Y})$$

preserves and reflects t -structures with respect to the perverse t -structures. Namely, for any n and any $\mathcal{F} \in \text{Shv}_{\text{gr},c}(\mathcal{Y})$, $\mathcal{F} \in \text{Shv}_{\text{gr},c}(\mathcal{Y})^{t \leq n}$ (resp. $\mathcal{F} \in \text{Shv}_{\text{gr},c}(\mathcal{Y})^{t \geq n}$) if and only if $\text{oblv}_{\text{gr}}(\mathcal{F}) \in \text{Shv}_c(\mathcal{Y})^{t \leq n}$ (resp. $\text{oblv}_{\text{gr}}(\mathcal{F}) \in \text{Shv}_c(\mathcal{Y})^{t \geq n}$). We have a similar statement for gr .

Proposition 1.3.7 (Proposition 5.5.2). *Let \mathcal{Y} and \mathcal{Y}_n be as above. The functor $\text{gr} : \text{Shv}_{m,c}(\mathcal{Y}_n) \rightarrow \text{Shv}_{\text{gr},c}(\mathcal{Y})$ preserves and reflect weights, where we use Frobenius weights on $\text{Shv}_{m,c}(\mathcal{Y}_n)$ and our weight structure on $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. Namely, for any k and $\mathcal{F} \in \text{Shv}_{m,c}(\mathcal{Y})$, then $\mathcal{F} \in \text{Shv}_{m,c}(\mathcal{Y})^{w \leq k}$ (resp. $\mathcal{F} \in \text{Shv}_{m,c}(\mathcal{Y})^{w \geq k}$) if and only if $\text{gr}(\mathcal{F}) \in \text{Shv}_{\text{gr},c}(\mathcal{Y})^{w \leq k}$ (resp. $\text{gr}(\mathcal{F}) \in \text{Shv}_{\text{gr},c}(\mathcal{Y})^{w \geq k}$).*

We will now relate our theory and previous attempts in the literature. Let $\text{Shv}_{\infty,c}(\mathcal{Y})$ denote the smallest full triangulate subcategory of $\text{Shv}_c(\mathcal{Y})$ containing the essential image of $\text{Shv}_{m,c}(\mathcal{Y}_m)$ under pullbacks for all k_m -forms of \mathcal{Y} (for all m). We show that this category inherits the perverse t -structure from $\text{Shv}_c(\mathcal{Y})$ whose heart is denoted by $\text{Perv}_{\infty,c}(\mathcal{Y})$.

Theorem 1.3.8 (Corollaries 5.6.4 and 5.6.5). *$\text{Perv}_{\text{gr},c}(\mathcal{Y})$ is a graded version of $\text{Perv}_{\infty,c}(\mathcal{Y})$ in the sense of [BGS96, Defn. 4.3.1.(1)].*

$\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is a mixed version of $\text{Shv}_{\infty,c}(\mathcal{Y})$ in the sense of [Rid13, Defn. 4.2].

1.3.9. Application to Hecke categories and Soergel bimodules. We conclude the paper with an application to representation theory. Let G be a reductive group with a fixed pair $T \subseteq B$ of Borel B and a maximal torus. Let W be the Weyl group of G and consider the monoidal DG-category of bounded chain complexes of Soergel bimodules $\text{Ch}^b(\text{SBim}_W)$. We show that this category has a natural geometric incarnation using graded sheaves.

Theorem 1.3.10 (Theorem 6.4.1). *We have an equivalence of monoidal categories*

$$\text{Shv}_{\text{gr},c}(B \backslash G/B) \simeq \text{Ch}^b(\text{SBim}_W).$$

As graded sheaves interact seamlessly with the usual theory of sheaves/mixed sheaves, the theorem above is obtained by readily using known results about $B \backslash G/B$ proved earlier in [Soe90, BY13].

1.4. An outline of the paper. In §2 we outline some of the applications we have in mind for the theory of graded sheaves. The mathematical content of the paper starts in §3 where we review the necessary background regarding DG-categories. This is followed by §4 where the categories of graded sheaves are constructed and their formal properties are established. In §5, using the theory developed by Bondarko in [Bon12], we show that the category of constructible graded sheaves on any Artin stack admits a transversal weight and t -structures. We show that the weight structure is compatible with Frobenius weights in the theory of mixed sheaves and that the t -structure is compatible with the perverse t -structure of [BBDG18]. Finally, in §6, we show that the category of constructible graded sheaves on the finite Hecke stack $B \backslash G/B$ is equivalent, as a monoidal DG category, to $\text{Ch}^b(\text{SBim}_W)$, the category of bounded chain complexes of Soergel bimodules.

2. FUTURE WORK

We will now describe some of the applications we have in mind which will be visited in subsequent papers. In what follows, statements are listed under “Theorem (in progress)” and “Expectation.” The former implies that we have a strategy, or in some case, even a fairly complete proof whereas the latter means that we are confident about the general shape of the statement but have not thought through the details.

2.1. Trace and center of Soergel bimodules. For a monoidal DG-category \mathcal{A} , the *categorical trace* and the *categorical center* (or *Drinfeld center*) of \mathcal{A} are $\mathrm{Tr}(\mathcal{A}) := \mathcal{A} \otimes_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}} \mathcal{A}$ and $Z(\mathcal{A}) := \mathrm{Hom}_{\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{A})$, respectively. There are canonical maps $\mathrm{tr} : \mathcal{A} \rightarrow \mathrm{Tr}(\mathcal{A})$ and $z : Z(\mathcal{A}) \rightarrow \mathcal{A}$.

In [BN09], Ben-Zvi and Nadler identify both the trace and center of the *unmixed*, i.e., *non-graded*, Hecke category $\mathrm{Shv}(B \backslash G/B)$ with the category $\mathrm{Ch}^u(G)$ of unipotent character sheaves.³ We will apply their argument in the graded setting to obtain the following theorem.

Theorem 2.1.1 (in progress). *The trace $\mathrm{Tr}(\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B))$ and Drinfeld center $Z(\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B))$ of $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)$ coincide and are identified with the full subcategory of $\mathrm{Shv}_{\mathrm{gr},c}(\frac{G}{G})$ generated by the image of $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)$ under the horocycle correspondence*

$$(2.1.2) \quad \begin{array}{ccc} & \frac{G}{B} & \\ \swarrow & & \searrow \\ B \backslash G/B & & \frac{G}{G} \end{array}$$

This category, denoted by $\mathrm{Ch}_{\mathrm{gr}}^u(G)$, will be referred to as the category of *graded unipotent character sheaves*. Under this identification, the natural functors $\mathrm{tr} : \mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B) \rightarrow \mathrm{Ch}_{\mathrm{gr}}^u(G)$ and $z : \mathrm{Ch}_{\mathrm{gr}}^u(G) \rightarrow \mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)$ are mutually adjoints and are given by going through the correspondence above in the appropriate directions.

Combined with the identification $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B) \simeq \mathrm{Ch}^b(\mathrm{SBim}_W)$ of Theorem 6.4.1, we obtain the following result.

Corollary 2.1.3 (in progress). *The trace $\mathrm{Tr}(\mathrm{Ch}^b(\mathrm{SBim}_W))$ and Drinfeld center $Z(\mathrm{Ch}^b(\mathrm{SBim}_W))$ coincide and are equivalent to $\mathrm{Ch}_{\mathrm{gr}}^u(G)$. Moreover, we have the following pair of adjoint functors*

$$\mathrm{tr} : \mathrm{Ch}^b(\mathrm{SBim}_W) \rightleftarrows \mathrm{Ch}_{\mathrm{gr}}^u(G) : z.$$

The category of *ungraded* unipotent character sheaves $\mathrm{Ch}^u(G)$ is explicitly calculated by the second author in [Li18]. We will give a similar description in the graded case. For simplicity, we will now assume that $G = \mathrm{GL}_n$, and hence, $W = S_n$.

Theorem 2.1.4 (in progress). *We have an equivalence of DG-categories*

$$\mathrm{Tr}(\mathrm{Ch}^b(\mathrm{SBim}_{S_n})) \simeq Z(\mathrm{Ch}^b(\mathrm{SBim}_{S_n})) \simeq \overline{\mathbb{Q}}_\ell[S_n] \ltimes \overline{\mathbb{Q}}_\ell[x_1, \dots, x_n, \alpha_1, \dots, \alpha_n]\text{-perf}_{\mathrm{gr}}$$

with x_i and α_i living in degree $(2, 2)$ and $(2, 1)$, respectively, where the first (resp. second) number denotes weight (resp. cohomological) degree.

Note that the algebra appearing on the right hand side, up to a Koszul dual and forgetting degree and weight, is precisely the endomorphism algebra of the Procesi bundle on the Hilbert scheme of n points on \mathbb{A}^2 . We will thus further identify the trace/center of $\mathrm{Ch}^b(\mathrm{SBim}_{S_n})$ with the category of coherent sheaves on the Hilbert scheme of n points on \mathbb{A}^2 , appropriately sheared. The relation between Hilbert scheme of points, Soergel bimodules, and triply graded HOMFLY-PT link homology appear as the main conjecture of [GNR21].

One of the key ingredients for Theorem 2.1.4 is the formality of the Grothendieck–Springer sheaf of the group G , [Li18]. We will use it to deduce the formality of graded version of the Grothendieck–Springer

³Strictly speaking, they work with D -modules. However, a variant of their argument also yields the result in the ℓ -adic sheaves setting.

sheaf as well. Note that by definition, the Grothendieck–Springer sheaf is simply $\mathrm{tr}(1)$. Therefore, the argument sketched above gives a proof of the following conjecture of Gorsky–Hogancamp–Wedrich, which is important in the study of derived annular link invariants.

Conjecture 2.1.5 ([GHW20]). The DG-algebra $\mathrm{End}_{\mathrm{Tr}(\mathrm{Ch}^b(\mathrm{SBim}_W))}(\mathrm{tr}(1))$ is formal.

2.2. Graded affine Hecke category and Langlands duality. Let \check{G} denote the Langlands dual group of G . A theorem of Kazhdan–Lusztig and independently of Ginzburg provides an isomorphism between the affine Hecke algebra of G and the K-group of equivariant coherent sheaves on the Steinberg variety of \check{G}

$$(2.2.1) \quad H_{\mathrm{aff}} \simeq K_{\check{G} \times \mathbb{G}_m}(\mathrm{St}).$$

Let LG be the loop group of G , and $I \subset LG$ the Iwahori subgroup. Then, Bezrukavnikov proved in [Bez16] that we have an equivalence of monoidal categories

$$(2.2.2) \quad \mathrm{Shv}_c(I \backslash LG / I) \simeq \mathrm{Coh}(\mathrm{St} / \check{G}).$$

Strictly speaking, (2.2.2) is not a categorification of (2.2.1). Indeed, to recover the isomorphism (2.2.1) by passing to K-groups, one has to consider a graded version of (2.2.2). In fact, it was expected in [Bez16, §11.1] that an “appropriately defined mixed version” of $\mathrm{Shv}(I \backslash LG / I)$ ⁴ will yield the graded version. The theory of graded sheaves developed in this paper provides an answer to this question. Indeed, the techniques developed by Bezrukavnikov can be transported to our setting and we will prove the following result.

Theorem 2.2.3 (in progress). *There is an equivalence of monoidal DG-categories*

$$\mathrm{Shv}_{\mathrm{gr},c}(I \backslash LG / I) \simeq \mathrm{Coh}(\mathrm{St} / (\check{G} \times \mathbb{G}_m)).$$

By the general formalism of weight structure, the K-group on the LHS is isomorphic to the split K-group of the weight heart, which is easily identified with the affine Hecke algebra H_{aff} . Therefore, taking K-group of Theorem 2.2.3 gives (2.2.1) precisely.

Note that $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G / B)$ is naturally a full subcategory of $\mathrm{Shv}_{\mathrm{gr},c}(I \backslash LG / I)$. On the other hand, $\mathrm{Coh}(\mathrm{St} / (\check{G} \times \mathbb{G}_m))$ is equivalent to a certain category of matrix factorizations, see [AG15, Appendix H]. We thus expect that the theorem above is closely related to a conjecture of Oblomkov–Rozansky [OR20, Conjecture 7.3.1] stating that there is an equivalence between $\mathrm{Ch}^b(\mathrm{SBim})$ and a certain category of matrix factorizations.

Remark 2.2.4. Since the weight heart $\mathrm{Shv}_{\mathrm{gr},c}(I \backslash LG / I)^{\heartsuit_w}$ is classical, it follows that $\mathrm{Shv}_{\mathrm{gr},c}(I \backslash LG / I)$ is equivalent to the more familiar DG-category of bounded chain complexes of pure objects, which is the DG-category $\mathrm{Ch}^b(\mathrm{SBim}_{\overline{W}})$ of bounded chain complexes of affine Soergel bimodules, mirroring Theorem 6.4.1.⁵ The work of Ben-Zvi–Nadler–Preygel [BNP17] could then be used to get a handle on the trace and Drinfeld center of $\mathrm{Ch}^b(\mathrm{SBim}_{\overline{W}})$ by using the coherent incarnation, i.e., the RHS of the equivalence in Theorem 2.2.3.

2.2.5. We will also consider a graded version of the monodromic Bezrukavnikov’s equivalence. Let $I_0 \subset I$ denote the (pro-)unipotent radical, and $\mathrm{Shv}_{\mathrm{gr},c}^u(I_0 \backslash LG / I_0) \subset \mathrm{Shv}_{\mathrm{gr},c}(I_0 \backslash LG / I_0)$ be the full subcategory consisting of unipotent-monodromic sheaves, i.e., it is the full subcategory generated by the essential image of the pullback along $I_0 \backslash LG / I_0 \rightarrow I \backslash LG / I$. On the other hand, denote $\mathrm{St}_{\check{\mathfrak{g}}} = \check{\mathfrak{g}} \times_{\check{\mathfrak{g}}} \check{\mathfrak{g}}$ the Lie algebra version of Steinberg variety, and $\mathrm{Coh}(\mathrm{St}_{\check{\mathfrak{g}}} / (\check{G} \times \mathbb{G}_m))_{\mathrm{St}} \subset \mathrm{Coh}(\mathrm{St}_{\check{\mathfrak{g}}} / (\check{G} \times \mathbb{G}_m))$ the full subcategory of coherent sheaves supported on St .

Theorem 2.2.6 (in progress). *There is an equivalence of DG-categories*

$$\mathrm{Shv}_{\mathrm{gr},c}^u(I_0 \backslash LG / I_0) \simeq \mathrm{Coh}(\mathrm{St}_{\check{\mathfrak{g}}} / (\check{G} \times \mathbb{G}_m))_{\mathrm{St}}.$$

⁴Or more precisely, its monodromic version below, see also Remark 2.2.7.

⁵See also Propositions 5.6.10 and 6.1.5 for where the classical-ness is established in the finite case.

Remark 2.2.7. Unlike the case of $\mathrm{Shv}_{\mathrm{gr},c}(I \backslash LG/I)$, the weight heart $\mathrm{Shv}_{\mathrm{gr},c}(I_0 \backslash LG/I_0)^{\nabla_w}$ is not classical and hence, $\mathrm{Shv}_{\mathrm{gr},c}(I_0 \backslash LG/I_0)$ is a genuinely new category. Despite the apparently easier statement of Theorem 2.2.3, we expect that its proof should more naturally go through Theorem 2.2.6 as this is the case for the ungraded situation [Bez16, §9.3.1].

2.2.8. Following [BN07], we expect to recover from this an ungraded version of the Koszul duality proved in [BGS96, BY13]. As explained in [BN12], this can be done using periodic localization.

The category on the RHS admits an action of $\mathrm{Perf}(B\mathbb{G}_a/\mathbb{G}_m)$ (viewed as an algebra object in $\mathrm{Perf}(B\mathbb{G}_m)$ -module categories), given by the presentation of the Steinberg stack as the unipotent loop space of $\check{B} \backslash \check{G} / \check{B}$. Moreover, its periodic localization can be identified with a certain category of D -modules $D(\check{B} \backslash \check{G} / \check{B})$. On the LHS, the group \mathbb{G}_m naturally acts on $I_0 \backslash LG/I_0$ by loop rotation. This induces an action of $\mathrm{Shv}_{\mathrm{gr},c}^u(\mathbb{G}_m) \simeq \mathrm{Perf}(B\mathbb{G}_a/\mathbb{G}_m)$.⁶ Its periodic localization can be identified as a certain category of sheaves on $U \backslash G/U$.

Expectation 2.2.9. The equivalence in Theorem 2.2.6 is naturally compatible with $\mathrm{Perf}(B\mathbb{G}_a/\mathbb{G}_m)$ -actions, and therefore induces an ungraded version of Koszul duality after periodic localization.

2.2.10. Following Soergel [Soe01] and Ben-Zvi–Nadler [BN07], the above statements can be extended to Real groups, and could lead to a categorification of Langlands duality for real groups. We shall not review the story here but simply remark that $\mathrm{Shv}_{\mathrm{gr}}(K \backslash G/B)$ (and its monodromic variant) gives a geometric construction of the graded category of Harish-Chandra modules, as expected in [BV21, Remark 5.5].

3. GENERALITIES ON DG-CATEGORIES

DG-categories play an instrumental role in the paper. While there are many different ways to think about DG-categories, we find the one worked out in [GR17, Vol. I, Chap. 1, §10] the most well-documented and most convenient to use, despite the fact that it is based on formidable machinery of ∞ -categories developed in [Lur17a, Lur17b]. For example, as far as we know, relative tensors of categories, which already appear in the definition of the category of graded sheaves, are only developed in this context. More generally, constructions involving limits and colimits can be performed in a much more streamlined way in this setting.

Having said that, since the machinery is packaged in such a convenient and intuitive way, most of it can be taken as a black box. This section thus provides a quick review the important aspects of the theory that are used throughout the paper, focusing on DG-categories, module categories, and relative tensors thereof. We do not aim to review all the concepts used in the paper as it is impossible to do so. Rather, the main goal is introduce the notations, to provide intuition, and to familiarize the readers with the language and references of the subject. We do, however, hope that even for readers who are unfamiliar with the theory, this section still provides sufficient background to follow the main ideas of the paper. We note that most results we recall below hold true in more generality than the way we phrase it. We choose to forfeit generality to keep things simple; the interested readers can consult the accompanying references.

Most of the materials written here are either developed in [Lur17a, GR17] or are direct consequences of the results therein. Because of that, no proof will be given here. This is with the exception of §3.5 where we prove an induction formula for the enriched Hom-spaces, crucial in our study of graded sheaves. Although the result seems to be well-known among experts, we cannot find a proof in the literature.

Throughout this section, we fix a field Λ of characteristics 0. For the purpose of constructing the theory of graded sheaves, $\Lambda = \overline{\mathbb{Q}_\ell}$. However, results stated in this section hold for any field Λ of characteristic 0. Thus, except for this section, all DG-categories appearing in this paper are linear over \mathbb{Q}_ℓ .

3.1. Stable presentable categories. We will now quickly review the main features of stable presentable (∞ -)categories.

⁶This is an equivalence of algebras over $\mathrm{Shv}_{\mathrm{gr},c}(\mathrm{pt}) \simeq \mathrm{Perf}(B\mathbb{G}_m)$.

3.1.1. *∞ -categories.* The theory of ∞ -category is indispensable for our purposes. The theory was developed in great details in [Lur17b, Lur17a], but shorter accounts exist, see [GR17, Vol. I, Chap. 1] and [Cis19]. One virtue of the theory is that, in the words of [Cis19], ∞ -categories allow for a union between category theory and homotopical algebra. For example, (homotopical) limits and colimits, which are ubiquitous in homotopical algebra, is a natural part of the theory of ∞ -categories.

We let \mathbf{Spc} denote the ∞ -category of spaces, or ∞ -groupoids. For any ∞ -category \mathcal{C} , and $c_1, c_2 \in \mathcal{C}$, we use $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$ to denote the *space* of homomorphisms between c_1 and c_2 in \mathcal{C} . Hom being a space, rather than just a set, is one of the main distinguishing features of the theory of ∞ -categories compared to classical category theory.

In this paper, unless otherwise specified, by a category, we always mean an ∞ -category. A classical category, i.e., a 1-category, can be viewed as an ∞ -category where the Hom -space is now just a discrete set.

3.1.2. *Presentable categories.* Roughly speaking, a *presentable category* is a “large” category (i.e., the collection of objects forms a class rather than a set) but is, in a precise sense, “generated” by a set of objects. The theory is worked out in details in [Lur17b, §5.5]. We let Pr^L denote the category of presentable categories where morphisms are continuous functors, i.e., they commute with colimits. For $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L$, we use $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to denote the full subcategory consisting of *continuous functors*, i.e., those that commute with colimits.⁷

Presentable categories enjoy many nice features. For example, they admit all (small) limits and colimits [Lur17b, Def. 5.5.0.1 and Cor. 5.5.2.4] and they satisfy the adjoint functor theorem [Lur17b, Cor. 5.5.2.9]. Moreover, Pr^L itself admits small limits and colimits [Lur17b, §5.5.3] as well as a symmetric monoidal structure (also known as the Lurie tensor product), [Lur17a, Prop. 4.8.1.15] (see also [GR17, Vol. I, Chap. 1, Thm. 6.1.2]). The category \mathbf{Spc} is the unit of the symmetric monoidal structure on Pr^L .

3.1.3. Due to the importance of this tensor product, let us quickly recall its characterization. Let \mathcal{C} and \mathcal{D} be presentable categories. Then, $\mathcal{C} \otimes \mathcal{D}$ is initial among presentable categories \mathcal{E} which receive a functor from $\mathcal{C} \times \mathcal{D}$ that is continuous in each variable. In particular, for any such \mathcal{E} , the category of continuous functors from $\mathcal{C} \otimes \mathcal{D}$ to \mathcal{E} is identified with the category of functors $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ that is continuous in each variable. We thus have a canonical functor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \otimes \mathcal{D}$ that is continuous in each variable. For $c \in \mathcal{C}$ and $d \in \mathcal{D}$, we write $c \boxtimes d$ to denote the image of $(c, d) \in \mathcal{C} \times \mathcal{D}$ under this functor.

3.1.4. *Stable categories.* The language of ∞ -category allows for an elegant formulation of triangulated categories which are called *stable* (∞ -)categories, [Lur17a, Defn. 1.1.1.9]. More precisely, an ∞ -category \mathcal{C} is stable if

- It is pointed, i.e., it has a 0 object (which is both final and initial).
- Every morphism $f : X \rightarrow Y$ in \mathcal{C} can be completed into a pullback and a pushout square, respectively, as follows

$$(3.1.5) \quad \begin{array}{ccc} F & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow f & & \downarrow \\ Y & \longrightarrow & C \end{array}$$

We call the first (resp. second) square a fiber (resp. co-fiber) sequence. Moreover, F and C are referred to as the fiber and co-fiber of f , respectively.

- The squares in (3.1.5) above are simultaneously pullback and pushout squares.⁸

⁷Note that this differs from the convention used in [GR17] where continuous functors are required to commute with only filtered colimits. However, when working in the setting of stable categories and exact functors, which is where all our categorical constructions happen, there is no difference between the two [GR17, Vol. I, Chap. 1, §5.1.6].

⁸More generally, pullback squares and pushout squares are the same in a stable category [Lur17a, Prop. 1.1.3.4].

3.1.6. Despite the simple formulation, the homotopy category $h\mathcal{C}$ of a stable category \mathcal{C} is a triangulated category [Lur17a, Thm. 1.1.2.14 and Rmk. 1.1.2.15]. Moreover, in this language,

$$F \simeq \text{coCone}(X \rightarrow Y) \simeq \text{Cone}(X \rightarrow Y)[-1] \quad \text{and} \quad C \simeq \text{Cone}(X \rightarrow Y)$$

where F and C are as in (3.1.5). In other words, these squares become distinguished triangles in the resulting triangulated category.

3.1.7. Let \mathcal{C} and \mathcal{D} be stable categories. We use $\text{Fun}_{\text{ex}}(\mathcal{C}, \mathcal{D})$ to denote the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ consisting of functors preserve pullbacks, or equivalently, pushouts. It is easy to see that these functors also preserve finite limits and colimits.

We let $\text{Pr}^{L, \text{st}}$ denote the full subcategory of Pr^L consisting of stable presentable categories. It is closed under limits, colimits, and the tensor products in Pr^L . The category Sptr of spectra is the unit of the symmetric monoidal structure on $\text{Pr}^{L, \text{st}}$. For $\mathcal{C}, \mathcal{D} \in \text{Pr}^{L, \text{st}}$, $\text{Fun}_{\text{cont}}(\mathcal{C}, \mathcal{D})$ consists of continuous functors that also preserve small colimits.

3.1.8. *Compact generation.* Among the presentable stable categories, compactly generated ones are particularly manageable. Fortunately, all of the stable categories appearing in this paper are of this type. We start by reviewing the notion of compactness and generation.

An object c in a category \mathcal{C} is said to be *compact* if

$$\text{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \rightarrow \text{Spc}$$

commutes with filtered colimits. We let \mathcal{C}^c denote the full subcategory of \mathcal{C} containing compact objects.

Let $\mathcal{C} \in \text{Pr}^{L, \text{st}}$. A collection of objects $\{c_\alpha\}$ in \mathcal{C} is said to *generate* if

$$\text{Hom}_{\mathcal{C}}(c_\alpha, c[i]) = 0, \forall \alpha, \forall i = 0, 1, \dots \Rightarrow c \simeq 0.$$

A category $\mathcal{C} \in \text{Pr}^{L, \text{st}}$ is said to be *compactly generated* if it admits a set of compact generators.

3.1.9. *Ind-completion.* Given any small stable category \mathcal{C}_0 , we let $\text{Ind}(\mathcal{C}_0)$ denote the smallest full-subcategory of $\text{Fun}(\mathcal{C}_0^{\text{op}}, \text{Spc})$ that contains the image of \mathcal{C}_0 under the Yoneda embedding and is closed under all filtered colimits. Equivalently, we can define $\text{Ind}(\mathcal{C}_0)$ to be the full-subcategory containing all functors that preserve finite limits (or equivalently, fiber products) [Lur17b, Cor. 5.3.5.4] or [GR17, Vol. I, Chap. 1, Def. 7.1.3].

We call $\text{Ind}(\mathcal{C}_0)$ the *ind-completion* of \mathcal{C}_0 . It has the following universal property: for any stable category \mathcal{D} that admits all (small) colimits (for example, when \mathcal{D} is stable presentable),

$$\text{Fun}_{\text{cont}}(\text{Ind}(\mathcal{C}_0), \mathcal{D}) \simeq \text{Fun}_{\text{ex}}(\mathcal{C}_0, \mathcal{D}).$$

In other words, continuous functors out of $\text{Ind}(\mathcal{C}_0)$, a *large* category, is determined by its restriction to \mathcal{C}_0 , a *small* category.

3.1.10. $\text{Ind}(\mathcal{C}_0)$ is stable, presentable, and compactly generated. Moreover, the Yoneda embedding $\mathcal{C}_0 \rightarrow \text{Ind}(\mathcal{C}_0)$ factors through $\text{Ind}(\mathcal{C}_0)^c$ and the essential image of \mathcal{C}_0 forms a collection of compact generators of $\text{Ind}(\mathcal{C}_0)$. Furthermore, all elements in $\text{Ind}(\mathcal{C}_0)^c$ are direct summands of (the Yoneda embeddings of) objects in \mathcal{C}_0 [GR17, Vol. I, Chap. 1, Lem. 7.2.4]. In other words, $\text{Ind}(\mathcal{C}_0)^c$ is the *idempotent completion* of \mathcal{C}_0 .

By [GR17, Vol. I, Chap. 1, Lem. 7.2.4.(3')], all compactly generated categories \mathcal{C} are of the form $\text{Ind}(\mathcal{C}_0)$ for some \mathcal{C}_0 .

We record the following relation between continuity of functors and compactness.

Lemma 3.1.11 ([GR17, Vol. I, Chap. 1, Lem. 7.1.5]). *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor in $\text{Pr}^{L, \text{st}}$ and F^R its right adjoint (which exists by the adjoint functor theorem). Suppose that \mathcal{C} is compactly generated. Then F^R is continuous if and only if F preserves compactness, i.e., F sends \mathcal{C}^c to \mathcal{D}^c .*

3.1.12. *Compact generation of tensors.* Tensor products of compactly generated categories are themselves compactly generated.

Proposition 3.1.13 ([GR17, Vol. I, Chap. 1, Prop. 7.4.2]). *Let $\mathcal{C}, \mathcal{D} \in \text{Pr}^{L, \text{st}}$ such that they are both compactly generated. Then, $\mathcal{C} \otimes \mathcal{D}$ is compactly generated by objects of the form $c_0 \boxtimes d_0$ where $c_0 \in \mathcal{C}^c$ and $d_0 \in \mathcal{D}^c$.*

3.2. Module categories and enriched Hom-spaces.

3.2.1. *Algebra and module objects.* For any symmetric monoidal category \mathcal{A} , we use $\text{ComAlg}(\mathcal{A})$ to denote the category of commutative algebra objects in \mathcal{A} . For any such commutative algebra $A \in \text{ComAlg}(\mathcal{A})$, we use $\text{Mod}_A(\mathcal{A})$ to denote the category of A -module objects in \mathcal{A} . See [GR17, Vol. I, Chap. 1, §3] for a more detailed discussion.

The Lurie tensor product on $\text{Pr}^{L, \text{st}}$ allows us to talk about commutative algebra objects in $\text{Pr}^{L, \text{st}}$ and modules over such an object. In other words, it makes sense to talk about $\text{ComAlg}(\text{Pr}^{L, \text{st}})$ and for any $A \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$, we can talk about A -module categories, which form a category $\text{Mod}_A(\text{Pr}^{L, \text{st}})$. For brevity's sake, unless confusion is likely to happen, we will write Mod_A to denote $\text{Mod}_A(\text{Pr}^{L, \text{st}})$ in the situation above.

3.2.2. Let us quickly unwind the definitions. A commutative algebra object $A \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ is a symmetric monoidal stable presentable category such that the tensor product $A \times A \rightarrow A$ is continuous in each variable (which induces a continuous functor $A \otimes A \rightarrow A$). We will use $\text{mult}_A : A \otimes A \rightarrow A$ (and sometimes, also $\text{mult}_A : A \times A \rightarrow A$) to denote the operation of taking the tensor product. In other words, for $a, b \in A$, $a \otimes b := \text{mult}_A(a, b) = \text{mult}_A(a \boxtimes b)$.

3.2.3. Similarly, an $\mathcal{M} \in \text{Mod}_A$ is equipped with a “multiplication” map $A \times \mathcal{M} \rightarrow \mathcal{M}$ that is continuous on each variable (and hence, induces a continuous functor $A \otimes \mathcal{M} \rightarrow \mathcal{M}$) along with higher compatibilities. We will use $\text{act}_{A, \mathcal{M}} : A \otimes \mathcal{M} \rightarrow \mathcal{M}$ (and sometimes, also $\text{act}_{A, \mathcal{M}} : A \times \mathcal{M} \rightarrow \mathcal{M}$) to denote the action of A on \mathcal{M} given by the module structure. For $a \in A$ and $m \in \mathcal{M}$, we will also use $a \otimes m$ to denote $a \otimes m := \text{act}_{A, \mathcal{M}}(a, m) = \text{act}_{A, \mathcal{M}}(a \boxtimes m)$.

Since A is symmetric monoidal, a left-module structure is the same as a right-module structure. Thus, in the above, we also sometimes use $m \otimes a$ to denote the action.

When $F : A \rightarrow B$ is a symmetric monoidal functor between symmetric monoidal categories. Then B obtains the structure of an A -module. In particular, for $a \in A$ and $b \in B$, we can talk about $a \otimes b = F(a) \otimes b$, where the first and second tensor products come from the A -module structure and the monoidal structure on B , respectively. Similarly to the above, since the monoidal structures are symmetric, we will also write $b \otimes a = b \otimes F(a)$.

3.2.4. *Lax functors.* Let $A \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ and $\mathcal{M}, \mathcal{N} \in \text{Mod}_A$. Then one can talk about left/right-lax A -module functors $F : \mathcal{M} \rightarrow \mathcal{N}$ [GR17, Vol. I, Chap. 1, §3.5.1]. Roughly speaking, if F is left-lax, then for any $a \in A, m \in \mathcal{M}$, we have a natural map

$$F(a \otimes m) \rightarrow a \otimes F(m).$$

Similarly, if F is right-lax, we have a map in the opposite direction. These maps are equivalences if F is a strict (rather than lax) functor of A -modules.

Lemma 3.2.5 ([Lur17a, Cor. 7.3.2.7] or [GR17, Vol. I, Chap. 1, Lem. 3.5.3]). *Let $A, \mathcal{M}, \mathcal{N}$ be as above and $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a pair of adjoint functors. Then, the structure on F of a left-lax functor of A -modules is equivalent to the structure on G of a right-lax functor of A -modules.*

3.2.6. *Enriched Hom-spaces.* For any (∞) -category \mathcal{C} and two objects $c_1, c_2 \in \mathcal{C}$, recall that $\text{Hom}_{\mathcal{C}}(c_1, c_2) \in \text{Spc}$ denotes the the space of maps between c_1 and c_2 . The theory of module categories allows for a richer notion of Hom-space that we will now turn to.

Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ and $\mathcal{M} \in \text{Mod}_{\mathcal{A}}$. For any two objects $m, n \in \mathcal{M}$, we consider the following functor

$$\begin{aligned} \mathcal{A}^{\text{op}} &\rightarrow \text{Spc} \\ a &\mapsto \text{Hom}_{\mathcal{M}}(a \otimes m, n). \end{aligned}$$

By assumption, this functor preserves limits and hence, by presentability of all categories involved, we know that this functor is representable [Lur17b, Prop. 5.5.2.2.]. We denote the representing object $\underline{\text{Hom}}_{\mathcal{M}}^{\mathcal{A}}(m, n)$, the \mathcal{A} -enriched Hom-space between m and n . This object is also called the *relative inner Hom* in [GR17, Vol. I, Chap. 1, §3.6].

