Some Notes on Trigonometric Sums

Quoc P. Ho
University of Chicago

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Contents

1 Trigonometric Sums

The main goal of this note is to establish certain bounds of Gauss and Kloosterman sums using étale cohomology.

1.1 Gauss Sums

Definition 1.1 (Gauss Sum)**.** *Let $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be an additive character and let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a multiplicative character, then the Gauss sum is defined to be*

$$\tau(\chi,\psi) = -\sum_{x\in\mathbb{F}_q^\times} \psi(x)\chi^{-1}(x).$$

We will prove the following two theorems.

Theorem 1.2. *Let ψ and χ as above, then*

$$|\tau(\chi,\psi)| = q^{1/2}.$$

Theorem 1.3 (Hasse-Davenport)**.** *Let $\mathbb{F}_{q^N}$ be a degree N field extension of $\mathbb{F}_q$, and denote* $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{F}_{q^N}/\mathbb{F}_q}$ *and* $N = \mathrm{N}_{\mathbb{F}_{q^N}/\mathbb{F}_q}$ *the trace and the norm resp. of this extension. Then,*

$$\tau(\chi \circ \mathrm{N}, \psi \circ \mathrm{Tr}) = -\sum_{x \in \mathbb{F}_{q^N}^{\times}} \psi(\mathrm{Tr}(x))\chi^{-1}(\mathrm{N}(x)) = (\tau(\chi,\psi))^n.$$

1.2 Kloosterman Sums

Definition 1.4 (Kloosterman Sum)**.** *Let $\psi : \mathbb{F}_q \to \mathbb{C}^{\times}$ be a non-trivial additive character and let $a \in \mathbb{F}_q$, then the Kloosterman sum is defined as*

$$K_{n,a} = \sum_{x_1 x_2 \ldots x_n = a} \psi(x_1 + x_2 + \cdots + x_n).$$

We will prove the following theorem.

Theorem 1.5. *We have the following estimates*

(i) When $a = 0$, then $K_{n,0} = (-1)^{n-1}$.

(ii) When $a \neq 0$, then $|K_{n,a}| \leq nq^{(n-1)/2}$.

Remark 1.6. These trigonometric sums are classically defined to take values in $\mathbb{C}$. However, since we hope to use étale cohomology to analyze them, we think of them as a number inside $\overline{\mathbb{Q}}_l$ by fixing an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$.

2 Geometrization

2.1 A Lemma on Torsors

We will move between different torsors of different groups. The following lemma is basic, but helpful when one thinks about such situations.

Lemma 2.1. *Let X be a scheme and G, G' be smooth group schemes over X. Let T, T' be G and G' torsors resp. over X. Let $\varphi_G : G \to G'$ be a morphism of group schemes over X and $\varphi_T : T \to T'$ a morphism of schemes over X compatible with φ_G in the obvious way. Let Y be an X-scheme, on which G' (and hence G) acts, then*

$$Y \times^G T \cong Y \times^{G'} T'.$$

Proof. Obvious: just write down the descent datum. □

Corollary 2.2. *Suppose we have the following sequence*

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/H \arrow[r] \arrow[d, "\varphi"'] & 1 \\
1 \arrow[r] & H' \arrow[r] & G' \arrow[r] & G'/H' \arrow[r] & 1
\end{tikzcd}$$

where G, H, G', H' are algebraic groups over a field k. Let X be a scheme on which H', and hence H, acts, then

$$G \times^H X \cong (\varphi^* G') \times^{H'} X$$

as X-bundles over G/H.

Proof. Note that $G \to G/H$ and $G' \to G'/H'$ are H and H'-torsors over G/H and G'/H' respectively. This is a direct consequence of the above. □

2.2 Artin-Shreier Sheaves

Goal: produce a sheaf whose local Frobenius traces are precisely the summands in the Gauss/Kloosterman sums, so we hope to use the other side of the trace formula to analyze these sums.

Definition. Let G_0 be a commutative, connected algebraic group over $\mathbb{F}_q$, where the group operation is written multiplicatively. Then *Lang isogeny* is defined to be

$$\begin{aligned} \mathscr{L} : G_0 &\to G_0 \\ g &\mapsto Fg \cdot g^{-1} \end{aligned}$$

We see easily that this map is an étale map. Thus, its image is an open subgroup of G_0. But since G_0 is connected, this is actually an étale covering. We have the following exact sequence

$$0 \longrightarrow G_0(\mathbb{F}_q) \longrightarrow G_0 \longrightarrow G_0 \longrightarrow 0.$$

Examples. Applied to the case $\mathbb{G}_a$ and $\mathbb{G}_m$, we get the following

$$0 \longrightarrow \mathbb{F}_q \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

$$0 \longrightarrow \mu_{q-1} \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0.$$

Frobenius Trace. Note that $\mathscr{L} : G_0 \to G_0$ is a smooth sheaf, and hence, we can talk about trace of the Frobenius at closed points of G_0.

Let $\gamma \in G_0(\mathbb{F}_q)$, then for any $g \in \mathscr{L}^{-1}(\gamma)$, we have $Fg = Fgg^{-1}g = \mathscr{L}(g)g = \gamma g$. Thus, the action of the geometric Frobenius on the stalk at γ is multiplication by $\gamma^{-1} : g \mapsto g\gamma^{-1}$.

