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Probability and Statistics

Testing Simple Hypotheses

Outline I

1. Introduction
2. Tests About a Population Mean
3. Other distribution - Large Sample Size
4. Summary

1 Introduction

Test of hypotheses I

- A statistical hypothesis, or just hypothesis, is a claim or assertion either about the value of a single parameter, about the values of several parameters, or about the form of an entire probability distribution.

$$H_0 : \mu = 50 \text{ cm/s} \quad \text{and} \quad H_1 : \mu \neq 50 \text{ cm/s}$$

  - The statement $H_0 : \mu = 50$ is called the null hypothesis.
  - The statement $H_1 : \mu \neq 50$ is called the alternative hypothesis.
- The null hypothesis, denoted by H_0, is the claim that is initially assumed to be true.
- The alternative hypothesis, denoted by H_1, is the assertion that is contradictory to H_0.
- A test of hypotheses is a method for using sample data to decide whether the null hypothesis should be rejected.

Test of hypotheses II

- Two-sided Alternative Hypotheses

$$H_0 : \mu = 50\,\text{cm/s} \quad \text{and} \quad H_1 : \mu \neq 50\,\text{cm/s}$$

- One-sided Alternative Hypotheses

$$H_0 : \mu \leq 50\,\text{cm/s} \quad \text{and} \quad H_1 : \mu > 50\,\text{cm/s}$$

or

$$H_0 : \mu \geq 50\,\text{cm/s} \quad \text{and} \quad H_1 : \mu < 50\,\text{cm/s}$$

Test of a Hypothesis

- A procedure leading to a decision about a particular hypothesis
- The two possible conclusions: reject H_0 or fail to reject H_0.

Test procedure

- A test statistic, a function of the sample data on which the decision (reject H_0 or do not reject H_0) is to be based.
- A rejection region, the set of all test statistic values for which H_0 will be rejected.

H_0 is rejected $\iff$ the test statistic $\in$ the rejection region.

Definition

The P-value is the smallest significance level α at which the null hypothesis can be rejected.

Error

Decision	H_0 iS True	H_0 iS False
Fail to reject H_0	No error	Type II error
Reject H_0	Type I error	No error

Definition

- *A type I error consists of rejecting H_0 when it is true.*
- *A type II error involves not rejecting H_0 when H_0 is false.*

Error

Definition

- *The probability of a type I error is called the significance level of the test and is usually denoted by* α.

$$\alpha = P(\text{ type I error }) = P(H_0 \text{ is rejected when it is true})$$

- *The probability that the null hypothesis is rejected when it is false is called the power of the test, and equals* $1-\beta$

$$\beta = P(\text{type II error}) = P(H_0 \text{ is not rejected when it is false})$$

- *A type I error is usually more serious than a type II error.*

Example

Write the claim as a mathematical statement. State the null and alternative hypotheses in words and in symbols, and identify which represents the claim. Then determine whether the hypothesis test is a left-tailed test, right-tailed test, or two-tailed test. Sketch a normal sampling distribution and shade the area for the $\mathbf{p}_{\mathrm{v}}$. How should you interpret your decision if you reject H_0? If you fail to reject H_0?

1. A school publicizes that the proportion of its students who are involved in at least one extracurricular activity is 61%.
2. A car dealership announces that the mean time for an oil change is less than 15 minutes.
3. A company advertises that the mean life of its furnaces is more than 18 years.

2 Tests About a Population Mean

- Normal Population + Known σ
- Normal Population + Unknown σ

Normal Population + Known σ

First of all, compute a statistic

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}.$$

Then apply the following decision rule

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Example

50 smokers were questioned about the number of hours they sleep each day. It was found that the sample mean is 7.5 hours, and suppose that the population has normal distribution with standard deviation 0.5 hours. Test at the 5% significance level the hypothesis that the smokers need less sleep than the general public which needs an average of 7.7 hours of sleep.

Solution

We want to test the following hypotheses

$$H_0 : \mu \geq 7.7 \quad vs. \quad H_1 : \mu < 7.7.$$

Compute the statistic

$$z = \frac{7.5 - 7.7}{0.5/\sqrt{50}} = -2.828.$$

Since $c = -z_{0.05} = -1.645$ and $z < c$, we can reject H_0.

