Laboratory 1

Integral Riemann

Variant №7

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<u>Exercises:</u>

Make an integral sum for the Riemann integral of a given function over a given interval. Calculate the integral through the limit of integral sums. **Prove** that the corresponding integral exists. **Check** using the Newton–Leibniz formula. Write a program (any language) that **calculates** and **draws** integral sums for a given function on a given segment. Input data for the program: the number of split points, the method of choosing the base (left, right, middle). **Find** the estimation error, compare it with the theoretical error (**derive** the formulas using the Taylor formula with the remainder in the form of Lagrange). The partition is uniform.

<u>The task report should contain:</u>

1. Analytical part: proof of the existence of the Riemann integral; obtaining an integral sum (for one case of equipment); finding its limit; comparison with the value of the integral found by the Newton—Leibniz formula.

2. Screenshots of the results of the program with comments. There should be several graphs of integral sum terms (step figures) for various partitions (n = 10, 100, for example) and various equipment (4 graphs are enough). The value of the corresponding integral sum must be specified for each graph.

3. Link to the repository with the program (for example, github).

4. Derivation of the formula of the theoretical error of the integral and its approximation.

1. Problem statement

Make an integral sum for the Riemann integral of a given function over a given interval. Calculate the integral through the limit of integral sums. **Prove** that the corresponding integral exists. **Check** using the Newton–Leibniz formula. Write a program (any language) that **calculates** and **draws** integral sums for a given function on a given segment. Input data for the program: the number of split points, the method of choosing the base (left, right, middle). **Find** the estimation error, compare it with the theoretical error (**derive** the formulas using the Taylor formula with the remainder in the form of Lagrange). The partition is uniform.

Given function:

$$f(x) = 3^x, [1,2]$$

2. Analysis part

* Function: $f(x) = 3^x$, $[1,2]$

We prove that the existence of Riemann integral for our function on this given segment

Let: $\begin{cases} x_k = a + (\Delta x) i \\ \Delta x = \frac{b-a}{n} = \frac{1}{n} \\ x_k = 1 + \frac{1}{n} \cdot i = 1 + \frac{i}{n} \end{cases}$

$$\to \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} 3^{1+\frac{i}{n}} \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{3}{n} \cdot 3^{i/n} = \frac{3}{n} \sum_{i=1}^{n} 3^{i/n}$$

$$S_n = \frac{u_1(q^n - 1)}{q-1} = \frac{3^{1/n}\left((3^{1/n})^n - 1\right)}{3^{1/n} - 1} = \frac{3^{1/n} \cdot 2}{3^{1/n} - 1}$$

So we have: $\displaystyle\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \frac{6 \cdot 3^{1/n}}{n(3^{1/n} - 1)}$ (1)

Now we consider $\lim_{t\to\infty} t\,(3^{1/t}-1) = I$

$$I = \lim_{t\to\infty} \frac{-1+3^{1/t}}{1/t} \xmapsto[\text{rule}]{\text{l'Hospital}} \lim_{t\to\infty} \frac{3^{1/t}.\ln 3.\frac{-1}{t^2}}{-1/t^2}$$

$$= \lim_{t\to\infty} 3^{1/t}.\ln 3 = 3^0.\ln 3 = \ln 3 \neq 0 \quad (2)$$

From (1) and (2) we can use the property

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \quad \text{if } \lim g(x) \neq 0$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_{n\to\infty} \frac{6.\,3^{1/n}}{n(3^{1/n}-1)} = \frac{\lim_{n\to\infty} 6.3^{1/n}}{\lim_{n\to\infty} n(3^{1/n}-1)}$$

$$= \frac{6.\,3^0}{\ln 3} = \frac{6}{\ln 3}$$

$\rightarrow$ conclusion: $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \boxed{\frac{6}{\ln 3}}$

*, Now we will find the same integral using Newton-Leibniz formula:

