CATEGORICAL TRACE

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Part 1. On a conjecture of Gorsky–Negut–Rasmussen [GNR]

In [GNR], a conjecture relating the trace/center of the (finite) Hecke category and the (flag) Hilbert scheme of points on $\mathbb{C}^2$. In [GHW, §1.5], a slightly different conjecture was stated (but is still attributed to [GNR]) on the relation between the monoidal trace $\mathsf{Tr}(\mathsf{SBim}_n)$ and $\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))$, the category of quasi-coherent sheaves on the Hilbert scheme of points on $\mathbb{C}^2$ (everything is derived, as usual).

In [GHW], it is speculated that the two categories are related by Koszul duality. We will show that this is indeed the case.

Notation/Convention. Everything is (∞-)derived (or dg-enhanced) by default. For example, Vect is the dg-category of (unbounded) chain complexes of vector spaces over some fixed field k.

1. Motivation

We will recall necessary background in §1.1 and §1.2 to motivate the statement we would like to prove.

1.1. **Hilbert scheme of points on $\mathbb{C}^2$.** We will now recall the basic materials regarding the Hilbert scheme side.

Date: May 5, 2020.

1.1.1. *Procesi bundle.* In [H], Haiman constructed a rank $n!$ vector bundle $\mathcal{P}$ on $\mathsf{Hilb}_n(\mathbb{C}^2)$, the Procesi bundle, which is a compact generator of $\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))$. Thus, the functor

$$\begin{aligned} \mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2)) &\to \mathsf{Vect} \\ M &\mapsto \mathcal{M}\mathsf{ap}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P}, M) \end{aligned}$$

where $\mathcal{M}\mathsf{ap}_{\mathsf{QCoh}}(-,-)$ is the Vect-enriched $\mathsf{Map}_{\mathsf{QCoh}}$, induces an equivalence of categories fitting into the following commutative diagram

$$\begin{tikzcd} \mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2)) \arrow[r, "\mathcal{M}\mathsf{ap}^{\mathsf{enh}}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P},-)", "\simeq"'] \arrow[dr, "\mathcal{M}\mathsf{ap}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P},-)"'] & \mathsf{End}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P})\text{-}\mathsf{mod}(\mathsf{Vect}) \arrow[d, "\mathsf{oblv}"] \\ & \mathsf{Vect} \end{tikzcd}$$

1.1.2. It is known that

$$\mathsf{End}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P}) \simeq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes \mathbb{C}[S_n]$$

where everything is in cohomological degree 0.[1] If we let R denote the ring on the RHS, then we the discussion above yields the following equivalence of categories

$$\mathcal{M}\mathsf{ap}^{\mathsf{enh}}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P}, -) : \mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2)) \to R\text{-}\mathsf{mod}(\mathsf{Vect}).$$

1.1.3. *Gradings.* It seems that in both [GNR; GHW], an extra grading is introduced, possibly via the torus actions on the two coordinates of $\mathbb{C}^2$. In this extra grading, both x_i's and y_i's are of graded degree 2. The graded equivalence is thus

$$\mathcal{M}\mathsf{ap}^{\mathsf{enh}}_{\mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2))}(\mathcal{P}, -) : \mathsf{QCoh}(\mathsf{Hilb}_n(\mathbb{C}^2)/\mathbb{G}_m) \to R\text{-}\mathsf{mod}(\mathsf{Vect}^{\mathsf{gr}}). \tag{1.1.4}$$

Question 1.1.5. Should there be two extra gradings instead of one? Moreover, with respect to these two gradings, x_i's should live in graded degrees $(2, 0)$ and y_i's in $(0, 2)$?

1.1.6. From now on, we will forget about the Hilbert scheme of points and will only remember the RHS of the equivalence (1.1.4).

1.2. **Monoidal/horizontal trace of** SBim_n**.** In [GHW], the notions of the trace and center of a monoidal dg-category are defined by hand. In particular, it is not known if it agrees with the general notion of trace defined in Part **??**. In this part, however, we will abuse notation and still use $\mathsf{Tr}(\mathcal{C})$ to denote the (Karoubi completion of the) monoidal trace defined in [GHW]. It would be very interesting to see if the main computation of [GHW] still holds for the usual notion of trace.