Twisting. Let $f_0 : X_0 \to G_0$ be a morphism and $\chi : G_0(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l^\times$ be a character. Then one can form a $\overline{\mathbb{Q}}_l$-sheaf on X_0 by twisting $\overline{\mathbb{Q}}_l$ with the Lang torsor using the inverse action χ^{-1} on $\overline{\mathbb{Q}}_l$. This inverse is to cancel out the inverse in the Frobenius action. And thus, if we denote this sheaf $\mathscr{F}(\chi, f_0)$, then the action of the Frobenius at a point $\gamma \in G_0(\mathbb{F}_q)$ is $\chi(\gamma)$. Since χ is a character of a finite group, we see easily that $|\chi(\gamma)| = 1$ and hence, we see that all Artin-Shreier sheaves are pure of weight 0.

Functorialities. All of these are natural consequences of the lemma on torsors above.

(i) $\mathscr{F}(\chi, f_0' \cdot f_0'') = \mathscr{F}(\chi, f_0') \otimes \mathscr{F}(\chi, f_0'')$.

(ii) $\mathscr{F}(\chi' \cdot \chi'', f_0) = \mathscr{F}(\chi', f_0) \otimes \mathscr{F}(\chi'', f_0)$.

(iii) Let $u_0 : G_0 \to H_0$ be a morphism of groups, and $\chi : H(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l^\times$, then

$$\mathscr{F}(\chi, u_0 f_0) \cong \mathscr{F}(\chi u_0, f_0).$$

(iv) Let $G_0 = \prod_{i \in I} G_0^i$, $\chi = (\chi_i)_{i \in I}$, and $f_0 = (f_0^i)_{i \in I}$, then

$$\mathscr{F}(\chi, f_0) = \bigotimes_{i \in I} \mathscr{F}(\chi_i, f_0^i).$$

Base Field Extension. Let $G_1 = G_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, then we have the following morphism of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & G_0(\mathbb{F}_{q^n}) \arrow[r] \arrow[d, "\mathrm{N}"] & G_1 \arrow[r, "\mathscr{L}_{(q^n)}"] \arrow[d, "\prod_{i=1}^{n-1} F_{(q)}^i"] & G_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_0(\mathbb{F}_q) \arrow[r] & G_1 \arrow[r, "\mathscr{L}_{(q)}"] & G_1 \arrow[r] & 0
\end{tikzcd}$$

and hence, $\mathscr{L}_{(q)} = \mathrm{N}\mathscr{L}_{(q^n)}$.

Abuse of Notation. Instead of writing $\mathscr{F}(\chi, f_0)$, we write $\mathscr{F}(\chi f_0)$. The properties above allows no ambiguity. Moreover, we also write

$$\mathscr{F}(\prod_i \chi_i f_0^i) = \mathscr{F}(\chi, f_0) = \bigotimes_{i \in I} \mathscr{F}(\chi_i, f_0^i).$$

The case of $\mathbb{A}_0^1$. Recall that when ψ is non-trivial, then $\mathscr{L}(\psi)$ is a smooth sheaf on $\mathbb{A}_0^1$ of rank 1, with Swan conductor 1 at ∞. Moreover, $H^*(\mathbb{A}^1, \mathscr{F}(\psi)) = 0$. In particular, the monodromy at ∞ is totally wildly ramified.

The case of $\mathbb{G}_m$**.** In this case, $Fg \cdot g = g^{q-1}$. Thus, we get a covering of $\mathbb{G}_m$ that is tamely ramified at both 0 and ∞.

In general, for n relatively prime to p, we have the following exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{g^n} \mathbb{G}_m \longrightarrow 0.$$

For any character $\chi : \mu_n \to \overline{\mathbb{Q}}_l$, we can form the *Kummer sheaf* $\mathscr{K}_n(\chi)$ that twists $\overline{\mathbb{Q}}_l$ with the μ_n-torsor $\mathbb{G}_m \to \mathbb{G}_m$ via χ^{-1}. $\mathscr{K}_n(\chi)$ is a smooth sheaf on $\mathbb{G}_m$. When χ is non-trivial, it is tamely ramified at both 0 and ∞ (since essentially, it's just from the extension $k(t)[x]/(x^n - t)$.

Geometrization of Gauss Sums. Let χ and ψ as in the definition of Gauss sum. From what we have said above,

$$\tau(\chi,\psi) = -\sum_{x\in\mathbb{F}_q^\times} \chi^{-1}(x)\psi(x) = -\sum_{x\in\mathbb{G}_m(\mathbb{F}_q)} \mathrm{Tr}(F, \mathscr{F}(\chi^{-1}) \otimes \mathscr{F}(\psi)) = -\sum \mathrm{Tr}(F, \mathscr{F}(\chi^{-1}\psi)),$$

where $\mathscr{F}(\chi^{-1})$ is the Artin-Shreier sheaf on $\mathbb{G}_m$ associated to χ, and $\mathscr{F}(\psi)$ the restriction of the Artin-Shreier sheaf on $\mathbb{A}_0^1$ associated to ψ. Note that the last equality is due to our convention (of notation abuse).