3.2.7. By definition, we have

$$(3.2.8) \quad \text{Hom}_{\mathcal{M}}(a \otimes m, n) \simeq \text{Hom}_{\mathcal{A}}(a, \underline{\text{Hom}}_{\mathcal{M}}^{\mathcal{A}}(m, n)).$$

In particular, evaluating at $a = 1_{\mathcal{A}}$, the monoidal unit of \mathcal{A} , we recover the usual Hom-space from the \mathcal{A} -enriched one

$$\text{Hom}_{\mathcal{M}}(m, n) \simeq \text{Hom}_{\mathcal{A}}(1_{\mathcal{A}}, \underline{\text{Hom}}_{\mathcal{M}}^{\mathcal{A}}(m, n)).$$

3.2.9. Equation (3.2.8) also implies that for any $m \in \mathcal{M}$, we have a pair of adjoint functors

$$\mathcal{A} \begin{array}{c} \xleftarrow{-\otimes m} \\ \xrightarrow{\underline{\text{Hom}}_{\mathcal{M}}^{\mathcal{A}}(m, -)} \end{array} \mathcal{M}.$$

3.2.10. When $\mathcal{M} = \mathcal{A}$, then for any $a_1, a_2 \in \mathcal{A}$, we obtain the inner-Hom: $\underline{\text{Hom}}_{\mathcal{A}}^{\mathcal{A}}(a_1, a_2) \in \mathcal{A}$. For brevity's sake, we will omit the superscript \mathcal{A} and simply write $\underline{\text{Hom}}_{\mathcal{A}}(a_1, a_2)$.

3.2.11. *Relative tensors of module categories.* Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ and $\mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{A}}$. Then, one can form the relative tensor ([Lur17a, §4.4] and [GR17, Vol. I, Chap. 1, §4.2.1])

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} = |\mathcal{M} \otimes \mathcal{A}^{\otimes \bullet} \otimes \mathcal{N}| \in \text{Mod}_{\mathcal{A}}.$$

Here, $\mathcal{M} \otimes \mathcal{A}^{\otimes \bullet} \otimes \mathcal{N}$ denotes the simplicial object obtained by the two-sided bar construction, and $|-|$ denotes the geometric realization of a simplicial object (i.e., taking colimit). This construction equips $\text{Mod}_{\mathcal{A}}$ with the structure of a symmetric monoidal category.

We have canonically defined functors

$$\mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}.$$

For $m \in \mathcal{M}$ and $n \in \mathcal{N}$, we write $m \boxtimes_{\mathcal{A}} n \in \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ to denote the image of $(m, n) \in \mathcal{M} \times \mathcal{N}$ under this functor. When no confusion is likely to occur, we will simply write $m \boxtimes n \in \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ for $m \boxtimes_{\mathcal{A}} n$.

3.2.12. *Compact generation of relative tensors.* We have the following generalization of Proposition 3.1.13.

Proposition 3.2.13 ([GR17, Vol. I, Chap. 1, Prop. 8.7.4]). *Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ and $\mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{A}}$ such that $\mathcal{A}, \mathcal{M}, \mathcal{N}$ are all compactly generated. Suppose that*

$$\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad \mathcal{A} \otimes \mathcal{N} \rightarrow \mathcal{N}, \quad \mathcal{M} \otimes \mathcal{A} \rightarrow \mathcal{M}$$

preserve compactness. Then, the functor $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ also preserves compactness. Moreover, $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is compactly generated by objects of the form $m_0 \boxtimes n_0$ where $m_0 \in \mathcal{M}^c$ and $n_0 \in \mathcal{N}^c$.

3.3. Duality. We will review basic general pattern of duality. The most important concept is that of a dualizable object. This is an important finiteness condition in a monoidal category and which frequently appears in knot theory and topological quantum field theory. The materials presented in this section come from [GR17, Vol. I, Chap. 1, §4].

3.3.1. *Dualizability.* Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ and $a \in \mathcal{A}$. Then, we obtain an endofunctor $a \otimes - : \mathcal{A} \rightarrow \mathcal{A}$. We say that a admits a dual a^\vee if $a^\vee \otimes -$ is adjoint to $a \otimes -$. In this case, we say that a is *dualizable*.

Note that most of [GR17, Vol. I, Chap. 1, §4] distinguishes between left and right duals (or dualizability), which correspond to specifying precisely which adjoint (i.e., left vs. right) we have. However, in a symmetric monoidal category, the two notions coincide. Indeed, given a dualizable object $a \in \mathcal{A}$ with dual a^\vee , for any $b, c \in \mathcal{A}$, we have

$$(3.3.2) \quad \text{Hom}_{\mathcal{A}}(a \otimes b, c) \simeq \text{Hom}_{\mathcal{A}}(b, a^\vee \otimes c)$$

and

$$\text{Hom}_{\mathcal{A}}(b, a \otimes c) \simeq \text{Hom}_{\mathcal{A}}(a^\vee \otimes b, c).$$

3.3.3. *Dual object and internal-Hom.* Just as in linear algebra, dual objects, in general, can be expressed as the (internal) Hom into the monoidal unit object. Let $a \in \mathcal{A}$ be a dualizable object as above. By §3.2.6, for any $b, c \in \mathcal{A}$, we have

$$(3.3.4) \quad \text{Hom}_{\mathcal{A}}(a \otimes b, c) \simeq \text{Hom}_{\mathcal{A}}(b, \underline{\text{Hom}}_{\mathcal{A}}(a, c)).$$

Comparing with (3.3.2), we get

$$\underline{\text{Hom}}_{\mathcal{A}}(a, c) \simeq a^\vee \otimes c.$$

In particular, taken $c = 1_{\mathcal{A}}$, we get

$$a^\vee \simeq \underline{\text{Hom}}_{\mathcal{A}}(a, 1_{\mathcal{A}}).$$

Moreover, the functor $\underline{\text{Hom}}_{\mathcal{A}}(a, -) \simeq a^\vee \otimes -$ commutes with both limits and colimits.

3.4. **Rigidity.** Rigid monoidal categories are stable monoidal categories which exhibit strong finiteness properties and behave extremely nicely with respect to duality and module category structures. In the stable categorical context, it was introduced in [GR17, Vol. I, Chap. 1, §9] which is the reference for this subsection.

3.4.1. *Compactly generated rigid symmetric monoidal categories.*

Definition 3.4.2. Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ such that \mathcal{A} is compactly generated. We say that \mathcal{A} is *rigid* if

- The monoidal unit $1_{\mathcal{A}}$ is compact.
- The functor $\text{mult}_{\mathcal{A}}$ sends $\mathcal{A}^c \times \mathcal{A}^c$ to \mathcal{A}^c , i.e., it preserves compactness.
- Every compact object of \mathcal{A} is dualizable.

We note that rigidity, in general, does *not* require the category involved to be compactly generated nor the monoidal structure to be symmetric. The general definition can be found at [GR17, Vol. I, Chap. 1, Def. 9.1.2]. However, we do not need this generality in the paper. The fact that this is equivalent to the above is [GR17, Vol. I, Chap. 1, Lem. 9.1.5].

Example 3.4.3. We let Vect denote the (∞) -derived category of chain complexes of vector spaces over a field Λ . The category Vect has a natural symmetric monoidal structure. It is compactly generated with compact objects being perfect complexes. In fact, it is compactly generated by Λ . From this description, we see that Vect is a compactly generated rigid symmetric monoidal category.

Example 3.4.4. Let Vect^{gr} denote the category of graded chain complexes. Formally speaking, let \mathbb{Z} be the discrete category with the underlying set of objects given by the set of integers. Then, $\text{Vect}^{\text{gr}} = \text{Fun}(\mathbb{Z}, \text{Vect})$. We think of objects in Vect^{gr} as graded chain complexes. Informally, any $V \in \text{Vect}^{\text{gr}}$ is of the form $V = \bigoplus_{i \in \mathbb{Z}} V_i$ where V_i is the i -th graded component of V .

Vect^{gr} has a natural symmetric monoidal structure given by Day convolution [Lur17a, §2.2.6]. More informally, given $V, W \in \text{Vect}^{\text{gr}}$,

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j.$$

Vect^{gr} is compactly generated with compact objects being graded chain complexes supported on finitely many degrees and each graded degree is perfect. Any compact object $V \in (\text{Vect}^{\text{gr}})^c$ is dualizable

with dual given by $(V^\vee)_i = (V_{-i})^\vee$. From this description, it we see that Vect^{gr} is a compactly generated rigid symmetric monoidal category.

For $V \in \text{Vect}$, we will also abuse notation and view it as an object in Vect^{gr} where V is placed in degree 0. For any $V \in \text{Vect}^{\text{gr}}$ and $k \in \mathbb{Z}$, we write $V\langle k \rangle$ to denote a grading shift of V , i.e., $(V\langle k \rangle)_n = V_{n+k}$. In particular, for $V \in \text{Vect}$, $V\langle k \rangle$ denotes the object in Vect^{gr} obtained by putting V put in graded degree $-k$. The category Vect^{gr} is compactly generated by $\{\Lambda\langle k \rangle\}_{k \in \mathbb{Z}}$.

3.4.5. Interaction with lax functors. Lax functors between module categories over a rigid monoidal category are automatically strict.

Lemma 3.4.6 ([GR17, Vol. I, Chap. 1, Lem. 9.3.6]). *Let \mathcal{A} be a rigid monoidal category. Any continuous right-lax or (left-lax) functor between \mathcal{A} -module categories is strict.*

Combining this result and Lemma 3.2.5, we obtain the following result.

Corollary 3.4.7. *Let \mathcal{A} be a rigid monoidal category, $\mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{A}}$. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a pair of adjoint functors between the underlying categories. Then, the following are equivalent*

- (i) *F is a left-lax functor of \mathcal{A} -modules.*
- (ii) *G is a right-lax functor of \mathcal{A} -modules.*
- (iii) *F is a strict functor of \mathcal{A} -modules.*
- (iv) *G is a strict functor of \mathcal{A} -modules.*

3.4.8. Interaction with compactness. Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ and $\mathcal{M} \in \text{Mod}_{\mathcal{A}}$. Then, an object $m \in \mathcal{M}$ is said to be *compact relative to \mathcal{A}* if

$$\underline{\mathcal{H}\text{om}}_{\mathcal{M}}^{\mathcal{A}}(m, -) : \mathcal{M} \rightarrow \mathcal{A}$$

commutes with filtered colimits (equivalently, all colimits) [GR17, Vol. I, Chap. 1, Def. 8.8.2].

The situation is especially nice when \mathcal{A} is rigid.

Lemma 3.4.9 ([GR17, Vol. I, Chap. 1, Lem. 9.3.4]). *Let \mathcal{A} be a rigid monoidal category and $\mathcal{M} \in \text{Mod}_{\mathcal{A}}$. Then, $m \in \mathcal{M}$ is compact if and only if it is compact relative to \mathcal{A} .*

3.4.10. Compact generation of relative tensors. When \mathcal{A} is rigid, relative tensors of compactly generated \mathcal{A} -modules are always compactly generated.

Lemma 3.4.11 ([GR17, Vol. I, Chap. 1, Lem. 9.3.2]). *Let \mathcal{A} be a rigid monoidal category and $\mathcal{M} \in \text{Mod}_{\mathcal{A}}$. Then, $\text{act}_{\mathcal{A}, \mathcal{M}} : \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{M}$ admits a continuous right adjoint.*

Combining with Lemma 3.1.11 and Proposition 3.1.13, we obtain the following

Corollary 3.4.12. *Let \mathcal{A} and \mathcal{M} be as in Lemma 3.4.11. Suppose that \mathcal{A} and \mathcal{M} are compactly generated. Then $\mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{M}$ preserves compactness.*

Combining with Proposition 3.2.13, we get

Corollary 3.4.13. *Let $\mathcal{A} \in \text{ComAlg}(\text{Pr}^{L, \text{st}})$ that is rigid and compactly generated. Let $\mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{A}}$ be compactly generated as well. Then, $\mathcal{M} \otimes \mathcal{N} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ preserves compactness. Moreover, $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is compactly generated by objects of the form $m_0 \boxtimes n_0$ where $m_0 \in \mathcal{M}^c$ and $n_0 \in \mathcal{N}^c$.*

3.5. An induction formula for enriched Hom-spaces. Relative tensor products over a compactly generated rigid symmetric monoidal category are quite explicit. Corollary 3.4.13 gives us a set of compact generators. But in fact, enriched Hom-spaces are also explicit.

Proposition 3.5.1. *Let \mathcal{A} be a compactly generated rigid symmetric monoidal category, $\mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{A}}$ that are both compactly generated. Then, $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is compactly generated. Moreover, for $(m_0, n_0) \in \mathcal{M}^c \times \mathcal{N}^c$ and $(m, n) \in \mathcal{M} \times \mathcal{N}$,*

$$\underline{\mathcal{H}\text{om}}_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}^{\mathcal{A}}(m_0 \boxtimes n_0, m \boxtimes n) \simeq \underline{\mathcal{H}\text{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, m) \otimes \underline{\mathcal{H}\text{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, n).$$

Proof. From §3.2.9, we have a pair of adjoint functors

$$(3.5.2) \quad \mathcal{A} \xrightleftharpoons[\underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)]{-\otimes n_0} \mathcal{N}.$$

By Lemma 3.4.9, we know that the right adjoint $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)$ is continuous. Moreover, since $-\otimes n_0$ is compatible with \mathcal{A} -module structures on both sides, so is $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)$, by Corollary 3.4.7. Thus, applying $\mathcal{M} \otimes_{\mathcal{A}} -$ to the above, we obtain the following pair of adjoint functors

$$(3.5.3) \quad \mathcal{M} \xrightleftharpoons[\mathrm{id}_{\mathcal{M}} \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)]{-\boxtimes n_0} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}.$$

Here, $\mathrm{id}_{\mathcal{M}} \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)$ is the functor given by

$$m \boxtimes n \mapsto m \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, n),$$

where $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, n) \in \mathcal{A}$ and we have used the \mathcal{A} -module structure on \mathcal{M} to form the tensor, see §3.2.3.

Similarly to (3.5.2), we have the following pair of adjoint functors

$$\mathcal{A} \xrightleftharpoons[\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, -)]{m_0 \otimes -} \mathcal{M}.$$

Composing this with (3.5.3), we obtain a pair of adjoint functors

$$\mathcal{A} \xrightleftharpoons[\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, -\otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -))]{-\otimes(m_0 \boxtimes n_0)} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}.$$

Now, note that

$$(3.5.4) \quad \underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, -\otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)) \simeq \underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, -) \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, -)$$

since $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, -)$ is compatible with \mathcal{A} -module structures by Corollary 3.4.7. On the other hand, $-\otimes(m_0 \boxtimes n_0)$ also admits a right adjoint given by $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}^{\mathcal{A}}(m_0 \boxtimes n_0, -)$, again by §3.2.9. This implies that for all $m \boxtimes n \in \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$, we have

$$\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}^{\mathcal{A}}(m_0 \boxtimes n_0, m \boxtimes n) \simeq \underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, m) \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{N}}^{\mathcal{A}}(n_0, n)$$

as desired. \square

We have the following variant of the proposition above.

Proposition 3.5.5. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a symmetric monoidal functor between compactly generated rigid symmetric monoidal categories. Let $\mathcal{M} \in \mathrm{Mod}_{\mathcal{A}}$. Then, for $(b_0, m_0) \in \mathcal{B}^c \times \mathcal{M}^c$ and $(b, m) \in \mathcal{B} \times \mathcal{M}$, we have (see the end of §3.2.3 for the notation)*

$$\begin{aligned} \underline{\mathcal{H}\mathrm{om}}_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}}^{\mathcal{B}}(b_0 \boxtimes m_0, b \boxtimes m) &\simeq \underline{\mathcal{H}\mathrm{om}}_{\mathcal{B}}(b_0, b) \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, m) \\ &\simeq \underline{\mathcal{H}\mathrm{om}}_{\mathcal{B}}(b_0, b) \otimes F(\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, m)). \end{aligned}$$

In particular, if we let $F_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}$ be the natural functor, i.e., $F_{\mathcal{M}}(m) = 1_{\mathcal{B}} \boxtimes m$, then

$$\underline{\mathcal{H}\mathrm{om}}_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}}^{\mathcal{B}}(F_{\mathcal{M}}(m_0), F_{\mathcal{M}}(m)) \simeq F(\underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, m)).$$

Proof. The second part follows from the first part by noticing that $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{B}}(1_{\mathcal{B}}, 1_{\mathcal{B}}) \simeq 1_{\mathcal{B}}$.

The proof of the first part is similar to the one above. Namely, we have the following pair of adjoint functors

$$\mathcal{B} \xrightleftharpoons[\underline{\mathcal{H}\mathrm{om}}_{\mathcal{B}}(b_0, -) \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, -)]{-\otimes(b_0 \boxtimes m_0)} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}.$$

On the other hand, $-\otimes(b_0 \boxtimes m_0)$ also admits a right adjoint given by $\underline{\mathcal{H}\mathrm{om}}_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}}^{\mathcal{B}}(b_0 \boxtimes m_0, -)$, by §3.2.9. Thus, for all $(b, m) \in \mathcal{B} \times \mathcal{M}$, we have

$$\underline{\mathcal{H}\mathrm{om}}_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}}^{\mathcal{B}}(b_0 \boxtimes m_0, b \boxtimes m) \simeq \underline{\mathcal{H}\mathrm{om}}_{\mathcal{B}}(b_0, b) \otimes \underline{\mathcal{H}\mathrm{om}}_{\mathcal{M}}^{\mathcal{A}}(m_0, m)$$

as desired. \square

Corollary 3.5.6. *In the situation of Proposition 3.5.5, suppose further that F is conservative. Then, $F_{\mathcal{M}}|_{\mathcal{M}^c}$ is also conservative.*

Proof. We will show that if $m \in \mathcal{M}^c$ such that $m \neq 0$ then $F_{\mathcal{M}}(m) \neq 0$. Now, note that $m \neq 0$ if and only if $\underline{\mathcal{H}\text{om}}_{\mathcal{M}}^A(m, m) \neq 0$. But since

$$\underline{\mathcal{H}\text{om}}_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}}^B(F_{\mathcal{M}}(m), F_{\mathcal{M}}(m)) \simeq F(\underline{\mathcal{H}\text{om}}_{\mathcal{M}}^A(m, m))$$

and F is conservative, this is equivalent to $\underline{\mathcal{H}\text{om}}_{\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}}^B(F_{\mathcal{M}}(m), F_{\mathcal{M}}(m)) \neq 0$. This is itself equivalent to $F_{\mathcal{M}}(m) \neq 0$. Hence, we are done. \square

3.6. DG-categories. The theory of DG-categories fits nicely in the framework of stable categories reviewed above. The materials in this subsection are from [GR17, Vol. I, Chap. 1, §10].

A DG-category is a stable presentable category equipped with a module structure over Vect , i.e., it is an object in $\text{Mod}_{\text{Vect}} = \text{Mod}_{\text{Vect}}(\text{Pr}^{L, \text{st}})$. We let $\text{DGCat}_{\text{pres, cont}} = \text{Mod}_{\text{Vect}}$ denote the category of DG-categories with morphisms being continuous functors (compatible with Vect -module structures).

As already seen above, $\text{DGCat}_{\text{pres, cont}}$ is a symmetric monoidal category where the tensor structure is given by relative tensor over Vect . In this paper, given $\mathcal{C}, \mathcal{D} \in \text{DGCat}_{\text{pres, cont}}$, we will never take the “absolute tensor” $\mathcal{C} \otimes \mathcal{D}$. On the other hand, we will make extensive use of the relative tensors $\mathcal{C} \otimes_{\text{Vect}} \mathcal{D}$. Because of that, from this point forward, following [GR17, Vol. I, Chap. 1, §10.6], for we will adopt the following convention

- When $\mathcal{C}, \mathcal{D} \in \text{DGCat}_{\text{pres, cont}}$, $\mathcal{C} \otimes \mathcal{D}$ is used to denote $\mathcal{C} \otimes_{\text{Vect}} \mathcal{D}$. Moreover, for $c \in \mathcal{C}$ and $d \in \mathcal{D}$, $c \boxtimes d$ is used to denote the element in $\mathcal{C} \otimes_{\text{Vect}} \mathcal{D}$.
- For $c_1, c_2 \in \mathcal{C}$ where $\mathcal{C} \in \text{DGCat}_{\text{pres, cont}}$, $\mathcal{H}\text{om}_{\mathcal{C}}(c_1, c_2)$ is used to denote $\underline{\mathcal{H}\text{om}}_{\mathcal{C}}^{\text{Vect}}(c_1, c_2)$. In particular, $\text{Hom}_{\mathcal{C}}(c_1, c_2) = \text{Hom}_{\text{Vect}}(\Lambda, \mathcal{H}\text{om}_{\mathcal{C}}(c_1, c_2))$.
- All functors between DG-categories are, unless otherwise specified, compatible with the Vect -module structures.

3.7. Large vs. small categories. Up to now, we have been working with *large* categories, in the sense that the collection of objects is not a set. As mentioned above, working in this context allows many convenient theorems regarding existence of adjoints and limits/colimits to hold true. On the other hand, working with small categories is usually more concrete. In fact, familiar constructions in representation theory usually involve small categories rather than large ones. In this sense, large categories could be viewed as a technical device: we usually perform our construction in the large category setting whenever it is convenient to do so, and then extract the small category out of that.

We will now review constructions involving small categories and how they are related to those in the large category setting.

3.7.1. Small DG-categories. Let $\text{DGCat}_{\text{idem, ex}}$ denote the category whose objects are idempotent complete stable infinity categories, equipped with an action of Vect^c , and whose morphisms are given by exact functors (i.e., functors that commute with finite limits and colimits) compatible with the action of Vect^c . Unless otherwise specified, all small DG-categories in the current paper belong to $\text{DGCat}_{\text{idem, ex}}$.

The procedure of taking Ind-completion provides a functor

$$\text{Ind} : \text{DGCat}_{\text{idem, ex}} \rightarrow \text{DGCat}_{\text{pres, cont}}.$$

The essential image of Ind is precisely those DG-categories that are compactly generated. Indeed, given such a category \mathcal{C} , the full subcategory \mathcal{C}^c spanned by the set of compact objects is an object of $\text{DGCat}_{\text{idem, ex}}$. Moreover, $\text{Ind}(\mathcal{C}^c) \simeq \mathcal{C}$ when \mathcal{C} is compactly generated, see also §3.1.10.

3.7.2. Tensors of small DG-categories. Similarly to [BFN10, Prop. 4.4], $\text{DGCat}_{\text{idem, ex}}$ admits a symmetric monoidal structure given by

$$\mathcal{C}_1 \otimes \mathcal{C}_2 := (\text{Ind}(\mathcal{C}_1) \otimes \text{Ind}(\mathcal{C}_2))^c, \quad \mathcal{C}_1, \mathcal{C}_2 \in \text{DGCat}_{\text{idem, ex}}.$$

Moreover, the functor Ind is symmetric monoidal. In particular, when $\mathcal{C}_1, \mathcal{C}_2 \in \text{DGCat}_{\text{pres,cont}}$ are compactly generated,

$$(\mathcal{C}_1 \otimes \mathcal{C}_2)^c = \mathcal{C}_1^c \otimes \mathcal{C}_2^c.$$

Using this monoidal structure, it is possible to make sense of $\text{ComAlg}(\text{DGCat}_{\text{idem,ex}})$. Namely, $\mathcal{A} \in \text{ComAlg}(\text{DGCat}_{\text{idem,ex}})$ is a symmetric monoidal small DG-category such that the tensor product is compatible with the Vect^c -action and with finite colimits in each variable. For such \mathcal{A} , it is also possible to make sense of $\text{Mod}_{\mathcal{A}}(\text{DGCat}_{\text{idem,ex}})$. Given $\mathcal{A} \in \text{ComAlg}(\text{DGCat}_{\text{idem,ex}})$, we will also use the notation $\text{Mod}_{\mathcal{A}} := \text{Mod}_{\mathcal{A}}(\text{DGCat}_{\text{idem,ex}})$ when no confusion is likely to occur. Note that this notation is similar to the one introduced in §3.2.1; however, the fact that \mathcal{A} is *small* should be clear from the context.

3.7.3. Module structures on large vs. small categories. For $\mathcal{A} \in \text{ComAlg}(\text{DGCat}_{\text{pres,cont}})$ and $\mathcal{M} \in \text{Mod}_{\mathcal{A}}$ such that \mathcal{A} is rigid compactly generated and \mathcal{M} is compactly generated, Corollary 3.4.12 implies that $\mathcal{M}^c \in \text{Mod}_{\mathcal{A}^c}$. Moreover, by continuity, the \mathcal{A} -module structure on \mathcal{M} is obtained by ind-extending the \mathcal{A}^c -module structure on \mathcal{M}^c .

3.7.4. Relative tensors of small DG-categories. Similarly to the above, but in the relative setting, let $\mathcal{A} \in \text{ComAlg}(\text{DGCat}_{\text{pres,cont}})$ be compactly generated and rigid, and $\mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{A}}$ that are compactly generated. Then, we define

$$\mathcal{M}^c \otimes_{\mathcal{A}^c} \mathcal{N}^c = (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})^c.$$

Here, the LHS is performed in $\text{DGCat}_{\text{idem,ex}}$ using the tensor product described above.

3.7.5. Enriched Hom. Let $\mathcal{A}_0 \in \text{ComAlg}(\text{DGCat}_{\text{idem,ex}})$ and $\mathcal{M}_0 \in \text{Mod}_{\mathcal{A}_0}(\text{DGCat}_{\text{idem,ex}})$, then we can make sense of $\mathcal{A} := \text{Ind}(\mathcal{A}_0)$ -enriched Hom-spaces between objects in \mathcal{M}_0 by embedding $\mathcal{M}_0 \hookrightarrow \mathcal{M} := \text{Ind}(\mathcal{M}_0)$ and use the discussion above for presentable categories. Here, the \mathcal{A} -module structure on \mathcal{M} is obtained by left Kan extending $\mathcal{A}_0 \otimes \mathcal{M}_0 \rightarrow \mathcal{M}_0$ along $\mathcal{A}_0 \otimes \mathcal{M}_0 \rightarrow \mathcal{A} \otimes \mathcal{M}$.

4. GRADED SHEAVES: CONSTRUCTION AND FORMAL PROPERTIES

In this section, we will construct the category of graded sheaves on schemes/stacks and prove their formal properties (see §1.2 for our conventions regarding schemes/stacks). In preparation for the construction, we start, in Sections 4.1 to 4.3, with a brief review of the theory of ℓ -adic sheaves and mixed sheaves on stacks in the form that is useful for our purposes. We construct the category of graded sheaves in §4.4 and study their functoriality in §4.5. The streamlined construction of the category of graded sheaves mean that six-functor formalism carries over to the graded setting in a straightforward manner. In §4.6, we study the Hom-spaces between graded sheaves. Despite being an easy application of §3.5, the result in this subsection serves as the foundation for most of the important results concerning graded sheaves in this paper. Possibly most interestingly, we show in §4.7 that in a precise sense, the category of graded sheaves is invariant under extensions of scalars. In other words, for a stack \mathcal{Y}_n defined over $\text{pt}_n = \text{Spec } \mathbb{F}_{q^n}$, the category of graded sheaves on it only depends on \mathcal{Y} , the base change of \mathcal{Y}_n to $\overline{\mathbb{F}}_q$. Finally, in §4.8, we explain how the results of [LZ17a, LZ17b] can be used to upgrade our theory of graded sheaves to a functor out of the category of correspondences on stacks. This captures all the base change isomorphisms in a homotopy coherent way, which is useful when one wants to construct an ∞ -monoidal category using convolutions, a pattern often seen in geometric representation theory.

4.1. ℓ -adic sheaves. We start with a quick review of the theory of ℓ -adic sheaves. The materials presented here are developed in details in [Gai15, GL19, HRS21] and [AGK⁺20, Appendix C.4].

4.1.1. ℓ -adic sheaves on schemes. For any scheme $S \in \text{Sch}_{\mathbb{F}}$ (see §1.2), we let $\text{Shv}(S)$ denote the ind-completion of the category $\text{Shv}_c(S)$ of constructible ℓ -adic ($\overline{\mathbb{Q}}_{\ell}$) sheaves on S . In other words, we have

$$\text{Shv}(S) = \text{Ind}(\text{Shv}_c(S)) \quad \text{and} \quad \text{Shv}_c(S) = \text{Shv}(S)^c.$$

For example, for $\text{pt} \in \text{Sch}_k$, where, following our conventions in §1.2, $k = \overline{\mathbb{F}}_q$, $\text{Shv}(\text{pt}) = \text{Vect}$ and $\text{Shv}_c(\text{pt}) = \text{Vect}^c$. Here, Vect denotes the DG-category of chain complexes of vector spaces over $\overline{\mathbb{Q}}_{\ell}$, and Vect^c the full subcategory spanned by compact objects, which are perfect complexes.

Note that since all functors in the six-functor formalism respect constructibility, the resulting functors between the large categories (i.e., after ind-completion) are obtained by ind-extension. Thus, all of these functors are continuous and compactness preserving.

Using the pullback functors $(-)^*$ and $(-)^!$, we obtain

$$\mathrm{Shv}^? : \mathrm{Sch}_{\mathbb{R}}^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{pres}, \mathrm{cont}}$$

where $?$ is either $*$ or $!$. Similarly, for the *small* versions, we have

$$\mathrm{Shv}_c^? : \mathrm{Sch}_{\mathbb{R}}^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{idem}, \mathrm{ex}}.$$

4.1.2. *ℓ -adic sheaves on stacks.* Right Kan extending $\mathrm{Shv}^?$ along $\mathrm{Sch}_{\mathbb{R}}^{\mathrm{op}} \hookrightarrow \mathrm{Stk}_{\mathbb{R}}^{\mathrm{op}}$, we obtain

$$\mathrm{Shv}^? : \mathrm{Stk}_{\mathbb{R}}^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{pres}, \mathrm{cont}}.$$

More concretely, for $\mathcal{Y} \in \mathrm{Stk}_{\mathbb{R}}$,

$$\mathrm{Shv}^?(\mathcal{Y}) = \lim_{S \in \mathrm{Sch}_{\mathbb{R}/\mathcal{Y}}^{\mathrm{op}}} \mathrm{Shv}(S),$$

where the transition functors are either $(-)^*$ or $(-)^!$. Since the property of satisfying descent is preserved by right Kan extension [Gai11, Prop. 6.4.3], the new sheaf theories satisfy smooth descent. In particular, if $h : Y \rightarrow \mathcal{Y}$ is a smooth atlas where Y is a scheme, then

$$(4.1.3) \quad \mathrm{Shv}^?(\mathcal{Y}) = \mathrm{Tot}(\mathrm{Shv}^?(\check{\mathrm{Cech}}^\bullet(Y/\mathcal{Y})))$$

where $\check{\mathrm{Cech}}^\bullet(Y/\mathcal{Y})$ is the (simplicial) Čech nerve of the covering $Y \rightarrow \mathcal{Y}$, and Tot is the procedure of taking totalization, i.e., limit, of a co-simplicial object.

4.1.4. As in [Gai11, Prop. 11.4.3], for any $\mathcal{Y} \in \mathrm{Stk}_{\mathbb{R}}$, $\mathrm{Shv}^!(\mathcal{Y}) \simeq \mathrm{Shv}^*(\mathcal{Y})$. This is essentially due to the fact that the diagram used to compute the totalization (4.1.3) for the two theories are equivalent, given by a shift. Thus, we will simply write $\mathrm{Shv}(\mathcal{Y})$ from now on. In particular, for $f : \mathcal{Y} \rightarrow \mathcal{Z}$ a morphism in $\mathrm{Stk}_{\mathbb{R}}$, we have functors

$$(4.1.5) \quad f^*, f^! : \mathrm{Shv}(\mathcal{Z}) \rightarrow \mathrm{Shv}(\mathcal{Y}).$$

By construction, f^* is continuous, and thus, has a right adjoint f_* . $f^!$ is co-continuous, inherited from same property of $(-)^!$ functors between schemes, and thus has a left adjoint $f_!$.⁹

Remark 4.1.6. Let $\mathrm{PreStk}_{\mathbb{R}} = \mathrm{Fun}(\mathrm{Sch}_{\mathbb{R}}^{\mathrm{op}}, \mathrm{Spc})$ be the category of prestacks. The same considerations yield two sheaf theories $\mathrm{Shv}^!$ and Shv^* on prestacks, which are useful, for instance, in the studies of affine Grassmannians as well as their factorizable versions. In this generality, however, (4.1.5) does not always hold. We will revisit this theory in a subsequent paper.

4.1.7. By [GR17, Vol. I, Chap. 1, Prop. 2.5.7], we obtain an alternative description of $\mathrm{Shv}^!(\mathcal{Y})$ from (4.1.3)

$$\mathrm{Shv}(\mathcal{Y}) = |\mathrm{Shv}(\check{\mathrm{Cech}}^\bullet(Y/\mathcal{Y}))|,$$

where we now use the $(-)_!$ -functor to move between different schemes. Moreover, $|-|$ denotes the procedure of taking geometric realization, i.e., colimit, of a simplicial object. Since all the functors used in the colimit preserve compactness, the category $\mathrm{Shv}(\mathcal{Y})$ is compactly generated with a set of compact generators given by $h_!(\mathrm{Shv}_c(Y))$, by [DG15, Cor. 1.9.4]. It is important that the colimit is taken inside $\mathrm{DGCat}_{\mathrm{pres}, \mathrm{cont}}$.

⁹ $f^!$ is in fact also continuous. Thus, it also has a right adjoint. However, this right adjoint is rarely used and in particular, we do not make use of this functor in this paper.

4.1.8. *Ind-constructible sheaves on stacks.* The constant sheaves on most stacks are not compact, see Example 4.1.12 below. On the other hand, it is constructible. The *renormalization* procedure described in this subsection enlarges the category of sheaves so that constructible sheaves become compact in this new category. The difference between the *renormalized* version and the usual version is parallel to the difference between coherent complexes and perfect complexes, or more precisely, between ind-coherent sheaves and quasi-coherent sheaves since we are working in the *large* category setting. A more detailed discussion can be found in [AGK⁺20, Appendix C.4].

Definition 4.1.9. For $\mathcal{Y} \in \text{Stk}_{\mathbb{F}}$, we let $\text{Shv}_c(\mathcal{Y})$ denote the full-subcategory of $\text{Shv}(\mathcal{Y})$ consisting of *constructible* objects, defined to be those whose pullbacks along $s : S \rightarrow \mathcal{Y}$ either via s^* or $s^!$ are compact/constructible for any scheme S .

We use $\text{Shv}(\mathcal{Y})^{\text{ren}} := \text{Ind}(\text{Shv}_c(\mathcal{Y}))$ to denote the renormalized category of sheaves on \mathcal{Y} .

Remark 4.1.10. When \mathcal{Y} is a scheme rather than a stack, $\text{Shv}(\mathcal{Y})$ and $\text{Shv}(\mathcal{Y})^{\text{ren}}$ coincide by construction.

