13 Design and Analysis of Single-Factor Experiments: The Analysis of Variance

CHAPTER OUTLINE

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13-1: Designing Engineering Experiments

Every experiment involves a sequence of activities:

1. Conjecture – the original hypothesis that motivates the experiment.
2. Experiment – the test performed to investigate the conjecture.
3. Analysis – the statistical analysis of the data from the experiment.
4. Conclusion – what has been learned about the original conjecture from the experiment. Often the experiment will lead to a revised conjecture, and a new experiment, and so forth.

13-2: The Completely Randomized Single-Factor Experiment

13-2.1 An Example

A manufacturer of paper used for making grocery bags is interested in improving the tensile strength of the product. Product engineering thinks that tensile strength is a function of the hardwood concentration in the pulp and that the range of hardwood concentrations of practical interest is between 5 and 20%. A team of engineers responsible for the study decides to investigate four levels of hardwood concentration: 5%, 10%, 15%, and 20%. They decide to make up six test specimens at each concentration level, using a pilot plant. All 24 specimens arc tested on a laboratory tensile tester, in random order. The data from this experiment are shown in Table 13-1.

13-2: The Completely Randomized Single-Factor Experiment

13-2.1 An Example

Table 13-1 Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations						Totals	Averages
	1	2	3	4	5	6		
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

13-2: The Completely Randomized Single-Factor Experiment

13-2.1 An Example

- The levels of the factor are sometimes called **treatments**.
- Each treatment has six observations or **replicates**.
- The runs are run in **random** order.

13-2: The Completely Randomized Single-Factor Experiment

13-2.1 An Example

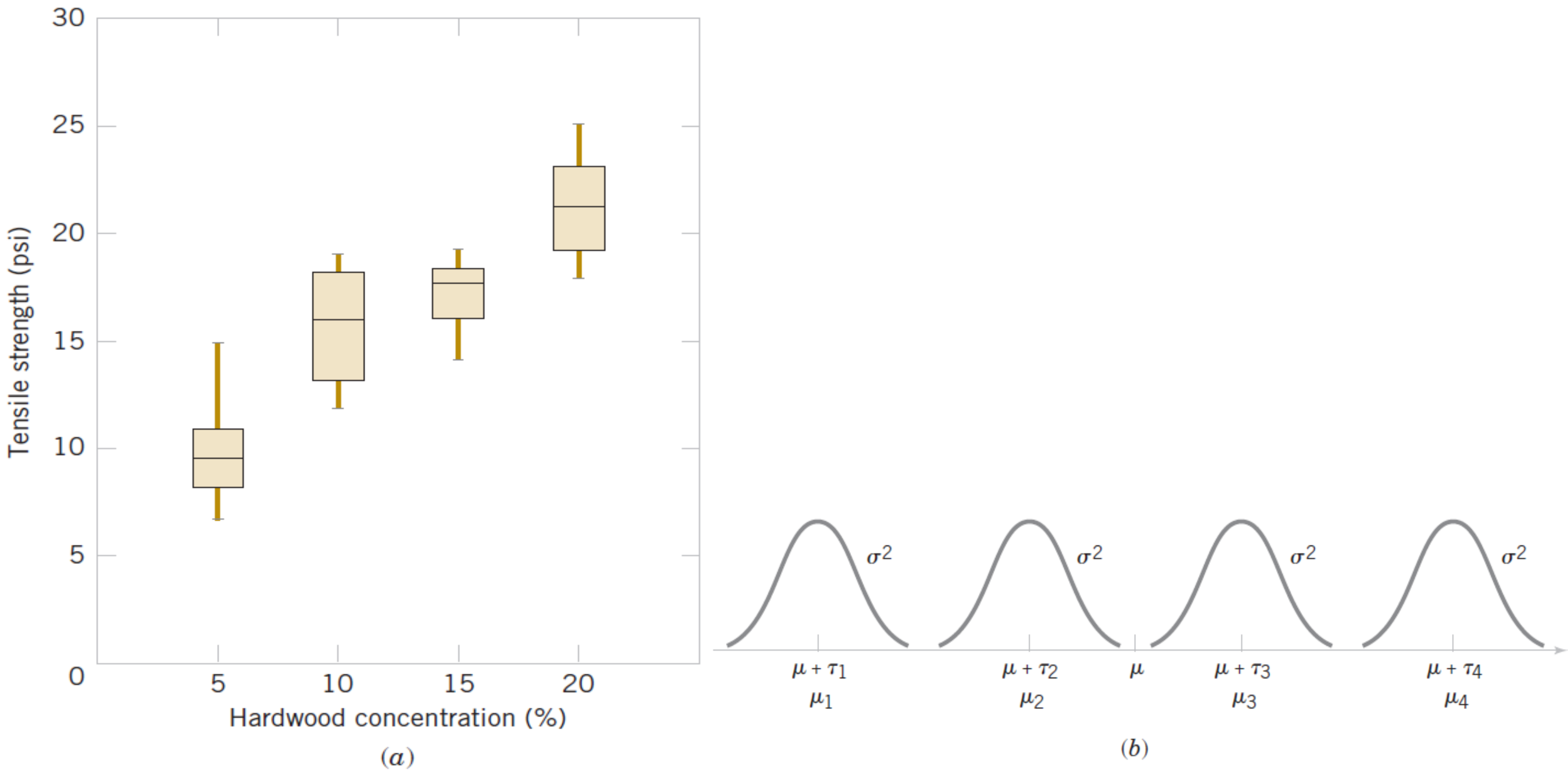

Figure 13-1 (a) Box plots of hardwood concentration data. (b) Display of the model in Equation 13-1 for the completely randomized single-factor experiment.