Geometrization of Kloosterman Sums. For $a \in \mathbb{F}_q$, let $V_a \subset \mathbb{A}_0^n$ defined by $x_1 x_2 \cdots x_n = a$. Let $\sigma : \mathbb{A}_0^n \to \mathbb{A}_0^1$ defined by the sum of the coordinates. Then

$$K_{n,a} = \sum_{x\in V_a(\mathbb{F}_q)} \mathrm{Tr}(F, \mathscr{F}(\psi\sigma)).$$

2.3 Some Cohomological Results

Theorem 2.3. *Let X_0 be a smooth, connected curve over a finite field k of characteristic p, U_0 an open subscheme of X and $\mathscr{F}_0$ an ℓ-adic sheaf on U_0 such that the natural map $j_!\mathscr{F}_0 \to j_*\mathscr{F}_0$ is an isomorphism. Then, $j_!\mathscr{F}_0 \to Rj_*\mathscr{F}_0$ is also an isomorphism.*

Proof. It suffices to show that for all $x \in X - U$, $0 = (R^i j_* \mathscr{F})_x \cong H^i(\mathrm{Spec}\,\mathscr{O}_x^{\mathrm{sh}}, \mathscr{F})$. But since $\mathscr{F}_x$ is supported at the generic point of $\mathrm{Spec}\,\mathscr{O}_x^{\mathrm{sh}}$, this cohomology is just

$$H^i(I, \mathscr{F}_{\bar\eta}).$$

where I is the inertia group at x. Since $H^i(P, \mathscr{F}_{\bar\eta}) = 0$ for all $j > 0$ (since it's the cohomology of a pro p group in a pro ℓ thing–look at finite subquotients of P, since that's how cohomology of a profinite group is computed, and see that the cohomology must be both p and ℓ torsion), the spectral sequence

$$H^i(I/P, H^j(P, \mathscr{F}_{\bar\eta})) \Rightarrow H^{i+j}(I, \mathscr{F}_{\bar\eta})$$

implies that

$$H^i(I, \mathscr{F}_{\bar\eta}) \cong H^i(I/P, \mathscr{F}_{\bar\eta}^P).$$

Now, I/P is a pro-infinite cyclic group, its cohomology concentrates at degree 0 and 1: there is a resolution of 2 terms $A \to A$, like in the infinite cyclic case. Thus, $\dim H^0 = \dim H^1$. But $\dim H^0 = 0$ since $j_! \cong j_*$. Thus, so is H^1, and we are done. □

Corollary 2.4. *Let $X_0, U_0, \mathscr{F}_0$ be as above, such that $j_!\mathscr{F}_0 \cong j_*\mathscr{F}_0$. Then the natural map $H^i_c(U,\mathscr{F}) \to H^i(U,\mathscr{F})$ is an isomorphism for all i.*

Proof. From the previous theorem 2.3, we know that $j_!\mathscr{F} \cong Rj_*\mathscr{F}$. Thus,

$$R\Gamma_c(U,\mathscr{F}) = R\Gamma(X, j_!\mathscr{F}) \cong R\Gamma(X, Rj_*\mathscr{F}) \cong R\Gamma(U,\mathscr{F}).$$

□

The following theorem is a cohomological reflection of the following fact: let $\chi : G \to \overline{\mathbb{Q}}_l^\times$ be a non-trivial character of a finite group, then

$$\sum_{g\in G}\chi(g) = \sum_{g\in G}\chi(gh) = \chi(h)\sum_{g\in G}\chi(g).$$

Since χ is non-trivial, we can choose h such that $\chi(h) \neq 1$. Thus,

$$(\chi(h)-1)\sum_{g\in G}\chi(g) = 0$$

and hence

$$\sum_{g\in G}\chi(g) = 0.$$

Theorem 2.5. *Let G_0 be a connected commutative group over $\mathbb{F}_q$ and $\chi : G_0(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l^\times$ a non-trivial character. Then $H^*_c(G,\mathscr{F}(\chi)) = 0$. Hence, $H^*(G,\mathscr{F}(\chi)) = 0$, by Poincaré duality.*

Proof. Let $x \in G_0(\mathbb{F}_q)$, and denote t_x the translation by x, then we have

$$\mathscr{L} \circ t_x = \mathscr{L},$$

since $\mathscr{L}(x) = 1$. Thus, t_x is a morphism of the Lang torsor $\mathscr{L}$ of G_0. This induces a morphism on $\mathscr{F}(\chi)$ by multiplication by $\chi(g)^{-1}$, which also induces a morphism on $H^*(G,\mathscr{F}(\chi))$ by multiplication by $\chi(g)^{-1}$. Since χ is non-trivial, we can choose g such that $\chi(g) \neq 1$. Thus, if we can show that the action of $\chi(g)$ is the same as the action of $\chi(e) = 1$, then we must have $H^*(G,\mathscr{F}(\chi)) = 0$. This is the cohomological reflection of the identity above.

