1 Definition

1.1 Functor of points

Suppose we have a scheme G over $\mathbb{k}$, then a $\mathbb{k}$-point is just a map $\operatorname{Spec}\mathbb{k} \to G$. Thus we can think of the set $G(\mathbb{k})$ as $\operatorname{Hom}(\operatorname{Spec}\mathbb{k}, G)$. Generalizing this, for any scheme G we can define a functor of points

$$h_G : (\text{Affine schemes}/\mathbb{k})^{\text{op}} \to \text{Sets}, \qquad X \mapsto \operatorname{Mor}_{\mathbb{k}}(X, G)$$

and Yoneda's lemma says that the functor $G \mapsto h_G$ is fully faithful, i.e., a scheme is determined up to isomorphism by its functor of points. Now, we say that a functor

$$F : (\text{Affine schemes}/\mathbb{k})^{\text{op}} \to \text{Sets}$$

is **representable** if it is isomorphic to h_G for some scheme G.

Theorem 1.1. *Such a functor is F representable if and only if F admits an open cover by representable functors and F is a sheaf with respect to the Zariski topology on the category of schemes.*

Note 1.1. The motivation for this topology comes from gluing sheaves. We say that a functor $F : \text{Sch}^{\text{op}} \to \text{Sets}$ satisfies the sheaf property if for every scheme T and every open covering $T = \bigcup_\alpha U_\alpha$ we have an exact complex:

$$0 \to F(T) \to \prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_T U_\beta)$$

Example 1.2. The functor $X \mapsto H^0(X, \mathcal{O}_X)$ is represented by $\mathbb{A}^1$. The functor $X \mapsto H^0(X, \mathcal{O}_X)^*$ is represented by $\mathbb{G}_m$.

For a group scheme G we just ask h_G to factor, i.e.,

$$h_G : \quad \begin{tikzcd} (\text{Sch}/\mathbb{k})^{\text{op}} \arrow[rr] \arrow[dr] & & \text{Sets} \\ & \text{Grps} \arrow[ur] & \end{tikzcd}$$

so we can think of a group scheme G over $\mathbb{k}$ as a functor $(\text{Sch}/\mathbb{k})^{\text{op}} \to \text{Grps}$.

1.2 Jacobian functor

Let C be a complete nonsingular curve over $\mathbb{k}$. Recall that a Weil divisor is just a formal sum of points

$$D = \sum_{j=1}^{n} n_j P_j, \qquad \deg D = \sum_{j=1}^{n} n_j [\mathbb{k}(P_j) : \mathbb{k}]$$

and we have a correspondence between divisors and line bundles on C. We defined $\operatorname{Pic}^0(C)$ to be the group of degree 0 line bundles on C; this is not necessarily a scheme.

Let T be a connected scheme over $\mathbb{k}$, look at the fiber product

$$\pi : C \times_{\mathbb{k}} T \to T, \qquad C_t = \pi^{-1}(t)$$

and for $\mathcal{L} \in \operatorname{Pic}(C \times_{\mathbb{k}} T)$ we define $\mathcal{L}_t = \mathcal{L}\big|_{C_t}$. Then we have that the map $t \mapsto \chi(C_t, \mathcal{L}_t)$ is locally constant (this is an example of a flat family of curves). By Riemann Roch, this implies that $\deg(\mathcal{L}_t)$ is independent of $t \in T$. This degree is also invariant under base change, so we can define a functor $J : (\mathrm{Sch}/\mathbb{k})^{\mathrm{op}} \to \mathrm{Grps}$,

$$J(T) = \left\{ \mathcal{L} \in \operatorname{Pic}(C \times_{\mathbb{k}} T) \,\middle|\, \deg(\mathcal{L}_t) = 0 \forall\ t \in T \right\} \Big/ \pi^* \operatorname{Pic}(T)$$

and we can think of $h_J(T)$ as the group of degree 0 line bundles on C parametrized by T, modulo the trivial family. Notice that $J(\mathbb{k}) = \operatorname{Pic}^0(C)$.

Definition 1.3. If J is representable, then we call the representative scheme $\operatorname{Jac}(C)$.

