HCMC University of Technology
Dung Nguyen

Probability and Statistics

Some Special Distributions

The Binomial Distributions $\mathrm{B}(n,p)$

Definition

- *Bernoulli trial* $\mathrm{B}(p)$:

u	1	0
$P(Z=u)$	p	$q=1-p$

$\mathrm{E}(Z)=p$ *and* $\mathrm{V}(Z)=pq$.

- *Binomial random variable = the # of successes in* n *Bernoulli trials (the probability of success in each trial is* $0\le p\le 1$*).*

Examples:

- The # of defective items among 20 independent items with the defective rate 5%.
- The # of winning tickets among 11 independent lottery tickets with the winning rate 1%.
- The # of patients reporting symptomatic relief with a specific medication with the effective rate 80%.

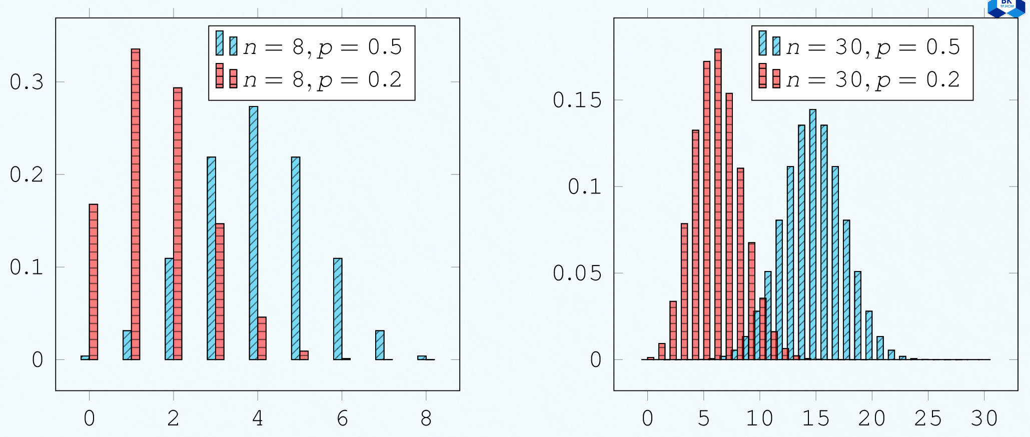

Figure: Pmf of $\mathrm{B}(n,p)$

Proposition

Let $X\sim \mathrm{B}(n,p)$. Then

1. X takes values in $\Omega=\{0,1,\dots,n\}$ such that
$$f(k)=P(X=k)=\mathrm{C}_n^k p^k q^{n-k}.$$
   - $k=0$: $\boxed{q}\ \boxed{q}\ \boxed{q}\ \boxed{\cdots}\ \boxed{q} \implies P(X=0)=q^n$.
   - $k=n$: $\boxed{p}\ \boxed{p}\ \boxed{p}\ \boxed{\cdots}\ \boxed{p} \implies P(X=n)=p^n$.
   - $k=1$: $\boxed{p}\ \boxed{q}\ \boxed{q}\ \boxed{\cdots}\ \boxed{q} \implies P(X=1)=\mathrm{C}_n^1 pq^{n-1}$.
   - $k=2$: $\boxed{p}\ \boxed{p}\ \boxed{q}\ \boxed{\cdots}\ \boxed{q} \implies P(X=2)=\mathrm{C}_n^2 p^2q^{n-2}$.
2. X is a sum of n independent Bernoulli random variables.
$$X=Z_1+Z_2+\cdots+Z_n,$$
where $Z_i=\begin{cases}1, & \text{if the i-th trial is successful}\\ 0, & \text{otherwise}\end{cases}$.
3. $\mathrm{E}(X)=np$ and $\mathrm{V}(X)=npq$.

Example

1. Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant.

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   c. Now determine the probability that $3 \leq X < 7$.

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- (a) Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.
- (b) Determine the probability that at least 4 samples contain the pollutant.
- (c) Now determine the probability that $3 \le X < 7$.

2 A certain electronic system contains 10 components. Suppose that the probability that each individual component will fail is 0.2 and that the components fail independently of each other. Given that at least one of the components has failed, what is the probability that at least two of the components have failed?