Lemma 4.1.11. Let $\mathcal{Y} \in \text{Stk}_{\mathbb{F}}$ and $h : Y \rightarrow \mathcal{Y}$ a smooth atlas. Then $\mathcal{F} \in \text{Shv}(\mathcal{Y})$ is constructible if and only if $h^*\mathcal{F}$ or $h^!\mathcal{F}$ is constructible.

Proof. We will prove the statement for h^* ; the one for $h^!$ is done analogously. The only if direction is clear by definition. We will now prove the if direction.

Let $f : S \rightarrow \mathcal{Y}$ be an arbitrary morphism, where $S \in \text{Sch}_{\mathbb{F}}$. Consider the following pullback square

$$\begin{array}{ccc} S \times_{\mathcal{Y}} Y & \xrightarrow{f'} & Y \\ \downarrow h_S & & \downarrow h \\ S & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where h_S is a smooth surjective map between schemes. Since $h_S^* f^* \mathcal{F} \simeq f'^* h^* \mathcal{F}$ is constructible, so is $f^* \mathcal{F}$ and we are done. \square

By definition, $\text{Shv}_c(\mathcal{Y}) \simeq (\text{Shv}(\mathcal{Y})^{\text{ren}})^c$ since $\text{Shv}_c(\mathcal{Y})$ is easily seen to be idempotent complete. Unlike the case of schemes, however, $\text{Shv}_c(\mathcal{Y})$ is in general different from $\text{Shv}(\mathcal{Y})^c$, i.e., constructibility and compactness do not necessarily coincide for a general stack \mathcal{Y} (see Example 4.1.12 below).

Example 4.1.12. Consider the constant sheaf $\overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m} \in \text{Shv}(B\mathbb{G}_m)$ which is evidently constructible. It is, however, not compact in $\text{Shv}(B\mathbb{G}_m)$. Indeed, to see that, it suffices to show that

$$\pi_* = \mathcal{H}\text{om}_{\text{Shv}(B\mathbb{G}_m)}(\overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m}, -) : \text{Shv}(B\mathbb{G}_m) \rightarrow \text{Shv}(\text{pt}) \simeq \text{Vect}$$

does not commute with filtered colimits, where $\pi : B\mathbb{G}_m \rightarrow \text{pt}$ is the structure map. Consider

$$\mathcal{H}\text{om}_{\text{Shv}(B\mathbb{G}_m)}(\overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m}, \overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m}) = \pi_* \overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m} \simeq \overline{\mathbb{Q}}_{\ell}[\beta]$$

where β is in cohomological degree 2. Let

$$\mathcal{F} = \text{colim}(\overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m} \xrightarrow{\beta} \overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m}[2] \xrightarrow{\beta} \cdots) \in \text{Shv}(B\mathbb{G}_m).$$

Let $h : \text{pt} \rightarrow B\mathbb{G}_m$ be the canonical map. Then, by descent, h^* is continuous and conservative, i.e., it does not kill any object. But we see that

$$h^* \mathcal{F} = \text{colim}(\overline{\mathbb{Q}}_{\ell} \xrightarrow{\beta} \overline{\mathbb{Q}}_{\ell}[2] \xrightarrow{\beta} \cdots) \simeq 0.$$

Hence, by conservativity of h^* , $\mathcal{F} = 0$, which means

$$\pi_* \text{colim}(\overline{\mathbb{Q}}_{\ell} \xrightarrow{\beta} \overline{\mathbb{Q}}_{\ell}[2] \xrightarrow{\beta} \cdots) \simeq \pi_* \mathcal{F} \simeq 0.$$

On the other hand,

$$\text{colim}(\pi_* \overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m} \xrightarrow{\beta} \pi_* \overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m}[2] \xrightarrow{\beta} \cdots) = \overline{\mathbb{Q}}_{\ell}[\beta, \beta^{-1}] \neq 0 \simeq \pi_* \text{colim}(\overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m} \xrightarrow{\beta} \overline{\mathbb{Q}}_{\ell, B\mathbb{G}_m}[2] \xrightarrow{\beta} \cdots)$$

and we are done.

4.1.13. *ren and unren.* The categories $\mathrm{Shv}(\mathcal{Y})$ and $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ are related by a pair of adjoint functors $\mathrm{ren} \dashv \mathrm{unren}$ which we will now turn to. For more details, see [AGK⁺20, Appendix C.4].

First, note that compact objects in $\mathrm{Shv}(\mathcal{Y})$ are constructible, i.e., $\mathrm{Shv}(\mathcal{Y})^c \subset \mathrm{Shv}_c(\mathcal{Y})$. Thus, by ind-extension, we obtain a continuous functor

$$\mathrm{ren} : \mathrm{Shv}(\mathcal{Y}) \rightarrow \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}.$$

Similarly, the functor $\mathrm{Shv}_c(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y})$ induces a continuous functor

$$\mathrm{unren} : \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \rightarrow \mathrm{Shv}(\mathcal{Y}).$$

These two functors form a pair of adjoints $\mathrm{ren} \dashv \mathrm{unren}$.

By definition, we have the following commutative diagrams

$$\begin{array}{ccc} \mathrm{Shv}(\mathcal{Y})^c & & \mathrm{Shv}_c(\mathcal{Y}) \\ \downarrow & \searrow & \swarrow \downarrow \\ \mathrm{Shv}(\mathcal{Y}) & \xrightarrow{\mathrm{ren}} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} & \xleftarrow{\mathrm{unren}} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} & \downarrow \\ & & \mathrm{Shv}(\mathcal{Y}) & & \mathrm{Shv}(\mathcal{Y}) \end{array}$$

4.1.14. *t-structures.* The categories $\mathrm{Shv}(\mathcal{Y})$ and $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ are naturally equipped with a standard and a perverse *t*-structure, both obtained from ind-extending the corresponding *t*-structure on the full subcategory of compact objects, see [AGH19, Prop. 2.13] or [Lur18, Lem. C.2.4.3]. Moreover, the functor unren , restricted to the bounded below subcategories, induces an equivalence of categories [AGK⁺20, Appendix C.4.1], i.e.

$$\mathrm{unren}|_{\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren},+}} : \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren},+} \xrightarrow{\simeq} \mathrm{Shv}(\mathcal{Y})^+.$$

4.1.15. *Functoriality of ind-constructible sheaves on stacks.* Since pulling back along any morphism $f : \mathcal{Y} \rightarrow \mathcal{Z}$ in $\mathrm{Stk}_{\mathbb{F}}$ preserves constructibility, we automatically get continuous functors $f_{\mathrm{ren}}^!$ and f_{ren}^* obtained by ind-extending the restrictions of $f^!$ and f^* to the constructible full-subcategories.¹⁰ By construction, f_{ren}^* is continuous, and hence, it admits a right adjoint, denoted by $f_{*,\mathrm{ren}}$. Moreover, since f_{ren}^* preserves compactness by definition, $f_{*,\mathrm{ren}}$ is also continuous.

When $f_!$ preserves constructibility (for example, when f is representable), we have a pair of adjoint functors

$$f_! : \mathrm{Shv}_c(\mathcal{Y}) \rightleftarrows \mathrm{Shv}_c(\mathcal{Z}) : f^!.$$

Ind-extending these, we obtain also a pair of adjoint functors

$$f_{!,\mathrm{ren}} : \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \rightleftarrows \mathrm{Shv}(\mathcal{Z})^{\mathrm{ren}} : f_{\mathrm{ren}}^!.$$

Note that in this case, both $f_{\mathrm{ren}}^!$ and $f_{!,\mathrm{ren}}$ are continuous.

f^* and f_{ren}^* are *t*-exact with respect to the standard *t*-structures. Hence, f_* and $f_{*,\mathrm{ren}}$, being the right adjoints, are left *t*-exact with respect to the standard *t*-structures. In particular, f_* and $f_{*,\mathrm{ren}}$ preserve eventual co-connectivity (the property of being bounded below). Since the $(-)^!$ functor for schemes preserve eventual co-connectivity, so is $f^!$ and $f_{\mathrm{ren}}^!$ (between stacks).

4.1.16. *Renormalized vs. usual functors.* We will now compare the various pullback/pushforward functors with their renormalized counterparts. Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ in $\mathrm{Stk}_{\mathbb{F}}$. We have the following (a priori not necessarily commutative) diagram, where parallel arrows are adjoints

$$\begin{array}{ccc} \mathrm{Shv}(\mathcal{Y}) & \xrightleftharpoons[\mathrm{unren}]{\mathrm{ren}} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \\ \uparrow f^* & & \uparrow f_{\mathrm{ren}}^* \\ \mathrm{Shv}(\mathcal{Z}) & \xrightleftharpoons[\mathrm{unren}]{\mathrm{ren}} & \mathrm{Shv}(\mathcal{Z})^{\mathrm{ren}} \\ \downarrow f_* & & \downarrow f_{*,\mathrm{ren}} \end{array}$$

¹⁰Note that on stacks, f^* and $f^!$ do not necessarily preserve compactness. See Example 4.1.12.

Proposition 4.1.17. *We have an equivalence of functors $f^* \circ \text{unren} \simeq \text{unren} \circ f_{\text{ren}}^*$. Moreover, for any $\mathcal{F} \in \text{Shv}_c(\mathcal{Y}) \subseteq \text{Shv}(\mathcal{Y})^{\text{ren}}$, we have a natural equivalence*

$$f_*(\text{unren}(\mathcal{F})) \simeq \text{unren}(f_{*,\text{ren}}(\mathcal{F})).$$

Proof. For the first statement, note that since all functors are continuous, it suffices to prove the statement when restricted to $\text{Shv}_c(\mathcal{Z})$. But then, it follows from the definition of f_{ren}^* .

For the second, we first define a functor

$$f_{\bullet,\text{ren}} : \text{Shv}_c(\mathcal{Y}) \rightarrow \text{Shv}(\mathcal{Z})^+ \hookrightarrow \text{Shv}(\mathcal{Z})^{\text{ren}},$$

as the following composition

$$\text{Shv}_c(\mathcal{Y}) \xrightarrow{f_*} \text{Shv}(\mathcal{Z})^+ \simeq \text{Shv}(\mathcal{Z})^{\text{ren},+} \hookrightarrow \text{Shv}(\mathcal{Z})^{\text{ren}}.$$

Ind-extending, we obtain a (eponymous) continuous functor

$$f_{\bullet,\text{ren}} : \text{Shv}(\mathcal{Y})^{\text{ren}} \rightarrow \text{Shv}(\mathcal{Z})^{\text{ren}}.$$

By construction, for any $\mathcal{F} \in \text{Shv}_c(\mathcal{Y})$, $\text{unren}(f_{\bullet,\text{ren}}(\mathcal{F})) \simeq f_*(\text{unren}(\mathcal{F}))$. It thus remains to show that $f_{\bullet,\text{ren}} \simeq f_{*,\text{ren}}$. Or equivalently, it suffices to show that $f_{\bullet,\text{ren}}$ is the right adjoint of f_{ren}^* .

To do that, we produce a natural equivalence

$$\text{Hom}_{\text{Shv}(\mathcal{Z})^{\text{ren}}}(\mathcal{F}, f_{\bullet,\text{ren}}\mathcal{G}) \simeq \text{Hom}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(f_{\text{ren}}^*\mathcal{F}, \mathcal{G}), \quad \forall \mathcal{F} \in \text{Shv}(\mathcal{Z})^{\text{ren}}, \forall \mathcal{G} \in \text{Shv}(\mathcal{Y})^{\text{ren}}.$$

In fact, since the functors involved are continuous, it suffices to assume that \mathcal{F} and \mathcal{G} are constructible. But now, we conclude using the following sequence of equivalences

$$\begin{aligned} \text{Hom}_{\text{Shv}(\mathcal{Z})^{\text{ren}}}(\mathcal{F}, f_{\bullet,\text{ren}}\mathcal{G}) &\simeq \text{Hom}_{\text{Shv}(\mathcal{Z})^+}(\mathcal{F}, f_*\mathcal{G}) \\ &\simeq \text{Hom}_{\text{Shv}(\mathcal{Y})^+}(f^*\mathcal{F}, \mathcal{G}) \\ &\simeq \text{Hom}_{\text{Shv}(\mathcal{Y})^{\text{ren},+}}(f_{\text{ren}}^*\mathcal{F}, \mathcal{G}) \\ &\simeq \text{Hom}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(f_{\text{ren}}^*\mathcal{F}, \mathcal{G}) \end{aligned}$$

where we made use of the fact that unren induces an equivalence between the bounded below parts of the usual and renormalized categories, see §4.1.14. \square

Remark 4.1.18. If f^* preserves compactness or equivalently, if f_* is continuous (for example, when f is representable), then the right adjoints in the diagram above commute in general, i.e., $f_* \circ \text{unren} \simeq \text{unren} \circ f_{*,\text{ren}}$. Indeed, this follows from Proposition 4.1.17, using continuity of f_* .

As a result, the left adjoints also commute, i.e., $\text{ren} \circ f^* \simeq f_{\text{ren}}^* \circ \text{ren}$.

4.1.19. Now, when $f_!$ preserves constructibility, consider the following (a priori not necessarily commutative) diagram

$$\begin{array}{ccc} \text{Shv}(\mathcal{Y}) & \xrightleftharpoons[\text{unren}]{\text{ren}} & \text{Shv}(\mathcal{Y})^{\text{ren}} \\ f_! \uparrow \downarrow f^! & & f_{!,\text{ren}} \uparrow \downarrow f_{\text{ren}}^! \\ \text{Shv}(\mathcal{Z}) & \xrightleftharpoons[\text{unren}]{\text{ren}} & \text{Shv}(\mathcal{Z})^{\text{ren}} \end{array}$$

Lemma 4.1.20. *Consider f as in Proposition 4.1.17. Then $\text{unren} \circ f_{\text{ren}}^! \simeq f^! \circ \text{unren}$.*

Moreover, if $f_!$ preserves constructibility (so that $f_{!,\text{ren}}$ is defined), then $f_! \circ \text{unren} \simeq \text{unren} \circ f_{!,\text{ren}}$. In this case, switching to the left adjoints of the previous statement also yields an equivalence $f_{!,\text{ren}} \circ \text{ren} \simeq \text{ren} \circ f_!$.

Proof. For the first statement, if $\mathcal{F} \in \text{Shv}_c(\mathcal{Z})$, the definition of $f_{\text{ren}}^!$ implies that $\text{unren}(f_{\text{ren}}^!(\mathcal{F})) \simeq f^!(\text{unren}(\mathcal{F}))$. But since all functors involved are continuous, the same conclusion applies to all $\mathcal{F} \in \text{Shv}(\mathcal{Z})$.

The second statement is proved similarly. \square

Remark 4.1.21. Due to Proposition 4.1.17, when working with constructible sheaves, there is no ambiguity between f^*, f_* and their renormalized versions $f_{\text{ren}}^*, f_{*,\text{ren}}$, respectively. By Lemma 4.1.20, the same statement also applies to $f^!$ vs. $f_{\text{ren}}^!$ in general, and also to $f_!$ vs. $f_{!,\text{ren}}$ when $f_!$ preserves constructibility. Therefore, in these situations, when working with constructible sheaves, we will use the two versions interchangeably.

4.1.22. *Symmetric monoidal structures.* For each $S \in \text{Sch}_{\mathbb{F}}$, $\text{Shv}(S)$ is equipped with a symmetric monoidal structure given by tensor products of sheaves. This tensor product preserves compactness, or equivalently (since we are working with schemes), constructibility. Moreover, for any $f : S \rightarrow T$ in $\text{Sch}_{\mathbb{F}}$, f^* is symmetric monoidal.

This structure induces symmetric monoidal structures on $\text{Shv}(\mathcal{Y})$ and $\text{Shv}(\mathcal{Y})^{\text{ren}}$ for any $\mathcal{Y} \in \text{Stk}_{\mathbb{F}}$. Moreover, for any $f : \mathcal{Y} \rightarrow \mathcal{Z}$ in $\text{Stk}_{\mathbb{F}}$, f^* and f_{ren}^* are symmetric monoidal. In particular, $\text{Shv}(\mathcal{Y})$ and $\text{Shv}(\mathcal{Y})^{\text{ren}}$ are equipped with the structures of module categories over $\text{Shv}(\text{pt}_{\mathbb{F}}) \simeq \text{Shv}(\text{pt}_{\mathbb{F}})^{\text{ren}}$.

4.2. **Mixed sheaves.** Working over a finite field $k_n = \mathbb{F}_{q^n}$, the theory of ℓ -adic sheaves has a “refinement” using weights of Frobenius.

4.2.1. *Mixed sheaves on schemes.* Following [BBDG18, §5.1.5], for a scheme $X_n \in \text{Sch}_{k_n}$, we let $\text{Shv}_{m,c}(X_n)$ denote the category of constructible mixed complexes, which is a full subcategory of $\text{Shv}_c(X_n)$.¹¹ We let

$$\text{Shv}_m(X_n) := \text{Ind}(\text{Shv}_{m,c}(X_n))$$

and obtain a fully faithful embedding

$$\text{Shv}_m(X_n) \hookrightarrow \text{Shv}(X_n).$$

4.2.2. *The case of a point.* The category $\text{Shv}_m(\text{pt}_n)$ consists of complexes of continuous Frobenius modules whose eigenvalues are algebraic numbers of absolute values $(q^n)^{w/2}$. The category $\text{Shv}_m(\text{pt}_n)$ thus breaks up into a direct sum of categories

$$\text{Shv}_m(\text{pt}_n) \simeq \bigoplus_{w \in \mathbb{Z}} \text{Shv}_m(\text{pt}_n)_w.$$

In particular, for $V \in \text{Shv}_m(\text{pt}_n)$, we have a natural decomposition

$$V \simeq \bigoplus_{w \in \mathbb{Z}} V_w$$

where V_w has *naive weight* w , i.e., all eigenvalues have absolute values $(q^n)^{w/2}$.

Remark 4.2.3. We caution the reader here that V_w does not have weight w in the sense of [BBDG18]. For example, consider $\mathbb{Q}_{\ell} \in \text{Shv}_m(\text{pt}_n)$. Then, $\mathbb{Q}_{\ell}[-2]$ has weight 2 in the sense of [BBDG18] but *naive weight* 0 in the sense above.

4.2.4. The category $\text{Shv}_m(\text{pt}_n)$ is thus equipped with a symmetric monoidal functor, given by forgetting the Frobenius module structure (but still remembering the naive weights)

$$(4.2.5) \quad \text{gr} : \text{Shv}_m(\text{pt}_n) \rightarrow \text{Vect}^{\text{gr}} = \text{Fun}(\mathbb{Z}, \text{Vect}).$$

In the notation above, $\text{gr}(V) = \bigoplus_{w \in \mathbb{Z}} \text{gr}(V)_w$ where $\text{gr}(V)_w = V_w$ and, as the notation suggests, is put in graded degree w .

We can think of this functor as semi-simplifying the Frobenius action. Though simple, it plays a crucial role in our construction of the category of graded sheaves. In fact, we will see that $\text{Shv}_{\text{gr}}(\text{pt}_n) \simeq \text{Vect}^{\text{gr}}$.

¹¹The category $\text{Shv}_{m,c}(X_n)$ is written as $D_m^b(X_n)$ in [BBDG18].

4.2.6. *Mixed sheaves on stacks.* For $\mathcal{Y}_n \in \text{Stk}_{k_n}$, we let $\text{Shv}_m(\mathcal{Y}_n)$ denote the full subcategory of $\text{Shv}(\mathcal{Y}_n)$ consisting of those \mathcal{F} whose $*$ -, or $!$ -, pullback along $s_n : S_n \rightarrow \mathcal{Y}_n$ for any $S_n \in \text{Sch}_{k_n}$ is mixed. Since mixedness is preserved under pullbacks [BBDG18, §5.1.6], it suffices to check mixedness on a stack by pulling back to a smooth atlas as in Lemma 4.1.11. See also [Sun12b, Prop. 2.8 and Rmk. 2.12].

This description also implies that the category $\text{Shv}_m(\mathcal{Y}_n)$ could also be obtained by right Kan extending the corresponding theory for schemes as in §4.1.2, where Shv is replaced by Shv_m .

Similarly to §4.1.8, we let $\text{Shv}_{m,c}(\mathcal{Y}_n)$ denote the full subcategory of $\text{Shv}_m(\mathcal{Y}_n)$ consisting of constructible objects. Moreover, we let $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} := \text{Ind}(\text{Shv}_{m,c}(\mathcal{Y}_n))$ be the associated compactly generated category. By construction, we also have a fully faithful embedding $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \hookrightarrow \text{Shv}_m(\mathcal{Y}_n)$.

4.2.7. *Functoriality.* Tensor products and the various pullback and pushforward functors preserve mixedness. Indeed, it is proved for the case of schemes in [BBDG18, §5.1.6] and the case of stacks follows by a spectral sequence argument, see, for example [Sun12b, Thm. 2.11 and Rmk. 2.12]. Thus, $\text{Shv}_m(\mathcal{Y}_n)$, $\text{Shv}_{m,c}(\mathcal{Y}_n)$, and $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ are symmetric monoidal. Moreover, for any morphism $f : \mathcal{Y}'_n \rightarrow \mathcal{Y}_n$ between objects in Stk_{k_n} , the various pullback and pushforward functors coming from f preserve mixedness, whenever these functors are defined. In particular, using the structure map $\pi : \mathcal{Y}_n \rightarrow \text{pt}_n = \text{Spec } k_n = \text{Spec } \mathbb{F}_{q^n}$, we obtain symmetric monoidal functors

$$\pi^* : \text{Shv}_m(\text{pt}_n) \rightarrow \text{Shv}_m(\mathcal{Y}_n) \quad \text{and} \quad \pi_{\text{ren}}^* : \text{Shv}_m(\text{pt}_n) \rightarrow \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}.$$

These functors equip $\text{Shv}_m(\mathcal{Y}_n)$ and $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ with structures of objects in $\text{ComAlg}(\text{Mod}_{\text{Shv}_m(\text{pt}_n)})$.

Remark 4.2.8. In the rest of the paper, we will work exclusive with the mixed version. All functors on sheaves are thus understood to operate on the mixed level by default.

4.3. **$\text{Shv}_m(\text{pt}_n)$ -module structures.** We have seen above that $\text{Shv}_m(\mathcal{Y}_n)$ and $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ admit $\text{Shv}_m(\text{pt}_n)$ -module structures for any $\mathcal{Y}_n \in \text{Stk}_{k_n}$. We will now show that all the usual functors are compatible with this structure and deduce various consequences from this fact.

We note that a large part of the materials here has been treated more systematically in [LZ17b, LZ17a], which will actually be used in §4.8 to obtain finer homotopy coherence structures for our sheaf theory. Using [GR17], the discussion below is a slightly different take of [LZ17b, LZ17a], included for the sake of completeness, especially for readers who are not familiar with [LZ17b, LZ17a] or are not interested the finer homotopy coherence aspects of correspondences.

4.3.1. *The case of schemes.* We start with the case of schemes. The arguments used here are the same as [GR17, Vol. I, Chap. 6]. We only indicate the main ideas here. The interested reader should consult [GR17, Vol. I, Chap. 6] for more details.

Throughout, we will make use of the following observation.

Lemma 4.3.2. *$\text{Shv}_m(\text{pt}_n)$ is a compactly generated rigid symmetric monoidal category.*

Proof. $\text{Shv}_m(\text{pt}_n)$ is compactly generated, by definition, see §4.1. Compact objects are perfect $\overline{\mathbb{Q}}_\ell$ complexes with continuous $\hat{\mathbb{Z}}$ -action with integral weights. The tensor product clearly preserves compactness. Finally, compact objects are all dualizable, whose duals are simply $\overline{\mathbb{Q}}_\ell$ -linear duals. \square

Lemma 4.3.3. *Let $f : Y_n \rightarrow Z_n$ where $Y_n, Z_n \in \text{Sch}_{k_n}$. Then, the functors f^* , f_* , $f^!$, and $f_!$ are strict functors of $\text{Shv}_m(\text{pt}_n)$ -modules.*

Proof. By definition, the functor f^* is a strict functor of $\text{Shv}_m(\text{pt}_n)$ -modules. By rigidity of $\text{Shv}_m(\text{pt}_n)$, and by the fact that f_* is continuous, we know that the right adjoint f_* is also strict, see Corollary 3.4.7.

We turn to the pair $f_! \dashv f^!$. We will treat the case $f^!$; the claim for $f_!$ follows from Corollary 3.4.7. We factor f as $Y_n \xrightarrow{j} \overline{Y}_n \xrightarrow{\bar{f}} Z_n$ where j is an open embedding and \bar{f} is proper. Then, $f^! \simeq j^! \circ \bar{f}^! \simeq j^* \circ \bar{f}^!$. Now, $\bar{f}^!$ is a right adjoint to $\bar{f}_! = \bar{f}_*$, which is a strict functor of $\text{Shv}_m(\text{pt}_n)$ -modules. Moreover, the case of $j^! = j^*$ already follows from the above. Arguing as in [GR17, Vol. I, Chap. 5] using [GR17, Vol. I, Chap. 7, Thm. 5.2.4], we see that this structure is independent of the choice of a factorization and we are done. \square

4.3.4. *The case of stacks.* We will now move to the case of stacks.

Proposition 4.3.5. *Let $f : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ where $\mathcal{Y}_n, \mathcal{Z}_n \in \text{Stk}_{k_n}$. Then, the functors f^* , f_* , $f^!$, and $f_!$ have the structures of strict functors of $\text{Shv}_m(\text{pt}_n)$ -modules. Moreover, the same statements apply to the renormalized sheaf theory.*

Proof. Using Lemma 4.3.3, we can define Shv_m on stacks by right Kan extending along $\text{Sch}_{k_n}^{\text{op}} \rightarrow \text{Stk}_{k_n}^{\text{op}}$ the following functor

$$\text{Shv}_m^? : \text{Sch}_{k_n}^{\text{op}} \rightarrow \text{Mod}_{\text{Shv}_m(\text{pt}_n)}$$

instead of the one in §4.1.2; see also the discussion in §4.2.6. The resulting object agrees with the category $\text{Shv}_m(\mathcal{Y}_n)$ defined above since the forgetful functor $\text{Mod}_{\text{Shv}_m(\text{pt}_n)} \rightarrow \text{DGCat}_{\text{pres,cont}}$, being a right adjoint, commutes with limits. In particular, for $f : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ in Stk_{k_n} , f^* and $f^!$ upgrade to strict functors of $\text{Shv}_m(\text{pt}_n)$ -modules. By rigidity of $\text{Shv}_m(\text{pt}_n)$ and Corollary 3.4.7, we obtain the corresponding statements for f_* and $f_!$ as well.

The desired conclusion also holds for the renormalized versions f_{ren}^* and $f_{\text{ren}}^!$ since the $\text{Shv}_m(\text{pt}_n)$ -module structures on $\text{Shv}_m(-)^{\text{ren}}$ is obtained by ind-extending the $\text{Shv}_{m,c}(\text{pt})$ -module structures on $\text{Shv}_{m,c}(-)$ (see §3.7.3) and f_{ren}^* and $f_{\text{ren}}^!$ are constructed by ind-extending the corresponding functors on the constructible part. By the rigidity of $\text{Shv}_m(\text{pt}_n)$ and the fact that $f_{*,\text{ren}}$ is continuous, we obtain the statement for $f_{*,\text{ren}}$. Finally, when $f_!$ preserves constructibility (for example, when f is representable), we have a pair of adjoint functors $f_{!,\text{ren}} \dashv f_{\text{ren}}^!$, which implies the same statement for $f_{!,\text{ren}}$ as well. \square

4.3.6. *$\text{Shv}_m(\text{pt}_n)$ -enriched Hom.* By the discussion in §3.2.6, for any $\mathcal{F}, \mathcal{G} \in \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ (resp. $\mathcal{F}, \mathcal{G} \in \text{Shv}_m(\mathcal{Y}_n)$), we can consider the internal Hom (which is usually called the *sheaf Hom* in the literature),

$$\underline{\mathcal{H}\text{om}}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G}) \in \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \quad (\text{resp. } \underline{\mathcal{H}\text{om}}_{\text{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G}) \in \text{Shv}_m(\mathcal{Y}_n))$$

as well as the $\text{Shv}_m(\text{pt}_n)$ -enriched Hom. To keep things less cluttered, we will use the following notation to denote the $\text{Shv}_m(\text{pt}_n)$ -enriched Hom

$$(4.3.7) \quad \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}, \mathcal{G}) := \underline{\mathcal{H}\text{om}}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^{\text{Shv}_m(\text{pt}_n)}(\mathcal{F}, \mathcal{G}) \in \text{Shv}_m(\text{pt}_n),$$

and similarly for the non-renormalized, i.e., usual, version.

In what follows, we will mostly focus on the renormalized case due to the applications we have in mind. In most cases, however, the proof for the usual version is verbatim.

4.3.8. Internal Hom and $\text{Shv}_m(\text{pt}_n)$ -enriched Hom are related in the expected way by the following lemma.

Lemma 4.3.9. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$ and $\mathcal{F}, \mathcal{G} \in \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ (resp. $\mathcal{F}, \mathcal{G} \in \text{Shv}_m(\mathcal{Y}_n)$). Then*

$$\begin{aligned} \pi_{*,\text{ren}} \underline{\mathcal{H}\text{om}}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G}) &\simeq \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}, \mathcal{G}) \\ (\text{resp. } \pi_* \underline{\mathcal{H}\text{om}}_{\text{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G}) &\simeq \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)}^m(\mathcal{F}, \mathcal{G})) \end{aligned}$$

Here $\pi : \mathcal{Y}_n \rightarrow \text{pt}_n$ denotes the structure map.

Proof. We will prove the statement for the renormalized sheaf theory. The proof of the other case is the same. For any $V \in \text{Shv}_m(\text{pt}_n)$, we have the following sequence of natural equivalences which yield the desired conclusion by Yoneda lemma

$$\begin{aligned} (4.3.10) \quad & \text{Hom}_{\text{Shv}_m(\text{pt}_n)}(V, \pi_{*,\text{ren}} \underline{\mathcal{H}\text{om}}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G})) \\ & \simeq \text{Hom}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\pi_{\text{ren}}^* V \otimes \mathcal{F}, \mathcal{G}) \\ & \simeq \text{Hom}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(V \otimes \mathcal{F}, \mathcal{G}) \\ & \simeq \text{Hom}_{\text{Shv}_m(\text{pt}_n)}(V, \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}, \mathcal{G})). \end{aligned}$$

In (4.3.10), the tensor denotes the action of $\text{Shv}_m(\text{pt}_n)$ on $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$, and the equivalence there is due to how the action is defined, i.e., via π_{ren}^* . \square

We also have the following expected result.

Proposition 4.3.11. *Let $f : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ be a representable smooth morphism in Stk_{k_n} , and $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}$. Then, we have a natural equivalence*

$$f_{\mathrm{ren}}^* \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G}) \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(f_{\mathrm{ren}}^* \mathcal{F}, f_{\mathrm{ren}}^* \mathcal{G}).$$

Proof. Since f is smooth, f_{ren}^* coincides with $f_{\mathrm{ren}}^!$, up to a cohomological shift. Thus, f_{ren}^* admits a left adjoint, $\tilde{f}_{!,\mathrm{ren}}$ that differs from $f_{!,\mathrm{ren}}$ by a cohomological shift. In particular, $\tilde{f}_{!,\mathrm{ren}}$ satisfies base change theorem and projection formula.¹² Note that representability of f is used to guarantee the existence of $f_{!,\mathrm{ren}}$.

Let $\mathcal{T} \in \mathrm{Shv}_m(\mathcal{Y})^{\mathrm{ren}}$ be any test object. The desired conclusion follows from Yoneda lemma and the equivalences below

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{T}, f_{\mathrm{ren}}^* \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}}(\tilde{f}_{!,\mathrm{ren}} \mathcal{T}, \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}}(\tilde{f}_{!,\mathrm{ren}}(\mathcal{T}) \otimes \mathcal{F}, \mathcal{G}) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}}(\tilde{f}_{!,\mathrm{ren}}(\mathcal{T} \otimes f_{\mathrm{ren}}^* \mathcal{F}), \mathcal{G}) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{T} \otimes f_{\mathrm{ren}}^* \mathcal{F}, f_{\mathrm{ren}}^* \mathcal{G}) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{T}, \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(f_{\mathrm{ren}}^* \mathcal{F}, f_{\mathrm{ren}}^* \mathcal{G})). \end{aligned}$$

□

Remark 4.3.12. The same statement as in Proposition 4.3.11 also holds for the usual sheaf theory, except that we do not need to require f to be representable. This is because $f_!$ always exists.

Corollary 4.3.13. *Let $\mathcal{Y}_n \in \mathrm{Stk}_{k_n}$ and $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{m,c}(\mathcal{Y}_n)$. Then, both*

$$\underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G}) \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})$$

are also constructible.

Proof. When \mathcal{Y}_n is a scheme, this is well-known as it is part of the six-functor formalism. Note that for schemes, the two sheaf theories coincide.

For a general \mathcal{Y}_n , by Lemma 4.1.11, to show that $\underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G})$ is constructible, it suffices to show that its pullback to a smooth atlas is constructible. But then, Remark 4.3.12 allows us to reduce to the scheme case and the desired conclusion follows.

Since $\underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G}) \in \mathrm{Shv}_{m,c}(\mathcal{Y}_n)$, we can view it as an element of $\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}$. To show that $\underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})$ is constructible, it suffices to show that we have an equivalence

$$\underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G}) \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G})$$

as objects in $\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}$. But now, for any $\mathcal{T} \in \mathrm{Shv}_{m,c}(\mathcal{Y}_n)$, we have

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{T}, \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_{m,c}(\mathcal{Y}_n)}(\mathcal{T}, \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G})) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{T}, \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G})) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)}(\mathcal{T} \otimes \mathcal{F}, \mathcal{G}) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_{m,c}(\mathcal{Y}_n)}(\mathcal{T} \otimes \mathcal{F}, \mathcal{G}) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{T} \otimes \mathcal{F}, \mathcal{G}) \\ & \simeq \mathrm{Hom}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{T}, \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})), \end{aligned}$$

¹²Base change theorem for $f_{!,\mathrm{ren}}$ follows from the usual setting (i.e., non-renormalized): one restricts to compact objects, which lives in the usual setting, and then ind-extend, since all functors are continuous. As in the usual proof, projection formula is a consequence of base change theorem and Künneth formula. See also [LZ17b, Cor. 6.2.3]

where we have used the fact that \otimes preserves constructibility in the fourth equivalence. This implies that for all $\mathcal{T} \in \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$,

$$\text{Hom}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{T}, \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)}(\mathcal{F}, \mathcal{G})) \simeq \text{Hom}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{T}, \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G})).$$

We conclude the proof by Yoneda lemma. \square

4.4. Graded sheaves: the construction. We now construct the category of graded sheaves.

4.4.1. The construction. We start with the definition of the category of graded sheaves.

Definition 4.4.2. For $\mathcal{Y}_n \in \text{Stk}_{k_n}$, we let $\text{Shv}_{\text{gr}}(\mathcal{Y}_n) := \text{Shv}_m(\mathcal{Y}_n) \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}}$ and $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}} := \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}}$ be the categories of graded sheaves and, respectively, renormalized graded sheaves on \mathcal{Y}_n . Here, the symmetric monoidal functor $\text{gr} : \text{Shv}_m(\text{pt}_n) \rightarrow \text{Vect}^{\text{gr}}$ of (4.2.5) is used to form the relative tensor.