[1] Maybe there should be two gradings besides the cohomological one coming from the torus action on the two coordinates where x_i's and y_i's are "orthogonal" with respect to these two gradings, eg. $(1, 0)$ and $(0, 1)$.

1.2.1. The main result regarding Soergel bimodules that is of interest to us is.

Theorem 1.2.2 ([GHW, Theorem 1.7]). *We have an equivalence of dg-categories*

$$\mathrm{Tr}(\mathrm{SBim}_n) \simeq \mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] \rtimes \mathbb{C}[S_n]\text{-mod}^{\mathrm{perf}}(\mathrm{Vect}^{\mathrm{gr}}) \tag{1.2.3}$$

where x_i's and θ_i's live in graded degrees 2, x_i's live cohomological degree 0 and θ_i's in cohomological degree -1.

1.2.4. As the degrees of the θ_i's are off by one compared to the degree of y_i's in R, the authors of [GHW] naturally speculate that the two categories in related by a Koszul duality. We will show that up to renormalization, this is indeed the case.

2. Setting up the stage

The prototype of what we are about to prove is [L, Proposition 14.6.1.1]. Let us start by briefly recalling the main objects that appear there.

2.1. Koszul duality for modules d'apres Lurie.

In this subsection, we fix a base field k and an augmented, connective (i.e. living in cohomological degrees ≤ 0) Artinian k-algebra A such that the augmentation ideal $\mathfrak{m}_A$ lives in cohomological degrees < 0.

2.1.1. $A\text{-mod}^!$. Consider the full subcategory $A\text{-mod}^{\mathrm{f.g.}}$ of $A\text{-mod}$ spanned by A-modules whose image under the forgetful functor to Vect is perfect. Let $A\text{-mod}^! = \mathrm{Ind}(A\text{-mod}^{\mathrm{f.g.}})$. Recall the full subcategory $A\text{-mod}^{\mathrm{perf}}$ of $A\text{-mod}$ consisting of perfect A-modules (equivalently, compact objects in $A\text{-mod}$). Then, we know that $A\text{-mod} \simeq \mathrm{Ind}(A\text{-mod}^{\mathrm{perf}})$.

Remark 2.1.2. When A is commutative, the $A\text{-mod}^!$ is usually called the category of ind-coherent sheaves over $\mathrm{Spec}\, A$ (see [GR]).

Clearly, we have the following fully-faithful embeddings

$$\begin{tikzcd} A\text{-mod}^{\mathrm{perf}} \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] & A\text{-mod}^{\mathrm{f.g.}} \arrow[d, hook] \arrow[dl, hook] \\ A\text{-mod} & A\text{-mod}^! \end{tikzcd}$$

Using the universal property of Ind-completion, one can show that there exists a pair of adjoint functors

$$\Xi_A : A\text{-mod} \rightleftarrows A\text{-mod}^! : \Psi_A$$

such that Ψ_A is t-exact and moreover, it induces an equivalence between the bounded below parts of the two categories [L, Proposition 14.5.4.3]

$$\Psi_A^+ : A\text{-mod}^{!,+} \xrightarrow{\simeq} A\text{-mod}^+$$

2.1.3. *Koszul duality.* Consider the following functor

$$\mathrm{triv}_A : \mathrm{Vect} \to A\text{-mod}^!$$

which is continuous and, by definition, preserves compact objects, and hence, admits a continuous right adjoint

$$\mathrm{inv}_A : A\text{-mod}^! \to \mathrm{Vect}$$

given by $\mathrm{inv}_A(-) \simeq \mathcal{M}\mathrm{ap}_{A\text{-mod}^!}(k, -)$.

It is important that we work with A-mod$^!$ and not A-mod for the right adjoint to be continuous. Note also that the image of triv generates A-mod$^!$ by design.