1.3 Obstruction to representability

Suppose J is representable by a group scheme $\operatorname{Jac}(C)$, and let $K/\mathbb{k}$ be a Galois extension with group Γ. Then

$$J(K) = \operatorname{Mor}_{\mathbb{k}}(\operatorname{Spec} K, \operatorname{Jac}(C)) \simeq \operatorname{Mor}_K(\operatorname{Spec} K, \operatorname{Jac}(C) \times_{\mathbb{k}} K)$$

Note 1.2. Let's convince myself of the affine case, i.e., to show $\operatorname{Hom}_{\mathbb{k}}(A, K) = \operatorname{Hom}_K(A \otimes_{\mathbb{k}} K, K)$. This comes from the fact that tensor product is a pushout, i.e., we have a diagram

$$\begin{tikzcd}
K & & \\
& A \otimes_{\mathbb{k}} K \arrow[ul, dashed] & K \arrow[l] \arrow[ull, bend right, "\mathrm{Id}"'] \\
& A \arrow[u] \arrow[uul, bend left] & \mathbb{k} \arrow[l] \arrow[u]
\end{tikzcd}$$

Here (on K) we have a Galois action by Γ. Since

$$\operatorname{Mor}_K(\operatorname{Spec} K, \operatorname{Jac}(C) \times_{\mathbb{k}} K)^{\Gamma} \simeq \operatorname{Mor}_{\mathbb{k}}(\operatorname{Spec} \mathbb{k}, \operatorname{Jac}(C))$$

Note 1.3. Once again, easy to prove for affine case. The Galois action on $\operatorname{Spec} A \times_k K \simeq \operatorname{Spec}(A \otimes_{\mathbb{k}} K)$ is just $1 \otimes \sigma$ for $\sigma \in \Gamma$.

we have that $J(K)^{\Gamma} = J(\mathbb{k})$. In other words, we would expect

$$\operatorname{Pic}^0(C \times_{\mathbb{k}} K)^{\Gamma} = \operatorname{Pic}^0(C)$$

but this is not true in general. In fact, we can measure the failure of this equality by an exact sequence

$$0 \to \operatorname{Pic}(C) \to \operatorname{Pic}(C \times_{\mathbb{k}} K)^{\Gamma} \to \operatorname{Br}(\mathbb{k})$$

where $\operatorname{Br}(\mathbb{k})$ is the Brauer group of $\mathbb{k}$.

Example 1.4. Consider $C = V(x^2+y^2+z^2) \in \mathbb{P}^2_{\mathbb{R}}$, which is empty. Now, $C \times_{\mathbb{R}} \mathbb{C}$ is a conice in $\mathbb{P}^2_{\mathbb{C}}$, hence isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. If $\operatorname{Pic}(C) = \operatorname{Pic}(C \times_{\mathbb{R}} \mathbb{C})^{\mathbb{Z}/2\mathbb{Z}}$ then $\operatorname{Pic}(C)$ is a subgroup of index at most 2 in $\operatorname{Pic}(C \times_{\mathbb{R}} \mathbb{C}) = \mathbb{Z}$, but this is impossible.

The possible issue here is that a line bundle $\mathcal{L} \in \mathrm{Pic}(C \times_{\Bbbk} K)^{\Gamma}$ has too many automorphisms (coming from Γ), and they have to satisfy some compatible conditions for $\mathcal{L}$ to descend to $\mathrm{Pic}(C)$. Fortunately, if $C(\Bbbk)$ is nonempty then everything works.

Theorem 1.5. *Suppose C has a $\Bbbk$-point. Then the functor J can be represented by a group scheme $\mathrm{Jac}(C)$, called the Jacobian variety of C.*

The idea here is that if we include the $\Bbbk-$point in our data, then we kill all the automorphisms. The forgetful functor getting rid of the extra data is actually an isomorphism, so we are good.

Example 1.6. $\mathrm{Jac}(\mathbb{P}^1) = \mathrm{Spec}\,\Bbbk$, since there is no nontrivial divisor of degree 0 ($\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, two points are linearly equivalent). The Jacobian of an elliptic curve is isomorphic to the elliptic curve itself.