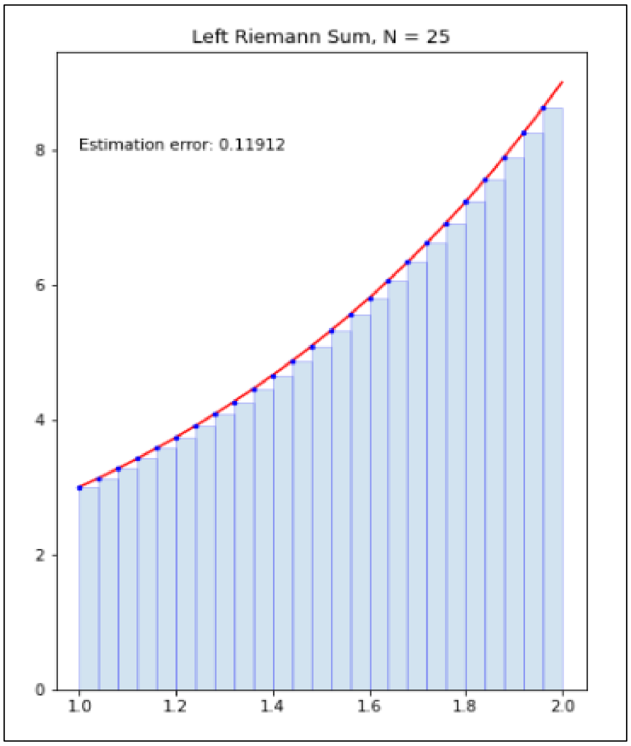
$$\int_1^2 3^x\,dx = \frac{3^x}{\ln 3}\Big|_1^2 = \frac{9}{\ln 3} - \frac{3}{\ln 3} = \boxed{\frac{6}{\ln 3}}$$

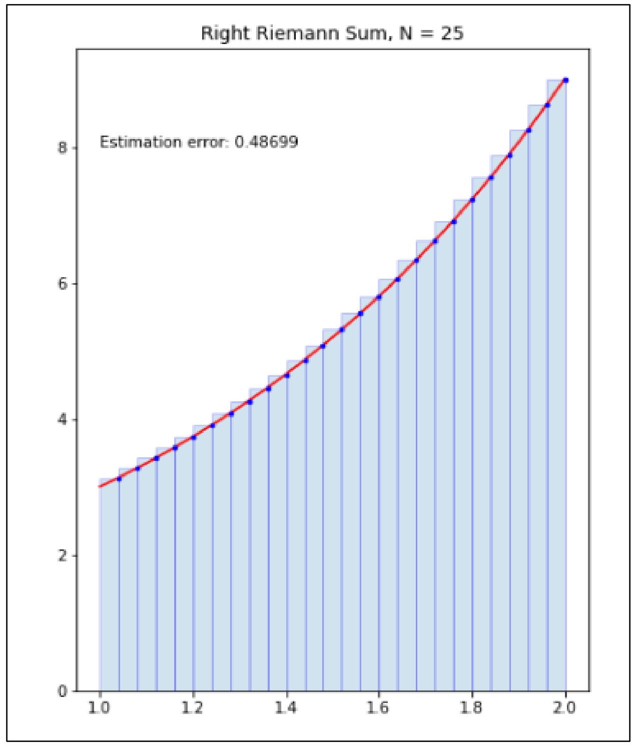
the two values match. Therefore the limit of the integral sum was calculated correctly.

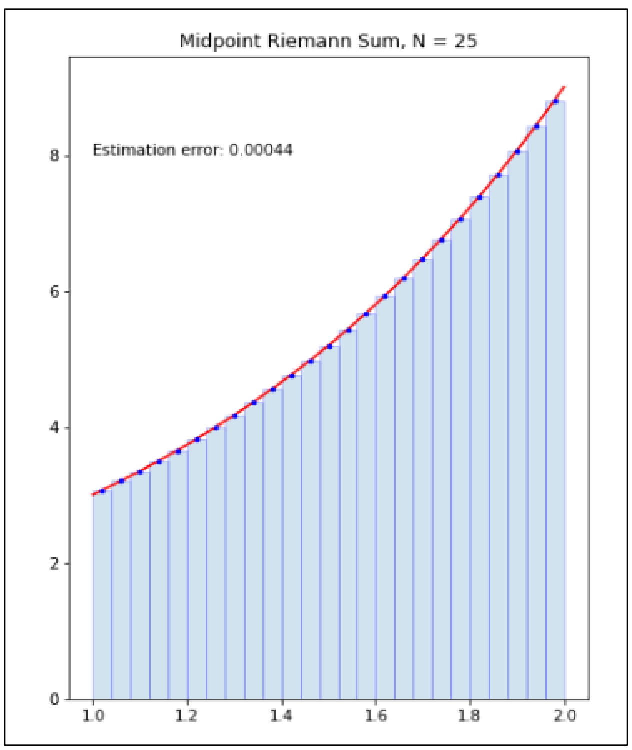
3. Graphics obtained as a result of the program