The full-subcategory of graded constructible sheaves $\text{Shv}_{\text{gr},c}(\mathcal{Y}_n)$ is defined to be the full subcategory spanned by the compact objects $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren},c}$.

Direct from the construction, we have the following observation.

Lemma 4.4.3. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$. Then, $\text{Shv}_{\text{gr}}(\mathcal{Y}_n), \text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}} \in \text{ComAlg}(\text{Vect}^{\text{gr}})$, i.e., they are symmetric monoidal DG-categories whose tensor products are compatible with Vect^{gr} -actions. Moreover, we have a natural symmetric monoidal continuous and compact preserving functor $\text{gr}_{\mathcal{Y}_n} : \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \rightarrow \text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$ given by $\text{gr}_{\mathcal{Y}_n}(\mathcal{F}) = \mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell$ and similarly for the non-renormalized version.*

When \mathcal{Y}_n is clear from the context or when $\mathcal{Y}_n = \text{pt}_n$, we simply write gr in place of $\text{gr}_{\mathcal{Y}_n}$.

A couple of remarks are in order.

Remark 4.4.4. Due to the applications we have in mind, we will, from now on, restrict ourselves to the renormalized case. As we already saw above, this case is equipped with better functoriality compared to the non-renormalized one. Note that many of the results below can also be proved in the same way for the non-renormalized case. And of course, in the case of schemes, there is no distinction between renormalized and non-renormalized theory. Thus, for $S_n \in \text{Sch}_{k_n}$, we can simply write $\text{Shv}_{\text{gr}}(S_n)$ to mean either one of the two theories.

Remark 4.4.5. By definition, an object $\mathcal{Y}_n \in \text{Stk}_{k_n}$ is equipped, as part of the definition, with a structure map $\mathcal{Y}_n \rightarrow \text{pt}_n = \text{Spec } \mathbb{F}_{q^n}$. It is this map that allows us to equip $\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ with the structure of a $\text{Shv}_m(\text{pt}_n)$ -module used in the definition of $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$. It is important to note that the resulting category of graded sheaves $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$ is sensitive to this structure.

For example, consider $\text{pt}_2 = \text{Spec } \mathbb{F}_{q^2}$. It can be viewed as an object of either Stk_{k_2} or Stk_{k_1} . We thus have two categories

$$\text{Shv}_m(\text{pt}_2) \otimes_{\text{Shv}_m(\text{pt}_1)} \text{Vect}^{\text{gr}} \quad \text{and} \quad \text{Shv}_m(\text{pt}_2) \otimes_{\text{Shv}_m(\text{pt}_2)} \text{Vect}^{\text{gr}} \simeq \text{Vect}^{\text{gr}}.$$

It is easy to see that these two categories are distinct. For example, using Proposition 4.6.2 below, one can see that the Vect^{gr} -enriched Hom of the constant sheaf in two cases are different: $\overline{\mathbb{Q}}_\ell^2$ for the first case and $\overline{\mathbb{Q}}_\ell$ for the second. This should not be surprising since roughly speaking, the first category is related to $\text{pt}_2 \times_{\text{pt}_1} \text{pt} \simeq \text{pt} \sqcup \text{pt}$.

Because of this, when using the notation $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$, unless specified otherwise, the subscript n in \mathcal{Y}_n is used to indicate the fact that we are viewing \mathcal{Y}_n as an object in Stk_{k_n} in defining the category of graded sheaves.

Remark 4.4.6. Instead of the construction $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}} = \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}}$, which has the effect of remembering only the weights, we could instead construct the category of ω -graded sheaves $\text{Shv}_\omega(\mathcal{Y}_n)^{\text{ren}} := \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^\omega$. Here, Vect^ω is the category of chain complexes indexed by the set of algebraic integers with (complex) absolute values $(q^n)^{w/2}$ for $w \in \mathbb{Z}$, i.e., those numbers that can appear as eigenvalues of Frobenius actions on objects in $\text{Shv}_m(\text{pt}_n)$, see also [BBDG18, §5.1.5]. Moreover, the symmetric monoidal functor $\text{Shv}_m(\text{pt}_n) \rightarrow \text{Vect}^\omega$ forgets the Frobenius action but still

remembers the eigenvalues. This construction thus literally means semi-simplifying Frobenius actions. Note also that $\mathrm{Shv}_m(\mathrm{pt}_n) \rightarrow \mathrm{Vect}^{\mathrm{gr}}$ factors through this functor.

All results in this paper can be made to work with this variant in a straightforward manner. Since taking trace of Frobenius forgets non-semisimplicity behaviors, the usual Grothendieck–Lefschetz trace formula also factors through this construction. For the applications we have in mind, however, we are primarily interested in the weight grading rather than the actual eigenvalues. Moreover, dealing with gradings somewhat simplifies the notation and the exposition. We will, as a result, not pursue this line of investigation in the current paper.

4.4.7. Grading shifts. Recall from Example 3.4.4 that an object $V \in \mathrm{Vect}^{\mathrm{gr}}$ could be written as a direct sum $V = \bigoplus_n V_n$ where $V_n \in \mathrm{Vect}$ is placed in graded degree n . For any integer k , we use $V\langle k \rangle$ to denote a grading shift of V , i.e., $V\langle k \rangle_n = V_{n+k}$. For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}$, we write $\mathcal{F}\langle k \rangle := \mathcal{F} \otimes \overline{\mathbb{Q}}_\ell\langle k \rangle$, where $\overline{\mathbb{Q}}_\ell\langle k \rangle$ is $\overline{\mathbb{Q}}_\ell$ placed in graded degree $-k$.

This notation is compatible Tate twist. Namely, if $\mathcal{F} \in \mathrm{Shv}_m(\mathcal{Y})^{\mathrm{ren}}$, then

$$\mathrm{gr}(\mathcal{F}\langle k \rangle) \simeq \mathrm{gr}(\mathcal{F})\langle 2k \rangle$$

where the factor 2 is due to the fact that the sheaf $\overline{\mathbb{Q}}_\ell(-1) \in \mathrm{Shv}_m(\mathrm{pt}_n)$ has weight 2.

From Corollary 3.4.13, we know that $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}_n)$ is compactly generated by $\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell\langle k \rangle$ for $\mathcal{F} \in \mathrm{Shv}_{m,c}(\mathcal{Y})$ and $k \in \mathbb{Z}$. But this is equivalent to $\mathcal{F}\langle k/2 \rangle \boxtimes \overline{\mathbb{Q}}_\ell = \mathrm{gr}(\mathcal{F}\langle k/2 \rangle)$. Thus, the image of $\mathrm{Shv}_{m,c}(\mathcal{Y}_n)$ under gr compactly generates $\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}$.

4.5. Functoriality of graded sheaves. We will now study functoriality of graded sheaves, which follows in a straightforward manner from that of mixed sheaves.

4.5.1. Pull and push functors. Let $f : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ where $\mathcal{Y}_n, \mathcal{Z}_n \in \mathrm{Stk}_{k_n}$. Then, recall that we have functors

$$f_{\mathrm{ren}}^*, f_{\mathrm{ren}}^! : \mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}} \rightarrow \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} \quad \text{and} \quad f_{*,\mathrm{ren}} : \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} \rightarrow \mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}}$$

where $f_{\mathrm{ren}}^* \dashv f_{*,\mathrm{ren}}$. Moreover, when $f_!$ preserves constructibility, $f_{\mathrm{ren}}^!$ admits a left adjoint $f_{!,\mathrm{ren}}$.

By Proposition 4.3.5, all of these functors are strict functors of $\mathrm{Shv}_m(\mathrm{pt}_n)$ -modules. Thus, applying $-\otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathrm{Vect}^{\mathrm{gr}}$, we obtain functors

$$f_{\mathrm{ren}}^*, f_{\mathrm{ren}}^! : \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Z}_n)^{\mathrm{ren}} \rightarrow \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}} \quad \text{and} \quad f_{*,\mathrm{ren}} : \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}} \rightarrow \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Z}_n)^{\mathrm{ren}}$$

where $f_{\mathrm{ren}}^* \dashv f_{*,\mathrm{ren}}$. As above, when $f_!$ (for mixed sheaves) preserves constructibility, $f_{\mathrm{ren}}^!$ admits a left adjoint $f_{!,\mathrm{ren}}$. Note that to avoid clutter, we do not include in the notation of the various pull and push functors any extra decoration to specify that we are operating with graded sheaves.

4.5.2. Compatibility with functoriality for mixed/non-mixed sheaves. The various pull and push functors for graded sheaves are compatible with those on mixed/non-mixed sheaves in a precise sense.

4.5.3. Let $\mathcal{Y}_n \in \mathrm{Stk}_{k_n}$ and $\mathcal{Y} \in \mathrm{Stk}_k$ its base change to k . Let $h_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y}_n$ and $h : \mathrm{pt} \rightarrow \mathrm{pt}_n$ denote the canonical maps, and $\pi_n : \mathcal{Y}_n \rightarrow \mathrm{pt}_n$ and $\pi : \mathcal{Y} \rightarrow \mathrm{pt}$ denote the structure maps. Then, we have the following commutative diagram in $\mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{pres},\mathrm{cont}})$, where in the bottom right, $\mathrm{Vect} \simeq \mathrm{Shv}(\mathrm{pt})$

$$\begin{array}{ccc} \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{h_{\mathcal{Y},\mathrm{ren}}^*} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \\ \pi_{n,\mathrm{ren}}^* \uparrow & & \uparrow \pi_{\mathrm{ren}}^* \\ \mathrm{Shv}_m(\mathrm{pt}_n) & \xrightarrow{\mathrm{gr}} \mathrm{Vect}^{\mathrm{gr}} \xrightarrow{\mathrm{oblv}_{\mathrm{gr}} \simeq \oplus} & \mathrm{Vect} \\ & \searrow h^* & \end{array}$$

Here, $\mathrm{oblv}_{\mathrm{gr}}$ denotes the functor of taking the direct sum of all graded components.

This implies the following lemma.

Lemma 4.5.4. *In the situation above, the functor $\mathrm{gr}_{\mathcal{Y}_n} : \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} \rightarrow \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}$ of Lemma 4.4.3 fits into the following commutative diagram in $\mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{pres}, \mathrm{cont}})$ where the square on the left is a pushout square in $\mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{pres}, \mathrm{cont}})$*

$$\begin{array}{ccccc}
 & & h_{\mathcal{Y}, \mathrm{ren}}^* & & \\
 & \nearrow & & \searrow & \\
 \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{gr}_{\mathcal{Y}_n}} & \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{oblv}_{\mathrm{gr}, \mathcal{Y}_n}} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \\
 \uparrow \pi_{n, \mathrm{ren}}^* & & \uparrow \pi_{n, \mathrm{ren}}^* & & \uparrow \pi_{\mathrm{ren}}^* \\
 \mathrm{Shv}_m(\mathrm{pt}_n) & \xrightarrow{\mathrm{gr}} & \mathrm{Vect}^{\mathrm{gr}} & \xrightarrow{\mathrm{oblv}_{\mathrm{gr}}} & \mathrm{Vect} \\
 & \searrow & h^* \simeq \mathrm{oblv}_{\mathrm{Frob}_n} & \nearrow &
 \end{array}$$

As before, when \mathcal{Y}_n is clear from the context, we will also use $\mathrm{oblv}_{\mathrm{gr}}$ to denote $\mathrm{oblv}_{\mathrm{gr}, \mathcal{Y}_n}$.

Compatibility between $\mathrm{gr}_{\mathcal{Y}_n}$ and various pull and push functors is given in the following result.

Proposition 4.5.5. *Let $f_n : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ where $\mathcal{Y}_n, \mathcal{Z}_n \in \mathrm{Stk}_{k_n}$, $f : \mathcal{Y} \rightarrow \mathcal{Z}$ its base change to k , $h_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y}_n$ and $h_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{Z}_n$ the canonical maps. Then we have the following commutative diagrams*

$$\begin{array}{ccccc}
 & & h_{\mathcal{Y}, \mathrm{ren}}^* & & \\
 & \nearrow & & \searrow & \\
 \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{gr}_{\mathcal{Y}_n}} & \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{oblv}_{\mathrm{gr}, \mathcal{Y}_n}} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \\
 \uparrow f_{n, \mathrm{ren}}^* \text{ (resp. } f_{n, \mathrm{ren}}^! \text{)} & & \uparrow f_{n, \mathrm{ren}}^* \text{ (resp. } f_{n, \mathrm{ren}}^! \text{)} & & \uparrow f_{\mathrm{ren}}^* \text{ (resp. } f_{\mathrm{ren}}^! \text{)} \\
 \mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{gr}_{\mathcal{Z}_n}} & \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Z}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{oblv}_{\mathrm{gr}, \mathcal{Z}_n}} & \mathrm{Shv}(\mathcal{Z})^{\mathrm{ren}} \\
 & \searrow & h_{\mathcal{Z}, \mathrm{ren}}^* & \nearrow &
 \end{array}$$

and

$$\begin{array}{ccccc}
 & & h_{\mathcal{Y}, \mathrm{ren}}^* & & \\
 & \nearrow & & \searrow & \\
 \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{gr}_{\mathcal{Y}_n}} & \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{oblv}_{\mathrm{gr}, \mathcal{Y}_n}} & \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}} \\
 \downarrow f_{n, \mathrm{ren}}^* \text{ (resp. } f_{n, \mathrm{ren}}^! \text{)} & & \downarrow f_{n, \mathrm{ren}}^* \text{ (resp. } f_{n, \mathrm{ren}}^! \text{)} & & \downarrow f_{\mathrm{ren}}^* \text{ (resp. } f_{\mathrm{ren}}^! \text{)} \\
 \mathrm{Shv}_m(\mathcal{Z}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{gr}_{\mathcal{Z}_n}} & \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Z}_n)^{\mathrm{ren}} & \xrightarrow{\mathrm{oblv}_{\mathrm{gr}, \mathcal{Z}_n}} & \mathrm{Shv}(\mathcal{Z})^{\mathrm{ren}} \\
 & \searrow & h_{\mathcal{Z}, \mathrm{ren}}^* & \nearrow &
 \end{array}$$

Note that here, the $(-)_!, \mathrm{ren}$ case is only defined when f_1 preserves constructibility.

Proof. The left squares in the diagrams above commute by naturality of

$$\mathrm{gr}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathrm{Vect}^{\mathrm{gr}}, \quad \mathcal{C} \in \mathrm{Mod}_{\mathrm{Shv}_m(\mathrm{pt}_n)}.$$

Moreover, the outer square also commutes by smooth base change, and are compatible with the module structures over $\mathrm{Shv}_m(\mathrm{pt}_n)$ and Vect . By the factorization

$$\mathrm{Shv}_m(\mathrm{pt}_n) \rightarrow \mathrm{Vect}^{\mathrm{gr}} \rightarrow \mathrm{Vect},$$

the commutativity of the squares on the right follows from universal property of “tensoring up.” \square

Remark 4.5.6. Similarly, for any $\mathcal{Y}_n \in \mathrm{Stk}_{k_n}$, we have functors

$$\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} \xrightarrow{\mathrm{gr}_{\mathcal{Y}_n}^{\omega}} \mathrm{Shv}_{\omega}(\mathcal{Y}_n)^{\mathrm{ren}} \xrightarrow{\mathrm{oblv}_{\mathrm{gr}_{\mathcal{Y}_n}^{\omega}}} \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}.$$

where $\mathrm{Shv}_{\omega}(\mathcal{Y}_n)^{\mathrm{ren}}$ is described in Remark 4.4.6.

Beilinson communicated the following observation to us. Let $A_m = \overline{\mathbb{Q}}_{\ell}[t] \in \mathrm{ComAlg}(\mathrm{Shv}_m(\mathrm{pt}_n))$ where Frobenius acts by $t \mapsto t + 1$ and $A \in \mathrm{ComAlg}(\mathrm{Shv}(\mathrm{pt}))$ its pullback to $\mathrm{Shv}(\mathrm{pt})$. Then, one can show that $\mathrm{Shv}_{\omega}(\mathrm{pt}_n) \simeq \mathrm{Mod}_{A_m}(\mathrm{Shv}_m(\mathrm{pt}_n))$, which implies, more generally, that $\mathrm{Shv}_{\omega}(\mathcal{Y}_n)^{\mathrm{ren}} \simeq \mathrm{Mod}_{A_m}(\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}})$.

Under this equivalence, $\mathrm{gr}_{\mathcal{Y}_n}^\omega$ is identified with $\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} \xrightarrow{A_m \otimes -} \mathrm{Mod}_{A_m}(\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}})$ and $\mathrm{oblv}_{\mathrm{gr}_{\mathcal{Y}_n}^\omega}$ factors as follows

$$\mathrm{Mod}_{A_m}(\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}) \xrightarrow{\mathrm{oblv}_m} \mathrm{Mod}_A(\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}) \xrightarrow{- \otimes_A \overline{\mathbb{Q}}_\ell} \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}},$$

where the commutative algebra map $A \rightarrow \overline{\mathbb{Q}}_\ell$ is given by sending $t \mapsto 0$. We will not make use of this in the current paper.

4.5.7. Base change results. Other functorial properties of graded sheaves also follow from the corresponding results for mixed sheaves in a straightforward manner.

Theorem 4.5.8. *The theory of graded sheaves satisfies projection formula and smooth and proper base change. Namely,*

- (i) *Projection formula: Let $f : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ be a morphism in Stk_{k_n} such that $f_{!,\mathrm{ren}}$ is defined (for example, when $f_!$ for mixed sheaves preserves constructibility). Then, for any $\mathcal{F} \in \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}, \mathcal{G} \in \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Z}_n)^{\mathrm{ren}}$, we have a natural equivalence*

$$f_{!,\mathrm{ren}}(\mathcal{F} \otimes f_{\mathrm{ren}}^* \mathcal{G}) \simeq f_{!,\mathrm{ren}} \mathcal{F} \otimes \mathcal{G}.$$

- (ii) *Consider the following Cartesian square in Stk_{k_n}*

$$\begin{array}{ccc} \mathcal{Y}'_n & \xrightarrow{h'} & \mathcal{Y}_n \\ \downarrow v' & & \downarrow v \\ \mathcal{Z}'_n & \xrightarrow{h} & \mathcal{Z}_n \end{array}$$

- *Proper base change: When $v_{!,\mathrm{ren}}$ and $v'_{!,\mathrm{ren}}$ are defined then we have a natural equivalence of functors $h_{\mathrm{ren}}^* v_{!,\mathrm{ren}} \simeq v'_{!,\mathrm{ren}} h_{\mathrm{ren}}'^*$.*
- *Smooth base change: When h is smooth, we have a natural equivalence of functors $h_{\mathrm{ren}}^* v_{*,\mathrm{ren}} \simeq v'_{*,\mathrm{ren}} h_{\mathrm{ren}}'^*$.*
- *Smooth base change (variant): We have an equivalence of functors $h_{\mathrm{ren}}^! v_{*,\mathrm{ren}} \simeq v'_{*,\mathrm{ren}} h_{\mathrm{ren}}'^!$.*

Proof. Since all functors involved are continuous and all categories involved are compactly generated, it suffices to check on a set of compact objects. It thus remains to show that these statements hold for constructible mixed sheaves, viewed as objects in the category of renormalized mixed sheaves.

All operations in the projection formula and proper base change statement preserve constructibility. Thus, the usual results for mixed sheaves imply those in the renormalized categories. For the last two statements involving smooth base change, we note that even though renormalized $*$ -pushforward functors do not preserve constructibility, they do preserve the property of being bounded below in the usual t -structure (in fact, they are left t -exact). Similarly, the renormalized $*$ -pullback functors also preserve the property of being bounded below, since they are t -exact. Now, the functor unren induces an equivalence between the full subcategories of bounded below objects in the usual and renormalized categories of sheaves. Proposition 4.1.17 and Lemma 4.1.20 then allow us to apply the usual smooth base change theorem for mixed sheaves and the proof concludes. \square

4.5.9. Smooth descent. Like Shv_m , the sheaf theory $\mathrm{Shv}_{\mathrm{gr}}$ satisfies smooth descent.

Proposition 4.5.10. *$\mathrm{Shv}_{\mathrm{gr}}$ satisfies smooth descent. I.e., let $\mathcal{Z}_n \rightarrow \mathcal{Y}_n$ be a smooth morphism in Stk_{k_n} and let $\check{\mathrm{Cech}}^\bullet(\mathcal{Z}_n/\mathcal{Y}_n)$ be the associated Čech nerve. Then the pullback functor (either $*$ or $!$) induces an equivalence of categories*

$$\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n) \simeq \mathrm{Tot}(\mathrm{Shv}_{\mathrm{gr}}(\check{\mathrm{Cech}}^\bullet(\mathcal{Z}_n/\mathcal{Y}_n))).$$

Similarly, $\mathrm{Shv}_{\mathrm{gr},c}$ also satisfies smooth descent, i.e., we have

$$\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}_n) \simeq \mathrm{Tot}(\mathrm{Shv}_{\mathrm{gr},c}(\check{\mathrm{Cech}}^\bullet(\mathcal{Z}_n/\mathcal{Y}_n))).$$

Proof. Since $\mathbf{Vect}^{\mathrm{gr}}$ is compactly generated, it is also dualizable as a presentable stable ∞ -category, by [GR17, Vol. I, Chap. 1, Prop. 7.3.2]. Since $\mathrm{Shv}_m(\mathrm{pt}_n)$ is rigid, $\mathbf{Vect}^{\mathrm{gr}}$ is also dualizable as an object in $\mathrm{Mod}_{\mathrm{Shv}_m(\mathrm{pt}_n)}$, by [GR17, Vol. I, Chap. 1, Prop. 9.4.4]. But now, the relative tensor $-\otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr}}$ commutes with limits, by [GR17, Vol. I, Chap. 1, §4.3.2]. Hence, descent for mixed sheaves imply that for graded sheaves

$$\begin{aligned} \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n) &\simeq \mathrm{Shv}_m(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr}} \\ &\simeq \mathrm{Tot}(\mathrm{Shv}_m(\check{\mathrm{Cech}}^\bullet(\mathcal{Z}_n/\mathcal{Y}_n))) \otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr}} \\ &\simeq \mathrm{Tot}(\mathrm{Shv}_m(\check{\mathrm{Cech}}^\bullet(\mathcal{Z}_n/\mathcal{Y}_n)) \otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr}}) \\ &\simeq \mathrm{Tot}(\mathrm{Shv}_{\mathrm{gr}}(\check{\mathrm{Cech}}^\bullet(\mathcal{Z}_n/\mathcal{Y}_n))). \end{aligned}$$

Since pulling back preserves compactness by construction, the second statement follows from the first. \square

Remark 4.5.11. The same proof shows that $\mathrm{Shv}_{\mathrm{gr}}$, as a functor out of $\mathrm{Stk}_{k_n}^{\mathrm{op}}$, can be obtained by right Kan extending $\mathrm{Shv}_{\mathrm{gr}}$ as a functor out of $\mathrm{Sch}_{k_n}^{\mathrm{op}}$.

Remark 4.5.12. $\mathrm{Shv}_{\mathrm{gr}}(-)^{\mathrm{ren}}$ does not satisfy smooth descent. In fact, $\mathrm{Shv}(-)^{\mathrm{ren}}$ already fails to satisfy descent. Consider the canonical morphism $h : \mathrm{pt} \rightarrow B\mathbb{G}_m$ as in Example 4.1.12. If $\mathrm{Shv}(-)^{\mathrm{ren}}$ satisfied smooth descent, h_{ren}^* would be conservative, i.e., it would not kill any object. This is, in fact, not the case.

As in Example 4.1.12, let $\pi : B\mathbb{G}_m \rightarrow \mathrm{pt}$ denote the structure morphism and

$$\mathcal{F} = \mathrm{colim}(\overline{\mathbb{Q}}_\ell \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell[2] \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell[4] \xrightarrow{\beta} \cdots).$$

Since π_{ren}^* preserves compactness, $\pi_{*,\mathrm{ren}}$ is continuous. Proposition 4.1.17 and the calculation in Example 4.1.12 implies that

$$\pi_{*,\mathrm{ren}} \mathcal{F} \simeq \mathrm{colim}(\pi_{*,\mathrm{ren}} \overline{\mathbb{Q}}_\ell \xrightarrow{\beta} \pi_{*,\mathrm{ren}} \overline{\mathbb{Q}}_\ell[2] \xrightarrow{\beta} \pi_{*,\mathrm{ren}} \overline{\mathbb{Q}}_\ell[4] \xrightarrow{\beta} \cdots) \simeq \overline{\mathbb{Q}}_\ell[\beta, \beta^{-1}] \neq 0.$$

In particular, $\mathcal{F} \neq 0$.

On the other hand, as in Example 4.1.12, $h_{\mathrm{ren}}^*(\mathcal{F}) \simeq 0$.

4.5.13. *Verdier duality.* The category of mixed sheaves is equipped with a duality functor

$$D_{\mathrm{Ver}} : \mathrm{Shv}_{m,c}(\mathcal{Y}_n) \xrightarrow{\sim} \mathrm{Shv}_{m,c}(\mathcal{Y}_n)^{\mathrm{op}}$$

which is compatible with the linear dual on a point, which is a symmetric monoidal functor

$$(-)^\vee : \mathrm{Shv}_{m,c}(\mathrm{pt}_n) \rightarrow \mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}}.$$

We thus get an induced functor

$$\mathrm{Shv}_{m,c}(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)} \mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}} \rightarrow \mathrm{Shv}_{m,c}(\mathcal{Y}_n)^{\mathrm{op}},$$

and hence, a functor

$$\mathrm{Shv}_{m,c}(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)} \mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}} \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}}} \mathbf{Vect}^{\mathrm{gr},c,\mathrm{op}} \rightarrow \mathrm{Shv}_{m,c}(\mathcal{Y}_n)^{\mathrm{op}} \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}}} \mathbf{Vect}^{\mathrm{gr},c,\mathrm{op}}$$

Note that the RHS is equivalent to

$$(\mathrm{Shv}_{m,c}(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr},c})^{\mathrm{op}} \simeq \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\mathrm{op}}$$

since $(-)^{\mathrm{op}}$ is a symmetric monoidal auto-equivalence of $\mathrm{DGCat}_{\mathrm{idem},\mathrm{ex}}$. Moreover, we have the following sequence of functors to the LHS

$$\begin{aligned} \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) &\simeq \mathrm{Shv}_{m,c}(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr},c} \xrightarrow{\mathrm{id} \otimes (-)^\vee} \mathrm{Shv}_{m,c}(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)} \mathbf{Vect}^{\mathrm{gr},c,\mathrm{op}} \\ &\simeq \mathrm{Shv}_{m,c}(\mathcal{Y}_n) \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)} \mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}} \otimes_{\mathrm{Shv}_{m,c}(\mathrm{pt}_n)^{\mathrm{op}}} \mathbf{Vect}^{\mathrm{gr},c,\mathrm{op}} \end{aligned}$$

We thus obtain the corresponding Verdier duality functor for graded sheaves

$$D_{\text{Ver}} : \text{Shv}_{\text{gr},c}(\mathcal{Y}) \xrightarrow{\simeq} \text{Shv}_{\text{gr},c}(\mathcal{Y})^{\text{op}},$$

compatible with the duality functor $(-)^{\vee}$ on Vect^{gr} .

4.5.14. Unwinding the definition, we see that for $\mathcal{F} \boxtimes V \in \text{Shv}_{\text{gr},c}(\mathcal{Y})$ where $\mathcal{F} \in \text{Shv}_{m,c}(\mathcal{Y}_n)$ and $V \in \text{Vect}^{\text{gr},c}$, $D_{\text{Ver}}(\mathcal{F} \boxtimes V) \simeq D_{\text{Ver}}(\mathcal{F}) \boxtimes V^{\vee}$. It is also easy to see that we get all the expected properties of the Verdier duality functor for graded sheaves from the Verdier duality functor for mixed sheaves.

4.6. **(Graded) Hom-spaces between graded sheaves.** We will now study the Hom-spaces, both Vect and Vect^{gr} -enriched, inside $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$. Following Lemma 4.3.9, we will employ the following notation to denote the Vect^{gr} -enriched Hom

$$\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}^{\text{gr}}(\mathcal{F}, \mathcal{G}) := \underline{\mathcal{H}\text{om}}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}^{\text{Vect}^{\text{gr}}}(\mathcal{F}, \mathcal{G}) \in \text{Vect}^{\text{gr}}.$$

In particular, when $\mathcal{Y}_n = \text{pt}_n$ and $V, W \in \text{Vect}^{\text{gr}} \simeq \text{Shv}_{\text{gr}}(\text{pt}_n)$, we write

$$\mathcal{H}\text{om}_{\text{Vect}^{\text{gr}}}^{\text{gr}}(V, W) := \underline{\mathcal{H}\text{om}}_{\text{Vect}^{\text{gr}}}^{\text{Vect}^{\text{gr}}}(V, W) \in \text{Vect}^{\text{gr}}$$

to denote the internal Hom.

We start with the following counterpart of Lemma 4.3.9, whose proof is identical to Lemma 4.3.9 and hence, will be omitted.

Lemma 4.6.1. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$ and $\mathcal{F}, \mathcal{G} \in \text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$. Then,*

$$\pi_{*,\text{ren}} \underline{\mathcal{H}\text{om}}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}^{\text{gr}}(\mathcal{F}, \mathcal{G}).$$

The next result concerns computation of Hom between graded sheaves.

Proposition 4.6.2. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$, $(\mathcal{F}_c, V_c) \in \text{Shv}_{m,c}(\mathcal{Y}_n) \times \text{Vect}^{\text{gr},c}$, and $(\mathcal{F}, V) \in \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \times \text{Vect}^{\text{gr}}$. Then,*

$$\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}^{\text{gr}}(\mathcal{F}_c \boxtimes V_c, \mathcal{F} \boxtimes V) \simeq \mathcal{H}\text{om}_{\text{Vect}^{\text{gr}}}^{\text{gr}}(V_c, V) \otimes \text{gr}(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}_c, \mathcal{F})).$$

Consequently,¹³

$$(4.6.3) \quad \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}(\text{gr}(\mathcal{F}_c), \text{gr}(\mathcal{F})) \simeq \text{gr}(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}_c, \mathcal{F}))_0 \simeq \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}_c, \mathcal{F})_0,$$

where the subscript 0 denotes the graded 0 part, or equivalently, the naive weight 0 part.¹⁴

Proof. The first part is an application of Proposition 3.5.5.

For the second part, from the definition of enriched Hom, we conclude via the following sequence of equivalences

$$\begin{aligned} & \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}(\text{gr}(\mathcal{F}_c), \text{gr}(\mathcal{F})) \\ & \simeq \mathcal{H}\text{om}_{\text{Vect}^{\text{gr}}}(\overline{\mathbb{Q}}_{\ell}, \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}^{\text{gr}}(\text{gr}(\mathcal{F}_c), \text{gr}(\mathcal{F}))) \\ & \simeq \mathcal{H}\text{om}_{\text{Vect}^{\text{gr}}}(\overline{\mathbb{Q}}_{\ell}, \text{gr} \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}_c, \mathcal{F})) \\ & \simeq \text{gr} \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}_c, \mathcal{F})_0 \\ & \simeq \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}^m(\mathcal{F}_c, \mathcal{F})_0, \end{aligned}$$

where the second equivalence is due to the first part. □

¹³Recall that for a DG-category \mathcal{C} , $\mathcal{H}\text{om}_{\mathcal{C}}$ denotes the Hom-complex between two objects. See §3.6.

¹⁴See also §4.2.2 for the notation.

Corollary 4.6.4. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$, $\mathcal{F}_n, \mathcal{G}_n \in \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}$ where \mathcal{F}_n is compact, $\mathcal{Y}, \mathcal{F}, \mathcal{G}$ their base changes to $\text{pt} = \text{Spec } k$, and $h : \mathcal{Y} \rightarrow \mathcal{Y}_n$ the canonical map. Then, $\text{obl}_{\text{gr}, \mathcal{Y}_n}$ (see Lemma 4.4.3) induces a map*

$$\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}(\text{gr}(\mathcal{F}_n), \text{gr}(\mathcal{G}_n)) \rightarrow \mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})$$

which realizes the former as a direct summand of the latter. In particular, if

$$\alpha \in H^0(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}}(\text{gr}(\mathcal{F}_n), \text{gr}(\mathcal{G}_n))),$$

then it is 0 if and only if its image in $\mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})$ is 0.

Proof. The second statement is a direct consequence of the first. The first statement is itself a consequence of (4.6.3) and the fact that

$$\mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G}) \simeq \text{obl}_{\text{Frob}_n} \mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y})^{\text{ren}}}^m(\mathcal{F}_n, \mathcal{G}_n).$$

□

4.7. Invariance under of extensions of scalars. The relative tensor $-\otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}}$ used in the definition of $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$ can be thought of as a categorical way to base change \mathcal{Y}_n to k (while still remembering some weight information). Thus, at least intuitively, we should expect that $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$ is invariant under base changing \mathcal{Y}_n to $\mathcal{Y}_m \in \text{Stk}_{k_m}$. This is the content of Proposition 4.7.9 below. By a spreading argument, this implies that $\text{Shv}_{\text{gr}}(-)^{\text{ren}}$ is a sheaf theory on Stk_k , see Theorem 4.7.12.