13-2.2 The Analysis of Variance

Definition

The **sum of squares identity** is

$$\sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij}-\bar{y}_{..}\right)^2 = n\sum_{i=1}^{a}\left(\bar{y}_{i.}-\bar{y}_{..}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij}-\bar{y}_{i.}\right)^2 \qquad (13\text{-}1)$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_E \qquad (13\text{-}2)$$

13-2.2 The Analysis of Variance

The expected value of the treatment sum of squares is

$$E(SS_{\text{Treatments}}) = (a-1)\sigma^2 + n\sum_{i=1}^{a}\tau_i^2$$

and the expected value of the error sum of squares is

$$E(SS_E) = a(n-1)\sigma^2$$

The ratio $MS_{\text{Treatments}} = SS_{\text{Treatments}}/(a-1)$ is called the **mean square for treatments**.

13-2.2 The Analysis of Variance

The appropriate test statistic is

$$F_0 = \frac{SS_{\text{Treatments}}/(a-1)}{SS_E/[a(n-1)]} = \frac{MS_{\text{Treatments}}}{MS_E} \qquad (13\text{-}3)$$

We would reject H_0 if $f_0 > f_{\alpha,a-1,a(n-1)}$

13-2.2 The Analysis of Variance

Definition

The sums of squares computing formulas for the ANOVA with equal sample sizes in each treatment are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N} \qquad (13\text{-}4)$$

and

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N} \qquad (13\text{-}5)$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}} \qquad (13\text{-}6)$$

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13-2: The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

Analysis of Variance Table

Table 13-3 The Analysis of Variance for a Single-Factor Experiment, Fixed-Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$MS_{\text{Treatments}}$	$\dfrac{MS_{\text{Treatments}}}{MS_E}$
Error	SS_E	$a(n - 1)$	MS_E	
Total	SS_T	$an - 1$		

EXAMPLE 13-1 Tensile Strength ANOVA Consider the paper tensile strength experiment described in Section 13-2.1. This experiment is a CRD. We can use the analysis of variance to test the hypothesis that different hardwood concentrations do not affect the mean tensile strength of the paper.

The hypotheses are

$$H_0\colon \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$$
$$H_1\colon \tau_i \neq 0 \ \text{ for at least one } i$$

13-2: The Completely Randomized Single-Factor Experiment

Example 13-1

We will use $\alpha = 0.01$. The sums of squares for the analysis of variance are computed from Equations 13-4, 13-5, and 13-6 as follows:

$$SS_T = \sum_{i=1}^{4} \sum_{j=1}^{6} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N}$$

$$= (7)^2 + (8)^2 + \cdots + (20)^2 - \frac{(383)^2}{24} = 512.96$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{i\cdot}^2}{n} - \frac{y_{\cdot\cdot}^2}{N}$$

$$= \frac{(60)^2 + (94)^2 + (102)^2 + (127)^2}{6} - \frac{(383)^2}{24}$$

$$= 382.79$$

$$SS_E = SS_T - SS_{\text{Treatments}}$$

$$= 512.96 - 382.79 = 130.17$$

13-2: The Completely Randomized Single-Factor Experiment

Example 13-1

The ANOVA is summarized in Table 13-4. Since $f_{0.01,3,20} =$ 4.94, we reject H_0 and conclude that hardwood concentration in the pulp significantly affects the mean strength of the paper. We can also find a P-value for this test statistic as follows:

$$P = P(F_{3,20} > 19.60) \simeq 3.59 \times 10^{-6}$$

Since $P \simeq 3.59 \times 10^{-6}$ is considerably smaller than α = 0.01, we have strong evidence to conclude that H_0 is not true.

Table 13-4 ANOVA for the Tensile Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-value
Hardwood concentration	382.79	3	127.60	19.60	3.59 E-6
Error	130.17	20	6.51		
Total	512.96	23			

Confidence Interval on a Treatment Mean

A 100(1 - α) percent confidence interval on the mean of the *i*th treatment μ_t is

$$\bar{y}_{i\cdot} - t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \leq \mu_i \leq \bar{y}_{i\cdot} + t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \qquad (13\text{-}7)$$

For 20% hardwood, the resulting confidence interval on the mean is

$$19.00 \text{ psi} \leq m_4 \leq 23.34 \text{ psi}$$

13-2: The Completely Randomized Single-Factor Experiment

Confidence Interval on a Difference in Treatment Means

A 100(1 - α) percent confidence interval on the difference in two treatment means μ_i - μ_j is

$$\bar{y}_{i\cdot} - \bar{y}_{j\cdot}\, t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i\cdot} - \bar{y}_{j\cdot} + t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \quad (13\text{-}8)$$

For the hardwood concentration example,

$$-1.74 \le m_3 - m_2 \le 4.40$$

An Unbalanced Experiment

The sums of squares computing formulas for the ANOVA with unequal sample sizes n_i in each treatment are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{..}^2}{N} \qquad (13\text{-}9)$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{N} \qquad (13\text{-}10)$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} \qquad (13\text{-}11)$$

13-2: The Completely Randomized Single-Factor Experiment

13-2.3 Multiple Comparisons Following the ANOVA

The least significant difference (LSD) is

$$\text{LSD} = t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \qquad (13\text{-}12)$$

If the sample sizes are different in each treatment:

$$\text{LSD} = t_{\alpha/2,N-a}\sqrt{MS_E\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$$

Example 13-2

We will apply the Fisher LSD method to the hardwood concentration experiment. There are $a = 4$ means, $n = 6$, $MS_E = 6.51$, and $t_{0.025,20} = 2.086$. The treatment means are

$$\bar{y}_{1\cdot} = 10.00 \text{ psi}$$
$$\bar{y}_{2\cdot} = 15.67 \text{ psi}$$
$$\bar{y}_{3\cdot} = 17.00 \text{ psi}$$
$$\bar{y}_{4\cdot} = 21.17 \text{ psi}$$

The value of LSD is $\text{LSD} = t_{0.025,20}\sqrt{2MS_E/n} = 2.086\sqrt{2(6.51)/6} = 3.07$. Therefore, any pair of treatment averages that differs by more than 3.07 implies that the corresponding pair of treatment means are different.