The following is an application of Barr–Beck–Lurie.

Proposition 2.1.4 ([L, Proposition 14.6.1.1])**.** *The pair of adjoint functors* $\mathrm{triv}_A \dashv \mathrm{inv}_A$ *induces an equivalence of categories*

$$\mathrm{inv}_A^{\mathrm{enh}} : A\text{-mod}^! \rightleftarrows A^!\text{-mod} : L$$

where $A^! = \mathcal{M}\mathrm{ap}_{A\text{-mod}^!}(k, k)$*. Moreover,* $\mathrm{inv}_A^{\mathrm{enh}}$ *and* L *exchange free and trivial objects.*

2.1.5. *Example.* Consider $A = \mathrm{Sym}\, k^n[1]$, the free cdga generated by k^n at cohomological degree -1. Then,

$$\begin{aligned} A^! &\simeq \mathcal{M}\mathrm{ap}_{A\text{-mod}^!}(k, k) \\ &\simeq \mathcal{M}\mathrm{ap}_{A\text{-mod}}(k, k) \\ &\simeq \mathcal{M}\mathrm{ap}_{\mathrm{Vect}}(k \otimes_A k, k) \\ &\simeq (\mathrm{Sym}\, k^n[2])^\vee \\ &\simeq \mathrm{Sym}\, k^n[-2]. \end{aligned}$$

where the second equivalence is due to the fact that $k \in A$-mod$^{\mathrm{f.g.}}$, which is a full-subcategory of A-mod$^!$, so we can compute hom's in this category. But now, since A-mod$^{\mathrm{f.g.}}$ is a full subcategory of A-mod, we can also compute everything we want in side A-mod instead.

Proposition 2.1.4 then gives us an equivalence

$$\mathrm{Sym}\, k^n[1]\text{-mod}^! \simeq \mathrm{Sym}\, k^n[-2]\text{-mod}.$$

2.2. **The statements.** Motivate by the above, we will now state the main result we will prove. The main idea is that instead of working in Vect, we should work in $\mathbb{C}[x_1, \dots, x_n] \rtimes \mathbb{C}[S_n]$-mod(Vect$^{\mathrm{gr}}$) since this is the part that is unchanged after Koszul duality.

Some of the objects are not defined yet (though they are not hard to guess from what we've seen above), but we will define them shortly.

Theorem 2.2.1. *We have an equivalence of categories*

$$\begin{aligned} &\mathbb{C}[x_1, \dots, x_n, \theta_1, \dots, \theta_n] \rtimes \mathbb{C}[S_n]\text{-mod}^!(\mathrm{Vect}^{\mathrm{gr}}) \\ &\quad \simeq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes \mathbb{C}[S_n]\text{-mod}(\mathrm{Vect}^{\mathrm{gr}}) \end{aligned} \tag{2.2.2}$$

where x_i's live in degrees $(2,0)$, θ_i's in degrees $(2,-1)$, and y_i's in degrees $(2,2)$. Here, the first and second numbers indicate graded and cohomological degrees respectively. This equivalence restrict to

Moreover, under this equivalence of categories, $\mathbb{C}[x_1,\dots,x_n]\rtimes\mathbb{C}[S_n]$ (with the trivial actions of θ_i's) is sent to $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]\rtimes\mathbb{C}[S_n]$ and $\mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]$ to $\mathbb{C}[x_1,\dots,x_n]\rtimes\mathbb{C}[S_n]$ (with trivial actions of y_i's).

Finally, this equivalence restricts to an equivalence between the corresponding full subcategories consisting of bounded below objects.

Question 2.2.3. Maybe on the Hilbert scheme side, there are multiple graded degrees to play with, and we can use one of them to "shear" y_i's in the theorem above back to degree $(2,0)$ while keeping x_i's at degree $(2,0)$? See also question 1.1.5.

If one can make sense of the first part of the question, it seems that once we've sheared y_i's back to cohomological degree 0, the equivalence in Theorem 2.2.1 induces an equivalence between the corresponding full-subcategories of cohomologically bounded objects.