4.7.1. Consider $f : \text{pt}_m \rightarrow \text{pt}_n$ where $m \geq n$, which induces a symmetric monoidal functor

$$f^* : \text{Shv}_m(\text{pt}_n) \rightarrow \text{Shv}_m(\text{pt}_m),$$

compatible with the functor to Vect^{gr} . Namely, we have the following commutative diagram of symmetric monoidal categories

$$\begin{array}{ccc} \text{Shv}_m(\text{pt}_n) & \xrightarrow{f^*} & \text{Shv}_m(\text{pt}_m) \\ & \searrow \text{gr}_{\text{pt}_n} & \swarrow \text{gr}_{\text{pt}_m} \\ & \text{Vect}^{\text{gr}} & \end{array}$$

This induces a natural functor

$$\text{Shv}_{\text{gr}}(\text{pt}_n) \rightarrow \text{Shv}_{\text{gr}}(\text{pt}_m)$$

that is an equivalence of categories. In fact, both sides are naturally identified with Vect^{gr} and under this identification, the resulting functor is an identity functor.

4.7.2. We will now consider the general case. Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$ and consider the following pullback square

$$\begin{array}{ccc} \mathcal{Y}_m & \xrightarrow{g} & \mathcal{Y}_n \\ \downarrow \pi_m & & \downarrow \pi_n \\ \text{pt}_m & \xrightarrow{f} & \text{pt}_n \end{array}$$

which induces the following commutative square of categories, where all functors are symmetric monoidal

$$\begin{array}{ccc} \text{Shv}_m(\mathcal{Y}_m)^{\text{ren}} & \xleftarrow{g_{\text{ren}}^*} & \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \\ \uparrow \pi_{m,\text{ren}}^* & & \uparrow \pi_{n,\text{ren}}^* \\ \text{Shv}_m(\text{pt}_m) & \xleftarrow{f^*} & \text{Shv}_m(\text{pt}_n) \end{array}$$

The diagram above induces a morphism in $\text{ComAlg}(\text{Mod}_{\text{Shv}_m(\text{pt}_m)})$

$$(4.7.3) \quad \tilde{g}_{\text{ren}}^* : \widetilde{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}} := \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Shv}_m(\text{pt}_m) \rightarrow \text{Shv}_m(\mathcal{Y}_m)^{\text{ren}}.$$

Lemma 4.7.4. \tilde{g}_{ren}^* is fully faithful.

Proof. It suffices to show that for any $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{m,c}(\mathcal{Y}_n)$ and $V, W \in \mathrm{Shv}_{m,c}(\mathrm{pt}_m)$, $\tilde{g}_{\mathrm{ren}}^*$ induces an equivalence

$$\underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}^{\mathrm{Shv}_m(\mathrm{pt}_m)}(\mathcal{F} \boxtimes V, \mathcal{G} \boxtimes W) \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}}}^{\mathrm{Shv}_m(\mathrm{pt}_m)}(V \otimes g_{\mathrm{ren}}^* \mathcal{F}, W \otimes g_{\mathrm{ren}}^* \mathcal{G})$$

where the tensors on the right come from the $\mathrm{Shv}_m(\mathrm{pt}_m)$ -module structure of $\mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}}$.

We have

$$\begin{aligned} & \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}^{\mathrm{Shv}_m(\mathrm{pt}_m)}(\mathcal{F} \boxtimes V, \mathcal{G} \boxtimes W) \\ \text{(Proposition 3.5.5)} \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathrm{pt}_m)}(V, W) \otimes f^* \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}^{\mathrm{Shv}_m(\mathrm{pt}_n)}(\mathcal{F}, \mathcal{G}) \\ \text{(Lemma 4.3.9)} \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathrm{pt}_m)}(V, W) \otimes f^* \pi_{n*, \mathrm{ren}} \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G}) \\ \text{(smooth base change)} \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathrm{pt}_m)}(V, W) \otimes \pi_{m*, \mathrm{ren}} g_{\mathrm{ren}}^* \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G}) \\ \text{(Proposition 4.3.11)} \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathrm{pt}_m)}(V, W) \otimes \pi_{m*, \mathrm{ren}} \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}}}(\tilde{g}_{\mathrm{ren}}^* \mathcal{F}, g_{\mathrm{ren}}^* \mathcal{G}) \\ \text{(Lemma 4.3.9)} \quad & \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathrm{pt}_m)}(V, W) \otimes \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}}}^{\mathrm{Shv}_m(\mathrm{pt}_m)}(g_{\mathrm{ren}}^* \mathcal{F}, g_{\mathrm{ren}}^* \mathcal{G}) \\ & \simeq \underline{\mathrm{Hom}}_{\mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}}}^{\mathrm{Shv}_m(\mathrm{pt}_m)}(V \otimes g_{\mathrm{ren}}^* \mathcal{F}, W \otimes g_{\mathrm{ren}}^* \mathcal{G}) \end{aligned}$$

where the last equivalence is as in (3.5.4), using Corollary 3.4.7. \square

Lemma 4.7.5. $\tilde{g}_{\mathrm{ren}}^*$ is essentially surjective.

Proof. The functor $\tilde{g}_{\mathrm{ren}}^*$ fits into the following diagram of adjoints

$$(4.7.6) \quad \begin{array}{ccccc} & & g_{\mathrm{ren}}^* & & \\ & \nearrow & & \searrow & \\ \mathrm{Shv}_m(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightleftharpoons[\mathrm{can}^R]{\mathrm{can}} & \widetilde{\mathrm{Shv}}_m(\mathcal{Y}_n)^{\mathrm{ren}} & \xrightleftharpoons[\tilde{g}_{*, \mathrm{ren}}]{\tilde{g}_{\mathrm{ren}}^*} & \mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}} \\ & \nwarrow & & \nearrow & \\ & & g_{*, \mathrm{ren}} & & \end{array}$$

Here, can is used to denote the canonical functor. All the functors above admit right adjoints since they are all continuous; in fact, the right adjoints are also continuous since the left adjoints preserve compactness. We have $g_{\mathrm{ren}}^* = \tilde{g}_{\mathrm{ren}}^* \circ \mathrm{can}$ by construction, and hence, passing to right adjoints, we also have $g_{*, \mathrm{ren}} = \mathrm{can}^R \circ \tilde{g}_{*, \mathrm{ren}}$.

To show that $\tilde{g}_{\mathrm{ren}}^*$ is essentially surjective, it suffices to show that any object $\mathcal{F} \in \mathrm{Shv}_m(\mathcal{Y}_m)^{\mathrm{ren}}$ can be obtained as a colimit of objects in (the essential image of) $\widetilde{\mathrm{Shv}}_m(\mathcal{Y}_n)^{\mathrm{ren}}$.

Now, note that on the one hand, by base change,

$$g_{\mathrm{ren}}^* g_{*, \mathrm{ren}} \mathcal{F} \simeq \mathcal{F}^{\oplus r},$$

for some $r \in \mathbb{Z}$. In particular, \mathcal{F} is a retract of $g_{\mathrm{ren}}^* g_{*, \mathrm{ren}} \mathcal{F}$, and hence,

$$\mathcal{F} \simeq \mathrm{colim}(g_{\mathrm{ren}}^* g_{*, \mathrm{ren}} \mathcal{F} \xrightarrow{e} g_{\mathrm{ren}}^* g_{*, \mathrm{ren}} \mathcal{F} \xrightarrow{e} g_{\mathrm{ren}}^* g_{*, \mathrm{ren}} \mathcal{F} \xrightarrow{e} \dots)$$

where e is an idempotent which projects to one \mathcal{F} factor.

On the other hand, (4.7.6) implies that

$$g_{\mathrm{ren}}^* g_{*, \mathrm{ren}} \mathcal{F} \simeq \tilde{g}_{\mathrm{ren}}^* \mathrm{can} \mathrm{can}^R \tilde{g}_{*, \mathrm{ren}} \mathcal{F} = \tilde{g}_{\mathrm{ren}}^* \mathcal{G}$$

where $\mathcal{G} = \mathrm{can} \mathrm{can}^R \tilde{g}_{*, \mathrm{ren}} \mathcal{F} \in \widetilde{\mathrm{Shv}}_m(\mathcal{Y}_n)^{\mathrm{ren}}$. Hence,

$$\mathcal{F} \simeq \mathrm{colim}(\tilde{g}_{\mathrm{ren}}^* \mathcal{G} \xrightarrow{e} \tilde{g}_{\mathrm{ren}}^* \mathcal{G} \xrightarrow{e} \tilde{g}_{\mathrm{ren}}^* \mathcal{G} \xrightarrow{e} \dots),$$

and the proof concludes. \square

As a consequence of the two lemmas above, we obtain the following statement.

Proposition 4.7.7. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$ and \mathcal{Y}_m its base change to k_m . Let $g : \mathcal{Y}_m \rightarrow \mathcal{Y}_n$ be the canonical map and \tilde{g}_{ren}^* be defined as in (4.7.3). Then, \tilde{g}_{ren}^* is an equivalence of objects in $\text{ComAlg}(\text{Mod}_{\text{Shv}_m(\text{pt}_m)})$*

$$\tilde{g}_{\text{ren}}^* : \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Shv}_m(\text{pt}_m) \xrightarrow{\simeq} \text{Shv}_m(\mathcal{Y}_m)^{\text{ren}},$$

i.e., it is an equivalence of symmetric monoidal categories, compatible with the $\text{Shv}_m(\text{pt}_m)$ -module structures on both sides.

4.7.8. From \tilde{g}_{ren}^* , we obtain the following sequence of equivalences in $\text{ComAlg}(\text{Mod}_{\text{Vect}^{\text{gr}}})$

$$\begin{aligned} \text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}} &\simeq \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}} \\ &\simeq \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Shv}_m(\text{pt}_m) \otimes_{\text{Shv}_m(\text{pt}_m)} \text{Vect}^{\text{gr}} \\ &\simeq \widetilde{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}} \otimes_{\text{Shv}_m(\text{pt}_m)} \text{Vect}^{\text{gr}} \\ &\xrightarrow[\simeq]{\tilde{g}_{\text{ren}}^* \otimes \text{id}_{\text{Vect}^{\text{gr}}}} \text{Shv}_m(\mathcal{Y}_m)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_m)} \text{Vect}^{\text{gr}} \\ &\simeq \text{Shv}_{\text{gr}}(\mathcal{Y}_m)^{\text{ren}}, \end{aligned}$$

(Proposition 4.7.7)

whose composition is denoted by \bar{g}_{ren}^* . Proposition 4.7.7 implies that \bar{g}_{ren}^* is also an equivalence and we have the following result.

Proposition 4.7.9. *Let $\mathcal{Y}_n \in \text{Stk}_{k_n}$ and \mathcal{Y}_m its base change to k_m . Let $g : \mathcal{Y}_m \rightarrow \mathcal{Y}_n$ be the canonical map and \bar{g}_{ren}^* be defined as above. Then, \bar{g}_{ren}^* is an equivalence of objects in $\text{ComAlg}(\text{Mod}_{\text{Vect}^{\text{gr}}})$*

$$\bar{g}_{\text{ren}}^* : \text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}} \xrightarrow{\simeq} \text{Shv}_{\text{gr}}(\mathcal{Y}_m)^{\text{ren}},$$

i.e., it is an equivalence of symmetric monoidal categories, compatible with the Vect^{gr} -module structures on both sides.

Remark 4.7.10. It is important to note that in forming $\text{Shv}_{\text{gr}}(\mathcal{Y}_m)^{\text{ren}}$, we are viewing \mathcal{Y}_m as an object in Stk_{k_m} ; see also Remark 4.4.5. In particular, the functor \bar{g}_{ren}^* defined above does not fall into the purview of §4.5 since over there, we pull and push along maps of geometric objects defined over the same base.

4.7.11. By our finiteness condition, any $\mathcal{Y} \in \text{Stk}_k$ is a pullback of some $\mathcal{Y}_n \in \text{Stk}_{k_n}$, see [LM00, Chap. 4] or [KP21, Thm. 2.1.13]. If $\mathcal{Y}_{n_1} \in \text{Stk}_{k_{n_1}}$ and $\mathcal{Y}_{n_2} \in \text{Stk}_{k_{n_2}}$ such that they both pullback to \mathcal{Y} over $\text{pt} = \text{Spec } k$, then there exists $m \gg 0$ such that their pullbacks to pt_m agree. Moreover, any morphism $f : \mathcal{Y} \rightarrow \mathcal{Z}$ between objects in Stk_k is already defined over some k_n .

Thus, by Proposition 4.7.9, we can view $\text{Shv}_{\text{gr}}(-)^{\text{ren}}$ as a sheaf theory on Stk_k . In particular, it makes sense to talk about $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$ for any $\mathcal{Y} \in \text{Stk}_k$, equipped with the usual six-functor formalism described above. This can be made precise in the following theorem.

Theorem 4.7.12. *We can attach for each $\mathcal{Y} \in \text{Stk}_k$ the category of graded sheaves $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}} \in \text{ComAlg}(\text{Mod}_{\text{Vect}^{\text{gr}}})$ on \mathcal{Y} . Moreover, for each $f : \mathcal{Y} \rightarrow \mathcal{Z}$ where $\mathcal{Y}, \mathcal{Z} \in \text{Stk}_k$, we have the usual functors $f_{\text{ren}}^*, f_{*,\text{ren}}, f_{\text{ren}}^!, f_{!,\text{ren}}$, where $f_{\text{ren}}^* \dashv f_{*,\text{ren}}$. When $f_!$ (for $\text{Shv}(-)$) preserves constructibility, $f_{\text{ren}}^!$ admits a left adjoint $f_{!,\text{ren}}$.*

Proof. By [KP21, Thm. 2.1.13],

$$(4.7.13) \quad \text{Stk}_k \simeq \text{colim}_n \text{Stk}_{k_n}$$

where functors $\text{Stk}_{k_n} \rightarrow \text{Stk}_k$ and $\text{Stk}_{k_n} \rightarrow \text{Stk}_{k_m}$ are given by base changes. Here, the colimit is taken over the partially ordered set of finite extensions of k_1 , which is a filtered system. Moreover, Proposition 4.7.9 furnishes us with compatible functors

$$\begin{array}{ccccccc} \cdots & \rightarrow & \text{Stk}_{k_n} & \longrightarrow & \text{Stk}_{k_{n'}} & \longrightarrow & \text{Stk}_{k_{n''}} \rightarrow \cdots \\ & & & \searrow & \downarrow & \swarrow & \\ & & & & \text{Mod}_{\text{Vect}^{\text{gr}}} & & \end{array}$$

where $\text{Stk}_{k_n} \rightarrow \text{Mod}_{\text{Vect}^{\text{gr}}}$ encodes the $*$ -pushforward functor. (4.7.13) then implies that we obtain a functor

$$\text{Stk}_k \rightarrow \text{Mod}_{\text{Vect}^{\text{gr}}},$$

which encodes the $*$ -pushforward functor of graded sheaves on stacks over $\text{pt} = \text{Spec } k$. The rest of the pull/push functors are obtained similarly. \square

4.7.14. Change of notation. The construction of $\text{Shv}_{\text{gr}}(-)^{\text{ren}}$ on Stk_k above implies that all the properties that we have proved earlier for $\text{Shv}_{\text{gr}}(-)^{\text{ren}}$ on Stk_{k_n} automatically carry over. Thus, everywhere $\text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$ is used, we can replace it by $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$. From this point onward, we will thus uniformly use notations that reflect this. For example, instead of writing $\text{gr}_{\mathcal{Y}_n} : \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \rightarrow \text{Shv}_{\text{gr}}(\mathcal{Y}_n)^{\text{ren}}$, we will write $\text{gr}_{\mathcal{Y}_n} : \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \rightarrow \text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$.

4.7.15. We end this subsection with the following useful lemma.

Lemma 4.7.16. *The functor $\text{oblv}_{\text{gr}} : \text{Shv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Shv}_c(\mathcal{Y})$ is conservative.*

Proof. We factor oblv_{gr} as follows

$$(4.7.17) \quad \text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}} \simeq \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}^{\text{gr}} \rightarrow \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect} \hookrightarrow \text{Shv}(\mathcal{Y})^{\text{ren}}$$

where the last functor is fully faithful for the same reason as Lemma 4.7.4. It thus remains to show that

$$\text{Shv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Shv}_m(\mathcal{Y}_n)^{\text{ren}} \otimes_{\text{Shv}_m(\text{pt}_n)} \text{Vect}$$

is conservative. But this is the content of Corollary 3.5.6 and the proof concludes. \square

Remark 4.7.18. Together with Proposition 3.5.5, the sequence (4.7.17) also implies that for $(\mathcal{F}^c, \mathcal{G}) \in \text{Shv}_{\text{gr},c}(\mathcal{Y}) \times \text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$, we have the following expected equivalences

$$\begin{aligned} & \bigoplus_{k \in \mathbb{Z}} \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}^c, \mathcal{G}(k)) \\ & \simeq \bigoplus_{k \in \mathbb{Z}} \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}^{\text{gr}}(\mathcal{F}^c, \mathcal{G})_k \\ & =: \text{oblv}_{\text{gr}}(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}^{\text{gr}}(\mathcal{F}^c, \mathcal{G})) \\ & \simeq \mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\text{oblv}_{\text{gr}}(\mathcal{F}^c), \text{oblv}_{\text{gr}}(\mathcal{G})). \end{aligned}$$

4.8. Functoriality of graded sheaves via correspondences. We will now describe how $\text{Shv}_{\text{gr}}(-)^{\text{ren}}$ can be enhanced to a functor out of the category of correspondences in Stk_k . This structure encodes various base change results of Theorem 4.5.8 in a homotopy coherent way, which allows us to construct monoidal structures coming from convolutions. This is necessary since we are dealing with ∞ -categories, where all compatibilities, such as associativity and commutativity, contain an infinite amount of data. The statements in this subsection are thus technical in nature. Fortunately, due to the way the theory of graded sheaves is set up, everything we need follows from the usual theory of ℓ -adic sheaves and has already been established in [LZ17a, LZ17b]. The readers who are only interested in monoidal structures on triangulated categories can safely skip this subsection.

4.8.1. Category of correspondences. Let \mathcal{C} be any ∞ -category. The ∞ -category $\text{Corr}(\mathcal{C})$ of correspondences in \mathcal{C} is defined in [GR17, Vol. I, Chap. 7]. We will quickly recall the ideas here. Roughly speaking, $\text{Corr}(\mathcal{C})$ has the same collection of objects as \mathcal{C} . Moreover, given $c_1, c_2 \in \mathcal{C}$, a morphism from c_1 to c_2 is given by the following diagram in \mathcal{C}

$$(4.8.2) \quad \begin{array}{ccc} c & \xrightarrow{h} & c_1 \\ \downarrow v & & \\ & & c_2 \end{array}$$

where $c \in \mathcal{C}$, and where compositions are given by Cartesian squares.

More generally, let vert and horiz be two collections of morphisms in \mathcal{C} such that vert (resp. horiz) is closed under pulling back along a morphism in horiz (resp. vert). Then, we let $\text{Corr}(\mathcal{C})_{\text{vert}, \text{horiz}}$ be the (non-full) subcategory of $\text{Corr}(\mathcal{C})$ containing the same collection of objects but morphisms are given by (4.8.2) such that $v \in \text{vert}$ and $h \in \text{horiz}$. We will also write $\text{Corr}(\mathcal{C})_{\text{all}, \text{all}}$ to denote $\text{Corr}(\mathcal{C})$ where all means *all* morphisms are allowed.

When \mathcal{C} is closed under finite products such that vert and horiz are stable under these products, $\text{Corr}(\mathcal{C})_{\text{vert},\text{horiz}}$ is equipped with a symmetric monoidal structure given by taking products.

We use $\mathcal{C}_{\text{vert}}$ and $\mathcal{C}_{\text{horiz}}$ to denote the (non-full) subcategories of \mathcal{C} where only morphisms in vert and horiz , respectively, are allowed. Then, we have natural functors

$$\mathcal{C}_{\text{vert}} \rightarrow \text{Corr}(\mathcal{C})_{\text{vert},\text{horiz}} \quad \text{and} \quad \mathcal{C}_{\text{horiz}}^{\text{op}} \rightarrow \text{Corr}(\mathcal{C})_{\text{vert},\text{horiz}}.$$

These functors are symmetric monoidal with respect to the finite product monoidal structures, if they are available.

4.8.3. Let \mathcal{S} be any ∞ -category. Then a functor $\Phi : \text{Corr}(\mathcal{C})_{\text{vert},\text{horiz}} \rightarrow \mathcal{S}$ induces two functors

$$(4.8.4) \quad \Phi_{\text{vert}} : \mathcal{C}_{\text{vert}} \rightarrow \mathcal{S} \quad \text{and} \quad \Phi_{\text{horiz}} : \mathcal{C}_{\text{horiz}}^{\text{op}} \rightarrow \mathcal{S}.$$

Moreover, for each Cartesian square in \mathcal{C}

$$\begin{array}{ccc} c' & \xrightarrow{h'} & c \\ \downarrow v' & & \downarrow v \\ d' & \xrightarrow{h} & d \end{array}$$

where $v, v' \in \text{vert}$ and $h, h' \in \text{horiz}$, we are given (as part of the data of Φ) an equivalence

$$\Phi_{\text{vert}}(v') \circ \Phi_{\text{horiz}}(h') \xrightarrow{\cong} \Phi_{\text{horiz}}(h) \circ \Phi_{\text{vert}}(v),$$

which has the same form as the usual base change results. The functor Φ encodes this base change equivalence along with all the homotopy coherence data.

4.8.5. Suppose that \mathcal{S} is symmetric monoidal and $\text{Corr}(\mathcal{C})_{\text{vert},\text{horiz}}$ is equipped with a symmetric monoidal structure as above. Then, a lax symmetric monoidal functor $\Phi : \text{Corr}(\mathcal{C})_{\text{vert},\text{horiz}} \rightarrow \mathcal{S}$ induces lax symmetric monoidal structures on Φ_{vert} and Φ_{horiz} . In particular, for any $c_1, c_2 \in \mathcal{C}$, we are given a morphism

$$\Phi(c_1) \otimes \Phi(c_2) \xrightarrow{\boxtimes} \Phi(c_1 \times c_2).$$

Moreover, this morphism is natural in c_1 and c_2 via both Φ_{vert} and Φ_{horiz} . This is the shape that our sheaf theory will take.

4.8.6. *Mixed sheaves as functors out of the category of correspondences.* The theory developed in [LZ17b, LZ17a] provides us with a right-lax symmetric monoidal functor

$$(4.8.7) \quad \text{Shv}_{m,!}^* : \text{Corr}(\text{Stk}_{k_n})_{\text{all},\text{all}} \rightarrow \text{Mod}_{\text{Shv}_m(\text{pt}_n)}$$

which sends each $\mathcal{Y}_n \in \text{Stk}_{k_n}$ to $\text{Shv}_m(\mathcal{Y}_n)$ and which encodes $*$ -pullback (resp. $!$ -pushforward) along all maps as well as the proper base change theorem for mixed sheaves. Note that the right-lax symmetric monoidal structure encodes the procedure of taking box-tensor

$$\text{Shv}_m(\mathcal{Y}_1) \otimes \text{Shv}_m(\mathcal{Y}_2) \xrightarrow{\boxtimes} \text{Shv}_m(\mathcal{Y}_1 \times \mathcal{Y}_2), \quad \text{for all } \mathcal{Y}_1, \mathcal{Y}_2 \in \text{Stk}_{k_n}$$

as well its compatibility with $!$ -pushforwards (K nneth formula) and $*$ -pullbacks. Here, for $\mathcal{F}_i \in \text{Shv}_m(\mathcal{Y}_i), i \in \{1, 2\}$,

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2$$

where $p_i : \mathcal{Y}_1 \times \mathcal{Y}_2 \rightarrow \mathcal{Y}_i$ denotes the projection onto the i -th factor.

Remark 4.8.8. Note that the theory of ℓ -adic sheaves developed in [LZ17a] is different from the one in [GL19, Gai15, HRS21]. However, the two theories agree on the subcategories of constructible sheaves, which is what we use to construct the renormalized sheaf theory. Moreover, the general results of [LZ17b] can take as input the results of [GL19, Gai15, HRS21] and yield the desired correspondence-functoriality as already done in [LZ17a]. We thank Y. Liu for pointing this out to us. See also [Cho21] where the general theory of [LZ17a] is used to obtain correspondence-functoriality for motivic homotopy theory.

Before continuing, we need the following definition.

Definition 4.8.9. A morphism $f : \mathcal{Y} \rightarrow \mathcal{Z}$ in Stk_{k_n} is said to be *universally constructible* with respect to the $*$ -pushforward functor (resp. $!$ -pushforward functor) if for all $\mathcal{Z}' \rightarrow \mathcal{Z}$ in $\text{Stk}_{\mathbb{A}^1}$, $f_{\mathcal{Z}',*}$ (resp. $f_{\mathcal{Z}',!}$) preserves constructibility, where $f_{\mathcal{Z}'} : \mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{Z}'$ is the base change of f to \mathcal{Z}' . We also say that f is UC_* (resp. $\text{UC}_!$) in this case.

A morphism $f : \mathcal{Y} \rightarrow \mathcal{Z}$ in Stk_k is said to be UC_* (resp. $\text{UC}_!$) if f is the pullback of a UC_* (resp. $\text{UC}_!$) morphism in Stk_{k_n} .

Remark 4.8.10. The property of being UC_* or $\text{UC}_!$ is stable under pullbacks and finite extensions of scalars. This allows one to make sense of the second part of the definition above.

Remark 4.8.11. Representable maps are UC_* and $\text{UC}_!$. However, there are more UC_* and $\text{UC}_!$ morphisms than just these. For example, consider $B\mathbb{G}_a \rightarrow \text{pt}_n$ where $\mathbb{G}_a = \text{Spec } k_n[t]$ is the additive group scheme over k_n .

4.8.12. The functor (4.8.7) restricts to a functor

$$\text{Shv}_{m,!}^* : \text{Corr}(\text{Stk}_{k_n})_{\text{UC}_!, \text{all}} \rightarrow \text{Mod}_{\text{Shv}_m(\text{pt}_n)}$$

which induces the following right-lax symmetric monoidal functor by restricting to the full subcategories of constructible sheaves

$$\text{Shv}_{m,c,!}^* : \text{Corr}(\text{Stk}_{k_n})_{\text{UC}_!, \text{all}} \rightarrow \text{Mod}_{\text{Shv}_{m,c}(\text{pt}_n)}.$$

Taking Ind , which is symmetric monoidal, we obtain a right-lax symmetric monoidal functor

$$(4.8.13) \quad \text{Shv}_{m,!}^{\text{ren},*} : \text{Corr}(\text{Stk}_{k_n})_{\text{UC}_!, \text{all}} \rightarrow \text{Mod}_{\text{Shv}_m(\text{pt}_n)},$$

which encodes the renormalized $!$ -pushforward functors along $\text{UC}_!$ -morphisms and $*$ -pullback functors along all morphisms.

4.8.14. To formulate the dual we need the following result.

Proposition 4.8.15. *Let $f_i : \mathcal{Y}_i \rightarrow \mathcal{Z}_i$ be morphisms in Stk_{k_n} , where $i \in \{1, 2\}$. Then, $f_{i,*,\text{ren}}$ satisfy Künneth; namely, the following diagram,*

$$\begin{array}{ccc} \text{Shv}_m(\mathcal{Y}_1)^{\text{ren}} \otimes \text{Shv}_m(\mathcal{Y}_2)^{\text{ren}} & \xrightarrow{\boxtimes} & \text{Shv}_m(\mathcal{Y}_1 \times \mathcal{Y}_2)^{\text{ren}} \\ \downarrow f_{1,*,\text{ren}} \otimes f_{2,*,\text{ren}} & & \downarrow (f_1 \times f_2)_{*,\text{ren}} \\ \text{Shv}_m(\mathcal{Z}_1)^{\text{ren}} \otimes \text{Shv}_m(\mathcal{Z}_2)^{\text{ren}} & \xrightarrow{\boxtimes} & \text{Shv}_m(\mathcal{Z}_1 \times \mathcal{Z}_2)^{\text{ren}} \end{array}$$

which a priori commutes up to a 2-morphism given by

$$f_{1,*,\text{ren}} \mathcal{F}_1 \boxtimes f_{2,*,\text{ren}} \mathcal{F}_2 \rightarrow (f_1 \times f_2)_{*,\text{ren}} (\mathcal{F}_1 \boxtimes \mathcal{F}_2), \quad \text{for all } \mathcal{F}_i \in \text{Shv}_m(\mathcal{Y}_i)^{\text{ren}}, i \in \{1, 2\},$$

is actually commutative, i.e., the canonical morphism above is an equivalence.

Proof. Since all functors are continuous, it suffices to assume that $\mathcal{F}_i \in \text{Shv}_{m,c}(\mathcal{Y}_i) \subset \text{Shv}_m(\mathcal{Y}_i)^+$, $i \in \{1, 2\}$. By Proposition 4.1.17, combined with the fact that renormalized $*$ -pushforward functors are left exact and unren induces an equivalence $\text{Shv}_m(-)^{\text{ren},+} \xrightarrow{\cong} \text{Shv}_m(-)^+$, we can work with non-renormalized, i.e., usual, sheaves and functors. By smooth base change and descent, it suffices to prove the statement when \mathcal{Z}_i 's are schemes. But then, the result is proved in [GL19, Theorem 3.4.5.1]. \square

Remark 4.8.16. The analogous result for the renormalized $!$ -pullback functors can be shown in an analogous way.

Now, using [GR17, Vol. I, Chap. 12], we can pass (4.8.7) to right adjoints and obtain a weakly right-lax symmetric monoidal functor

$$\text{Shv}_{m,*}^! : \text{Corr}(\text{Stk}_{k_n})_{\text{all}, \text{all}} \rightarrow \text{Mod}_{\text{Shv}_m(\text{pt}_n)}.$$

Here, weakly right-lax symmetric monoidal means that the transformation

$$\text{Shv}_m(\mathcal{Y}_1) \otimes \text{Shv}_m(\mathcal{Y}_2) \xrightarrow{\boxtimes} \text{Shv}_m(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

is only natural in \mathcal{Y}_1 and \mathcal{Y}_2 in a lax way, i.e., up to a 2-morphism, see also [GR17, Vol. I, Chap. 10, §3.2.1]. Applying the regularization procedure of [Pre12, Thm. 4.3.6, Lem. 4.6.1] (which is a formal way to formulate the passage from $\mathrm{Shv}_m(-)$ to $\mathrm{Shv}_m(-)^{\mathrm{ren}}$), we obtain a weakly right-lax symmetric monoidal functor

$$(4.8.17) \quad \mathrm{Shv}_{m,*}^{\mathrm{ren},!} : \mathrm{Corr}(\mathrm{Stk}_{k_n})_{\mathrm{all},\mathrm{all}} \rightarrow \mathrm{Mod}_{\mathrm{Shv}_m(\mathrm{pt}_n)}.$$

By Proposition 4.8.15 and Remark 4.8.16, we know that $\mathrm{Shv}_{m,*}^{\mathrm{ren},!}$ is actually a right-lax symmetric monoidal functor.

Remark 4.8.18. Without using the regularization machinery of [Pre12], we can obtain the following right-lax symmetric monoidal functor in a more straightforward way

$$\mathrm{Shv}_{m,*}^{\mathrm{ren},!} : \mathrm{Corr}(\mathrm{Stk}_{k_n})_{\mathrm{UC}_*,\mathrm{all}} \rightarrow \mathrm{Mod}_{\mathrm{Shv}_m(\mathrm{pt}_n)},$$

following §4.8.12. This suffices for the applications in §6.

4.8.19. *Graded sheaves as functors out of the category of correspondences.* Composing (4.8.13) and (4.8.17) with

$$\mathrm{Mod}_{\mathrm{Shv}_m(\mathrm{pt}_n)} \xrightarrow{- \otimes_{\mathrm{Shv}_m(\mathrm{pt}_n)} \mathrm{Vect}^{\mathrm{gr}}} \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}}$$

and using §4.7, we obtain the following corresponding results for graded sheaves.

Theorem 4.8.20. *We have a right-lax symmetric monoidal functor*

$$\mathrm{Shv}_{\mathrm{gr},*}^{\mathrm{ren},!} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{UC}_!,\mathrm{all}} \rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}}$$

which, in particular, encodes the renormalized $!$ -pushforward functors along $\mathrm{UC}_!$ -morphisms and renormalized $*$ -pullback functors along all morphisms.

Similarly, we have a right-lax symmetric monoidal functor

$$\mathrm{Shv}_{\mathrm{gr},*}^{\mathrm{ren},!} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{all},\mathrm{all}} \rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}}$$

which, in particular, encodes the renormalized $*$ -pushforward and renormalized $!$ -pullback functors along all morphisms.

We also have “small category” versions of the two functors above

$$\begin{aligned} \mathrm{Shv}_{\mathrm{gr},c,*}^{\mathrm{ren},!} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{UC}_!,\mathrm{all}} &\rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr},c}}, \\ \mathrm{Shv}_{\mathrm{gr},c,*}^{\mathrm{ren},!} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{UC}_*,\mathrm{all}} &\rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr},c}}. \end{aligned}$$

4.8.21. We also have the following variant.

Theorem 4.8.22. *We have a right-lax symmetric monoidal functor*

$$(4.8.23) \quad \mathrm{Shv}_{\mathrm{gr},*}^{\mathrm{ren},*} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{all},\mathrm{sm}} \rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}}$$

which, in particular, encodes the renormalized $*$ -pushforward functors along all morphisms and renormalized $*$ -pullback functors along smooth morphisms.

Similarly, we have a right-lax symmetric monoidal functor

$$(4.8.24) \quad \mathrm{Shv}_{\mathrm{gr},*}^{\mathrm{ren},*} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{pr},\mathrm{all}} \rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}}$$

which, in particular, encodes the renormalized $*$ -pushforward functors along proper morphisms and renormalized $*$ -pullback functors along all morphisms.