Example 13-2

The comparisons among the observed treatment averages are as follows:

4 vs. 1 = 21.17 - 10.00 = 11.17 > 3.07
4 vs. 2 = 21.17 - 15.67 = 5.50 > 3.07
4 vs. 3 = 21.17 - 17.00 = 4.17 > 3.07
3 vs. 1 = 17.00 - 10.00 = 7.00 > 3.07
3 vs. 2 = 17.00 - 15.67 = 1.33 < 3.07
2 vs. 1 = 15.67 - 10.00 = 5.67 > 3.07

Conclusions: From this analysis, we see that there are significant differences between all pairs of means except 2 and 3. This implies that 10% and 15% hardwood concentration produce approximately the same tensile strength and that all other concentration levels tested produce different tensile strengths.

It is often helpful to draw a graph of the treatment means, such as in Fig. 13-2, with the means that are *not* different underlined. This graph clearly reveals the results of the experiment and shows that 20% hardwood produces the maximum tensile strength.

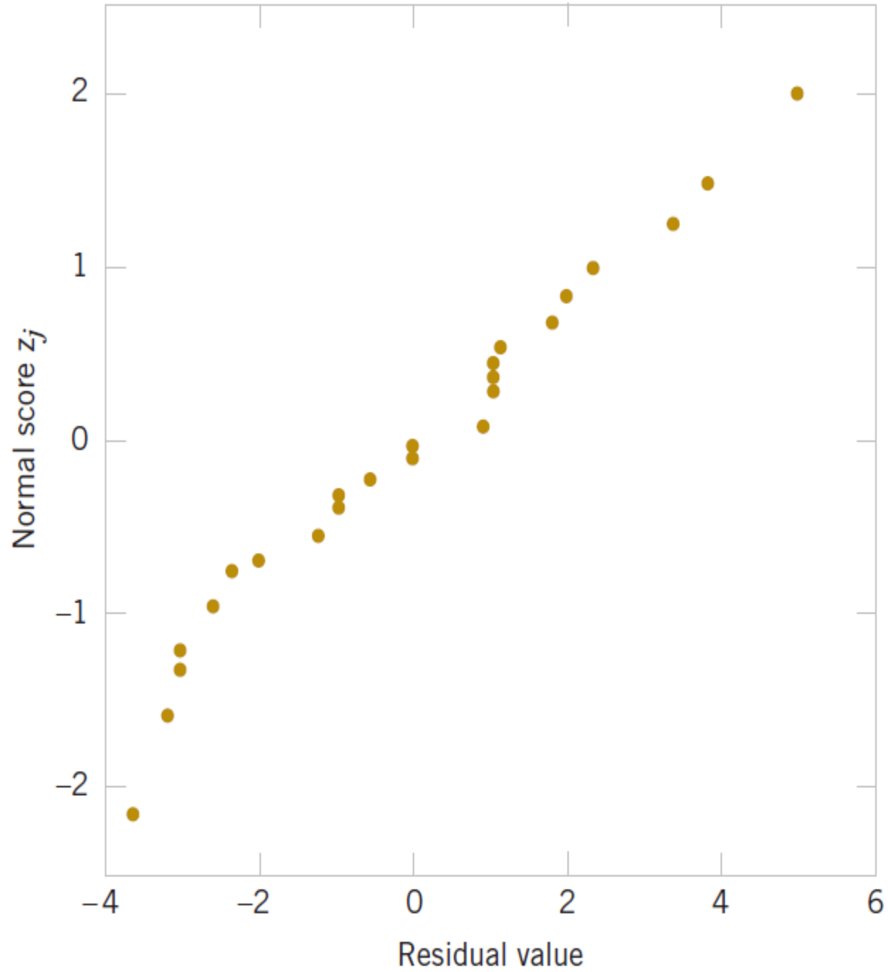
13-2.4 Residual Analysis and Model Checking

Table 13-6 Residuals for the Tensile Strength Experiment

Hardwood Concentration (%)	Residuals					
5	-3.00	-2.00	5.00	1.00	-1.00	0.00
10	-3.67	1.33	-2.67	2.33	3.33	-0.67
15	-3.00	1.00	2.00	0.00	-1.00	1.00
20	-2.17	3.83	0.83	1.83	-3.17	-1.17

13-2.4 Residual Analysis and Model Checking

Figure 13-4 Normal probability plot of residuals from the hardwood concentration experiment.