Example

3 A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. If 6 bits are transmitted, then how many bits, on average, will be received in error? Determine the corresponding variance.

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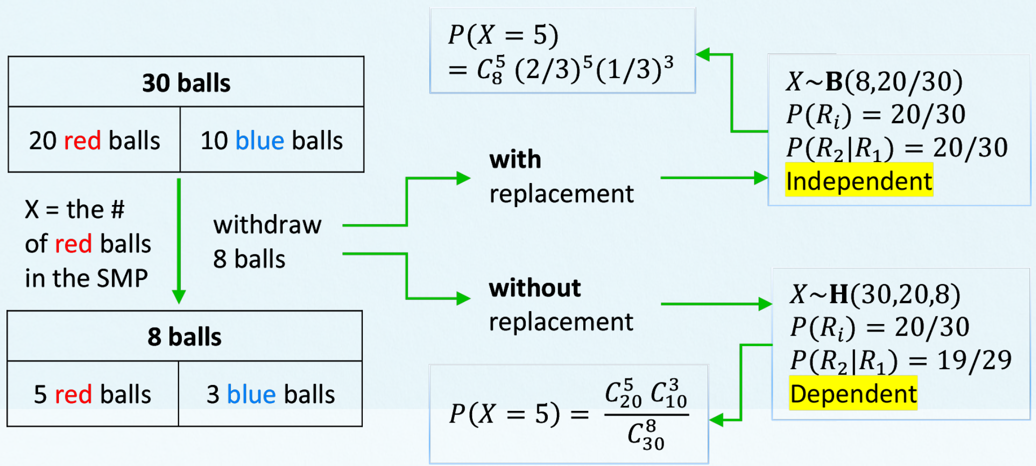
4 Three men A, B, and C shoot at a target. Suppose that A shoots three times and the probability that he will hit the target on any given shot is 1/8, B shoots five times and the probability that he will hit the target on any given shot is 1/4, and C shoots twice and the probability that he will hit the target on any given shot is 1/3. What is the expected number of times that the target will be hit?

Example 5 –



Generalization

N balls

m red balls	(N-m) blue balls

X = the # of red balls in the SMP

withdraw n balls

n balls

k red balls	(n-k) blue balls

with replacement

$X \sim \mathbf{B}(n, p = m/N)$
$P(R_i) = \mathrm{p}$
$P(R_2|R_1) = \mathrm{p}$
$\Rightarrow$ Independent

$P(X = \mathrm{k}) = C_n^{\mathrm{k}} p^{\mathrm{k}} q^{n\text{-}\mathrm{k}}$

without replacement

$X \sim \mathbf{H}(N, m, n)$
$P(R_i) = p = m/N$
$P(R_2|R_1) = \dfrac{m\text{-}1}{N\text{-}1}$
$\Rightarrow$ Dependent

$$P(X = \mathrm{k}) = \frac{C_m^{\mathrm{k}} C_{N\text{-}m}^{n\text{-}\mathrm{k}}}{C_N^n}$$

$P(R_i) = p$

N balls

m red balls	(N-m) blue balls

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withdraw n balls

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k red balls	(n-k) blue balls

- With replacement

$$P(R_1) = \frac{m}{N}$$
$$P(R_2|R_1) = P(R_2|B_1) = \frac{m}{N}$$
$$P(R_2) = \frac{m}{N}.$$

- Without replacement

$$P(R_1) = \frac{m}{N}$$
$$P(R_2|R_1) = \frac{m-1}{N-1}$$
$$P(R_2|B_1) = \frac{m}{N-1}$$
$$P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1) = \frac{m-1}{N-1}\frac{m}{N} + \frac{m}{N-1}\frac{N-m}{N} = \frac{m}{N}.$$

The Hypergeometric Distributions $\mathrm{H}(N, m, n)$

Definition

Suppose that there are n draws from a finite population of size N containing m successes without replacement. Let X be the number of successes. Then X is called a hypergeometric random variable or X has a hypergeometric distribution.