As in the situation of §2.1, we have an equivalence of categories

$$\mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}^!(\mathrm{Vect}^{\mathrm{gr}})^+ \simeq \mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}(\mathrm{Vect}^{\mathrm{gr}})^+$$

and similarly for the bounded version. Thus, assuming that the answer to question 2.2.3 is positive, we have the following Corollary.

Corollary 2.2.4. *We have the following equivalence of categories*

$$\mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}(\mathrm{Vect}^{\mathrm{gr}})^b \simeq \mathrm{QCoh}(\mathrm{Hilb}_n(\mathbb{C}^2)/\mathbb{G}_m)^b.$$

where $\mathbb{C}[x_1,\dots,x_n]\rtimes\mathbb{C}[S_n]$ is sent to the Procesi bundle $\mathcal{P}$. In particular, we have a fully faithful embedding

$$\mathrm{Tr}(\mathrm{SBim}_n)\hookrightarrow\mathrm{QCoh}(\mathrm{Hilb}_n(\mathbb{C}^2)/\mathbb{G}_m)^b, \tag{2.2.5}$$

Question 2.2.6. Under the identification (1.2.3), $\mathrm{Tr}(1)$ corresponds to a free module and hence, via (2.2.2), it corresponds to $\mathbb{C}[x_1,\dots,x_n]\rtimes\mathbb{C}[S_n]$. In particular, y_i's act on it trivially. Does this, in any way, correspond to the condition in [GNR] regarding the flag Hilbert scheme where y acts trivially on the subquotients?

2.3. **The players.** We will now define the objects that appear in the main result and prove some elementary properties about them. The concrete definitions without using stacks are given in §2.3.3. It's ok to skip to that section.

2.3.1. *Algebro-geometric point of view.* Before starting, we note the objects of interest could be phrased in algebro-geometric terms. We will not make much use of this in the actual proof.

Consider the stack $\mathrm{B}(S_n\times\mathbb{G}_m)$ and recall that

$$\mathrm{QCoh}(B(S_n\times\mathbb{G}_m))\simeq\mathbb{C}[S_n]\text{-mod}(\mathrm{Vect}^{\mathrm{gr}}).$$

Consider $\mathbb{A}^n$, equipped with the natural permutation action of S_n and weight two action of $\mathbb{G}_m$. Then

$$\mathrm{QCoh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m))\simeq\mathbb{C}[x_1,\dots,x_n]\rtimes\mathbb{C}[S_n]\text{-mod}(\mathrm{Vect}^{\mathrm{gr}}).$$

Finally, let $\mathbb{A}^n[-1] = \operatorname{Spec}\operatorname{Sym}\mathbb{C}_2^n[1]$. Then

$$\mathsf{QCoh}((\mathbb{A}^n[-1]\times\mathbb{A}^n)/(S_n\times\mathbb{G}_m)) \simeq \mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}(\mathsf{Vect}^{\mathrm{gr}}).$$

2.3.2. *Ind-coherent sheaves.* We will now phrase the result in terms of the theory of ind-coherent sheaves developed in [G]. This is mostly for aesthetic reasons. We will recast everything in concrete terms later.

First, observe that since $\mathbb{A}^n$ is smooth,

$$\mathsf{Coh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m)) \simeq \mathsf{Perf}(\mathbb{A}^n/(S_n\times G_m))$$

and hence,

$$\begin{aligned}\mathsf{IndCoh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m)) &:= \mathsf{Ind}(\mathsf{Coh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m)))\\ &\simeq \mathsf{Ind}(\mathsf{Perf}(\mathbb{A}^n)/(S_n\times\mathbb{G}_m))\\ &\simeq \mathsf{QCoh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m)).\end{aligned}$$

Consider the following diagram

$$\begin{tikzcd} \mathbb{A}^n/(S_n\times\mathbb{G}_m) \arrow[r, "s"] \arrow[dr, equal] & (\mathbb{A}^n[-1]\times\mathbb{A}^n)/(S_n\times\mathbb{G}_m) \arrow[d, "p"] \\ & \mathbb{A}^n/(S_n\times\mathbb{G}_m) \end{tikzcd}$$

where P is the natural projection and s the 0-section.