13-2.4 Residual Analysis and Model Checking

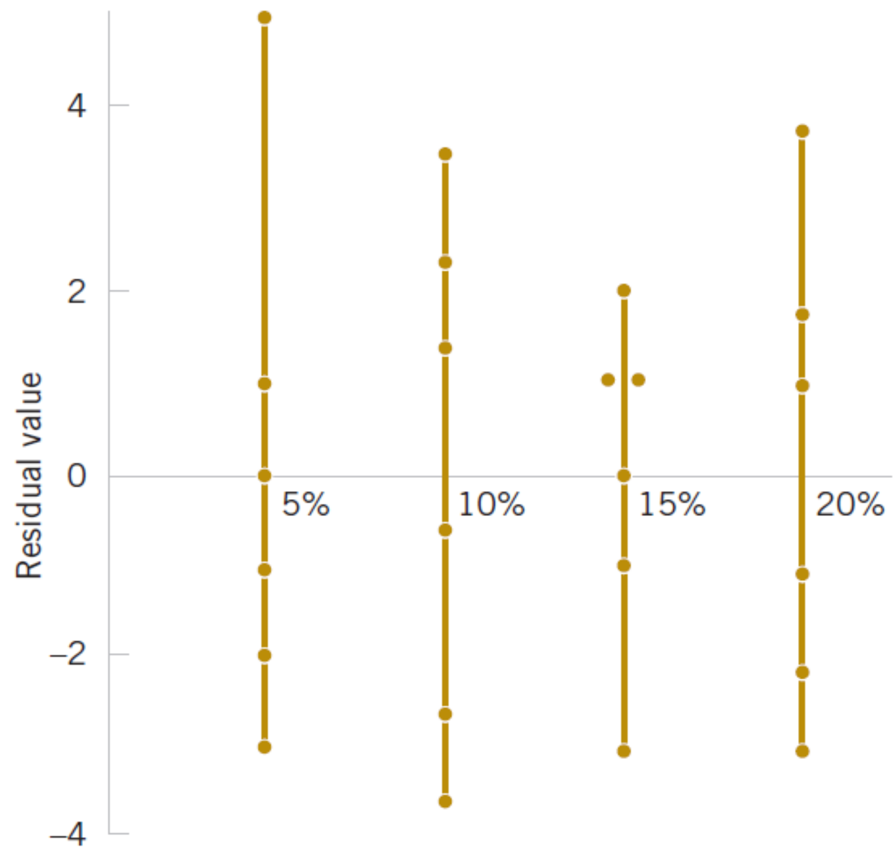
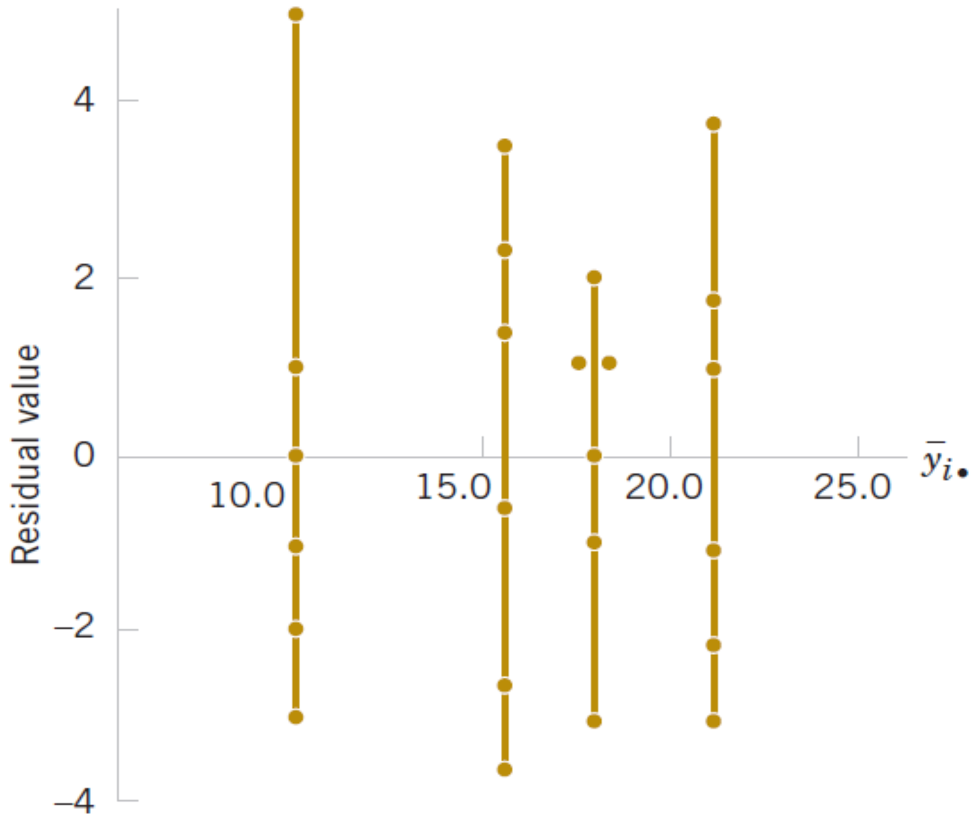


Figure 13-5 Plot of residuals versus factor levels (hardwood concentration).

13-2.4 Residual Analysis and Model Checking

Figure 13-6 Plot of residuals versus $\overline{y}_{i\cdot}$



13-3: The Random-Effects Model

13-3.2 ANOVA and Variance Components

The linear statistical model is

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

The variance of the response is $V(Y_{ij}) = \sigma_\tau^2 + \sigma^2$
Where each term on the right hand side is called a **variance component.**

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13-3: The Random-Effects Model

13-3.2 ANOVA and Variance Components

For a random-effects model, the appropriate hypotheses to test are:

$$H_0: \sigma_\tau^2 = 0$$
$$H_1: \sigma_\tau^2 > 0$$

The ANOVA decomposition of total variability is still valid:

$$SS_T = SS_{\text{Treatments}} + SS_E$$

13-3: The Random-Effects Model

13-3.2 ANOVA and Variance Components

The expected values of the mean squares are

In the random-effects model for a single-factor, completely randomized experiment, the expected mean square for treatments is

$$E(MS_{\text{Treatments}}) = E\left(\frac{SS_{\text{Treatments}}}{a-1}\right) = \sigma^2 + n\sigma_\tau^2 \qquad (13\text{-}13)$$

and the expected mean square for error is

$$E(MS_E) = E\left[\frac{SS_E}{a(n-1)}\right] = \sigma^2 \qquad (13\text{-}14)$$

13-3: The Random-Effects Model

13-3.2 ANOVA and Variance Components

The estimators of the variance components are

$$\hat{\sigma}^2 = MS_E \tag{13-15}$$

and

$$\hat{\sigma}_\tau^2 = \frac{MS_{\text{Treatments}} - MS_E}{n} \tag{13-16}$$

13-3: The Random-Effects Model

Example 13-4 Textile Manufacturing In *Design and Analysis of Experiments,* 7th edition (John Wiley, 2009), D. C. Montgomery describes a single-factor experiment involving the random-effects model in which a textile manufacturing company weaves a fabric on a large number of looms. The company is interested in loom-to-loom variability in tensile strength. To investigate this variability, a manufacturing engineer selects four looms at random and makes four strength determinations on fabric samples chosen