This is done by a homotopy argument. First, we put all these morphisms into a family:

$$\begin{tikzcd} G\times G \arrow[r, "(\mathrm{id},t_x)"] \arrow[d, "(\mathrm{id},\mathscr{L})"'] & G\times G \arrow[d, "(\mathrm{id},\mathscr{L})"] \\ G\times G \arrow[r, "(\mathrm{id},t_{\mathscr{L}(x)})"] & G\times G \end{tikzcd}$$

where x belongs to the first coordinate. The following lemma will finish the job. □

Lemma 2.6. *Let X, Y be two schemes over an algebraically closed field k, with X separated, of finite type, and Y connected. Let $\mathscr{F}$ be a sheaf over X and (ρ,ε) a family of endomorphisms of $(X,\mathscr{F})$ parametrized by Y:*

(i) $\rho : Y \times_k X \to Y \times_k X$ *is a Y-morphism.*

(ii) $\varepsilon : \rho^* \mathrm{pr}_2^* \mathscr{F} \to \mathrm{pr}_2^* \mathscr{F}$ *a morphism of sheaves.*

Suppose that ρ is proper. Then if we denote $\rho_H(y)^*$ the endomorphism of $H_c^*(X, \mathscr{F})$ induced by (ρ_y, ε_y), where $y \in Y(k)$, then $\rho_H(y)^*$ is independent of y.

Proof. By proper base change, we know that $R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathscr{F}$ is a constant sheaf on Y, whose fibers are $H^p(X, \mathscr{F})$. Now, $\rho_H(y)^*$ is the fiber of the following endomorphism

$$R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathscr{F} \xrightarrow{\rho^*} R^p \mathrm{pr}_{1!} \rho^* \mathrm{pr}_2^* \mathscr{F} \xrightarrow{\varepsilon^*} R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathscr{F}.$$

Now, note that a morphism of any locally constant sheaf is determined at a point (when the scheme is connected). □

Remark 2.7. There is an alternative proof of theorem 2.5 using the same strategy as for the Artin-Scheier sheaf over $\mathbb{A}^1$. First, note that via the equivalence of categories

$$\{\text{Smooth sheaves}\} \longleftrightarrow \{\text{representation of } \pi_1\},$$

if we have a finite connected étale Galois cover $\pi : Y \to X$, then π_* is the same as the induction $\mathrm{Ind}_{\pi_1(Y)}^{\pi_1(X)}$. Thus, applied to $\overline{\mathbb{Q}}_l$, the push forward is just the one corresponds to the regular representation of $\mathrm{Gal}(Y/X)$. Applied to $\mathscr{L} : G_0 \to G_0$ (connected, commutative group), and note that the Galois group of this is $G_0(\mathbb{F}_q)$

$$\mathscr{L}_* \overline{\mathbb{Q}}_l \cong \bigoplus_{\chi \in \widehat{G_0(\mathbb{F}_q)}} \mathscr{F}(\chi).$$

Then, the same proof as in the case of $\mathbb{A}_0^1$ carries over.

3 Estimates

3.1 A General Estimates from Weil Conjectures

As noted above, all the Artin-Shreier sheaves (and hence, also tensors, direct sums thereof) have weight 1. Thus, if $\mathscr{F}$ is just a sheaf, $H_c^i(X, \mathscr{F})$ has weight $\leq i$, by the Weil conjectures, and hence,

$$\left| \sum_{x \in X_0(\mathbb{F}_q)} \mathrm{Tr}(F, \mathscr{F}_{\bar{x}}) \right| = \left| \sum_i (-1)^i \mathrm{Tr}(F, H_c^i(X, \mathscr{F}) \right| \leq \sum_i q^{i/2} \dim H_c^i(X, \mathscr{F}).$$

But for Gauss sums and Kloosterman sums, we can get a more precise information about the dimension and the weights of the cohomology groups.

3.2 Gauss Sums

From the above, we have

$$\tau(\chi,\psi) = -\sum_{i=0}^{2}(-1)^i \mathrm{Tr}(F, H_c^i(\mathbb{G}_m, \mathscr{F}(\chi^{-1}\psi))).$$

We start with the following cohomological result.

Lemma 3.1. *Let $U_0 \subset X_0$ be an open subscheme of a projective smooth curve, and $\mathscr{F}$ an étale sheaf on U_0. Suppose $x \in X_0 - U_0$ such that $\mathscr{F}$ is totally ramified at x, then $H_c^i(U, \mathscr{F}) = 0$ for all i, except possibly at $i = 1$.*

Proof. Since X_0 is of dimension 1, we only need to worry about $i = 0, 1, 2$. We have the vanishing for $i = 0$ since U_0 is a proper open subscheme. For $i = 2$, by Poincare duality, we have

$$\dim H_c^2(U, \mathscr{F}) = \dim H^0(U, \check{\mathscr{F}}) = 0,$$

due to the fact that $\mathscr{F}$, and hence $\check{\mathscr{F}}$ is totally ramified. □

Theorem 3.2. *The cohomology of $\mathbb{G}_m$ with coefficient in $\mathscr{F}(\psi\chi^{-1})$ satisfies the following:*

(i) If χ is non-trivial, then $H_c^* \to H^*$ is an isomorphism.

(ii) $H_c^i = 0$ for $i \neq 1$ and $\dim H_c^1 = 1$.

(iii) F acts on H_c^1 via multiplication by $\tau(\chi,\psi)$.

Proof. Clearly, (iii) is a consequence of (i) and (ii). We will use corollary 2.4 to show (i). To do that, we need to show that $\mathscr{F}(\psi\chi^{-1})$ is totally ramified at 0 and ∞ when χ is non-trivial.