Proposition

Let $X \sim \mathrm{H}(N, m, n)$. Then

1. *X has the pmf:* $f(k) = \dfrac{C_m^k C_{N-m}^{n-k}}{C_N^n}$.
2. *X is a sum of n dependent Bernoulli random variables.*
$$X = Z_1 + Z_2 + \cdots + Z_n, \quad Z_i = \begin{cases} 1, & \text{if the i-th trial is successful} \\ 0, & \text{otherwise} \end{cases}.$$
3. $\mathrm{E}(X) = np$ *and* $\mathrm{V}(X) = npq \cdot \dfrac{N-n}{N-1}$ *with* $p = m/N$.

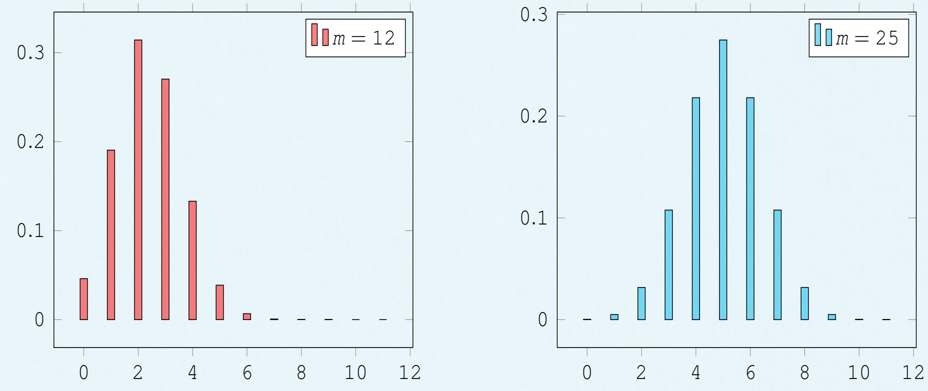

Figure: Pmf of $\mathrm{H}(40, m, 10)$

Example

6. A batch of parts contains 100 from a local supplier of tubing and 200 from a supplier of tubing in a nearby city.

Discrete Distributions The Hypergeometric Distributions $H(N, m, n)$

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(a) If four parts are selected randomly and without replacement, what is the probability they are all from the local supplier?

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Discrete Distributions The Hypergeometric Distributions $H(N, m, n)$

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7 Suppose that seven balls are selected at random without replacement from a box containing five red balls and ten blue balls. If X denotes the proportion of red balls in the sample, what are the mean and the variance of X?

Approximation property

If $m, N-m, n$ are large enough then $\mathsf{H}(N,m,n) \approx \mathsf{B}(n,p)$.

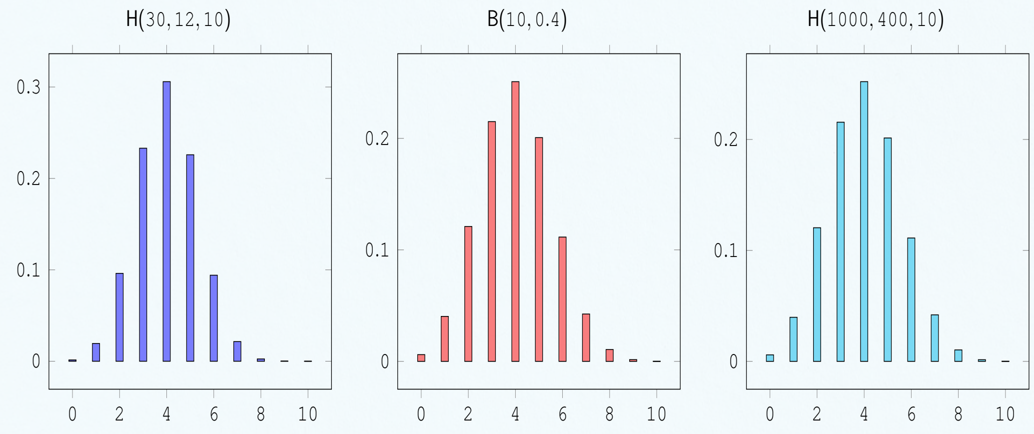
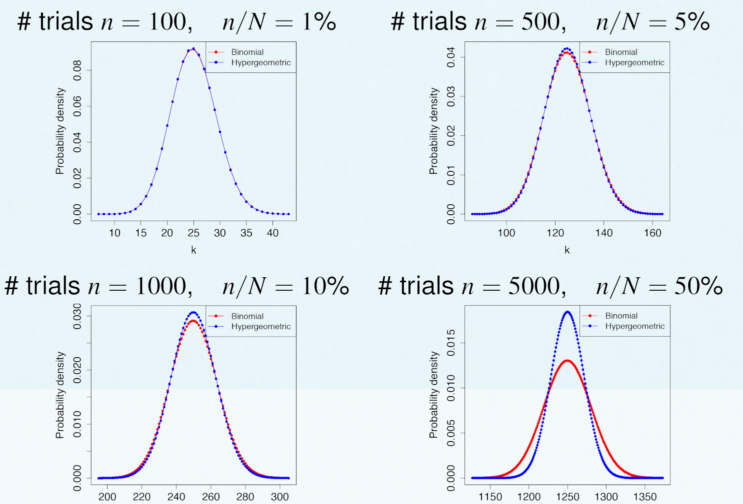