Table 13-7 Strength Data for Example 13-4

	Observations					
Loom	1	2	3	4	Total	Average
1	98	97	99	96	390	97.5
2	91	90	93	92	366	91.5
3	96	95	97	95	383	95.8
4	95	96	99	98	388	97.0
					1527	95.45

Table 13-8 Analysis of Variance for the Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	*P*-value
Looms	89.19	3	29.73	15.68	1.88 E-4
Error	22.75	12	1.90		
Total	111.94	15			

13-3: The Random-Effects Model

Example 13-4

From the analysis of variance, we conclude that the looms in the plant differ significantly in their ability to produce fabric of uniform strength. The variance components are estimated by $\hat{\sigma}^2 = 1.90$ and

$$\hat{\sigma}_\tau^2 = \frac{29.73 - 1.90}{4} = 6.96$$

Therefore, the variance of strength in the manufacturing process is estimated by

$$V(Y_{ij}) = \hat{\sigma}_\tau^2 + \hat{\sigma}^2 + 6.96 + 1.90 = 8.86$$

Conclusions: Most of the variability in strength in the output product is attributable to differences between looms.

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

The **randomized block design** is an extension of the paired t-test to situations where the factor of interest has more than two levels.

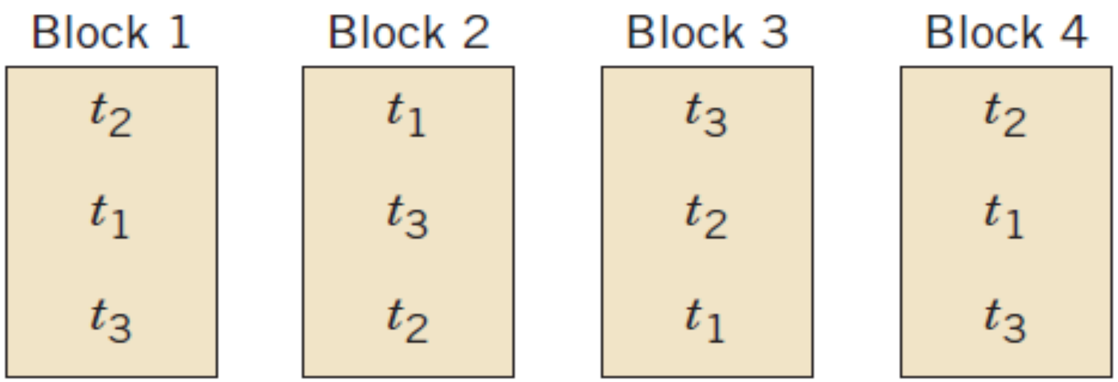

Figure 13-9 A randomized complete block design.

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

General procedure for a randomized complete block design:

Table 13-10 A Randomized Complete Block Design with *a* Treatments and *b* Blocks

	Blocks					
Treatments	1	2	…	b	Totals	Averages
1	y_{11}	y_{12}	…	y_{1b}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	…	y_{2b}	$y_{2.}$	$\bar{y}_{2.}$
⋮	⋮	⋮		⋮	⋮	⋮
a	y_{a1}	y_{a2}	…	y_{ab}	$y_{a.}$	$\bar{y}_{a.}$
Totals	$y_{.1}$	$y_{.2}$	…	$y_{.b}$	$y_{..}$	
Averages	$\bar{y}_{.1}$	$\bar{y}_{.2}$	…	$\bar{y}_{.b}$		$\bar{y}_{..}$

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

The appropriate linear statistical model:

$$Y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

We assume

- treatments and blocks are initially fixed effects
- blocks do not interact
- $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

We are interested in testing:

The **sum of squares identity for the randomized complete block design** is

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{.j}-\bar{y}_{i.}+\bar{y}_{..})^2 \tag{13-17}$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks}} + SS_E$$

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13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

The mean squares are:

$$MS_{\text{Treatments}} = \frac{SS_{\text{Treatments}}}{a-1}$$

$$MS_{\text{Blocks}} = \frac{SS_{\text{Blocks}}}{b-1}$$

$$MS_E = \frac{SS_E}{(a-1)(b-1)}$$

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

The expected values of these mean squares are:

$$E(MS_{\text{Treatments}}) = \sigma^2 + \frac{b \sum_{i=1}^{a} \tau_i^2}{a - 1}$$

$$E(MS_{\text{Blocks}}) = \sigma^2 + \frac{a \sum_{j=1}^{b} \beta_j^2}{b - 1}$$

$$E(MS_E) = \sigma^2$$

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

Definition

The computing formulas for the sums of squares in the analysis of variance for a randomized complete block design are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{ab} \qquad (13\text{-}18)$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i\cdot}^2 - \frac{y_{\cdot\cdot}^2}{ab} \qquad (13\text{-}19)$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{\cdot j}^2 - \frac{y_{\cdot\cdot}^2}{ab} \qquad (13\text{-}20)$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}} \qquad (13\text{-}21)$$

13-4: Randomized Complete Block Design

13-4.1 Design and Statistical Analysis

Table 13-11 ANOVA for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$\dfrac{SS_{\text{Treatments}}}{a-1}$	$\dfrac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b - 1$	$\dfrac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$(a-1)(b-1)$	$\dfrac{SS_E}{(a-1)(b-1)}$	
Total	SS_T	$ab - 1$		

13-4: Randomized Complete Block Design

Example 13-5 Fabric Strength An experiment was performed to determine the effect of four different chemicals on the strength of a fabric. These chemicals are used as part of the permanent press finishing process. Five fabric samples were selected, and a RCBD was run by testing each chemical type once in random order on each fabric sample. The data are shown in Table 13-12. We will test for differences in means using an ANOVA with a = 0.01.