This gives rise to the following commutative diagram

$$\begin{tikzcd} \mathsf{IndCoh}(\mathbb{A}^n/S_n\times\mathbb{G}_m) \arrow[r, shift left, "s_*"] \arrow[dr, equal] & \mathsf{IndCoh}((\mathbb{A}^n[-1]\times\mathbb{A}^n)/(S_n\times\mathbb{G}_m)) \arrow[l, shift left, "s^!"] \arrow[d, shift right, "p_*"'] \\ & \mathsf{IndCoh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m)) \arrow[u, shift right, "p^!"'] \end{tikzcd}$$

Consider

$$\mathcal{A} = \mathsf{Sym}(\mathcal{O}^n_{\mathbb{A}^n/(S_n\times\mathbb{G}_m)}(2)[-2]) \in \mathsf{ComAlg}(\mathsf{QCoh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m))).$$

Then, equivalence (2.2.2) says that the adjunction $s_* \dashv s^!$ induces the following equivalence of categories

$$\mathsf{IndCoh}((\mathbb{A}^n[-1]\times\mathbb{A}^n)/(S_n\times\mathbb{G}_m)) \simeq \mathcal{A}\text{-mod}(\mathsf{QCoh}(\mathbb{A}^n/(S_n\times\mathbb{G}_m))).$$

2.3.3. *The relevant categories in more concrete terms.* Consider

$$\mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}^{\mathrm{f.g.}}(\mathsf{Vect}^{\mathrm{gr}})$$

consisting of bounded chain complexes of modules such that all cohomology groups are finitely generated $\mathbb{C}[x_1,\dots,x_n]$-modules. Let

$$\begin{aligned}&\mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}^!(\mathsf{Vect}^{\mathrm{gr}})\\ &\qquad= \mathsf{Ind}(\mathbb{C}[x_1,\dots,x_n,\theta_1,\dots,\theta_n]\rtimes\mathbb{C}[S_n]\text{-mod}^{\mathrm{f.g.}}(\mathsf{Vect}^{\mathrm{gr}}))\end{aligned}$$

To keep the notation short, we write

$$\mathcal{C} = \mathbb{C}[x_1,\dots,x_n]\rtimes\mathbb{C}[S_n]\text{-mod}(\mathsf{Vect}^{\mathrm{gr}})$$

and,

$$\Lambda_n = \mathbb{C}[\theta_1, \ldots, \theta_n]$$

and, slightly abuse the notation,

$$\Lambda_n\text{-mod}^!(\mathcal{C}) \simeq \mathbb{C}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] \rtimes \mathbb{C}[S_n]\text{-mod}^!(\mathrm{Vect}^{\mathrm{gr}}).$$

2.3.4. We have a natural continous functor which preserves compactness

$$\mathrm{triv}_{\Lambda_n} : \mathcal{C} \to \Lambda_n\text{-mod}^!(\mathcal{C})$$

which induces a continuous right adjoint

$$\mathrm{inv}_{\Lambda_n}(-) = \mathcal{M}\mathrm{ap}_{\Lambda_n\text{-mod}^!(\mathcal{C})}(1_{\mathcal{C}}, -)$$

where $1_{\mathcal{C}} = \mathbb{C}[x_1, \ldots, x_n]$ is the monoidal unit of $\mathcal{C}$. In the algebro-geometric language above, $\mathrm{triv}_{\Lambda_n} \dashv \mathrm{inv}_{\Lambda_n}$ corresponds to $s_* \dashv s^!$. Here, $\mathcal{M}\mathrm{ap}$ denotes the $\mathcal{C}$-enriched hom.