We also have “small category” versions of the two functors above

$$\begin{aligned} \mathrm{Shv}_{\mathrm{gr},c,*}^{\mathrm{ren},*} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{pr},\mathrm{all}} &\rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr},c}}, \\ \mathrm{Shv}_{\mathrm{gr},c,*}^{\mathrm{ren},*} : \mathrm{Corr}(\mathrm{Stk}_k)_{\mathrm{UC}_*,\mathrm{sm}} &\rightarrow \mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr},c}}. \end{aligned}$$

Proof. By the adjunction $f_{\text{ren}}^* \dashv f_{*,\text{ren}}$ and the smooth and proper base change results of Theorem 4.5.8, the statements in the theorem, without the right-lax symmetric monoidal structures, are direct consequences of the universal property of the categories of correspondences proved in [GR17, Vol. 1, Chap. 7, §3] after ignoring non-invertible 2-morphisms.¹⁵ A variant of this result but with right-lax symmetric monoidal structures and only invertible 2-morphisms are proved in [LZ17b, §3.2]. Note that the latter uses the language of multisimplicial sets rather than correspondences. However, the two formulations are the same, see [LZ17b, §6.1] and [GR17, Vol. 1, Part. III, §1.3]. \square

5. WEIGHT STRUCTURE AND PERVERSE t -STRUCTURE FOR GRADED SHEAVES

In this section, we construct a perverse t -structure and a weight structure on the category of constructible graded sheaves $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ for any $\mathcal{Y} \in \text{Stk}_k$. Unlike previous sections, we work exclusively with *small* categories here. In particular, unless otherwise specified, DG-categories appearing in this section are assumed to be small and idempotent complete; see also §3.7. It is mostly due to convenience since most of the available literatures on weight structures, including our main source for this section [Bon12], operate in this setting. It is expected that this restriction on size can be lifted [Bon12, Rmk. 1.2.3].¹⁶ However, we will not pursue this direction, as it is not needed for the applications we have in mind.

We will review the basics of weight structures in §5.1. This is followed by technical preparations needed to actually construct a perverse t -structure and a weight structure on $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. More specifically, in §§5.2 and 5.3, we will construct categories of pure graded perverse sheaves of a given weight and show that these categories generate the whole category of constructible graded sheaves in a precise sense. The actual construction is given in §5.4, which follows directly from the work of Bondarko [Bon12]. In §5.5, various expected results regarding the interactions between the weight/ t -structure and functoriality of graded sheaves are established. These results follow naturally from the standard ones for mixed sheaves. Finally, in §5.6, we describe connections between our construction and various notions in the mixed geometry literature.

Since we work mostly with constructible sheaves in this section, in situations where there is no difference between the usual pull/push functors and their renormalized versions, we will, for brevity's sake, omit *ren* from the notation, see also Remark 4.1.21.

We note that the term *weight* is used to refer to either the Frobenius weight structures on mixed sheaves in the sense of [BBDG18] or to weights in the sense of weight structure (on an arbitrary category) in the sense of [Bon10, Pau08]. When we want to emphasize the former, we will use the term Frobenius weight.

5.1. Weight structures: a quick review. We will now give a brief review of weight structures (or co- t -structures) as discovered independently by Pauksztello and Bondarko [Pau08, Bon10]. For more modern treatments, using the language of stable ∞ -categories, the readers may consult [Sos19, Aok20, ES21b], which are our main sources for this subsection. Note that the indexing convention used in these papers is the reverse of that in [Bon10] but is the same as the one in [Bon12]. We will follow the convention used in [Bon12, Sos19, Aok20, ES21b]. We note that the proofs of all of the results stated here can easily be found in these papers.

Definition 5.1.1. A *weight structure* on a stable ∞ -category \mathcal{C} is the data of two retract-closed full additive subcategories $(\mathcal{C}^{w \leq 0}, \mathcal{C}^{w \geq 0})$ such that

- (i) $\mathcal{C}^{w \geq 0}[1] \subseteq \mathcal{C}^{w \geq 0}$ and $\mathcal{C}^{w \leq 0}[-1] \subseteq \mathcal{C}^{w \leq 0}$. We write

$$\mathcal{C}^{w \geq n} := \mathcal{C}^{w \geq 0}[n] \quad \text{and} \quad \mathcal{C}^{w \leq n} := \mathcal{C}^{w \leq 0}[n].$$

¹⁵Note that non-invertible 2-morphisms do not appear in our definition of the category of correspondences. In contrast, [GR17] allows non-invertible morphisms. Thus, to obtain our result from theirs, we just remove non-invertible 2-morphisms from the answer.

¹⁶[Lur18, Lem. C.2.4.3] allows one to *Ind*-extend a t -structure on a small stable ∞ -category \mathcal{C} to one on $\text{Ind}(\mathcal{C})$. On the other hand, we do not expect that this can be done with weight structures in general.

(ii) If $c \in \mathcal{C}^{w \leq 0}$ and $d \in \mathcal{C}^{w \geq 1}$, then¹⁷

$$\pi_0 \operatorname{Hom}_{\mathcal{C}}(c, d) \simeq 0.$$

(iii) For any object $c \in \mathcal{C}$, we have a cofiber sequence

$$w_{\leq 0}c \rightarrow c \rightarrow w_{\geq 1}c$$

where $w_{\leq 0}c \in \mathcal{C}^{w \leq 0}$ and $w_{\geq 1}c \in \mathcal{C}^{w \geq 1}$. Such a sequence is called a weight truncation of c .¹⁸

We will use the term *weight category* to refer to a stable ∞ -category equipped with a weight structure. We say that a weight structure on \mathcal{C} is *bounded* if

$$\mathcal{C} \simeq \bigcup_n (\mathcal{C}^{w \geq -n} \cap \mathcal{C}^{w \leq n}).$$

We let $\operatorname{Cat}_{\infty}^w$ and $\operatorname{Cat}_{\infty}^{w,b}$ denote the ∞ -categories of weight categories and bounded weight categories, respectively.

Remark 5.1.2. Note that the definition of a weight structure on a stable ∞ -category \mathcal{C} does not use any ∞ -categorical data. Thus, we could equivalently define what it means to have a weight structures on a triangulated category and then state that a weight structure on a stable ∞ -category \mathcal{C} is that on its homotopy category $h\mathcal{C}$. This approach is taken, for example, in [Aok20], and is parallel to how [Lur17a] defines a t -structure.

5.1.3. The heart of a weight structure. Let \mathcal{C} be a stable ∞ -category equipped with a weight structure. We let $\mathcal{C}^{\heartsuit_w} := \mathcal{C}^{w \leq 0} \cap \mathcal{C}^{w \geq 0}$ denote the weight heart of the weight structure. An object $c \in \mathcal{C}$ is said to be *pure of weight n* if $c \in \mathcal{C}^{\heartsuit_w}[n]$.

Unlike the case of t -structures described in [Lur17a, Rmk. 1.2.1.12], $\mathcal{C}^{\heartsuit_w}$ is not necessarily *classical*, i.e., $\mathcal{C}^{\heartsuit_w}$ is different from its homotopy category $h\mathcal{C}^{\heartsuit_w}$. In more concrete terms, let $c, d \in \mathcal{C}^{\heartsuit_w}$. Then, $\operatorname{Hom}_{\mathcal{C}}(c, d)$ might have non-trivial higher homotopy groups. In the setting of DG-categories, being non-classical means $\mathcal{H}\operatorname{om}_{\mathcal{C}}(c, d)$ might have non-vanishing negative cohomology groups.¹⁹

5.1.4. The weight heart $\mathcal{C}^{\heartsuit_w}$ is an additive ∞ -category in the sense that it has all finite products and co-products and moreover, its homotopy category $h\mathcal{C}^{\heartsuit_w}$ is an additive category in the usual sense. In particular, finite products and co-products in $\mathcal{C}^{\heartsuit_w}$ coincide. See [GGN16, §2] and [Lur18, Appx. C.1.5] for a more detailed discussion of additive ∞ -categories.

We use $\operatorname{Cat}_{\infty}^{\operatorname{add}}$ to denote the ∞ -category of additive ∞ -categories.

5.1.5. As mentioned above, when \mathcal{C} is equipped with a t -structure, the t -heart $\mathcal{C}^{\heartsuit_t}$ is classical. In particular, if $c, d \in \mathcal{C}$, $\pi_0 \operatorname{Hom}_{\mathcal{C}}(c, d[n]) \simeq 0$ for all $n < 0$. In the DG-category setting, this is equivalent to saying that $\mathcal{H}\operatorname{om}_{\mathcal{C}}(c, d) \in \operatorname{Vect}$ concentrates in non-negative cohomological degrees. The heart of a weight structure $\mathcal{C}^{\heartsuit_w}$ satisfies a dual condition. Namely, it is easy to see from Definition 5.1.1.(i) and (ii) that for $c, d \in \mathcal{C}^{\heartsuit_w}$, $\pi_0 \operatorname{Hom}_{\mathcal{C}}(c, d[n]) \simeq 0$ for all $n > 0$. In the DG-settings, this is equivalent to saying that $\mathcal{H}\operatorname{om}_{\mathcal{C}}(c, d) \in \operatorname{Vect}$ concentrates in non-positive cohomological degrees.

Remark 5.1.6. The statements above regarding DG-categories could have also been stated more generally for stable ∞ -categories using Hom-spectra instead of Vect-enriched ones. However, this generality is not needed in the paper and we expect that the readers are more likely to be familiar with Vect.

¹⁷When \mathcal{C} is a DG-category, this condition is equivalent to saying that $H^0(\mathcal{H}\operatorname{om}_{\mathcal{C}}(c, d)) = 0$.

¹⁸Note that unlike the case of t -structures, weight truncations are not canonical.

¹⁹Negative because we are using cohomological indexing convention.

5.1.7. *From additive ∞ -categories to weight categories.* The procedure of taking the weight heart forms a functor

$$(-)^{\heartsuit_w} : \text{Cat}_{\infty}^w \rightarrow \text{Cat}_{\infty}^{\text{add}}.$$

It is possible to go the other direction as well using the $(-)^{\text{fin}}$ construction, which we will now briefly review for the reader's convenience. We note that for our purposes, it is enough to know the existence of such a functor. For more details, see [ES21b, §2.2.7].

Let $\mathcal{A} \in \text{Cat}_{\infty}^{\text{add}}$. Consider the stable ∞ -category $\hat{\mathcal{A}} := \text{Fun}^{\times}(\mathcal{A}^{\text{op}}, \text{Sptr})$ consisting of functors that preserves finite products, i.e., those that turn finite co-products in \mathcal{A} to products in the category Sptr of spectra. The Yoneda lemma furnishes a fully faithful embedding $\mathcal{A} \rightarrow \hat{\mathcal{A}}$. Let \mathcal{A}^{fin} be the smallest stable ∞ -subcategory of $\hat{\mathcal{A}}$ containing the image of \mathcal{A} . The category \mathcal{A}^{fin} is equipped with a natural weight structure.

Theorem 5.1.8 ([ES21b, Thm. 2.2.9]). *We have an adjoint pair*

$$(-)^{\text{fin}} : \text{Cat}_{\infty}^{\text{add}} \rightleftarrows \text{Cat}_{\infty}^{w,b} : (-)^{\heartsuit_w}.$$

Moreover,

- (i) *the right adjoint $(-)^{\heartsuit_w}$ is fully faithful, and*
- (ii) *the adjoint pair restricts to a pair of mutually inverse equivalences of ∞ -categories of idempotent complete ∞ -categories on both sides.*

5.1.9. *Weight complex functor.* Let \mathcal{A} be a classical additive category. Then, we can form a stable ∞ -category of bounded chain complexes $\text{Ch}^b(\mathcal{A})$, see [Lur17a, §1.3.1]. Its homotopy category is the homotopy category $K^b(\mathcal{A})$ of bounded chain complexes in \mathcal{A} . The category $\text{Ch}^b(\mathcal{A})$ is equipped with a natural weight structure where weight truncations are given by stupid/brutal truncations of complexes, [Bon10, §1.1]. This is a proto-typical example of a weight structure. Note that for the axioms of Definition 5.1.1 to hold, the differentials in $\text{Ch}^b(\mathcal{A})$ is *homological*, i.e., they *decrease* the indices.

5.1.10. For any $\mathcal{C} \in \text{Cat}_{\infty}^{w,b}$, there exists a *weight complex functor*

$$(5.1.11) \quad \text{wt} : \mathcal{C} \rightarrow \text{Ch}^b(h\mathcal{C}^{\heartsuit_w}),$$

see [Sos19, Cor. 3.5] and [ES21b, Expl. 5.1.7]. This functor is the image of the natural functor $\mathcal{C}^{\heartsuit_w} \rightarrow h\mathcal{C}^{\heartsuit_w}$ under the following equivalence, coming from Theorem 5.1.8

$$\text{Hom}_{\text{Cat}_{\infty}^{\text{add}}}(\mathcal{C}^{\heartsuit_w}, h\mathcal{C}^{\heartsuit_w}) \simeq \text{Hom}_{\text{Cat}_{\infty}^{w,b}}(\mathcal{C}, \text{Ch}^b(h\mathcal{C}^{\heartsuit_w})).$$

In particular, when $\mathcal{C}^{\heartsuit_w}$ is classical, i.e., $\mathcal{C}^{\heartsuit_w} \simeq h\mathcal{C}^{\heartsuit_w}$, then the weight complex functor induces an equivalence of categories $\mathcal{C} \simeq \text{Ch}^b(h\mathcal{C}^{\heartsuit_w})$.

Remark 5.1.12. The construction of the weight complex functor given in [Sos19, Cor. 3.5] is an ∞ -categorical enhancement of the one given in [Bon10, §3], which has a more concrete description. Given an object $c \in \mathcal{C}$ where \mathcal{C} is a bounded weight category, by Definition 5.1.1.(iii), there exists a (non-canonical) finite filtration c_{\bullet} of c , such that the i -th associated graded piece $\text{assgr}_i(c_{\bullet})$ is pure of weight i . Thus, $\text{assgr}_i(c_{\bullet})[-i]$ is pure of weight 0, i.e., $\text{assgr}_i(c_{\bullet})[-i] \in \mathcal{C}^{\heartsuit_w}$. A standard construction in homological algebra, [Lur17a, Defn. 1.2.2.2 and Rmk. 1.2.2.3], gives us a chain complex

$$\cdots \rightarrow \text{assgr}_{i+1}(c_{\bullet})[-(i+1)] \rightarrow \text{assgr}_i(c_{\bullet})[-i] \rightarrow \text{assgr}_{i-1}(c_{\bullet})[-(i-1)] \rightarrow \cdots$$

The content of the weight complex functor is that this complex (up to homotopy) is canonical in c even though the weight filtration is *not* canonical.

We note that this construction takes the same shape as the construction of the chromatographic complex in [WW17, §3.5], which is not known to be a functor. As we will see, their construction can be realized as the composition of the canonical functor $\text{Shv}_{m,c}(\mathcal{Y}_n) \rightarrow \text{Shv}_{gr,c}(\mathcal{Y})$ and the weight complex functor on $\text{Shv}_{gr,c}(\mathcal{Y})$. In particular, this shows that the procedure of taking chromatographic complex is a functor.

5.1.13. *Compatibility with monoidal structures.* Let \mathcal{C} be a bounded weight category, equipped with a compatible monoidal structure in the sense that $\mathcal{C}^{w \leq 0}$ and $\mathcal{C}^{w \geq 0}$ are closed under tensor products. Here, the monoidal structure can be symmetric or not. In fact, more generally, \mathcal{C} can be E_n -monoidal in the sense of [Lur17a, §5.4], with E_1 -monoidal, E_2 -monoidal, and E_∞ -monoidal being the usual, braided, and symmetric monoidal structures, respectively. Such an E_n -monoidal structure gives rise to an E_n -monoidal structure on $\mathcal{C}^{\heartsuit_w}$ and hence, also on $(\mathcal{C}^{\heartsuit_w})^{\text{fin}}$ and $(h\mathcal{C}^{\heartsuit_w})^{\text{fin}} \simeq \text{Ch}^b(h\mathcal{C}^{\heartsuit_w})$. We have the following result.

Theorem 5.1.14 ([Aok20]). *Let \mathcal{C} be a bounded weight category, equipped with an E_n -monoidal structure such that $\mathcal{C}^{w \leq 0}$ and $\mathcal{C}^{w \geq 0}$ are preserved under tensor products. Then, $\mathcal{C}^{\heartsuit_w}$, and hence $(\mathcal{C}^{\heartsuit_w})^{\text{fin}}$ and $(h\mathcal{C}^{\heartsuit_w})^{\text{fin}}$ are equipped with natural E_n -monoidal structures. Moreover, the following natural functors*

$$\mathcal{C} \rightarrow (\mathcal{C}^{\heartsuit_w})^{\text{fin}} \rightarrow (h\mathcal{C}^{\heartsuit_w})^{\text{fin}} \simeq \text{Ch}^b(h\mathcal{C}^{\heartsuit_w})$$

are E_n -monoidal. In particular, the weight complex functor is E_n -monoidal.

Remark 5.1.15. The main result of [Aok20] is stated only for symmetric monoidal categories. However, the same proof works more generally. Indeed, the main tool used is [Nik16], which works more generally; see also [Nik16, Rmk. 6.11].

5.1.16. *The case of $\text{Vect}^{\text{gr},c}$.* As mentioned earlier, our goal is to equip $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ with a weight structure and a perverse t -structure for any $\mathcal{Y} \in \text{Stk}_k$. The remainder of this section will be devoted to this goal. As an warm up exercise, we will now equip $\text{Vect}^{\text{gr},c} \simeq \text{Shv}_{\text{gr},c}(\text{pt})$ with a bounded weight structure and a t -structure.

The t -structure is just the usual one, obtained from the usual t -structure on Vect^c . The weight structure is defined as follows: $\text{Vect}^{\text{gr},c,w \geq 0}$ (resp. $\text{Vect}^{\text{gr},c,w \leq 0}$) consists of graded perfect chain complexes $A = (A_i)$ where the i -th graded piece A_i concentrates in cohomological degrees $\leq i$ (resp. $\geq i$). It is easy to check that the pair $\text{Vect}^{\text{gr},c,w \leq 0}$ and $\text{Vect}^{\text{gr},c,w \geq 0}$ satisfies the conditions given in Definition 5.1.1.

From the description above, we see that $\text{Vect}^{\text{gr},c,\heartsuit_w} \simeq \bigoplus_i \text{Vect}^{c,\heartsuit_t}[-i]$. Namely, given any $A \in \text{Vect}^{\text{gr},c,\heartsuit_w}$, we have an equivalence $A \simeq \bigoplus_i H^i(A_i)\langle -i \rangle[-i]$. This is the “baby case” of the decomposition theorem for pure graded sheaves we will establish in §5.4.

From the description above, it is clear that $\text{Vect}^{\text{gr},c,\heartsuit_w}$ is classical. Thus, the weight complex functor provides an equivalence of symmetric monoidal categories

$$\text{Vect}^{\text{gr},c} \simeq \text{Ch}^b(\text{Vect}^{\text{gr},c,\heartsuit_w}) \simeq \bigoplus_i \text{Ch}^b(\text{Vect}^{c,\heartsuit_t}[-i]) \simeq \bigoplus_i \text{Ch}^b(\text{Vect}^{c,\heartsuit_t}),$$

which is evident. Note that the last equivalence is just a re-indexing.

5.2. Pure graded perverse sheaves. This subsection and the next make the necessary preparation to apply the results of [Bon12] to construct perverse t -structure and a weight structure on the category $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. Throughout, we let $\mathcal{Y}_n \in \text{Stk}_{k_n}$ whose base changes to k and k_m are \mathcal{Y} and \mathcal{Y}_m , respectively, for any $m \in n\mathbb{Z}_{>0}$.

More specifically, we will now construct various categories $\text{Perv}_{\text{gr},c}(\mathcal{Y}_m)^{w=\nu}$ of pure graded perverse sheaves of a fixed weight $w = \nu$ and study their formal properties. In the next subsection, we will show that these categories together generate $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ in the appropriate sense.

5.2.1. For any $\nu \in \mathbb{Z}$, we define the category of graded perverse sheaves of pure weight ν to be

$$(5.2.2) \quad \text{Perv}_{\text{gr},c}(\mathcal{Y}_m)^{w=\nu} := \text{Idem}(\text{Im}(\text{Perv}_{\text{m},c}(\mathcal{Y}_m)^{w=\nu} \xrightarrow{\text{gr}\mathcal{Y}_m} \text{Shv}_{\text{gr},c}(\mathcal{Y})))$$

where Idem is the functor of taking idempotent completion and $\text{Perv}_{\text{m},c}(\mathcal{Y}_m)^{w=\nu}$ is the full subcategory of the category of constructible mixed perverse sheaves $\text{Perv}_{\text{m},c}(\mathcal{Y}_m)$ of (Frobenius) weight $w = \nu$, in the sense of [BBDG18, LO09].²⁰ Since $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is idempotent complete, $\text{Perv}_{\text{gr},c}(\mathcal{Y}_m)^{w=\nu}$ is naturally a full subcategory of $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. The main goal of this subsection is to have a more explicit description of $\text{Perv}_{\text{gr},c}(\mathcal{Y}_m)^{w=\nu}$ for any m and ν .

²⁰What we write $\text{Perv}_{\text{m},c}(\mathcal{Y}_m)$ is more usually written as $\text{Perv}_m(\mathcal{Y}_m)$.

Remark 5.2.3. Directly from the construction, we see that the endofunctor $\langle k \rangle : \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}} \rightarrow \mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}$ described in §4.4.7 induces an equivalence of categories

$$\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu} \rightarrow \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu-k}.$$

Remark 5.2.4. Note that the term “pure graded perverse sheaves” does not a priori have a meaning. However, in §5.4, we will construct a perverse t -structure and a weight structure on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ that make sense of this term. Namely, the t -heart, $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})$, is obtained by assembling $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu}$ for various ν together. Moreover, $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu}$ consists precisely of objects in $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})$ of weight ν in the new weight structure.

5.2.5. Invariance under of extensions of scalars. As Remark 5.2.4 suggests, $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu}$ is also independent of m . This is indeed the case.

Lemma 5.2.6. *Let \mathcal{Y} , \mathcal{Y}_n , and \mathcal{Y}_m as above. Then, for any $\nu \in \mathbb{Z}$, $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_n)^{w=\nu}$ and $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu}$ coincide as full subcategories of $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$.*

Proof. The proof follows essentially the same strategy as that of Proposition 4.7.7. We have the following factorization where the horizontal arrow is given by pulling back along $\mathcal{Y}_m \xrightarrow{g} \mathcal{Y}_n$, which preserves perversity and Frobenius weights since it is a finite étale map²¹

$$(5.2.7) \quad \begin{array}{ccc} \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=\nu} & \xrightarrow{g^*} & \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_m)^{w=\nu} \\ & \searrow \mathrm{gr}_{\mathcal{Y}_n} & \downarrow \mathrm{gr}_{\mathcal{Y}_m} \\ & & \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \end{array}$$

which implies the inclusion $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_n)^{w=\nu} \subseteq \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_m)^{w=\nu}$ as full subcategories of $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$.

To show that the two coincide, it suffices to show that $\mathrm{gr}_{\mathcal{Y}_m}(\mathcal{F}) \in \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_n)^{w=\nu}$ for any $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_m)^{w=\nu}$. As in Lemma 4.7.5, $\mathrm{gr}_{\mathcal{Y}_m}(\mathcal{F})$ is a direct summand of $\mathrm{gr}_{\mathcal{Y}_n}(g_*\mathcal{F})$ where $g_*\mathcal{F} \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=\nu}$ since g is finite étale. We are thus done since $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_n)^{w=\nu}$ is idempotent complete, by definition. \square

As in the case of graded sheaves explained in §4.7.11, for any $\mathcal{Y} \in \mathrm{Stk}_n$, we can define

$$\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=\nu} := \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}_n)^{w=\nu} \subset \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}_n) \simeq \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$$

where $\mathcal{Y}_n \in \mathrm{Stk}_n$ is any choice such that its base change to pt is \mathcal{Y} . Indeed, Lemma 5.2.6 guarantees that this is well-defined.

5.2.8. Hom estimates and semisimplicity. We will now study morphisms between objects in $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=\nu}$ for various ν . As a consequence, we will obtain the fact that $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=\nu}$ is classical and semi-simple.

Proposition 5.2.9. *Let $\mathcal{F} \in \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=k}$ and $\mathcal{G} \in \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=l}$. Then, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})$ concentrates in cohomological degrees $[0, k-l]$. In particular, when $k-l < 0$, then $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G}) \simeq 0$.*

Proof. By construction, objects in $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=\nu}$ are direct summands of objects of the form $\mathrm{gr}_{\mathcal{Y}_n}(\mathcal{F})$ where $\mathcal{F} \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=\nu}$. It thus suffices to show the statement above for $\mathcal{H}_{\mathrm{gr}} := \mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}(\mathrm{gr}_{\mathcal{Y}_n}(\mathcal{F}_n), \mathrm{gr}_{\mathcal{Y}_n}(\mathcal{G}_n))$ where $\mathcal{F}_n \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=k}$ and $\mathcal{G}_n \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=l}$. We will use \mathcal{F} and \mathcal{G} to denote their pullbacks of \mathcal{Y} .

By [Sun12a, Prop. 3.9], we know that $\mathcal{H}_{\mathrm{m}} := \mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{m}}(\mathcal{Y}_n)^{\mathrm{ren}}}(\mathcal{F}_n, \mathcal{G}_n) \in \mathrm{Shv}_{\mathrm{m}}(\mathrm{pt}_n)$ has Frobenius weights $\geq l-k$; see also [BBDG18, Prop. 5.1.15] for the scheme version. By definition, this means $H^i(\mathcal{H}_{\mathrm{m}})$ has Frobenius weights $\geq i+l-k$. By Proposition 4.6.2, $\mathcal{H}_{\mathrm{gr}}$ is the naive weight 0 part of \mathcal{H}_{m} . Thus, $H^i(\mathcal{H}_{\mathrm{gr}}) \simeq 0$ when $i+l-k > 0$ or equivalently, when $i > k-l$. In other words, we have shown that $\mathcal{H}_{\mathrm{gr}}$ is cohomologically supported on $(-\infty, k-l]$.

We know that $\mathcal{H} := \mathcal{H}\mathrm{om}_{\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}}(\mathcal{F}, \mathcal{G})$ concentrates in cohomological degrees ≥ 0 since \mathcal{F} and \mathcal{G} lie in the heart of a t -structure, namely, the perverse t -structure, of $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$. By Corollary 4.6.4, $\mathcal{H}_{\mathrm{gr}}$ is a direct summand of \mathcal{H} . Thus, $\mathcal{H}_{\mathrm{gr}}$ is also supported in cohomological degrees $[0, \infty)$. Combined with the above, we know that $\mathcal{H}_{\mathrm{gr}}$ is supported in cohomological degrees $[0, k-l]$. \square

²¹Since we are working with constructible sheaves only, g^* and g_{ren}^* are the same.

The following result is a direct consequence of Proposition 5.2.9 above.

Corollary 5.2.10. *Let $\mathcal{F}, \mathcal{G} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$. Then, $\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})$ concentrates in cohomological degree 0. In particular, $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ is classical (as opposed to being a genuine ∞ -category) and semi-simple.*

Here, semi-simple means that any exact triangle $A \rightarrow B \rightarrow C$ splits, where $A, C \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$. This, in particular, implies that $B \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ as well since $B \simeq A \oplus C$.

Proposition 5.2.9 can be slightly strengthened. We start with a lemma involving mixed sheaves.

Lemma 5.2.11. *Let $\mathcal{F}, \mathcal{G} \in \text{Perv}_{m,c}(\mathcal{Y}_n)$ of Frobenius weights k and l respectively such that $k \neq l$. Then, $H^0(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G})) \in \text{Shv}_m(\text{pt}_n)^{\heartsuit}$ has no weight 0 component.*

Proof. Suppose $H^0(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G}))$ has non-zero weight 0-component. Without changing the weight of \mathcal{G} , we can twist it by a sheaf of weight 0 using a Frobenius eigenvalue of $H^0(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G}))$. We can thus assume that $H^0(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G})^{\text{ren}})^F \neq 0$ where F is the Frobenius. We thus obtain a non-zero morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ between perverse sheaves of different weights. We will show that such a φ cannot exist.

Since \mathcal{F} and \mathcal{G} have finite filtrations whose associated graded pieces are simple perverse sheaves, we can assume that \mathcal{F} and \mathcal{G} themselves are simple, without loss of generality. But then, φ being non-zero implies that φ is an isomorphism. This is not possible since they have different weights. The proof thus concludes. \square

Corollary 5.2.12. *Let $\mathcal{F} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=k}$ and $\mathcal{G} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=l}$ such that k and l are distinct. Then, $H^0(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})) \simeq 0$.*

Proof. As in the proof of Proposition 5.2.9, we can assume that \mathcal{F} and \mathcal{G} are of the form $\text{gr}(\mathcal{F})$ and $\text{gr}(\mathcal{G})$ respectively, where $\mathcal{F}, \mathcal{G} \in \text{Perv}_{m,c}(\mathcal{Y}_n)$ are of weights k and l respectively. Proposition 4.6.2 implies that $H^0(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\text{gr}(\mathcal{F}), \text{gr}(\mathcal{G})))$ is the weight 0 part of $H^0(\mathcal{H}\text{om}_{\text{Shv}_m(\mathcal{Y}_n)^{\text{ren}}}(\mathcal{F}, \mathcal{G}))$, which vanishes by Lemma 5.2.11. \square

Combining Proposition 5.2.9 and Corollary 5.2.12, we obtain the following result.

Theorem 5.2.13. *Let $\mathcal{F} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=k}$ and $\mathcal{G} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=l}$. Then, $\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})$ concentrates in cohomological degrees $(0, k-l]$ when $k \neq l$ and $\{0\}$ when $k = l$. In particular, when $k-l < 0$, then $\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G}) \simeq 0$.*

5.2.14. *Orthogonality.* Corollary 5.2.12 can be interpreted as a statement about orthogonality between $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ for different ν . We will now study the same question but within the same ν , see Proposition 5.2.17 below. We start with the following conservativity lemma.

Lemma 5.2.15. *Let $\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$, $\mathcal{F} = \text{oblv}_{\text{gr}}(\mathcal{F}_{\text{gr}})$, and $\mathcal{G} = \text{oblv}_{\text{gr}}(\mathcal{G}_{\text{gr}})$. Then $\mathcal{F}_{\text{gr}} \simeq \mathcal{G}_{\text{gr}}$ if and only if $\mathcal{F} \simeq \mathcal{G}$.*

Proof. The only if direction is clear. For the if direction, suppose $\mathcal{F} \simeq \mathcal{G}$. Then, by Remark 4.7.18,

$$\bigoplus_{k \in \mathbb{Z}} H^0(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}}\langle k \rangle)) \simeq H^0(\mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})).$$

But unless $k = 0$, \mathcal{F}_{gr} and $\mathcal{G}_{\text{gr}}\langle k \rangle$ can have no non-trivial morphism between them, by Theorem 5.2.13. Thus, $k = 0$, and we obtain an isomorphism

$$(5.2.16) \quad H^0(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}})) \simeq H^0(\mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})).$$

In particular, there exists a non-zero $\varphi_{\text{gr}} : \mathcal{F}_{\text{gr}} \rightarrow \mathcal{G}_{\text{gr}}$ such that $\text{oblv}_{\text{gr}}(\varphi_{\text{gr}}) : \mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism. The proof concludes by conservativity of oblv_{gr} , see Lemma 4.7.16. \square

Proposition 5.2.17. *In the situation of Lemma 5.2.15, suppose further that $\mathcal{F}, \mathcal{G} \in \text{Perv}_c(\mathcal{Y})$ are simple. Then*

$$\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}}) \simeq \begin{cases} \overline{\mathbb{Q}}_\ell, & \text{when } \mathcal{F}_{\text{gr}} \simeq \mathcal{G}_{\text{gr}} \text{ or equivalently, } \mathcal{F} \simeq \mathcal{G}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 5.2.13, we know that $\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}})$ concentrates in cohomological degree 0. In particular,

$$\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}}) \simeq H^0(\mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}_{\text{gr}}, \mathcal{G}_{\text{gr}})).$$

Thus we only need to deal with the cohomological degree 0 part.

By Lemma 5.2.15, $\mathcal{F}_{\text{gr}} \simeq \mathcal{G}_{\text{gr}}$ if and only if $\mathcal{F} \simeq \mathcal{G}$. Moreover, simplicity of \mathcal{F} and \mathcal{G} implies that

$$H^0(\mathcal{H}\text{om}_{\text{Shv}(\mathcal{Y})^{\text{ren}}}(\mathcal{F}, \mathcal{G})) \simeq \begin{cases} \overline{\mathbb{Q}}_\ell, & \text{when } \mathcal{F} \simeq \mathcal{G}, \\ 0, & \text{otherwise.} \end{cases}$$

We thus conclude using (5.2.16). \square

5.2.18. *Generators.* We will now describe the generators of $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$, which results in an explicit description of the category. To start, let $\widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ be the full subcategory of $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ spanned by finite direct sums of objects of the form $\text{gr}_{\mathcal{Y}_n}(\mathcal{F}_n)$ for $\mathcal{F}_n \in \text{Perv}_{\text{m},c}(\mathcal{Y}_n)^{w=\nu}$ for some n such that the pullback of \mathcal{F}_n to \mathcal{Y} is simple. By construction and Lemma 5.2.6, $\widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ is a full subcategory of $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$. By Proposition 5.2.17, we obtain

$$\widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu} = \bigoplus_{s \in S} \text{Vect}^{c,\heartsuit}$$

where $\text{Vect}^{c,\heartsuit}$ is the abelian category of finite dimensional vector spaces and S is the set of simple perverse sheaves $\mathcal{F} \in \text{Perv}_c(\mathcal{Y})$ such that \mathcal{F} is the pullback of some (necessarily simple) perverse sheaf \mathcal{F}_n over \mathcal{Y}_n for some n . It is clear from the description above that $\widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ is a semi-simple abelian category. In particular, it is also idempotent complete.

Theorem 5.2.19. $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} = \widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ as full subcategories of $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. In particular,

$$\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \simeq \bigoplus_{s \in S} \text{Vect}^{c,\heartsuit}$$

is a semi-simple abelian category, where S is the set of simple perverse sheaves $\mathcal{F} \in \text{Perv}_c(\mathcal{Y})$ such that \mathcal{F} is the pullback of some (necessarily simple) perverse sheaf \mathcal{F}_n over \mathcal{Y}_n for some n .

Before proving Theorem 5.2.19, we recall [BBDG18, Prop. 5.3.9.(ii)] below. While the result is stated for schemes there, the same proof works for stacks. Indeed, the only ingredient is [BBDG18, Prop. 5.1.2], which is also valid for stacks, see also [Sun12a, Proof of Thm. 3.11].