Link on github: https://github.com/quocanh34/ITMO-Mathematics-Laboratory

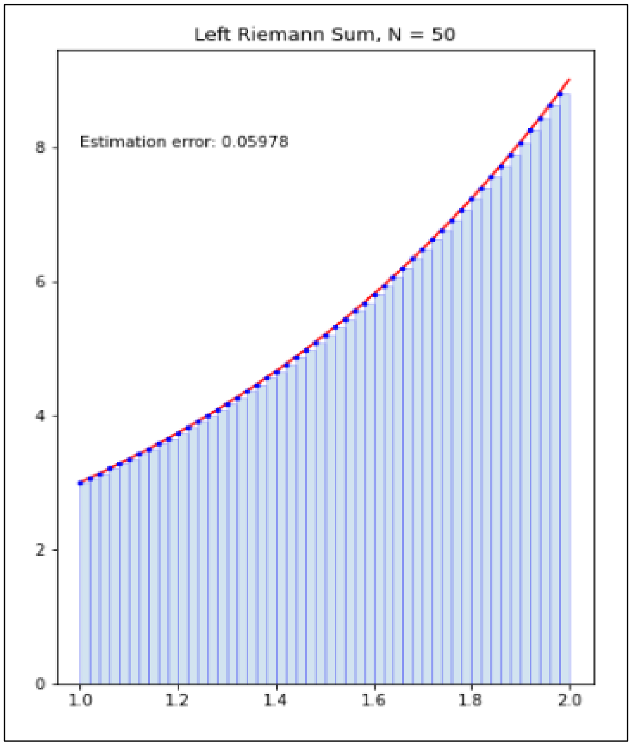
N = 25



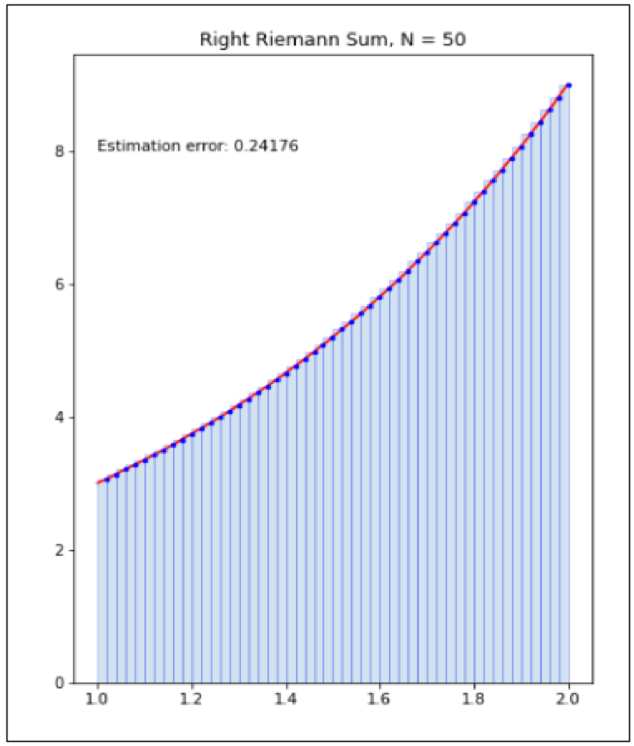




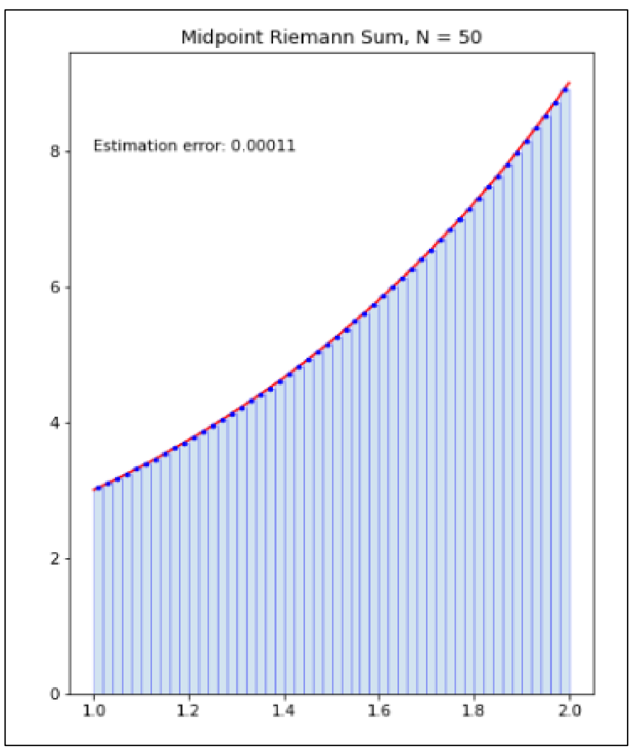
N = 50



Left Riemann Sum, N = 50
Estimation error: 0.05978
0
2
4
6
8
1.0
1.2
1.4
1.6
1.8
2.0



Right Riemann Sum, N = 50
Estimation error: 0.24176
0
2
4
6
8
1.0
1.2
1.4
1.6
1.8
2.0



Midpoint Riemann Sum, N = 50
Estimation error: 0.00011
0
2
4
6
8
1.0
1.2
1.4
1.6
1.8
2.0

4. Error calculation

* For more clarity, we consider interval [1,2] in an uniform partition consisting of n segments

$$a = x_0 < x_1 < \dots < x_n < x_{n+1} = b$$

+) The length of each segment: $l_i = \frac{l}{n}$ (l: total length)

The estimation error is the difference in modulus between the resulting value and the real one.