Figure: $\mathsf{B}(n,p)$ vs. $\mathsf{H}(N,m,n)$ with $p = m/N$

Hypergeometric distribution vs. Binomial distribution ($p = 0.25$, Population size $N = 10000$)



Example

8 A list of customer accounts at a large company contains 1,000 customers. Of these, 700 have purchased at least one of the company's products in the last 3 months. To evaluate a new product, 50 customers are sampled at random from the list. What is the probability that more than 45 of the sampled customers have purchased from the company in the last 3 months?

Applications of Poisson Distributions

- Electrical system example: the number of telephone calls arriving in a system in 1 second, the number of wrong connections to your phone number per day.
- Astronomy example: the number of photons arriving at a telescope in 1 microsecond.
- Biology example: the number of mutations on a strand of DNA per unit length, the number of bacteria on some surface or weed in the field,
- Management example: the number of customers arriving at a counter or call centre in 10 minutes.
- Civil engineering example: the number of cars arriving at a traffic light in 5 minutes.
- Finance and insurance example: the number of Losses/Claims occurring in a given period of time.

The Poisson Distributions

Poisson r.v. = the count of events that occur within an interval.

- Unknown: the # of trials n or the probability of success p
- Known: the average # of successes per time period $\lambda = np$.

$$\lim_{n\to\infty} P(X_n = k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = e^{-\lambda}\cdot\frac{\lambda^k}{k!}.$$

Poisson distribution:

$$\Omega = \{0,1,2,\ldots\} \quad \text{and} \quad f(k) = P(X=k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}, x\in\Omega.$$

The mean and variance of the Poisson model are the same.

$$\boxed{E(X) = \lambda \quad \text{and} \quad V(X) = \lambda.}$$

If $X_i \sim \text{Poisson}(\lambda_i)$ and are independent then

$$\sum_{i=1}^{n} X_i \sim \text{Poisson}\left(\sum_{i=1}^{n}\lambda_i\right).$$

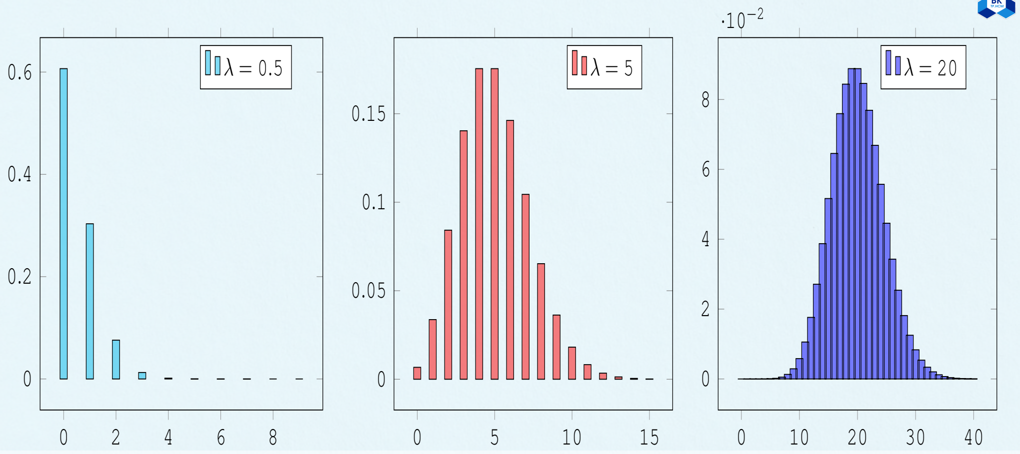


Figure: Pmf of Poisson(λ)