Proposition 5.2.20 ([BBDG18, Prop. 5.3.9.(ii)]). *Let $\mathcal{K}_n \in \text{Perv}_{\text{m},c}(\mathcal{Y}_n)$ be a simple perverse sheaf. Then, there exists $m = nd$ for some d and $\mathcal{L}_m \in \text{Perv}_{\text{m},c}(\mathcal{Y}_m)$ such that \mathcal{K}_n is the pushforward of \mathcal{L}_m under $f_{mn} : \mathcal{Y}_m \rightarrow \mathcal{Y}_n$. Moreover, the pullback of \mathcal{L}_m to \mathcal{Y} is simple.*

Proof of Theorem 5.2.19. It remains to show that $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \subseteq \widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$. Since the latter is idempotent complete, it suffices to show that $\text{gr}(\mathcal{F}_n) \in \widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ for any $\mathcal{F}_n \in \text{Perv}_{\text{m},c}(\mathcal{Y}_n)^{w=\nu}$. Since $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ is semi-simple by Corollary 5.2.10 and since any such \mathcal{F}_n has a finite filtration whose associated graded pieces are simple, $\text{gr}(\mathcal{F}_n)$ is a finite direct sum of objects of the form $\text{gr}(\mathcal{K}_n)$ where $\mathcal{K}_n \in \text{Perv}_{\text{m},c}(\mathcal{Y}_n)$ is simple. It remains to show that $\text{gr}(\mathcal{K}_n) \in \widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$.

Let $\mathcal{L}_m \in \text{Perv}_{\text{m},c}(\mathcal{Y}_m)$ be as in Proposition 5.2.20. Then, by definition, $\text{gr}_{\mathcal{Y}_m}(\mathcal{L}_m) \in \widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})$. Moreover, $f_{mn}^* \mathcal{K}_n \simeq \mathcal{L}_m^{\oplus d}$. By the factorization (5.2.7), we obtain that

$$\text{gr}_{\mathcal{Y}_n}(\mathcal{K}_n) \simeq \text{gr}_{\mathcal{Y}_m}(\mathcal{L}_m)^{\oplus d}.$$

Thus, $\text{gr}_{\mathcal{Y}_n}(\mathcal{K}_n) \in \widetilde{\text{Perv}}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ as desired. \square

5.3. Generation. We will now show that $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ for all ν together generate $\text{Shv}_{\text{gr},c}(\mathcal{Y})$, i.e., $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is the smallest full DG-subcategory (or equivalently, full stable ∞ -subcategory) of $\text{Shv}_{\text{gr}}(\mathcal{Y})^{\text{ren}}$ containing $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ for all ν .

Let \mathcal{C} be a triangulated/stable ∞ -DG-category and $\{S_i\}_{i \in I}$, where I is an indexing set, a family of collections of objects in \mathcal{C} or subcategories \mathcal{C} . Then, following [Bon12], we use $\langle S_i \rangle_{i \in I}$ to denote the smallest full triangulated/stable ∞ -DG-subcategory of \mathcal{C} containing S_i for all i . When I is a singleton, i.e., we simply have one S , then we simply write $\langle S \rangle$.

The rest of the current subsection will be devoted to the proof of the following result.

Theorem 5.3.1. *The categories $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$ together generate $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. More precisely,*

$$\text{Shv}_{\text{gr},c}(\mathcal{Y}) \simeq \langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in \mathbb{Z}}.$$

5.3.2. By definition, $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle \subseteq \text{Shv}_{\text{gr},c}(\mathcal{Y})$. To prove the other inclusion, it suffices to show that

- (i) $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in \mathbb{Z}}$ contains $\text{gr}(\mathcal{F}_n)$ for all $\mathcal{F}_n \in \text{Shv}_{\text{m},c}(\mathcal{Y}_n)$ for some fixed (and hence, all) n , and
- (ii) $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in \mathbb{Z}}$ is idempotent complete.

5.3.3. We will now prove the first item in §5.3.2. Since $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in \mathbb{Z}}$ is closed under extensions, using the perverse t -structure on $\text{Shv}_{\text{m},c}(\mathcal{Y})$, we reduce to the case where $\mathcal{F}_n \in \text{Perv}_{\text{m},c}(\mathcal{Y}_n)$. By [LO09, Theorem 9.2], \mathcal{F}_n has a weight filtration whose associated graded pieces are pure. This allows us to further reduce to the case where \mathcal{F}_n is a pure perverse sheaf of weight ν for some ν . But then, we are done since, $\text{gr}(\mathcal{F}_n) \in \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu}$.

5.3.4. We will prove the second item in §5.3.2 in the remainder of this subsection. It suffices to show that $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in [k,l]}$ is idempotent complete for any $k \leq l$ since $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in \mathbb{Z}}$ is the union of categories of this form. We will prove this by induction on the length $l - k$.

5.3.5. Base case. For the base case $k = l = \nu$, consider $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle$. Theorem 5.2.19 and Proposition 5.2.17 imply that

$$\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle \simeq \bigoplus_{s \in S} \text{Vect}^c,$$

which is idempotent complete.

5.3.6. Induction step. Suppose we know that $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in [k,l]}$ is idempotent complete. We will show that $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in [k,l+1]}$ is also idempotent complete. But first, we will need some preparation.

We start with the following lemma, which is a DG-categorical counterpart of [Bon12, Lem. 1.1.5].

Lemma 5.3.7. *Let \mathcal{C} be a DG-category and $\mathcal{C}_1, \mathcal{C}_2$ full DG-subcategories of \mathcal{C} . Let $\mathcal{C}^\circ = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ be the smallest full DG-subcategory of \mathcal{C} containing \mathcal{C}_1 and \mathcal{C}_2 . Suppose that for any $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$, $\mathcal{H}\text{om}_{\mathcal{C}}(c_1, c_2) \simeq 0$. Then, we have the following adjoint pairs $F_1 \dashv G_1$ and $F_2 \dashv G_2$ fitting into the following diagram*

$$\mathcal{C}_1 \begin{array}{c} \xleftarrow{F_1} \\ \xrightarrow{G_1} \end{array} \mathcal{C}^\circ \begin{array}{c} \xleftarrow{F_2} \\ \xrightarrow{G_2} \end{array} \mathcal{C}_2$$

such that for any $c \in \mathcal{C}^\circ$, we have an exact triangle

$$F_1 G_1 c \rightarrow c \rightarrow G_2 F_2 c.$$

Proof. Let \mathcal{D}° be the full subcategory of \mathcal{C}° spanned by objects d such that the functor

$$\begin{aligned} \mathcal{C}_1^{\text{op}} &\rightarrow \text{Vect} \\ c &\mapsto \mathcal{H}\text{om}_{\mathcal{C}}(F_1(-), d). \end{aligned}$$

is representable. Denoting the representing object $G_1(d)$, we obtain a functor $G_1 : \mathcal{D}^\circ \rightarrow \mathcal{C}_1$ which is a partial right adjoint to F_1 . It is easy to see that \mathcal{D}° contains \mathcal{C}_1 and \mathcal{C}_2 : $G_1(d) = d$ when $d \in \mathcal{C}_1$ and $G_1(d) = 0$ when $d \in \mathcal{C}_2$. It is also easy to check that \mathcal{D}° is closed under finite direct sums, shifts, and cones, or equivalently, G_1 is compatible with these operations. In particular, $\mathcal{D}^\circ = \mathcal{C}^\circ$ and hence, G_1 is defined on \mathcal{C}° as desired.

We have the following pairs of adjoint functors

$$\mathcal{C}_1 \xrightleftharpoons[G_1]{F_1} \mathcal{C}^\circ \xrightleftharpoons[G_2]{F_2} \ker G_1$$

where $\ker G_1$ is the full subcategory of \mathcal{C}° consisting of objects $c \in \mathcal{C}^\circ$ such that $G_1(c) \simeq 0$, and $F_2(c) = \text{Cone}(F_1 G_1 c \rightarrow c)$. We will now show that $\ker G_1 = \mathcal{C}_2$, which will conclude the proof.

As already seen above, $\mathcal{C}_2 \subseteq \ker G_1$. For the other inclusion, it suffices to show that $F_2 c \in \mathcal{C}_2$ for any $c \in \mathcal{C}^\circ$. For that, note that if we let \mathcal{E}° be the full subcategory of \mathcal{C}° consisting of objects c such that $F_2 c \in \mathcal{C}_2$, then \mathcal{E}° contains $\mathcal{C}_1, \mathcal{C}_2$ and is closed under finite direct sums, shifts, and cones. Thus, $\mathcal{E}^\circ = \mathcal{C}^\circ$ and we are done. \square

Lemma 5.3.8. *Consider the situation of Lemma 5.3.7. Suppose that \mathcal{C}_1 and \mathcal{C}_2 are idempotent complete. Then, so is $\mathcal{C}^\circ = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$.*

Proof. Replacing \mathcal{C} by its idempotent completion if necessary, we can assume that \mathcal{C} is also idempotent complete without changing $\mathcal{C}_1, \mathcal{C}_2$, or $\mathcal{C}^\circ = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. Let Idem denote the category given in [Lur17b, Defn. 4.4.5.2]. Consider $\rho : \text{Idem} \rightarrow \mathcal{C}^\circ$, which determines an object $c \in \mathcal{C}$ and a map $e : c \rightarrow c$ such that e^2 is homotopic to e . Choosing a left cofinal map $\mathbb{Z}_{\geq 0} \rightarrow \text{Idem}$, [Lur17b, Prop. 4.4.5.17], we are reduced to showing that the following diagram

$$c \xrightarrow{e} c \xrightarrow{e} c \xrightarrow{e} \dots$$

has a colimit in \mathcal{C}° .

From Lemma 5.3.7, we obtain functors $\mathbb{Z}_{\geq 0} \rightarrow \text{Idem} \rightarrow \mathcal{C}_1$ and $\mathbb{Z}_{\geq 0} \rightarrow \text{Idem} \rightarrow \mathcal{C}_2$ given by idempotents $e_1 = F_1 G_1 e$ on $F_1 G_1 c$ and $e_2[-1] = G_2 F_2 e[-1]$ on $G_2 F_2 c[-1]$, fitting into the following diagram

$$\begin{array}{ccccccc} G_2 F_2 c[-1] & \xrightarrow{e_2[-1]} & G_2 F_2 c[-1] & \xrightarrow{e_2[-1]} & G_2 F_2 c[-1] & \xrightarrow{e_2[-1]} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ F_1 G_1 c & \xrightarrow{e_1} & F_1 G_1 c & \xrightarrow{e_1} & F_1 G_1 c & \xrightarrow{e_1} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ c & \xrightarrow{e} & c & \xrightarrow{e} & c & \xrightarrow{e} & \dots \end{array}$$

where the columns are exact triangles. The colimits of the first two rows are in \mathcal{C}° and hence, so is the colimit of the last row. Thus, we are done. \square

Remark 5.3.9. The lemmas above hold equally true for stable ∞ -categories in general. The only change is that instead of $\mathcal{H}\text{om}_{\mathcal{C}}(c_1, c_2)$, we need to formulate our statement in terms of spectra-enriched Hom instead. We do not need this in the current paper.

Back to our induction step, by Theorem 5.2.13, it is easy to see that if we take $\mathcal{C}_1 = \langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in [k,l]}$ and $\mathcal{C}_2 = \langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=l+1} \rangle$, then $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C} = \text{Shv}_{\text{gr},c}(\mathcal{Y})$ satisfy the conditions of Lemma 5.3.8. Consequently, $\langle \text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=\nu} \rangle_{\nu \in [k,l+1]}$ is idempotent complete and the induction step concludes.

5.3.10. We have thus finished proving Theorem 5.3.1.

5.4. Weight and perverse t -structure for graded sheaves. In this subsection, we will finally construct a weight structure and a perverse t -structure on $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ for any $\mathcal{Y} \in \text{Stk}_k$. The preparation done in §§5.2 and 5.3 allows us to apply [Bon12] directly, yielding a *transversal weight* and t -structure on $\text{Shv}_{\text{gr},c}(\mathcal{Y})$.

5.4.1. *Transversal weight and t -structures.* We begin by recalling various results from [Bon12].

Definition 5.4.2 ([Bon12, Defn. 1.1.4]). (i) Let \mathcal{C} be a triangulated category. We say that a family $\{A_i\}_{i \in \mathbb{Z}}$ of full subcategories $A_i \subseteq \mathcal{C}$ is *semi-orthogonal* if for any i, j , $\text{Hom}_{\mathcal{C}}(A_i, A_j[*])$ is supported on $* \in [0, i - j]$. We will say that $\{A_i\}_{i \in \mathbb{Z}}$ is *strongly semi-orthogonal* if, in addition to the above, $\text{Hom}_{\mathcal{C}}(A_i, A_j) = 0$ when $i \neq j$.²²

²²Note that since we are working with a triangulated category \mathcal{C} , $\text{Hom}_{\mathcal{C}}(-, -)$ is a set, rather than a space.

- (ii) We will say that $\{A_i\}_{i \in \mathbb{Z}}$ is *generating* (in \mathcal{C}) if $\langle A_i \rangle_{i \in \mathbb{Z}} = \mathcal{C}$. See §5.3 for the definition of $\langle - \rangle$.²³

Theorem 5.4.3 ([Bon12, Thm. 1.2.1]). *Fix a triangulated category \mathcal{C} .*

Let $\{A_i\}_{i \in \mathbb{Z}}$ be a family of full subcategories of \mathcal{C} . Then the following are equivalent (all weight and t -structures appearing below are bounded)

- (i) *Each A_i is abelian semi-simple and $\{A_i\}_{i \in \mathbb{Z}}$ is a strongly semi-orthogonal generating family (see Definition 5.4.2) in \mathcal{C} .*
- (ii) *$\{A_i\}_{i \in \mathbb{Z}}$ is a semi-orthogonal family in \mathcal{C} such that if we let $\mathcal{C}^{t \leq 0}$ (resp. $\mathcal{C}^{t \geq 0}$) be the smallest extension-closed full subcategory of \mathcal{C} containing $\cup_{i \in \mathbb{Z}, j \geq 0} A_i[j]$ (resp. $\cup_{i \in \mathbb{Z}, j \leq 0} A_i[j]$), then $(\mathcal{C}^{t \leq 0}, \mathcal{C}^{t \geq 0})$ yields a t -structure on \mathcal{C} .*
- (iii) *$\{A_i\}_{i \in \mathbb{Z}}$ is a semi-orthogonal family in \mathcal{C} such that the smallest extension-closed full subcategory of \mathcal{C} containing $\cup_{i \in \mathbb{Z}} A_i$ is the heart of a t -structure.*
- (iv) *$\{A_i\}_{i \in \mathbb{Z}}$ is a semi-orthogonal family in \mathcal{C} and \mathcal{C} has a t -structure whose heart $\mathcal{C}^{\heartsuit_t}$ contains A_i 's such that for all $X \in \mathcal{C}^{\heartsuit_t}$ there exists an exhaustive separated increasing filtration by subobjects $W_{\leq i} X$ such that $W_{\leq i} X / W_{\leq i-1} X \in A_i$ for all i .*

The above conditions are equivalent to the following equivalent conditions

- (a) *\mathcal{C} has a weight and a t -structure, such that for all $i \in \mathbb{Z}$ and $X \in \mathcal{C}^{\heartsuit_t}$ there exists a weight truncation $w_{\leq i} X \rightarrow X \rightarrow w_{\geq i+1} X$ such that all terms are in $\mathcal{C}^{\heartsuit_t}$. We call such a truncation a nice decomposition.*
- (b) *\mathcal{C} has a weight and a t -structure such that nice decompositions exist for any object in $\mathcal{C}^{\heartsuit_t}$. Moreover, they are functorial in $X \in \mathcal{C}^{\heartsuit_t}$ (if we fix i) and the corresponding functors $X \mapsto w_{\leq i} X$ and $X \mapsto w_{\geq i+1} X$ are exact (as endo-functors on $\mathcal{C}^{\heartsuit_t}$).*
- (c) *\mathcal{C} has a weight and a t -structure such that for any $X \in \mathcal{C}^{\heartsuit_t}$ and any $w_{\leq i} X$ (that is part of a weight truncation of X), $\text{Im}(^t H^0(w_{\leq i} X \rightarrow X) \rightarrow X)$ extends to a nice decomposition of X .*

Moreover the $\{A_i\}_{i \in \mathbb{Z}}$ in the first set of equivalent conditions (indexed by Roman numerals) can be obtained from the weight and t -structures in the second set of equivalent conditions (indexed by Latin characters) as follows: $A_i = \mathcal{C}^{\heartsuit_t} \cap \mathcal{C}^{w=i}$. Namely, A_i consists of objects in the t -heart that has weight i .

Definition 5.4.4. Let \mathcal{C} be a triangulated category equipped with a weight w and a t -structure. If w and t satisfy the (equivalent) conditions of Theorem 5.4.3, we will say that t is transversal to w .

Remark 5.4.5. (i) Note that since weight and t -structures on a stable ∞ -category is defined as structures on the underlying triangulated category, all theorems about weight and t -structures on triangulated category (Theorem 5.4.3 above and Proposition 5.4.6 below, in particular) apply equally well to stable ∞ -categories.

- (ii) From the proof of Theorem 5.4.3, we see that the weight structure on \mathcal{C} is given by the pair $(\mathcal{C}^{w \leq 0}, \mathcal{C}^{w \geq 0})$ where $\mathcal{C}^{w \leq 0}$ (resp. $\mathcal{C}^{w \geq 0}$) is the idempotent-closure in \mathcal{C} of the smallest extension-closed full subcategory of \mathcal{C} containing finite direct sum of objects in $\cup_{i \in \mathbb{Z}, j, i+j \leq 0} A_i[j]$ (resp. $\cup_{i \in \mathbb{Z}, j, i+j \geq 0} A_i[j]$).
- (iii) $\mathcal{C}^{\heartsuit_w}$ is the full subcategory of \mathcal{C} spanned by finite direct sums of objects in $A_i[-i]$'s. In particular, any $X \in \mathcal{C}^{\heartsuit_w}$ is equivalent to $\bigoplus_i H^i(X)[-i]$ where $H^i(X)[-i] \in A_i[-i]$, see [Bon12, Rmk. 1.2.3.2]. This is an analog of the decomposition theorem of [BBDG18].

Proposition 5.4.6 ([Bon12, Prop. 1.2.4]). *Let \mathcal{C} be a triangulated category equipped with a weight structure and a t -structure such that t is transversal to w . Then,*

- (i) *t -truncations $X \mapsto \tau^{\leq i} X$ and $X \mapsto \tau^{\geq i} X$ are weight-exact for any i , i.e., they preserve $\mathcal{C}^{w \geq 0}$, $\mathcal{C}^{w \leq 0}$, and hence, $\mathcal{C}^{\heartsuit_w}$.*
- (ii) *Let $X \in \mathcal{C}$ and $i \in \mathbb{Z}$. Then, $X \in \mathcal{C}^{w \leq i}$ (resp. $X \in \mathcal{C}^{w \geq i}$) if and only if for all $j \in \mathbb{Z}$, $W_{\leq i+j} {}^t H^j(X) \simeq {}^t H^j(X)$ (resp. $W_{\leq i+j-1} {}^t H^j(X) \simeq 0$).*

²³Note that this notion is different from the one discussed in §3.1.8.

(iii) For $X \in \mathcal{C}^{\heartsuit_i}$, denote $W_{\geq i+1}X := X/W_{\leq i}X$. We have the following pairs of adjoint functors

$$\mathcal{C}^{w \leq i} \cap \mathcal{C}^{\heartsuit_i} \underset{W_{\leq i}}{\overset{\quad}{\rightleftarrows}} \mathcal{C}^{\heartsuit_i} \underset{W_{\leq i}}{\overset{W_{\geq i+1}}{\rightleftarrows}} \mathcal{C}^{w \geq i+1} \cap \mathcal{C}^{\heartsuit_i}$$

Moreover, both $W_{\leq i}$ and $W_{\geq i+1}$ are exact.

(iv) The functors $X \mapsto W_{\leq i}(W_{\geq i}X)$ and $X \mapsto W_{\geq i}(W_{\leq i}X)$ are canonically isomorphic as functors $\mathcal{C}^{\heartsuit_i} \rightarrow A_i$. For $X \in \mathcal{C}^{\heartsuit_i}$, we write $\mathrm{Gr}_i^W X := W_{\leq i}X/W_{\leq i-1}X \in \mathcal{C}^{\heartsuit_i}$.

(v) For $X \in \mathcal{C}^{\heartsuit_w}$, $X \in \mathcal{C}^{w \leq i}$ (resp. $X \in \mathcal{C}^{w \geq j}$) if and only if $\mathrm{Gr}_j^W X = 0$ for all $j > i$ (resp. $j < i$).

5.4.7. We will now apply the discussion above to the case of $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ where $\mathcal{Y} \in \mathrm{Stk}_k$.

Theorem 5.4.8. For any $\mathcal{Y} \in \mathrm{Stk}_k$, $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is equipped with a weight and a t -structure such that the t -structure is transversal to the weight structure.

This t -structure will be referred to as the perverse t -structure, whose heart is the category of graded perverse sheaves $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}) := \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_i}$.

Proof. Take $A_i = \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=i}$ as in (5.2.2). Theorems 5.2.13, 5.2.19 and 5.3.1 imply that the family $\{A_i\}_{i \in \mathbb{Z}}$ satisfies the conditions of Theorem 5.4.3. The proof thus concludes. \square

Remark 5.4.9. Theorem 5.4.3 gives another characterization of $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=k}$ defined in (5.2.2). Namely, $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=k} = \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y}) \cap \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{w=k}$ is precisely the category of graded perverse sheaves of pure weight k . We will also use $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ to denote $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=0}$, the category of pure graded perverse sheaves of weight 0.

Remark 5.4.10. The action of $\mathrm{Vect}^{\mathrm{gr},c} \simeq \mathrm{Shv}_{\mathrm{gr},c}(\mathrm{pt})$ on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is compatible with the weight and t -structures involved. Indeed, it suffices to check for a single vector space concentrated in one graded and cohomological degree. But now, it is easy to see since the action by such an object is simply given by a combination of taking finite direct sums, cohomological degree shift, and graded degree shifts. As a result, $\mathrm{Vect}^{\mathrm{gr},c,\heartsuit_w}$ acts on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ and $\mathrm{Vect}^{\mathrm{gr},c,\heartsuit_i}$ acts on $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})$. Thus, $\mathrm{Vect}^{\mathrm{gr},c} \simeq \mathrm{Ch}^b(\mathrm{Vect}^{\mathrm{gr},c,\heartsuit_w})$ also acts on $\mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w})$. Moreover, this action is compatible with the weight complex functors.

5.5. Formal properties. Since $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is equipped with a perverse t -structure, transversal to a weight structure, all statements in Theorem 5.4.3, Proposition 5.4.6, and Remark 5.4.5 apply to $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ as well. In this subsection, we are interested in the interaction between these structures and those on the categories of mixed sheaves \mathcal{Y}_n and constructible sheaves on \mathcal{Y} .

The general meta-theorem is that the perverse t -structure on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is compatible with those on $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ and $\mathrm{Shv}_c(\mathcal{Y})$. Similarly, the weight structure on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is compatible with the notion of weight on $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ as defined in [BBDG18, LO09]. Finally, results regarding weights in the theory of mixed sheaves have natural analogs in the theory of graded sheaves. We will collect some of these properties below. Our goal is not to be exhaustive; rather, we aim demonstrate how straightforward it is to adapt known results from the theory of (mixed) sheaves to the graded sheaf setting.

Proposition 5.5.1. Let $\mathcal{Y}_n \in \mathrm{Stk}_{k_n}$ and \mathcal{Y} its base change to k . Then, the functors

$$\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n) \xrightarrow{\mathrm{gr}} \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \xrightarrow{\mathrm{oblv}_{\mathrm{gr}}} \mathrm{Shv}_c(\mathcal{Y})$$

preserves and reflects t -structures with respect to the perverse t -structures. Namely, for any n and any $\mathcal{F} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$, $\mathcal{F} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{t \leq n}$ (resp. $\mathcal{F} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{t \geq n}$) if and only if $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}) \in \mathrm{Shv}_c(\mathcal{Y})^{t \leq n}$ (resp. $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}) \in \mathrm{Shv}_c(\mathcal{Y})^{t \geq n}$). We have a similar statement for gr .

Proof. We will now show that gr is t -exact, i.e., it preserves the t -structures involved; t -exactness for $\mathrm{oblv}_{\mathrm{gr}}$ can be argued similarly. It suffices to show that gr preserves the t -heart. By [LO09, Thm. 9.2], any mixed perverse sheaf admits a finite filtration whose associated graded pieces are pure perverse sheaves. Thus, it suffices to show that gr sends pure perverse sheaves to (pure) graded perverse sheaves. But this follows from the description of the t -heart given in Theorem 5.4.3.(iii) and the definition of $A_i = \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=i}$ given in (5.2.2).

Note the general fact that a conservative t -exact functor necessarily reflects the t -structure as well. The desired conclusion then follows from the conservativity of gr and $\mathrm{oblv}_{\mathrm{gr}}$, see Corollary 3.5.6 and Lemma 4.7.16. \square

Proposition 5.5.2. *Let \mathcal{Y} and \mathcal{Y}_n be as above. The functor $\mathrm{gr} : \mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n) \rightarrow \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ preserves and reflect weights, where we use Frobenius weights on $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ and our weight structure on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$. Namely, for any k and $\mathcal{F} \in \mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y})$, then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y})^{w \leq k}$ (resp. $\mathcal{F} \in \mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y})^{w \geq k}$) if and only if $\mathrm{gr}(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{w \leq k}$ (resp. $\mathrm{gr}(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{w \geq k}$).*

Proof. Perverse t -truncation is compatible with Frobenius weights on mixed sheaves, by [Sun12a, Thm. 3.5], and is compatible with the transversal weight structure on graded sheaves by Proposition 5.4.6. Moreover, gr is t -exact, by Proposition 5.5.1. It therefore suffices to consider the case where \mathcal{F} is perverse.

Now, \mathcal{F} has a finite filtration whose associated graded pieces are pure perverse sheaves, by [LO09, Thm. 9.2]. In particular, \mathcal{F} has Frobenius weight $\leq k$ (resp. $\geq k$) if and only if all the pure pieces have Frobenius weights $\leq k$ (resp. $\geq k$). This allows us to further reduce to the case where \mathcal{F} is pure. But then, this follows from the definition of $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=k}$ given in (5.2.2) and Remark 5.4.9. \square

Remark 5.5.3. Pre-composing the weight complex functor $\mathrm{wt} : \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \rightarrow \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w})$ (see (5.1.11)) with gr , we obtain a functor $\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n) \rightarrow \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w})$. Due to weight exactness of gr , Proposition 5.5.2, any (Frobenius) weight filtration on $\mathcal{F} \in \mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ yields a weight filtration on $\mathrm{gr}(\mathcal{F})$. Hence, using $\mathrm{Chr}(\mathcal{F}) \in \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=0})$ to denote the chromatographic complex of \mathcal{F} in the sense of [WW17, §3.5], we have

$$\mathrm{gr}(\mathrm{Chr}(\mathcal{F})) \simeq \mathrm{wt}(\mathrm{gr}(\mathcal{F})),$$

see also Remark 5.1.12 for an explicit description of the weight complex functor. This is the precise sense in which the weight complex functor is compatible with the chromatographic complex construction, answering the question posed in [WW17, Rmk. 2].

Corollary 5.5.4. *The weight structure and perverse t -structure on $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is symmetric with respect to the Verdier duality functor for graded sheaves. Namely, for any integer n , Verdier duality induces an equivalence of categories*

$$\begin{aligned} \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{t \leq n} &\xrightarrow[\simeq]{\mathrm{D}_{\mathrm{Ver}}} \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{t \geq -n} \\ \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{w \leq n} &\xrightarrow[\simeq]{\mathrm{D}_{\mathrm{Ver}}} \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{w \geq -n}. \end{aligned}$$

Proposition 5.5.5. *Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a morphism in Stk_k . Then, for any $k \in \mathbb{Z}$,*

- (i) f^* (resp. $f^!$) : $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Z}) \rightarrow \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ preserves the property of having weight $\leq k$ (resp. $\geq k$). Moreover, f^* and $f^!$ are weight exact when f is smooth.
- (ii) f_* (resp. $f_!$) : $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \rightarrow \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Z})$ preserves the property of having weight $\geq k$ (resp. $\leq k$). In particular, $f_* \simeq f_!$ is weight-exact when f is proper.
- (iii) When f_* and $f_!$ are t -exact when f is an affine open embedding.
- (iv) $f^*[d]\langle d \rangle \simeq f^![-d]\langle -d \rangle : \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Z}) \rightarrow \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})$ is weight and t -exact when f is smooth of relative dimension d .

Proof. We will show that f^* is weight exact when f is smooth. The rest can be argued in a similar manner. As before, by Proposition 5.4.6, it suffices to show that $f^*\mathcal{F}$ is pure when \mathcal{F} is a pure graded perverse sheaf. But we have a complete classification of pure graded perverse sheaves given in Theorem 5.2.19. The desired result then follows from the corresponding statement about Frobenius weight under smooth pullbacks, using compatibility between functoriality of graded sheaves and mixed sheaves, Proposition 4.5.5. \square

Proposition 5.5.6. *The category $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})$ is Artinian and Noetherian. Simple graded perverse sheaves are necessarily pure and moreover, $\mathcal{F} \in \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})$ is simple if and only if $\mathrm{oblv}_{\mathrm{gr}}(\mathcal{F}) \in \mathrm{Perv}_c(\mathcal{Y})$ is simple.*

Proof. As $\text{oblv}_{\text{gr}} : \text{Shv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Shv}_c(\mathcal{Y})$ is t -exact by Proposition 5.5.1 and conservative by Lemma 4.7.16, Artinian-ness and Noetherian-ness of $\text{Perv}_{\text{gr},c}(\mathcal{Y})$ follow from the same properties of $\text{Perv}_c(\mathcal{Y})$.

Simple objects are necessarily pure since otherwise, the weight filtration will provide a non-trivial filtration.

Now, let $\mathcal{F} \in \text{Perv}_{\text{gr},c}(\mathcal{Y})$. By conservativity of oblv_{gr} , Lemma 4.7.16, \mathcal{F} is simple if $\text{oblv}_{\text{gr}}(\mathcal{F})$ is. Conversely, suppose \mathcal{F} is simple, we would like to show that $\text{oblv}_{\text{gr}}(\mathcal{F})$ is also simple. As seen above, \mathcal{F} is necessarily pure. But now, the desired statement follows from the complete description of pure graded perverse sheaves given in Theorem 5.2.19. \square

We also have a version of the decomposition theorem of [BBDG18, Sun12a] in the graded setting.

Theorem 5.5.7. *Let $\mathcal{F} \in \text{Shv}_{\text{gr},c}(\mathcal{Y})^{\heartsuit_w}$. Then*

$$(5.5.8) \quad \mathcal{F} \simeq \bigoplus_i {}^p H^i(\mathcal{F})[-i] \simeq \bigoplus_i \bigoplus_k \mathcal{G}_{ik}[-i]$$

where the direct sums are finite and \mathcal{G}_{ik} are simple objects in $\text{Perv}_{\text{gr},c}(\mathcal{Y})^{w=i}$.

Moreover, if $\mathcal{F} \simeq \text{gr}\mathcal{F}'$ for some $\mathcal{F}' \in \text{Shv}_{\text{m},c}(\mathcal{Y}_n)^{w=0}$, for some $n \in \mathbb{Z}_{>0}$. Then,

$$(5.5.9) \quad \text{oblv}_{\text{gr}}(\mathcal{F}) \simeq \bigoplus_i \bigoplus_k \text{oblv}_{\text{gr}} \mathcal{G}_{ik}[-i]$$

where \mathcal{G}_{ik} are simple perverse sheaves on \mathcal{Y} . In other words, the decomposition above is compatible with the usual decomposition theorem of [BBDG18, Sun12a].

Proof. The first equivalence of (5.5.8) is by Remark 5.4.5.(iii). The second equivalence of (5.5.8) is by the complete description of simple graded perverse sheaves given in Theorem 5.2.19. Finally, (5.5.9) is a consequence of (5.5.8) and Proposition 5.5.6. \square

Remark 5.5.10. The decomposition in Theorem 5.5.7 above remembers the weights and thus could be thought of as an enhancement of the usual decomposition theorem.

5.5.11. Intermediate extensions. Intermediate extensions are one of the main reasons that make perverse sheaves special. The same construction can be applied to graded perverse sheaves as well. Let $j : \mathcal{U} \rightarrow \mathcal{X}$ be an open embedding in Stk_k and $\mathcal{F} \in \text{Perv}_{\text{gr},c}(U)$. Then, we can form

$$j_{!*}\mathcal{F} = \text{Im}({}^p H^0(j_!\mathcal{F}) \rightarrow {}^p H^0(j_*\mathcal{F})).$$

Due to t -exactness of oblv_{gr} and gr , and the compatibility between functoriality of graded sheaves and that of (mixed) sheaves, $j_{!*}$ is also compatible with the usual intermediate extension functor for (mixed) sheaves. Namely,

$$\text{oblv}_{\text{gr}} \circ j_{!*} \simeq j_{!*} \circ \text{oblv}_{\text{gr}} \quad \text{and} \quad j_{!*} \circ \text{gr} \simeq \text{gr} \circ j_{!*}.$$

Moreover, standard properties of the intermediate extension functor such as purity preserving and simplicity preserving etc. also hold in the graded setting. Moreover, these properties can be obtained easily from corresponding results in the theory of (mixed) sheaves.

5.5.12. Simple graded perverse sheaves and graded intersection complex. Suppose \mathcal{U} is smooth and $\mathcal{L} \in \text{Perv}_{\text{gr},c}(\mathcal{U})$ is a graded local system, i.e., $\text{oblv}_{\text{gr}}(\mathcal{L})$ is a local system on \mathcal{U} , appropriately cohomologically shifted to make it perverse. Then, we also write $\text{IC}_{\text{gr}}(\mathcal{L}) := j_{!*}\mathcal{L}$ to denote the graded intersection complex.

More generally, when \mathcal{U} is locally closed in \mathcal{X} with closure $\overline{\mathcal{U}}$, and \mathcal{L} is as before, we also write $\text{IC}_{\text{gr}}(\mathcal{L}) := \bar{j}_{!*}j_{!*}\mathcal{L}$, where j and \bar{j} fit into the following diagram

$$\mathcal{U} \xrightarrow{j} \overline{\mathcal{U}} \xrightarrow{\bar{j}} \mathcal{X}.$$

Proposition 5.5.6 implies that \mathcal{L} is irreducible if and on if $\text{oblv}_{\text{gr}}(\mathcal{L})$ is. We call such an object an irreducible graded local system. More generally, Theorem 5.2.19 implies the following expected statement.