+) The notation R_i as an error on the i segment $[x_i, x_{i+1}]$

+) The notation I_i as the real value

+) The notation S_i as the resulting value

Therefore $R_i = |I_i - S_i|$ $\begin{cases} I_i = \int_{x_i}^{x_{i+1}} f(x)\,dx \\ S_i = \frac{l}{n} \cdot f(\xi_i) \end{cases}$

(ξ_i: the selected point)

We can write $\frac{l}{n}$ as $\int_{x_i}^{x_{i+1}} dx$

$$\rightarrow R_i = \left| \int_{x_i}^{x_{i+1}} f(x)\,dx - \int_{x_i}^{x_{i+1}} f(\xi_i)\,dx \right| = R_i = \left| \int_{x_i}^{x_{i+1}} \left[f(x) - f(\xi_i) \right] dx \right|$$

Using Taylor's theorem (with Lagrange Remainder) for $n=0$:

$$f(x) = f(\xi_i) + f'(c_i)(x-\xi_i), \text{ where } c_i \in (\xi_i, x)$$

$$\rightarrow R_i = \left| \int_{x_i}^{x_{i+1}} f'(c_i)(x-\xi_i)\,dx \right| = f'(c_i)\frac{(x-\xi_i)^2}{2}\Bigg|_{x_i}^{x_{i+1}}$$

$$R_i = \left| f'(c_i) \cdot \frac{1}{2}\left[(x_{i+1}-\xi_i)^2 - (x_i-\xi_i)^2\right] \right|$$

$$R_i = \left| f'(c_i) \cdot \frac{(x_{i+1}-x_i)(x_{i+1}+x_i-2\xi_i)}{2} \right|$$

We have $R = \sum_{i=0}^{n} R_i$

$$\rightarrow R \le \max|f'(c_i)| \cdot \sum_{i=0}^{n} \frac{(x_{i+1}-x_i)(x_{i+1}+x_i-2\xi_i)}{2}$$

*, For left Rieman sum ($\xi_i = x_i$):

$$\rightarrow |R| \le \max|f'(c_i)| \sum_{i=0}^{n} \frac{(x_{i+1}-x_i)^2}{2} = \max|f'(c_i)| \cdot \frac{(b-a)^2}{2n}$$

$$\rightarrow \boxed{|R| \le \frac{9}{2n \cdot \ln 3}}$$

*, For right Riemann sum ($\xi_i = x_{i+1}$)

Similarly $\boxed{|R| \le \frac{9}{2n\ln 3}}$

*, For mid-point Riemann sum ($\xi_i = \frac{x_{i+1}+x_i}{2}$)

Let use Taylor's theorem (with Lagrange Remainder) for $n=1$:

$$f(x) = f(\xi_i) + f'(\xi_i)(x-\xi_i) + \frac{f''(c_i)(x-\xi_i)^2}{2}, \quad c_i \in (\xi_i, x)$$

$$\rightarrow f(x) - f(\xi_i) = f'(\xi_i)(x-\xi_i) + \frac{f''(c_i)(x-\xi_i)^2}{2}$$

$$\rightarrow R_i = \left| \int_{x_i}^{x_{i+1}} \left[f(x) - f(\xi_i) \right] dx \right| = \left| f'(\xi_i) \frac{(x-\xi_i)^2}{2} \Big|_{x_i}^{x_{i+1}} + f''(c_i) \frac{(x-\xi_i)^3}{6} \Big|_{x_i}^{x_{i+1}} \right|$$

$$= \left| \frac{f''(c_i)}{6} \right| \left| \left(\frac{x_{i+1} - x_i}{2} \right)^3 + \left(\frac{x_{i+1} - x_i}{2} \right)^3 \right| = \left| f''(c_i) \frac{(x_{i+1} - x_i)^3}{24} \right|$$

Therefore $|R| = \sum_{i=0}^{n} R_i \leq \max |f''(c_i)| \cdot \frac{(b-a)^3}{24n^2} =$

$$\rightarrow \boxed{|R| \leq \frac{9}{24 \cdot \ln^2 3 \cdot n^2}}$$

<u>Conclusion:</u> For self-checking, we can guarantee that they satisfy the estimate obtained from the program.