Example

9 Consider an experiment that consists of counting the number of α particles given off in a 1-second interval by 1 gram of radioactive material. If we know from past experience that, on average, 3.2 such α particles are given off, what is a good approximation to the probability that no more than 2 α particles will appear?

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(b) exactly 10 flaws in 5 mm of wire.

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(a) exactly 2 flaws in 1 mm of wire.

(b) exactly 10 flaws in 5 mm of wire.

(c) at least 1 flaw in 2 mm of wire.

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Approximation property

Let $Y \sim \text{Poisson}(\lambda)$ and $X_n \sim \text{B}(n, p_n)$ with $p_n = \lambda/n$. Then

$$\lim_{n\to\infty} \frac{P(Y=k)}{P(X_n=k)} = 1, \quad \forall k = 0, 1, \ldots$$

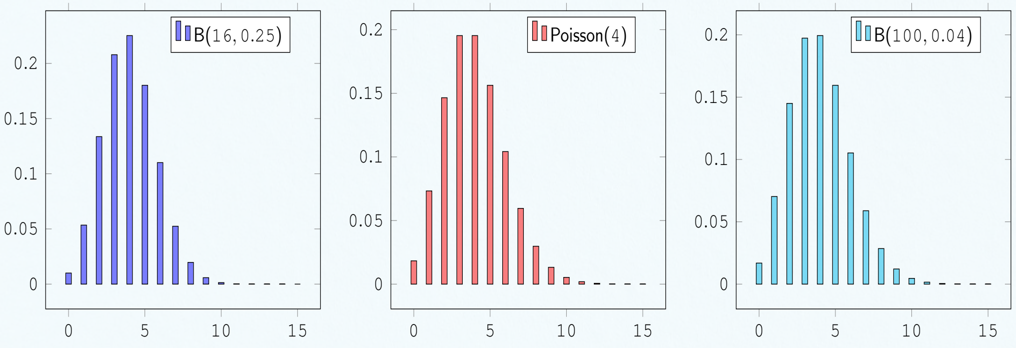


Figure: $\text{Poisson}(\lambda)$ vs. $\text{B}(n,p)$ with $np = \lambda$

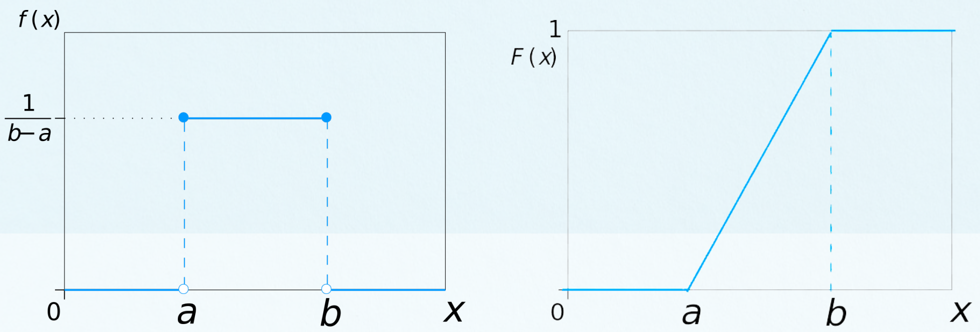
Example

11 Suppose that 1 in 5000 light bulbs are defective. What is the probability that there are at least 3 defective light bulbs in a group of size 10000?