2.3.5. Let

$$Y_n = \mathbb{C}[y_1, \ldots, y_n]$$

where y_i's are in degrees $(2, 2)$ and, again, slightly abusing the notation

$$Y_n\text{-mod}(\mathcal{C}) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes \mathbb{C}[S_n]\text{-mod}(\mathrm{Vect}^{\mathrm{gr}}).$$

Using the fact that both $\mathrm{triv}_{\Lambda_n}$ and inv_{Λ_n} are continuous, one can check that

$$\mathrm{inv}_{\Lambda_n} \circ \mathrm{triv}_{\Lambda_n} \simeq Y_n \otimes -$$

where $\otimes$ is the monoidal structure of $\mathcal{C}$. Indeed, this can be checked easily on perfect objects in $\mathcal{C}$, and this is sufficient since both sides are continuous.

2.4. **Sketch of proof.** We will now sketch the proof of the first part of Theorem 2.2.1.

2.4.1. General nonsense about monads then gives us the following diagram

$$\begin{tikzcd}
\Lambda_n\text{-mod}^!(\mathcal{C}) \arrow[d, shift right, "\mathrm{triv}_{\Lambda_n}"'] \arrow[drr, shift left, "\mathrm{inv}^{\mathrm{enh}}_{\Lambda_n}"] & & \\
\mathcal{C} \arrow[u, shift right, "\mathrm{inv}_{\Lambda_n}"'] \arrow[rr, shift left, "\mathrm{Free}_{Y_n}"] & & Y_n\text{-mod}(\mathcal{C}) \arrow[ll, shift left, "\mathrm{oblv}_{Y_n}"] \arrow[ull, shift left, "L"]
\end{tikzcd}$$

where $\mathrm{oblv}_{Y_n} \circ \mathrm{inv}^{\mathrm{enh}}_{\Lambda} \simeq \mathrm{inv}_{\Lambda_n}$ and L is the left adjoint to $\mathrm{inv}^{\mathrm{enh}}_{\Lambda}$. The right adjoints in the diagram commute, by design, and hence, so do the left adjoints.

2.4.2. We will now show that the pair of adjoint functors

$$\mathsf{inv}^{\mathsf{enh}}_{\Lambda_n} : \Lambda_n\text{-}\mathsf{mod}^!(\mathcal{C}) \rightleftarrows Y_n\text{-}\mathsf{mod}(\mathcal{C}) : L$$

are equivalences of categories.

First, we will prove that L is fully-faithful, which is equivalent to showing that the unit

$$\mathsf{id}_{Y_n\text{-}\mathsf{mod}(\mathcal{C})} \to \mathsf{inv}^{\mathsf{enh}}_{\Lambda_n} \circ L$$

is an equivalence. Since both sides are continuous and since $Y_n\text{-}\mathsf{mod}(\mathcal{C})$ is compactly generated by the image of Free_{Y_n}, it suffices to show that the natural transformation is an equivalent when evaluated on free objects. But now, we have

$$\mathsf{inv}^{\mathsf{enh}}_{\Lambda_n} \circ L \circ \mathsf{Free}_{Y_n} \simeq \mathsf{inv}^{\mathsf{enh}}_{\Lambda_n} \circ \mathsf{triv}_{\Lambda_n} \simeq \mathsf{Free}_{Y_n}.$$

It thus remains to show that $\mathsf{inv}^{\mathsf{enh}}_{\Lambda_n}$ is conservative. Since L and $\mathsf{inv}^{\mathsf{enh}}_{\Lambda_n}$ are continuous and since both categories are compactly generated, it suffices to show that $\Lambda_n\text{-}\mathsf{mod}^!(\mathcal{C})$ is generated by the image of $\mathsf{Free}_{Y_n} \circ L \simeq \mathsf{triv}_{\Lambda_n}$.

Recall that $\Lambda_n\text{-}\mathsf{mod}^!(\mathcal{C})$ is compactly generated by objects living finitely many cohomological degrees. In particular, the natural cohomological filtration (i.e. given by truncation $\tau^{\leq n}$) on any compact object has associated graded trivial objects (objects living in the image of $\mathsf{triv}_{\Lambda_n}$). Thus, we are done.

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