At 0, $\mathscr{F}(\psi)$ is unramified, and $\mathscr{F}(\chi^{-1})$ is totally tamely ramified (when χ is non-trivial). Thus, $\mathscr{F}(\psi\chi^{-1})$ is totally ramified at 0. At ∞, we know that $\mathscr{F}(\chi^{-1})$ is tamely ramified. If ψ is trivial, then $\mathscr{F}(\psi)$ is non-ramified and hence, $\mathscr{F}(\chi^{-1}\psi)$ is also totally tamely ramified. If ψ is non-trivial, then $\mathscr{F}(\psi)$ is totally wildly ramified (with Swan conductor 1). Thus, $\mathscr{F}(\chi^{-1}\psi)$ is also totally ramified at ∞. This implies that $j_!\mathscr{F} \cong j_*\mathscr{F}$ and hence, $H_c^* \to H^*$ is an isomorphism.

The vanishing of (ii) is guaranteed by lemma 3.1 above. For the second part, we recall that $\mathscr{F}(\psi\chi^{-1})$ is tamely ramified at 0 and wildly ramified at ∞ with Swan conductor 1. Therefore, the Grothendieck-Ogg-Shafarevich formula says that $\dim H_c^1 = 1$. □

Proof of 1.2. From theorem 3.2 above, we know that

$$\tau(\chi,\psi) = -\mathrm{Tr}(F, H_c^1(\mathbb{G}_m, \mathscr{F}(\psi\chi^{-1}))).$$

The isomorphism $H^i \cong H_c^i$ implies that the weight must be precisely 1, and hence, we have

$$|\tau(\chi,\psi)| = q^{1/2}.$$

□

Proof of 1.3. The Hasse-Davenport's identity is derived easily from the following

$$\begin{aligned}\tau(\chi \mathrm{N}, \psi \mathrm{Tr}) &= \mathrm{Tr}(F^n, H^1_c(\mathbb{G}_m, \mathscr{F}(\psi\chi^{-1})) \\ &= \mathrm{Tr}(F, H^1_c(\mathscr{G}_m, \mathscr{F}(\psi\chi^{-1})))^n \qquad\qquad \text{(due to 1-dimensionality)} \\ &= \tau(\chi, \psi)^n.\end{aligned}$$

□

Remark 3.3. The following theme is similar to the Kloosterman sums case:

(i) Cohomology concentrates at one middle degree. This suggests a link to perverse sheaves. (In the above, $j_!\mathscr{F}[1] = j_*\mathscr{F}[1] = j_{!*}\mathscr{F}[1]$ is perverse).

(ii) Purity follows from isomorphism $H^i_c \cong H^i$.

Remark 3.4. In the cohomology study above, we can replace $\mathscr{F}(\chi^{-1})$ by any $\mathscr{K}_n(\chi)$ as long as n is relative prime to p (to ensure tameness).

3.3 Kloosterman Sums

The following identity serves as the inspiration for our cohomological study of Kloosterman Sums:

$$\begin{aligned}K_{n,a} &= \sum_{x_1x_2\cdots x_n=a} \psi(x_1+x_2+\cdots+x_n) \\ &= \sum_{x_1}\psi(x_1) \sum_{x_2\cdots x_n=a/x_1} \psi(x_2+x_3+\cdots+x_n) \\ &= \sum_{x\in\mathbb{F}_q^\times} \psi(x)K_{n-1,a/x}.\end{aligned}$$

Recall: denote $\pi, \sigma : \mathbb{A}^n_0 \to \mathbb{A}^1_0$ defined by the product and sum of the coordinates respectively. Let ψ be an additive character, then the sheaf $\mathscr{F}(\sigma\psi)$ restricted to $V^{n-1}_a = \pi^{-1}(a)$ is the geometrization of our Kloosterman sum $K_{n,a}$.

As in the case of Gauss sums, the main estimate for $K_{n,a}$ comes from the following cohomology result.

Theorem 3.5. *The cohomology of V^{n-1}_a with coefficient in $\mathscr{F}(\psi\sigma)$ satisfies the following:*

(i) $H^i_c = 0$ for all $i \neq n-1$.

(ii) $H^*_c \cong H^*$.

(iii) For $a \neq 0$, $\dim H^{n-1}_c = n$.

(iv) For $a = 0$, $\dim H^{n-1}_c$ is canonically isomorphic to $\overline{\mathbb{Q}}_l$.

We will prove this theorem alongside with its global analogue.

Theorem 3.6. *We have the following global analog of 3.5*

(i) *The sheaf $R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ is smooth of rank n over $\mathbb{A}^1-\{0\}$.*

(ii) *The extension by* 0 *of $R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ from $\mathbb{A}^1$ to $\mathbb{P}^1$ is the same as the direct image to $\mathbb{P}^1$ of the restriction to $\mathbb{A}^1-\{0\}$.*

(iii) *At 0, the monodromy tame and unipotent, with exactly 1 Jordan block.*

(iv) *At ∞, the wild inertia acts without non-zero fixed point, and the Swan conductor is equal to 1.*

(v) *We have $R\pi_!\mathscr{F}(\psi\sigma)\cong R^i\pi_*\mathscr{F}(\psi\sigma)$.*

We will use the following notation: 3.5(n) and 3.6(n) are used to denote theorem 3.5 and respectively, 3.6 for case n. The general strategy is an induction argument (the case $n=1$ is trivial):

$$\text{Theorem 3.5(n)} \xRightarrow{(*)} \text{Theorem 3.6(n)} \xRightarrow{(**)} \text{Theorem 3.5(n+1)}.$$

One of the points of globalizing is that even though each fiber is not nice, together they form a very nice family. For instance, we can take advantage of the vanishing theorem 2.5.