Normal Population + Unknown σ

First of all, compute a statistic

$$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}.$$

Then apply the following decision rule

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Example 1 –

In a random sample of 20 components taken from a production line, the mean length of each component in this sample is 108.6 millimeters with a standard deviation of 6.3 millimeters. Given that each component should measure 105 millimeters and that the population distribution is normal, is there enough statistical evidence to show that the production line is producing components that are of an incorrect length? Test at 5 percent level of significance.

Solution

We want to test the following hypotheses

$$H_0 : \mu = 105 \quad vs. \quad H_1 : \mu \neq 105.$$

Compute the statistic

$$t = \frac{108.6 - 105}{6.3/\sqrt{20}} = 2.556.$$

Since $c = t_{0.025,19} = 2.093$ and $|t| > c$, we can reject H_0.

3 Other distribution - Large Sample Size

- Any Distribution
- Tests About a Population Proportion

Any Distribution - Large Sample Size

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Example 2 – Dynamic cone penetrometer

A dynamic cone penetrometer (DCP) is used for measuring material resistance to penetration as a cone is driven into pavement or subgrade. Suppose that for a particular application it is required that the true average DCP value for a certain type of pavement be less than 30. The pavement will not be used unless there is conclusive evidence that the specification has been met. Let's state and test the appropriate hypotheses (with $\alpha = 0.05$) using the following data

14.1	14.5	17.8	18.1	20.8	20.8	30.0	31.6	36.7	40.0
55.0	57.0	15.5	16.0	18.2	18.3	21.0	21.5	31.7	31.7
40.0	41.3	16.0	16.7	16.9	18.3	19.0	19.2	23.5	27.5
27.5	32.5	33.5	33.9	41.7	47.5	50.0	17.1	17.5	17.8
19.4	20.0	20.0	28.0	28.3	30.0	35.0	35.0	35.0	51.0
51.8	54.4								

Solution

The sample mean DCP is less than 30. However, there is a substantial amount of variation in the data, so the fact that the mean is less than the design specification cutoff may be a consequence just of sampling variability.

Notice that the histogram does not resemble at all a normal curve.

$H_1 : \mu < 30$, $n = 52, \overline{x} = 28.76, s = 12.2647$. Thus

$$z = \frac{28.76 - 30}{12.2647/\sqrt{52}} = -0.73$$

Since $-0.73 > -1.645$, H_0 cannot be rejected.

We do not have compelling evidence for concluding that $\mu < 30$; use of the pavement is not justified.

Population Proportion - Large Sample Size

Let $\hat{p}$ be the sample proportion. First of all, compute a statistic

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}.$$

Then apply the following decision rule

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Example 3 – Cork taint

Natural cork in wine bottles is subject to deterioration, and as a result wine in such bottles may experience contamination. An article reported that, in a tasting of commercial chardonnays, 16 of 91 bottles were considered spoiled to some extent by cork-associated characteristics. Does this data provide strong evidence for concluding that more than 15% of all such bottles are contaminated in this way? ($\alpha = 5\%$)

Solution

$H_1 : p > 0.15$, where p = the true proportion of all commercial chardonnay bottles considered spoiled...

$np_0 = 13.65 > 10$, $nq_0 = 77.35 > 10$, $\hat{p} = 16/91 = 0.1758$.

$z = (\hat{p} - 0.15)/\sqrt{(0.15)(0.85)/n} = 0.69 < z_{0.05} = 1.64$.

4 Summary

Summary

- Test one mean
  - Population distribution
    - Normal
      - Known σ^2: $\boxed{z = \frac{\widehat{\mu} - \mu_0}{\sigma/\sqrt{n}}}$ $\boxed{\mathsf{N}(0, 1)}$
      - Unknown σ^2: $\boxed{t = \frac{\widehat{\mu} - \mu_0}{s/\sqrt{n}}}$ $\boxed{\mathbf{t}(n-1)}$
    - Any, $n \gg 1$: $\boxed{z = \frac{\widehat{\mu} - \mu_0}{s/\sqrt{n}}}$ $\boxed{\mathsf{N}(0, 1)}$