Table 13-12 Fabric Strength Data—Randomized Complete Block Design

			Fabric Sample			**Treatment Totals**	**Treatment Averages**
Chemical Type	1	2	3	4	5	$y_{i\cdot}$	$\bar{y}_{\cdot j}$
1	1.3	1.6	0.5	1.2	1.1	5.7	1.14
2	2.2	2.4	0.4	2.0	1.8	8.8	1.76
3	1.8	1.7	0.6	1.5	1.3	6.9	1.38
4	3.9	4.4	2.0	4.1	3.4	17.8	3.56
Block totals $y_{\cdot j}$	9.2	10.1	3.5	8.8	7.6	39.2($y_{\cdot\cdot}$)	
Block averages $\bar{y}_{i\cdot}$	2.30	2.53	0.88	2.20	1.90		1.96($\bar{y}_{\cdot\cdot}$)

13-4: Randomized Complete Block Design

Example 13-5

The sums of squares for the analysis of variance are computed as follows:

$$SS_T = \sum_{i=1}^{4} \sum_{j=1}^{5} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{ab}$$

$$= (1.3)^2 + (1.6)^2 + \cdots + (3.4)^2 - \frac{(39.2)^2}{20} = 25.69$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{i\cdot}^2}{b} - \frac{y_{\cdot\cdot}^2}{ab}$$

$$= \frac{(5.7)^2 + (8.8)^2 + (6.9)^2 + (17.8)^2}{5}$$

$$- \frac{(39.2)^2}{20} = 18.04$$

13-4: Randomized Complete Block Design

Example 13-5

$$SS_{\text{Blocks}} = \sum_{j=1}^{5} \frac{y_{\cdot j}^2}{a} - \frac{y_{\cdot\cdot}^2}{ab}$$

$$= \frac{(9.2)^2 + (10.1)^2 + (3.5)^2 + (8.8)^2 + (7.6)^2}{4}$$

$$- \frac{(39.2)^2}{20} = 6.69$$

$$SS_E = SS_T - SS_{\text{Blocks}} - SS_{\text{Treatments}}$$

$$= 25.69 - 6.69 - 18.04 = 0.96$$

The ANOVA is summarized in Table 13-13.
Since $f_0 = 75.13 > f_{0.01,3,12} = 5.95$ (the *P*-value is 4.79×10^{-8}), we conclude that there is a significant difference in the chemical types so far as their effect on strength is concerned.

13-4: Randomized Complete Block Design

Example 13-5

Table 13-13 Analysis of Variance for the Randomized Complete Block Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	*P*-value
Chemical types (treatments)	18.04	3	6.01	75.13	4.79 E-8
Fabric samples (blocks)	6.69	4	1.67		
Error	0.96	12	0.08		
Total	25.69	19			

13-4: Randomized Complete Block Design

13-4.2 Multiple Comparisons

Fisher's Least Significant Difference for Example 13-5

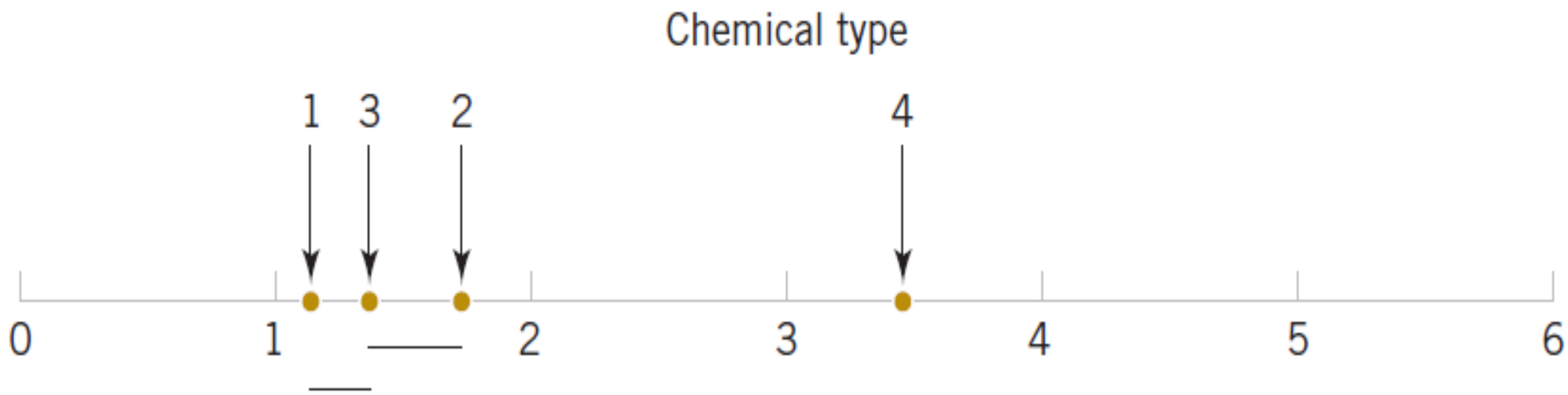
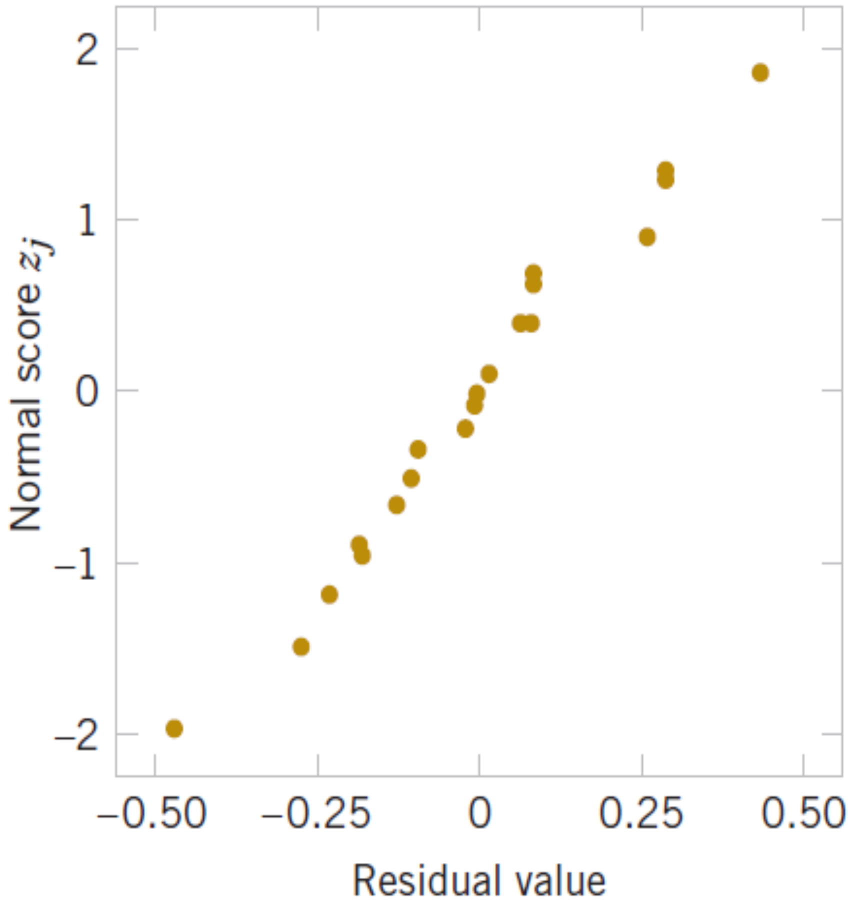