Proof of (*). By the proper base change theorem, we know that $R^i\pi_!\mathscr{F}(\psi\sigma)_a\cong H^i_c(V_a,\mathscr{F}(\psi\sigma))$. We also know that $R^i\pi_!\mathscr{F}(\psi\sigma)$ is a constructible sheaf, and hence, there exists an open dense subscheme of $\mathbb{A}^1$ such that $R^i\pi_!\mathscr{F}(\psi\sigma)$ is a locally constant sheaf. From generic base change theorem, and by shrinking U if necessary, we can assume that over U, $R^i\pi_*\mathscr{F}(\psi\sigma)_a\cong H^i(V_a,\mathscr{F}(\psi\sigma))$. Thus, theorem 3.5(n) gives us

Lemma 3.7.

(i) $R^i\pi_!\mathscr{F}(\sigma\psi)=0$*, for* $i\neq n-1$.

(ii) *For $i=n-1$, the stalks of this sheaf at all points $a\neq 0$ is of constant rank n. At* 0*, it's of rank 1.*

(iii) *On an open dense subscheme U, we have $R^i\pi_!\mathscr{F}(\psi\sigma)\cong R^i\pi_*\mathscr{F}(\psi\sigma)$.*

The lemma above shows that $H^p_c(\mathbb{A}^1,R^q\pi_!\mathscr{F}(\sigma\psi))=0$, except possibly when $q=n-1$. Thus, the Leray spectral sequence for π collapse. But from theorem 2.5, we know that $H^*_c(\mathbb{A}^n,\mathscr{F}(\psi\sigma))\cong H^*(\mathbb{A}^n,\mathscr{F}(\psi\sigma))=0$. We must therefore get the vanishing of the whole 2nd page of the spectral sequence.

In particular, $H^0_c(\mathbb{A}^n,R^{n-1}\pi_!\mathscr{F}(\sigma\psi))=0$. This means that $R^{n-1}\pi_!\mathscr{F}(\sigma\psi)$ doesn't have any isolated support at a point (no punctual support). But since the rank of $R^{n-1}\mathscr{F}(\psi\sigma)$ is constant on $\mathbb{A}^1-\{0\}$, $R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ must be locally constant on $\mathbb{A}^1-\{0\}$, and this finishes (i). **This argument is very nice! Motto: use cohomology with compact support to detect punctual support on an open curve.**

Next we will show (ii). First we worry about the point 0. Let $\mathscr{G}$ be the direct image of the restriction of $R^{n-1}\pi_!\mathscr{F}(\sigma\psi)$ to $\mathbb{A}^1-\{0\}$. Then, we have the following exact sequence

$$0\longrightarrow R^{n-1}\pi_!\mathscr{F}(\psi\sigma)\longrightarrow\mathscr{G}\longrightarrow\mathscr{Q}\longrightarrow 0,$$

where $\mathcal{Q}$ has support only at 0. Note that the injectivity comes from the fact that $R^{n-1}\pi_!\mathcal{F}$ doesn't have any punctual support. This gives

$$0 = H_c^0(\mathbb{A}^1, \mathcal{G}) \longrightarrow H_c^0(\mathbb{A}^1, \mathcal{Q}) \longrightarrow H_c^1(\mathbb{A}^1, R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) = 0,$$

where the first equality is from the fact that $\mathbb{A}^1$ is an open curve, and the second equality is from what we said above. Thus, $H_c^0(\mathbb{A}^1, \mathcal{Q}) = 0$, and hence, $\mathcal{Q} = 0$, which implies $R^{n-1}\pi_!\mathcal{F}(\psi\sigma) \cong \mathcal{G}$. This concludes (ii) for the point 0: $R^{n-1}\pi_!\mathcal{F}(\psi\sigma)$ is the direct image of its restriction to $\mathbb{A}^1 - \{0\}$.

For the point ∞, let $j : \mathbb{A}^1 \to \mathbb{P}^1$ and let Δ be the mapping cone of $j_!R\pi_!\mathcal{F}(\psi\sigma) \to Rj_*R\pi_*\mathcal{F}(\psi\sigma)$. Then, from lemma 3.7, we know that the cohomology sheaves of Δ has finite support. But observe that

$$\mathbb{H}^*(\mathbb{P}^1, j_!R\pi_!\mathcal{F}(\psi\sigma)) \cong \mathbb{H}_c^*(\mathbb{A}^1, R\pi_!\mathcal{F}(\psi\sigma)) \cong H_c^*(\mathbb{A}^n, \mathcal{F}(\psi\sigma)) = 0$$

and

$$\mathbb{H}^*(\mathbb{P}^1, Rj_*R\pi_*\mathcal{F}(\psi\sigma)) \cong \mathbb{H}^*(\mathbb{A}^1, R\pi_*\mathcal{F}(\psi\sigma)) \cong H^*(\mathbb{A}^n, \mathcal{F}(\psi\sigma)) = 0.$$