Proposition 5.5.13. *All simple perverse sheaves $\text{Perv}_{\text{gr},c}(\mathcal{X})$ are of the form $\text{IC}_{\text{gr}}(\mathcal{L})$ where $\mathcal{L} \in \text{Perv}_{\text{gr},c}(\mathcal{U})$ is an irreducible graded local system, appropriately shifted.*

5.6. Mixed geometry. As mentioned in the introduction, our theory of graded sheaves is designed to give a uniform construction of mixed categories in the sense of [BGS96]. In this subsection, we will describe how our category of graded sheaves fits into the spirit of mixed geometry in the literature.

5.6.1. For $\mathcal{Y} \in \text{Stk}_k$, we let $\text{Shv}_{\infty,c}(\mathcal{Y})$ be the smallest full DG-subcategory of $\text{Shv}_c(\mathcal{Y})$ containing the essential images under pullbacks of $\text{Shv}_{m,c}(\mathcal{Y}_m)$ for all k_m -forms \mathcal{Y}_m of \mathcal{Y} , for all m .

By Lemma 4.5.4, for all such m , the pullback functor factors through $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. Moreover, by Theorem 5.3.1, we know that $\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is generated as a DG-category by the essential image of $\text{Shv}_{m,c}(\mathcal{Y}_m)$ for all m . Thus, we obtain a functor

$$\text{oblv}_{\text{gr}} : \text{Shv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Shv}_{\infty,c}(\mathcal{Y}).$$

Proposition 5.6.2. *The functor $\text{oblv}_{\text{gr}} : \text{Shv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Shv}_{\infty,c}(\mathcal{Y})$ is essentially surjective.*

Proof. Recall that for any k_m -form \mathcal{Y}_m of \mathcal{Y} , we have a factorization

$$\text{Shv}_{m,c}(\mathcal{Y}_m) \xrightarrow{\text{gr } \mathcal{Y}_m} \text{Shv}_{\text{gr},c}(\mathcal{Y}) \xrightarrow{\text{oblv}_{\text{gr}}} \text{Shv}_{\infty,c}(\mathcal{Y}).$$

Thus, the essential image of $\text{Shv}_{m,c}(\mathcal{Y}_m)$ in $\text{Shv}_c(\mathcal{Y})$ for any m and \mathcal{Y}_m belongs to the essential image of oblv_{gr} . It thus remains to show that the essential image of oblv_{gr} is closed under finite direct sums, shifts, and cones. Only the last item needs elaboration.

For that, it suffices to show that any $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ in $\text{Shv}_{\infty,c}(\mathcal{Y})$, where $\mathcal{F} = \text{oblv}_{\text{gr}}(\mathcal{F}^{\text{gr}})$ and $\mathcal{G} = \text{oblv}_{\text{gr}}(\mathcal{G}^{\text{gr}})$ for some $\mathcal{F}^{\text{gr}}, \mathcal{G}^{\text{gr}} \in \text{Shv}_{\text{gr},c}(\mathcal{Y})$, admits a lift to $\text{Shv}_{\text{gr},c}(\mathcal{Y})$. But by Remark 4.7.18, such an α is the image of some morphism

$$\alpha_{\text{gr}} : \mathcal{F}^{\text{gr}} \rightarrow \mathcal{G}^{\text{gr}}[k], \quad \text{for some } k \in \mathbb{Z}.$$

The proof thus concludes. \square

5.6.3. Since oblv_{gr} is t -exact, Proposition 5.6.2 implies that $\text{Shv}_{\infty,c}(\mathcal{Y})$ inherits the perverse t -structure on $\text{Shv}_c(\mathcal{Y})$. Let $\text{Perv}_{\infty,c}(\mathcal{Y}) := \text{Shv}_{\infty,c}(\mathcal{Y})^{\heartsuit}$ denote the t -heart. Then, oblv_{gr} induces an essentially surjective exact functor

$$\text{oblv}_{\text{gr}} : \text{Perv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Perv}_{\infty,c}(\mathcal{Y})$$

which is faithful, by Remark 4.7.18.

Directly from the construction, we have a canonical natural equivalence

$$\text{oblv}_{\text{gr}}(\mathcal{F}[k]) \xrightarrow[\simeq]{\varepsilon} \text{oblv}_{\text{gr}}(\mathcal{F}), \quad \text{for all } \mathcal{F} \in \text{Shv}_{\text{gr}}(\mathcal{Y}), k \in \mathbb{Z}$$

We thus obtain the following result.

Corollary 5.6.4. *The functor $\text{oblv}_{\text{gr}} : \text{Perv}_{\text{gr},c}(\mathcal{Y}) \rightarrow \text{Perv}_{\infty,c}(\mathcal{Y})$ together with ε realizes $\text{Perv}_{\text{gr},c}(\mathcal{Y})$ as a grading on $\text{Perv}_{\infty,c}(\mathcal{Y})$ in the sense of [BGS96, Defn. 4.3.1.(1)].*

Proof. The only thing we have not mentioned so far is the fact that oblv_{gr} preserves semi-simplicity. In fact, a stronger statement is true: oblv_{gr} preserves simplicity, by Proposition 5.5.6. The proof concludes. \square

Similarly, we also have the following result.

Corollary 5.6.5. *$\text{Shv}_{\text{gr},c}(\mathcal{Y})$ is a mixed version of $\text{Shv}_{\infty,c}(\mathcal{Y})$ in the sense of [Rid13, Defn. 4.2].*

Remark 5.6.6. We in fact proved something stronger than the statement of Corollary 5.6.5. Namely, while [Rid13, Defn. 4.2] only requires that the essential image generates the target as a triangulated category, we do in fact proved essential surjectivity in Proposition 5.6.2.

Remark 5.6.7. The results above have natural extensions (with exactly the same proofs) where instead of $\text{Shv}_{\infty,c}(\mathcal{Y})$, we consider the smallest full DG-subcategory of $\text{Shv}_c(\mathcal{Y})$ generated by a collection of objects coming from $\text{Shv}_{m,c}(\mathcal{Y}_m)$ for some m 's. For example, we can fix a stratification and only allow objects that are constructible along the stratification.

Remark 5.6.8. Since $\mathrm{Shv}_{\infty,c}(\mathcal{Y})$ has a bounded t -structure, it is idempotent complete, by [AGH19, Cor. 2.14]. Thus, the natural functor $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \rightarrow \mathrm{Shv}_c(\mathcal{Y})$ induces an equivalence

$$\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \otimes_{\mathrm{Vect}^{\mathrm{gr},c}} \mathrm{Vect}^c \simeq \mathrm{Shv}_{\infty,c}(\mathcal{Y}),$$

which gives another characterization of $\mathrm{Shv}_{\infty,c}(\mathcal{Y})$.

5.6.9. As mentioned in the introduction, before the current paper (with the exception of [SW18, SVW18]), mixed versions are constructed explicitly as the DG-/homotopy category of bounded chain complexes in a chosen collection of “semi-simple” objects. Moreover, it can only be done when the mixed $\mathcal{H}\mathrm{om}$ complexes between those objects satisfy some purity condition. What we did above amounts to saying that a mixed version always exists, free from any extra purity condition. However, as we will see in Proposition 5.6.10 below, when a purity condition is satisfied, the weight heart $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ is classical and hence, the weight complex functor of §5.1.9 yields an equivalence of (weight) DG-categories $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y}) \simeq \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w})$. In other words, the mixed version assumes a more concrete realization as the category of chain complexes of pure objects of weight 0 in this case.

Proposition 5.6.10. *Let $\mathcal{Y} \in \mathrm{Stk}_k$ and $\mathcal{Y}_n \in \mathrm{Stk}_{k_n}$ be any k_n -form of \mathcal{Y} . Then, the following are equivalent:*

- (i) $\mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ is classical.
- (ii) For any $\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}(\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}}) \in \mathrm{Vect}$ concentrates in cohomological degree 0.
- (iii) For any $\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}^{\mathrm{gr}}(\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}}) \in \mathrm{Vect}^{\mathrm{gr}}$ is pure of weight 0, i.e. it concentrates in the diagonal or equivalently, the k -graded component concentrates in cohomological degree k .
- (iv) For any $\mathrm{IC}_1^{\mathrm{gr}}, \mathrm{IC}_2^{\mathrm{gr}} \in \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w} = \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{w=0}$ that are simple, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}^{\mathrm{gr}}(\mathrm{IC}_1^{\mathrm{gr}}, \mathrm{IC}_2^{\mathrm{gr}}) \in \mathrm{Vect}^{\mathrm{gr}}$ is pure of weight 0.
- (v) For any $\mathcal{F}_n, \mathcal{G}_n \in \mathrm{Shv}_{\mathrm{m},c}(\mathcal{Y}_n)$ that is pure of weight 0, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}}^{\mathrm{m}}(\mathcal{F}_n, \mathcal{G}_n) \in \mathrm{Shv}_{\mathrm{m}}(\mathrm{pt}_n)$ is pure of weight 0.
- (vi) For any $\mathrm{IC}_1, \mathrm{IC}_2 \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y}_n)^{w=0}$ that are simple, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}}^{\mathrm{m}}(\mathrm{IC}_1, \mathrm{IC}_2) \in \mathrm{Shv}_{\mathrm{m}}(\mathrm{pt}_n)$ is pure of weight 0.

Proof. By definition (i) \Leftrightarrow (ii). Moreover, it is easy to see that (iii) \Rightarrow (ii), (iv), and (v); and (v) \Rightarrow (vi). It remains to prove that (ii) \Rightarrow (iii), (iv) \Rightarrow (iii), and (vi) \Rightarrow (iv).

For (ii) \Rightarrow (iii), we note that if $\mathcal{G}^{\mathrm{gr}} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ then so is $\mathcal{G}^{\mathrm{gr}}\langle k \rangle[k]$ for any $k \in \mathbb{Z}$. Thus, for any $\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}} \in \mathrm{Shv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$, assuming (ii), we know that

$$\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}(\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}}\langle k \rangle[k]) \simeq \mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}^{\mathrm{gr}}(\mathcal{F}^{\mathrm{gr}}, \mathcal{G}^{\mathrm{gr}})_k[k] \in \mathrm{Vect}$$

concentrates in cohomological degree 0 for any k . (iii) thus follows.

We will now prove that (iv) \Rightarrow (iii). By the decomposition theorem for graded sheaves, Theorem 5.5.7, it suffices to prove (iii) when $\mathcal{F}^{\mathrm{gr}}$ and $\mathcal{G}^{\mathrm{gr}}$ are of the form $\mathrm{IC}_i^{\mathrm{gr}}\langle k_i \rangle[k_i]$ where $\mathrm{IC}_i^{\mathrm{gr}} \in \mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ are simple, $i \in \{1, 2\}$. But by the hypothesis (iv), $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}^{\mathrm{gr}}(\mathrm{IC}_1^{\mathrm{gr}}, \mathrm{IC}_2^{\mathrm{gr}}) \in \mathrm{Vect}^{\mathrm{gr}}$ is pure of weight 0, and hence, so is

$$\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}^{\mathrm{gr}}(\mathrm{IC}_1^{\mathrm{gr}}\langle k_1 \rangle[k_1], \mathrm{IC}_2^{\mathrm{gr}}\langle k_2 \rangle[k_2]) \simeq \mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y})^{\mathrm{ren}}}^{\mathrm{gr}}(\mathrm{IC}_1^{\mathrm{gr}}, \mathrm{IC}_2^{\mathrm{gr}})\langle k \rangle[k] \in \mathrm{Vect}^{\mathrm{gr}}, \quad k = k_2 - k_1.$$

We thus obtain (iii).

Finally, we will show that (vi) \Rightarrow (iv). Assuming (vi), observe that for any $\mathrm{IC}_1, \mathrm{IC}_2$ as in (vi), by Proposition 4.6.2, $\mathcal{H}\mathrm{om}_{\mathrm{Shv}_{\mathrm{gr}}(\mathcal{Y}_n)^{\mathrm{ren}}}^{\mathrm{gr}}(\mathrm{gr}(\mathrm{IC}_1), \mathrm{gr}(\mathrm{IC}_2)) \in \mathrm{Vect}^{\mathrm{gr}}$ is pure of weight 0. But now, by (5.2.2) and Theorem 5.2.19, any simple object in $\mathrm{Perv}_{\mathrm{gr},c}(\mathcal{Y})^{\heartsuit_w}$ is a direct summand of an object of the form $\mathrm{gr}(\mathrm{IC})$ where $\mathrm{IC} \in \mathrm{Perv}_{\mathrm{m},c}(\mathcal{Y})^{w=0}$ is simple. The proof thus concludes. \square

6. HECKE CATEGORIES

We will now apply the theory developed above to obtain a geometric realization of the DG-category of bounded chain complexes of Soergel bimodules. More precisely, fixing a reductive group G over k and subgroups $T \subset B \subset G$ where T is a maximal torus and B a Borel subgroup, the main result of this section, Theorem 6.4.1, states that we have an equivalence of monoidal categories

$$\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B) \simeq \mathrm{Shv}_{\mathrm{gr},c}(BB \times_{BG} BB) \simeq \mathrm{Ch}^b(\mathrm{SBim}_W),$$

compatible with the weight and monoidal structures on both sides, where the weight structure on $\mathrm{Ch}^b(\mathrm{SBim}_W)$ is given by the “stupid” truncation. In particular,

$$\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)^{\heartsuit_w} \simeq \mathrm{SBim}_W.$$

Here, SBim_W is the category of Soergel bimodules attached to the Coxeter system coming from G and W is the Weyl group of G . The main feature of this section is that known statements about sheaves on $B \backslash G/B$ can be readily applied to yield the desired result.

6.1. Geometric Hecke categories. We will now study the category of constructible graded sheaves on $B \backslash G/B \simeq BB \times_{BG} BB$.

6.1.1. Monoidal structure via convolution. Since $BB \times_{BG} BB$ is of the form $X \times_Y X$, it is naturally an algebra object in $\mathrm{Corr}(\mathrm{Stk}_k)$ where the multiplication map is given by the standard convolution correspondence

$$\begin{array}{ccc} BB \times_{BG} BB \times_{BG} BB & \longrightarrow & (BB \times_{BG} BB) \times (BB \times_{BG} BB) \\ \downarrow & & \\ BB \times_{BG} BB & & \end{array}$$

Applying $\mathrm{Shv}_{\mathrm{gr},!}^{\mathrm{ren},*}$ of Theorem 4.8.20, we obtain a monoidal structure on $\mathrm{Shv}_{\mathrm{gr}}(B \backslash G/B)^{\mathrm{ren}}$. More precisely, $\mathrm{Shv}_{\mathrm{gr}}(B \backslash G/B)^{\mathrm{ren}} \in \mathrm{Alg}(\mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}})$. Note that since the vertical map is proper and the horizontal map is smooth, we can, equivalently, apply $\mathrm{Shv}_{\mathrm{gr},*}^{\mathrm{ren},*}$ of (4.8.23) to obtain the same monoidal structure on $\mathrm{Shv}_{\mathrm{gr}}(B \backslash G/B)^{\mathrm{ren}}$.

For any $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Shv}_{\mathrm{gr}}(B \backslash G/B)^{\mathrm{ren}}$, we will use $\mathcal{F}_1 * \mathcal{F}_2 \in \mathrm{Shv}_{\mathrm{gr}}(B \backslash G/B)^{\mathrm{ren}}$ to denote the convolution of \mathcal{F}_1 and \mathcal{F}_2 .

6.1.2. Compatibility with weight structures. In forming the monoidal structure, we pull along smooth (and representable) morphisms and push along proper morphisms. Thus, the monoidal structure on $\mathrm{Shv}_{\mathrm{gr}}(B \backslash G/B)^{\mathrm{ren}}$ restricts to one on $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)$ which is also compatible with the weight structure. By Theorem 5.1.14, we obtain a monoidal functor

$$(6.1.3) \quad \mathrm{wt} : \mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B) \rightarrow \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)^{\heartsuit_w}),$$

compatible with the action of $\mathrm{Vect}^{\mathrm{gr},c}$, i.e., a morphism in $\mathrm{Alg}(\mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr}}})$.

6.1.4. Purity. We will now show that the weight complex functor wt of (6.1.3) is an equivalence of categories.

Proposition 6.1.5. $\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)^{\heartsuit_w}$ is classical. As a result, the weight complex functor gives an equivalence

$$\mathrm{wt} : \mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B) \xrightarrow{\sim} \mathrm{Ch}^b(\mathrm{Shv}_{\mathrm{gr},c}(B \backslash G/B)^{\heartsuit_w})$$

as objects in $\mathrm{Alg}(\mathrm{Mod}_{\mathrm{Vect}^{\mathrm{gr},c}})$, i.e., they are equivalent as monoidal categories and the equivalence is compatible with $\mathrm{Vect}^{\mathrm{gr},c}$ -actions.

Proof. The second part follows from the first by §5.1.9. Now, the hypothesis of Proposition 5.6.10.(v) is satisfied by [BY13, Lem. 3.1.3 and Lem. 3.1.5.(2)]. The proof thus concludes. \square

6.2. Soergel bimodules. We will now quickly recall the definition of the category of Soergel bimodules. We will make use of the following notation: for any $\mathcal{Y} \in \text{Stk}_k$ with the structure map $\pi : \mathcal{Y} \rightarrow \text{pt}$, we use

$$C_{\text{gr}}^*(\mathcal{Y}) := \pi_{*,\text{ren}} \overline{\mathbb{Q}}_\ell \in \text{ComAlg}(\text{Vect}^{\text{gr}})$$

to denote the *graded cohomology* of \mathcal{Y} , where $\pi_{*,\text{ren}}$ is the pushforward functor of graded sheaves. Note that this is just the usual cohomology of \mathcal{Y} , except that we turn Frobenius weights to the grading in Vect^{gr} .

6.2.1. We let $R^\Rightarrow := C_{\text{gr}}^*(BT) \simeq C_{\text{gr}}^*(BB)$. Explicitly,

$$R^\Rightarrow \simeq \text{Sym}(X^*(T) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell[-2]\langle -2 \rangle),$$

where $X^*(T)$ is the co-character lattice of T . Namely, R^\Rightarrow is a polynomial algebra generated by $\text{rank } X^*(T)$ many variables, put in graded degree 2 and cohomological degree 2.

Let sh^\Rightarrow and sh^\Leftarrow be monoidal auto-equivalences of Vect^{gr} given by²⁴

$$\text{sh}^\Rightarrow(V)_i = V_i[-i] \quad \text{and} \quad \text{sh}^\Leftarrow(V)_i = V_i[i], \quad \text{for all } V \in \text{Vect}^{\text{gr}},$$

Let $R := \text{sh}^\Leftarrow(R^\Rightarrow)$. Then, sh^\Leftarrow and sh^\Rightarrow induce equivalences of monoidal categories

$$\text{sh}^\Leftarrow : \text{BiMod}_{R^\Rightarrow}(\text{Vect}^{\text{gr}}) \xrightarrow{\sim} \text{BiMod}_R(\text{Vect}^{\text{gr}}) : \text{sh}^\Rightarrow$$

where BiMod denotes the category of bimodules.

6.2.2. Let W denote the Weyl group of G . Then, W acts on T , and hence, on R^\Rightarrow and R . The category of Soergel bimodules SBim_W is a full subcategory of $\text{BiMod}_R(\text{Vect}^{\text{gr},\heartsuit})$ (and hence, also of $\text{BiMod}_R(\text{Vect}^{\text{gr}})$) spanned by finite direct sums of graded degree shifts of direct summands of objects of the form

$$R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R$$

for any sequence of simple reflections $s_i \in W$. Equivalently, SBim_W is the smallest full subcategory of $\text{BiMod}_R(\text{Vect}^{\text{gr},\heartsuit})$ containing $R \otimes_{R^s} R$ for all simple reflections $s \in W$ and is closed under taking tensor products, graded degree shifts, finite direct sums, and direct summands.

We define $\text{SBim}_W^\Rightarrow := \text{sh}^\Rightarrow(\text{SBim}_W)$, which is a full subcategory of $\text{BiMod}_{R^\Rightarrow}(\text{Vect}^{\text{gr}})$. As above, we have mutually inverse equivalences of monoidal categories

$$\text{sh}^\Leftarrow : \text{SBim}_W^\Rightarrow \xrightarrow{\sim} \text{SBim}_W : \text{sh}^\Rightarrow.$$

Note also that this implies that $\text{SBim}_W^\Rightarrow$ is also a classical category which is closed under simultaneous shifts $[n]\langle n \rangle$ for any $n \in \mathbb{Z}$.

6.2.3. Finally, we let $\text{Ch}^b(\text{SBim}_W)$ and $\text{Ch}^b(\text{SBim}_W^\Rightarrow)$ denote the corresponding monoidal DG-categories of bounded chain complexes of Soergel bimodules. We have mutually inverse equivalences of monoidal categories

$$\text{sh}^\Leftarrow : \text{Ch}^b(\text{SBim}_W^\Rightarrow) \xrightarrow{\sim} \text{Ch}^b(\text{SBim}_W) : \text{sh}^\Rightarrow.$$

These functors are easily seen to be compatible with $\text{Vect}^{\text{gr},c}$ -actions.

6.3. $\text{Shv}_{\text{gr}}(BG)^{\text{ren}}$ as a category of modules. We will now relate the category $\text{Shv}_{\text{gr}}(BG)^{\text{ren}}$ and the category $\text{Mod}_{C_{\text{gr}}^*(BG)}(\text{Vect}^{\text{gr}})$ of graded modules over its graded cohomology $C_{\text{gr}}^*(BG)$. The material is standard and well-known. In the ungraded setting, a version of this also appeared in [WW08].

²⁴Note that the shear functors sh^\Rightarrow and sh^\Leftarrow are only monoidal rather than symmetric monoidal.

6.3.1. Identification of categories.

Proposition 6.3.2. *Let G be any connected algebraic group and $\pi : BG \rightarrow \text{pt}$ the structure map. Then, the functor of taking global sections $\pi_{*,\text{ren}}$ induces an equivalence of symmetric monoidal Vect^{gr} -module categories $\pi_{*,\text{ren}}^{\text{enh}}$ fitting into the following commutative diagram*

$$\begin{array}{ccc}
 \text{Shv}_{\text{gr}}(BG)^{\text{ren}} & & \\
 \pi_{\text{ren}}^* \uparrow \pi_{*,\text{ren}} & \swarrow \pi_{*,\text{ren}}^{\text{enh}} & \\
 \text{Vect}^{\text{gr}} & \xrightleftharpoons[\text{oblv}_{C_{\text{gr}}^*(BG)}]{C_{\text{gr}}^*(BG) \otimes -} & \text{Mod}_{C_{\text{gr}}^*(BG)}(\text{Vect}^{\text{gr}})
 \end{array}$$

Proof. We have a pair of adjoint functors

$$\pi_{\text{ren}}^* : \text{Vect}^{\text{gr}} \rightleftarrows \text{Shv}_{\text{gr}}(BG)^{\text{ren}} : \pi_{*,\text{ren}} \simeq \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(BG)^{\text{ren}}}(\overline{\mathbb{Q}}_\ell, -),$$

where π_{ren}^* is symmetric monoidal. We know that π_{ren}^* preserves compactness, and hence, $\pi_{*,\text{ren}} \simeq \mathcal{H}\text{om}_{\text{Shv}_{\text{gr}}(BG)^{\text{ren}}}(\overline{\mathbb{Q}}_\ell, -)$ is also continuous. Moreover, since G is connected, $\text{Shv}_{\text{gr}}(BG)^{\text{ren}}$ is compactly generated by the constant sheaf (along with graded degree shifts). As a result, π_{ren}^* generates the target and hence, by Barr–Beck–Lurie theorem, [Lur17a, Thm. 4.7.4.5], the functor $\pi_{*,\text{ren}}$ upgrades to an equivalence of Vect^{gr} -module categories

$$\pi_{*,\text{ren}}^{\text{enh}} : \text{Shv}_{\text{gr}}(BG)^{\text{ren}} \xrightarrow{\simeq} \text{Mod}_{\pi_{*,\text{ren}}\pi_{\text{ren}}^*}(\text{Vect}^{\text{gr}})$$

where $\pi_{*,\text{ren}}\pi_{\text{ren}}^*$ is a monad. Since all functors are strict functors of Vect^{gr} -modules, we have an equivalence of functors

$$\pi_{*,\text{ren}}\pi_{\text{ren}}^*(-) \simeq \pi_{*,\text{ren}}\pi_{\text{ren}}^*(\overline{\mathbb{Q}}_\ell) \otimes - \simeq C_{\text{gr}}^*(BG) \otimes -.$$

where the monad structure on $\pi_{*,\text{ren}}\pi_{\text{ren}}^*$ induces an algebra structure on $C_{\text{gr}}^*(BG)$. We thus get an equivalence of Vect^{gr} -module categories

$$(6.3.3) \quad \pi_{*,\text{ren}}^{\text{enh}} : \text{Shv}_{\text{gr}}(BG)^{\text{ren}} \xrightarrow{\simeq} \text{Mod}_{C_{\text{gr}}^*(BG)}(\text{Vect}^{\text{gr}}).$$

The symmetric monoidal structure on $\text{Shv}_{\text{gr}}(BG)^{\text{ren}}$ induces one on $\text{Mod}_{C_{\text{gr}}^*(BG)}$ and hence, by applying [Lur17a, Cor. 4.8.5.20], we can upgrade the algebra structure on $C_{\text{gr}}^*(BG)$ to a commutative algebra structure such that the tensor product on $\text{Mod}_{C_{\text{gr}}^*(BG)}(\text{Vect}^{\text{gr}})$ is just the relative tensor product over $C_{\text{gr}}^*(BG)$. The equivalence (6.3.3) thus upgrades to a symmetric monoidal equivalence. It remains to identify the commutative algebra structure on $C_{\text{gr}}^*(BG)$ obtained above with the usual one from cup products.

By construction, the right-lax symmetric monoidal structures on $\pi_{*,\text{ren}}$ and $\text{oblv}_{C_{\text{gr}}^*(BG)}$ are compatible under the identification (6.3.3). In particular, we have an equivalence of commutative algebras

$$\text{oblv}_{C_{\text{gr}}^*(BG)}(C_{\text{gr}}^*(BG)) \simeq \pi_{*,\text{ren}}\overline{\mathbb{Q}}_\ell.$$

But now, the commutative algebra structure on the right hand side is precisely given by the cup-product. Thus, we are done. \square

6.3.4. Functoriality. From the construction, suppose $h : G \rightarrow H$ is a homomorphism of connected algebraic groups, which induces a morphism $\tilde{h} : BG \rightarrow BH$ at the level of classifying stacks. Then, we have a morphism of objects in $\text{ComAlg}(\text{Vect}^{\text{gr}})$

$$C_{\text{gr}}^*(BH) \rightarrow C_{\text{gr}}^*(BG).$$

A similar argument to Proposition 6.3.2 above yields the following commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{gr}}(BG)^{\mathrm{ren}} & \xrightarrow{\simeq} & \mathrm{Mod}_{C_{\mathrm{gr}}^*(BG)}(\mathrm{Vect}^{\mathrm{gr}}) \\ \downarrow \tilde{h}_{*,\mathrm{ren}} & & \downarrow \mathrm{res}_{C_{\mathrm{gr}}^*(BH)}^{C_{\mathrm{gr}}^*(BG)} \\ \mathrm{Shv}_{\mathrm{gr}}(BH)^{\mathrm{ren}} & \xrightarrow{\simeq} & \mathrm{Mod}_{C_{\mathrm{gr}}^*(BH)}(\mathrm{Vect}^{\mathrm{gr}}) \end{array}$$

In other words, pushing forward is identified with restriction of scalar $\mathrm{res}_{C_{\mathrm{gr}}^*(BH)}^{C_{\mathrm{gr}}^*(BG)}$ along $C_{\mathrm{gr}}^*(BH) \rightarrow C_{\mathrm{gr}}^*(BG)$. Passing to left adjoints, we see that the pullback functor $\tilde{h}_{\mathrm{ren}}^*$ is identified with induction $-\otimes_{C_{\mathrm{gr}}^*(BH)} C_{\mathrm{gr}}^*(BG)$.

Proposition 6.3.5. *In the situation of Proposition 6.3.2, we have an equivalence of categories*

$$\mathrm{Shv}_{\mathrm{gr}}(BG \times BG)^{\mathrm{ren}} \simeq \mathrm{BiMod}_{C_{\mathrm{gr}}^*(BG)}(\mathrm{Vect}^{\mathrm{gr}}).$$

Moreover, the monoidal product on $\mathrm{BiMod}_{C_{\mathrm{gr}}^*(\mathrm{Vect}^{\mathrm{gr}})}$ is identified with the usual convolution monoidal structure on $\mathrm{Shv}_{\mathrm{gr}}(BG \times BG)^{\mathrm{ren}}$.

Proof. This follows from the discussion above. Indeed, the monoidal structure on $\mathrm{Shv}_{\mathrm{gr}}(BG \times BG)^{\mathrm{ren}}$ is given by applying (4.8.23) to the following the correspondence below, where we pull along the horizontal map and push along the vertical map

$$\begin{array}{ccc} BG \times BG \times BG & \xrightarrow{\mathrm{id} \times \Delta \times \mathrm{id}} & BG \times BG \times BG \times BG \\ \downarrow p_{13} & & \\ BG \times BG & & \end{array}$$

By the discussion above, in terms of $\mathrm{BiMod}_{C_{\mathrm{gr}}^*(BG)}(\mathrm{Vect}^{\mathrm{gr}})$, this is identified with the relative tensor $-\otimes_{C_{\mathrm{gr}}^*(BG)} -$ and the proof concludes. \square

6.4. Equivalence of categories. As promised, we will prove the following result.

Theorem 6.4.1. *We have an equivalence of monoidal categories*

$$\mathrm{Shv}_{\mathrm{gr},c}(B \setminus G/B) \simeq \mathrm{Ch}^b(\mathrm{SBim}_W^{\Rightarrow}) \simeq \mathrm{Ch}^b(\mathrm{SBim}_W).$$

Due to Proposition 6.1.5, it suffices to show that we have an equivalence $\mathrm{Shv}_{\mathrm{gr},c}(B \setminus G/B)^{\heartsuit_w} \simeq \mathrm{SBim}_W^{\Rightarrow}$. Note that, in particular, this implies that the equivalence stated in Theorem 6.4.1 is compatible with the weight structures.

The rest of this subsection is devoted to the proof of this statement.

6.4.2. Construction of functor. Consider the following commutative diagram (along with its higher analog, where we consider higher powers of $BB \times BB$ and $BB \times_{BG} BB$)

$$(6.4.3) \quad \begin{array}{ccccc} BB \times_{BG} BB \times BB \times_{BG} BB & \longleftarrow & BB \times_{BG} BB \times_{BG} BB & \longrightarrow & BB \times_{BG} BB \\ \downarrow & & \downarrow & & \downarrow \\ BB \times BB \times BB \times BB & \longleftarrow & BB \times BB \times BB & \longrightarrow & BB \times BB \end{array}$$

where the first square is Cartesian. Pushing forward, we obtain a monoidal functor

$$\mathrm{Shv}_{\mathrm{gr},c}(B \setminus G/B)^{\heartsuit_w} \hookrightarrow \mathrm{Shv}_{\mathrm{gr}}(B \setminus G/B)^{\mathrm{ren}} \rightarrow \mathrm{Shv}_{\mathrm{gr}}(BB \times BB)^{\mathrm{ren}} \simeq \mathrm{BiMod}_{R^{\Rightarrow}}(\mathrm{Vect}^{\mathrm{gr}}),$$

We will show that this functor factors through

$$(6.4.4) \quad \mathrm{Shv}_{\mathrm{gr},c}(B \setminus G/B)^{\heartsuit_w} \xrightarrow{S} \mathrm{SBim}_W^{\Rightarrow} \xrightarrow{\tilde{S}} \mathrm{BiMod}_{R^{\Rightarrow}}(\mathrm{Vect}^{\mathrm{gr}})$$

and moreover, the functor S is an equivalence.

6.4.5. *Factoring the functor.* For each $w \in W$, we let $X_w = B \backslash BwB/B$ denote the Schubert stack associated to w and \bar{X}_w its closure in $B \backslash G/B$. By the decomposition theorem for graded sheaves, Theorem 5.5.7, for any $\mathcal{F} \in \text{Shv}_{\text{gr},c}(B \backslash G/B)^{\circlearrowright}_w$, we have a finite decomposition

$$\mathcal{F} \simeq \bigoplus_{w,i} \text{IC}_{\bar{X}_w}[n_i]\langle n_i \rangle.$$

Thus, to show the factorization (6.4.4), it suffices to show that \tilde{S} sends $\text{IC}_{\bar{X}_w}$ to an object in $\text{SBim}_W^{\rightarrow}$.

When $w_1, w_2 \in W$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, the morphism $\bar{X}_{w_1} * \bar{X}_{w_2} \rightarrow \bar{X}_{w_1 w_2}$ is birational, where $\bar{X}_{w_1} * \bar{X}_{w_2} \subset BB \times_{BG} BB \times_{BG} BB$ is the closed substack corresponding to the preimage of $\bar{X}_{w_1} \times \bar{X}_{w_2} \subset BB \times_{BG} BB \times_{BG} BB$ under the top left map of (6.4.3). Thus, $\text{IC}_{\bar{X}_{w_1 w_2}}$ is a direct summand of $\text{IC}_{\bar{X}_{w_1}} * \text{IC}_{\bar{X}_{w_2}}$. Note that comparing to [BY13, Prop. 3.2.5], we do not need to invoke Frobenius-semisimplicity.

Since \tilde{S} is monoidal, the discussion above implies that it suffices to show that \tilde{S} sends $\text{IC}_{\bar{X}_s}$ to an object in $\text{SBim}_W^{\rightarrow}$ for any simple reflection $s \in W$. In this case, \bar{X}_s is smooth. Moreover, it is a classical computation that the cohomology of \bar{X}_s is simply $(R \otimes_{R^s} R)^{\rightarrow}$ which belongs to $\text{SBim}_W^{\rightarrow}$ by definition. We thus obtain the factorization (6.4.4).

Note that this discussion also implies that the functor S is essentially surjective.

6.4.6. *Fully-faithfulness.* To show that S is an equivalence of categories, it remains to show that S is fully-faithful. However, this is a consequence of [BY13, Prop. 3.1.6] and how Hom is computed in the category of graded sheaves, see Proposition 4.6.2. The proof of Theorem 6.4.1 concludes.

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