Figure 13-10 Results of Fisher's LSD method.

13-4: Randomized Complete Block Design

13-4.3 Residual Analysis and Model Checking

Figure 13-11 Normal probability plot of residuals from the randomized complete block design.



13-4: Randomized Complete Block Design

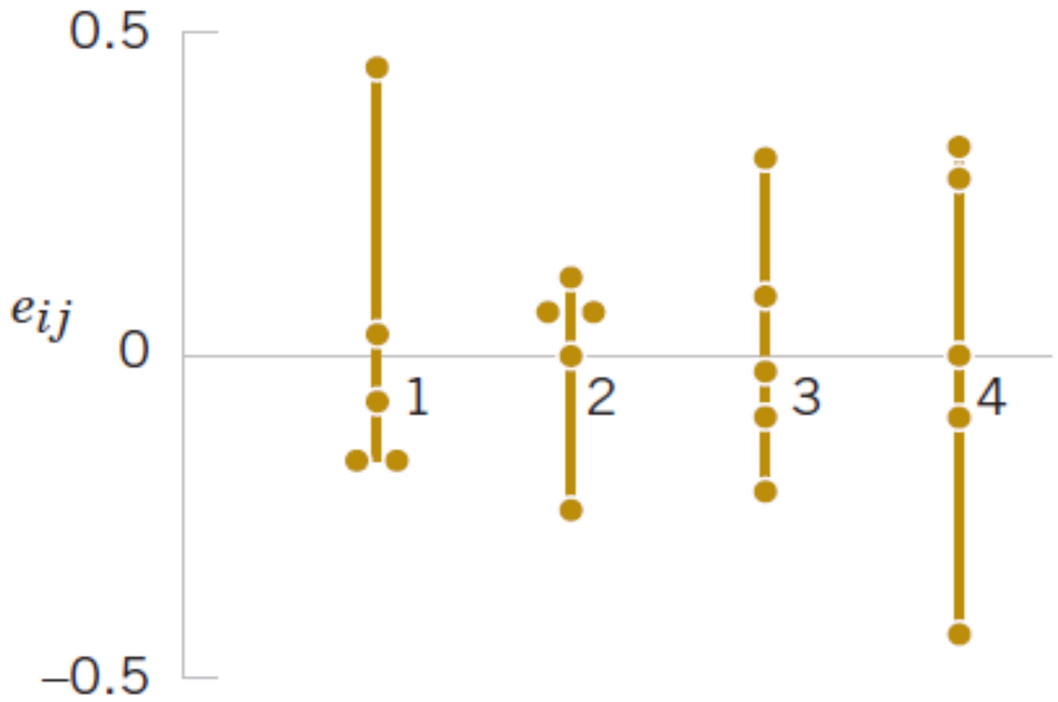


Figure 13-12 Residuals by treatment from the randomized complete block design.