This means $\mathbb{H}^*(\mathbb{P}^1, \Delta) = 0$. Using a spectral sequence for hypercohomology, we see that $H^0(\mathbb{P}^1, \mathcal{H}^*(\Delta)) = 0$, and hence, $\Delta = 0$ and we get

$$j_!R\pi_!\mathcal{F}(\psi\sigma) \cong Rj_*R\pi_*\mathcal{F}(\psi\sigma).$$

Thus, in particular, $R\pi_!\mathcal{F}(\psi\sigma) \cong R\pi_*(\psi\sigma)$ which concludes (v), and both concentrate at 1 degree, $n-1$. Therefore,

$$j_!R\pi_!\mathcal{F}(\psi\sigma) \cong Rj_*R\pi_*\mathcal{F}(\psi\sigma) \cong j_*R\pi_*\mathcal{F}(\psi\sigma) \cong j_*R\pi_!\mathcal{F}(\psi\sigma).$$

This concludes (ii).

Note that this kind of argument applies whenever the base is a curve and the top space has no cohomology.

We have seen above (at the beginning of this proof) that $H_c^*(\mathbb{A}^1, R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) = 0$, and hence, by Grothendieck-Ogg-Shafarevich formula (note that there is one dimensional stalk at 0), we have

$$\mathrm{Swan}_0(R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) + \mathrm{Swan}_\infty(R^{n-1}\pi_!\mathcal{F}(\psi\sigma)) = 1. \tag{1}$$

Thus, this sheaf is wildly ramified at exactly one point 0 or ∞.

Now, we use the following nice lemma (which is just linear algebria) to detect which one is which.

Lemma 3.8. *Let $D = \mathrm{Gal}\, K^{\mathrm{sep}}/K$ be the Galois group of a local field K, whose residue field is finite. Let I and P be the inertia, and wild inertia groups respectively. Let V be a $\overline{\mathbb{Q}}_l$-representation of D. Then*

- *(i) If $V^I = 0$ and $(V \otimes \chi)^I = 0$ for all characters χ of I that factors through I/P, then $V^P = 0$. In particular, V is wildly ramified.*

(ii) If V is tamely ramified, $\dim V^I = 1$ and $(V \otimes \chi)^{I/P} = 0$ for all χ as above. Then, the representation of I has to be unipotent with uniquely one Jordan block.

Proof. For (i), suppose $V^P \neq 0$, then V^P is a representation of D/P. We know that I acts quasi-unipotently, and thus, there is a character χ of I/P that makes the first entry of a Jordan block of $V^P \otimes \chi$ one. This means that $(V^P \otimes \chi)^{I/P} \neq 0$, and hence, $(V \otimes \chi)^I \neq 0$, which contradicts the hypothesis.

For (ii), we can argue in a similar way as above. □

Let $\mathscr{G}$ be a Kummer sheaf of rank 1 over $\mathbb{G}_m$, i.e. $\mathscr{G} = \mathscr{K}_n(\chi)$, such that $\chi \neq 1$. Using the computation in the Gauss sum section (cf. remark 3.4 as well) and Künneth formula, we get the following isomorphism

$$H_c^*(V^*, \pi^*\mathscr{G} \otimes \mathscr{F}(\psi\sigma)) \cong H^*(V^*, \pi^*\mathscr{G} \otimes \mathscr{F}(\psi\sigma)), \tag{2}$$

where $V^* = \pi^{-1}(\mathbb{G}_m) = \mathbb{G}_m^n$. Using Leray spectral sequence for π and projection formula, we get

$$\mathscr{G} \otimes R^{n-1}\pi_!\mathscr{F}(\psi\sigma) \cong \mathscr{G} \otimes R^{n-1}\pi_*\mathscr{F}(\psi\sigma). \tag{3}$$

Let Δ be the mapping cone of $i_!(\mathscr{G} \otimes R^{n-1}\pi_!\mathscr{F}(\psi\sigma)) \to Ri_*(\mathscr{G} \otimes R^{n-1}\pi_*\mathscr{F}(\psi\sigma))$, where i is the inclusion $\mathbb{G}_m \to \mathbb{P}^1$. Then, a similar argument as earlier implies that $\Delta = 0$.[1]

We can now conclude that $i_*(\mathscr{G} \otimes R^{n-1}\pi_!\mathscr{F}(\psi\sigma)) \cong i_!(\mathscr{G} \otimes R^{n-1}\pi_!\mathscr{F}(\psi\sigma))$. In particular, we have

(i) At 0 and ∞, $\mathscr{G} \otimes R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ is totally ramified.

(ii) From the lemma, we know that $R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ is wildly ramified. The equality (1) then implies that $R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ is totally wildly ramified, with Swan conductor 1.

(iii) Equality (1) then implies that $R^{n-1}\pi_!\mathscr{F}(\psi\sigma)$ is tamely ramified ($\mathrm{Swan}_0 = 0$). The lemma then implies that the action of I/P is unipotent, with a unique Jordan block.