Example 1.7. Let C be a projective curve over $\mathbb{F}_p$, and p a $\mathbb{F}_p-$point of C. Then $C\backslash\{p\}$ is affine, and the class group of its coordinate ring is $J(\mathbb{F}_p)$. The reason is that $\mathrm{Pic}(C\backslash\{p\}) = \mathrm{Pic}^0(C)$ by mapping $D \mapsto D - \deg D \cdot p$.

2 Properties and applications

Clearly, $J = \mathrm{Jac}(C)$ is a nonsingular abelian variety.

Proposition 2.1. *The tangent space T_0J is canonically isomorphic to $H^1(C, \mathcal{O}_C)$. Thus the dimension of J is equal to the genus of C.*

Definition 2.2. For each point $p \in C(\Bbbk)$ we can define a map $f_p : C \to \mathrm{Jac}(C)$ such that at the level of $\Bbbk-$points,

$$f_p : C(\Bbbk) \to \mathrm{Jac}(C)(\Bbbk) = \mathrm{Pic}^0(C), \quad x \mapsto [x - p]$$

Proposition 2.3. *The map $f_p^* : H^0(J, \Omega_J) \to H^0(C, \Omega_C)$ is an isomorphism.*

Proof. Essentially we need to show that the following diagram commutes

$$\begin{tikzcd}
H^0(J,\Omega_J) \arrow[r, "f_p^*"] \arrow[d, "\simeq"'] & H^0(C,\Omega_C) \\
(T_0J)^\vee \arrow[r, "\simeq"] & H^1(C,\mathcal{O}_C)^\vee \arrow[u, "\simeq"']
\end{tikzcd}$$

□

Note 2.1. What is this map $H^0(J, \Omega_J) \simeq (T_0J)^\vee$? It's just evaluating the 1-form at 0; the idea is that a group variety is homogeneous, so a vector X_0 in T_0J extends uniquely to a vector field X hence we get an isomorphism.

$$H^0(J, \Omega_J) \ni \omega \mapsto (X_0 \mapsto \omega_0(X_0))$$

Proposition 2.4. *The map f_p is a closed embedding.*

Proof. Field extensions are faithfully flat, so it suffices prove this for the case $\Bbbk = \overline{\Bbbk}$. Then we just need to show that the map separates points and tangents. For points, suppose $f_p(x) = f_p(y)$ then $[x - p] = [y - p]$ which implies x, y are linearly equivalent, but this is impossible on a curve of genus > 0. □

Now consider the map:

$$f_p^r : C^r \to J, \quad (p_1, ..., p_r) \mapsto [p_1 + ... + p_r - r \cdot p]$$

which descends to a map $f_p^{(r)} : C^{(r)} \to J$. The image $W^r = f_p^{(r)}(C^{(r)})$ is a closed subvariety of J, and thus $W^g = J$.

Note 2.2. Abel's theorem says that fibers of $f_p^{(r)}$ correspond to linear equivalence classes of effective divisors of degree r.

Theorem 2.5. *For all $r \leq g$, the map $f_p^{(r)} : C^{(r)} \to W^r$ is birational. In particular, J is the unique abelian variety birational to $C^{(g)}$.*

Example 2.6. Consider a curve C of genus 2. We have a double cover (by the canonical divisor) $\pi : C \to \mathbb{P}^1$ branched at 6 points. Each fiber $\pi^{-1}(x) = \{p, q\}$ (not necessarily distinct) defines a degree 2 divisor $p + q$. Since any 2 points on $\mathbb{P}^1$ are linearly equivalent, all these degree 2 divisors are linearly equivalent and get mapped to the same point by $f^{(2)}$.

So we have a family of degree 2 divisors (which is itself a divisor in $C^{(2)}$) which gets contracted in $J(C)$. In other words, $f^{(2)}$ is a blow down here.

Now let $\Theta = W^{g-1}$ then this is a divisor in J. This does depend on the chosen point p, but only up to translation. Such a divisor induces a map:

$$\phi_{\mathcal{L}(\Theta)} : J \to J^\vee, \quad x \mapsto \left[t_x^* \mathcal{L}(\Theta) \otimes \mathcal{L}(\Theta)^{-1}\right]$$

which is an isomorphism in this case. Hence (A, Θ) is a principally polarized abelian variety.

Theorem 2.7 (Torelli)**.** *C is determined, up to isomorphism, by its principally polarized Jacobian variety.*