13-4: Randomized Complete Block Design

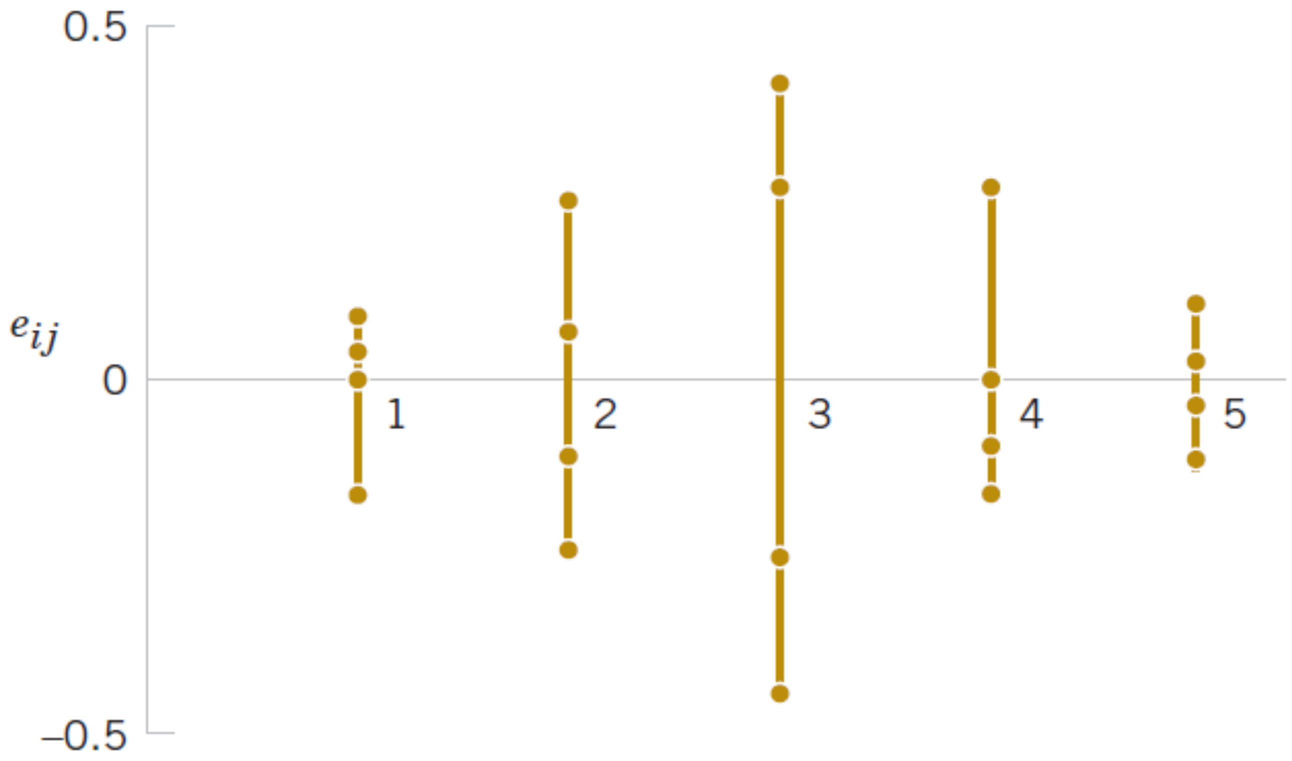


Figure 13-13 Residuals by block from the randomized complete block design.

13-4: Randomized Complete Block Design

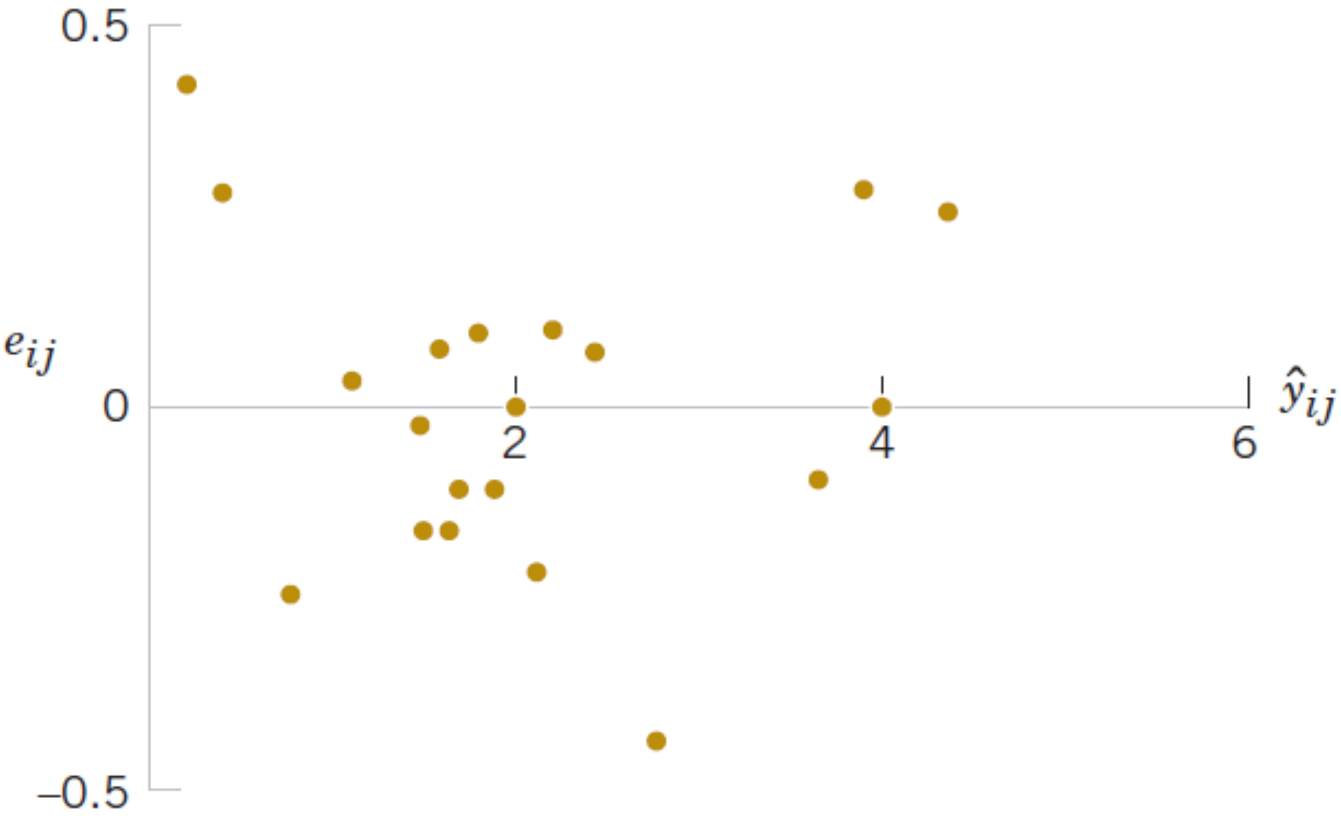


Figure 13-14 Residuals versus $\hat{y}_{ij}$ from the randomized complete block design.