This concludes the proof of (*). □

Proof of (**). The case where $a = 0$ is treated separately in a simple way using a spectral sequence argument. We will now deal with the case where $a \neq 0$.

We will give a cohomological reflection of the identity

$$K_{n+1,a} = \sum_{x \in \mathbb{F}_q^\times} \psi(x) K_{n,a/x}.$$

Denote $x_0, x_1, \ldots, x_n$ the coordinates of $\mathbb{A}^{n+1}$ and $V_a^n \subset \mathbb{A}^{n+1}$ defined by $x_0 x_1 \cdots x_n = a$. Let $g : \mathbb{A}^{n+1} \to \mathbb{A}^1$ defined by the projection onto the first coordinate x_0. By abuse of notation, we will write $g|_{V_a^n} = g$ as well, and note that $g|_{V_a} : V_a^n \to \mathbb{G}_m$. Let $\tau : \mathbb{G}_m \to \mathbb{G}_m$ be an involution defined by $x \mapsto ax^{-1}$, and let $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^1$ by $\pi(x_0, \ldots, x_n) = x_1 x_2 \cdots x_n$, then $g|_{V_a} = \tau\pi|_{V_a}$.

[1]The long exact sequence of hyper-cohomology has consecutive isomorphic terms due to 2. Δ has punctual support due to 3.

As suggested by the formula above, we will use the Leray spectral sequence for g. We write $\mathbb{A}^{n+1} = \mathbb{A}^1 \times \mathbb{A}^n$, and note that the Artin-Shreier sheaf of interest on V_a is $\mathscr{F}(\psi\sigma) \cong \mathscr{F}(\psi) \boxtimes \mathscr{F}(\psi\sigma')$, where σ is sum of all $n+1$ coordinates, and σ' is sum of the last n coordinates. By projection formula, we have (everything restricted to V_a^n)

$$Rg_*(\mathscr{F}(\psi) \boxtimes \mathscr{F}(\psi\sigma')) \cong \mathscr{F}(\psi) \otimes Rg_*\mathscr{F}(\psi\sigma') \cong \mathscr{F}(\psi) \otimes \tau^* R\pi_*\mathscr{F}(\psi\sigma'). \tag{4}$$

and

$$Rg_!(\mathscr{F}(\psi) \boxtimes \mathscr{F}(\psi\sigma')) \cong \mathscr{F}(\psi) \otimes \tau^* R\pi_!\mathscr{F}(\psi\sigma'). \tag{5}$$

Observe the following commutative diagram

$$\begin{tikzcd} V_a^n \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & \mathbb{G}_m^n \subset \mathbb{A}_m^n \arrow[dl, "\pi"] \\ & \mathbb{G}_m & \end{tikzcd}$$

and the sheaf $\mathscr{F}(\psi\sigma)$ on $\mathbb{A}^n$ is the same as the sheaf $\mathscr{F}(\psi\sigma')$ on V_a^n. Thus, we can use results in 3.6 here.

Using (4) and (5), we have the following spectral sequences:

$$'E_2^{pq} = H^p(\mathbb{G}_m, \mathscr{F}(\psi) \otimes \tau^* R^q\pi_*\mathscr{F}(\psi\sigma')) \Rightarrow H^{p+q}(V_a, \mathscr{F}(\psi\sigma)).$$

and

$$''E_2^{pq} = H_c^p(\mathbb{G}_m, \mathscr{F}(\psi) \otimes \tau^* R^q\pi_!\mathscr{F}(\psi\sigma')) \Rightarrow H_c^{p+q}(V_a, \mathscr{F}(\psi\sigma)).$$

From theorem 3.6, we see that $'E_2^{pq} \cong {}''E_2^{pq}$. Hence,

$$H_c^*(V_a, \mathscr{F}(\psi\sigma)) \cong H^*(V_a, \mathscr{F}(\psi\sigma)),$$

and this finishes (ii) of theorem 3.5(n+1).

By Poincaré duality and cohomological dimension of affine schemes, we get (i) for 3.5(n+1) as well.

For (iii), we first note that $R^{n-1}\pi_!\mathscr{F}(\psi\sigma')$ (all other ones vanish) is tamely ramified at 0 and wildly ramified at ∞ with Swan conductor 1, by theorem 3.6(n). Thus, $\tau^* R^{n-1}\pi_!\mathscr{F}(\psi\sigma')$ is tamely ramified at ∞ and wildly ramified at 0, with Swan conductor 1. But we know that $\mathscr{F}(\psi)$ is not ramified at 0, and wildly ramified at ∞ with Swan conductor 1. Thus, $\mathscr{F}(\psi) \otimes \tau^* R^{n-1}\pi_!\mathscr{F}(\psi\sigma')$ is totally wildly ramified at both 0 and ∞ with Swan conductor 1 and n (the rank of $\tau^* R^{n-1}\pi_*\mathscr{F}(\psi\sigma')$) respectively.

Observe that $''E_2^{pq} = 0$ unless $p = 1$ and $q = n-1$. Moreover, by Grothendieck-Ogg-Shafarevich, we know that $''E_2^{pq} = n+1$ and we are done.

Remark 3.9. We didn't really use the thing about Jordan block.

4 Reference

SGA $4\frac{1}{2}$.