The Continuous Uniform Distributions $\text{U}(a, b)$

This distribution has pdf and cdf

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & x \in [a,b) \\ 1, & x \geq b \end{cases}.$$



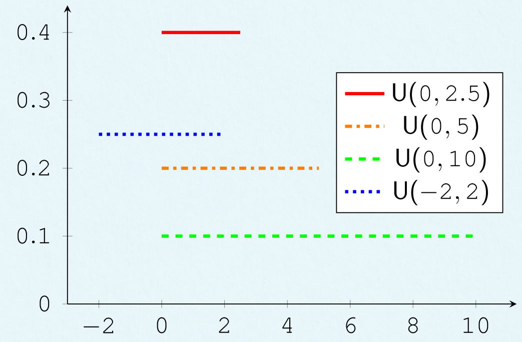


Figure: Pdf of $\text{U}(a, b)$

Proposition (Properties)

1. $\text{E}(X) = \dfrac{a+b}{2}$
2. $\text{Var}(X) = \dfrac{(b-a)^2}{12}$

Example

12 If X is uniformly distributed over (0, 10), calculate the probability that

Continuous Distributions | The Continuous Uniform Distributions U(a, b)

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ⓐ $X < 3$

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Continuous Distributions | The Continuous Uniform Distributions U(a, b)

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ⓒ $3 < X < 8$

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Continuous Distributions | The Continuous Uniform Distributions $U(a, b)$

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ⓐ What is the probability that the current is between 4.95 mA and 5.0 mA?

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ⓑ Calculate the mean and variance.

Introduction

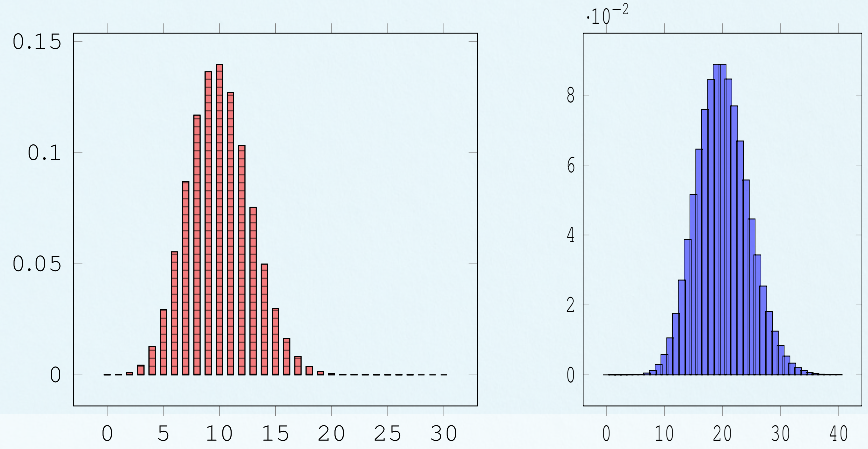


Figure: $B(50, 0.2)$ and $\text{Poisson}(20)$

The Normal Distributions $\mathrm{N}(m,\sigma^2)$

X is called to be of a normal distribution $N(m,\sigma^2)$ if its pdf satisfies

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, x \in \mathbb{R},$$

where $m = \mathrm{E}(X)$ and $\sigma^2 = \mathrm{V}(X)$.

We often standardize a normal distribution $X \sim N(m,\sigma^2)$ by

$$Y = \frac{X-m}{\sigma} \sim N(0,1)$$

In this case, Y is called a random variable of standard normal distribution, or simply a standard score. Its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in \mathbb{R}$$

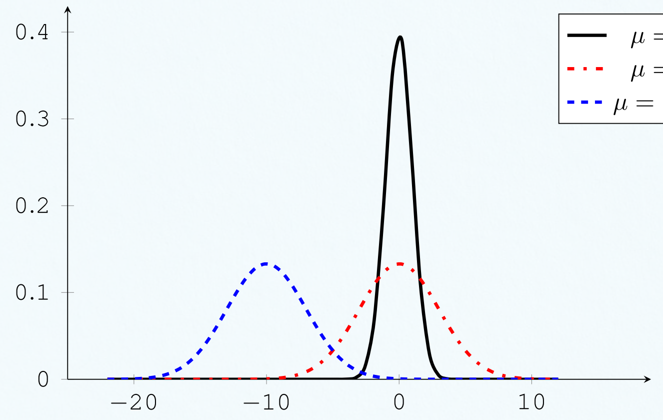


Figure: Pdf of $\mathrm{N}(\mu,\sigma^2)$

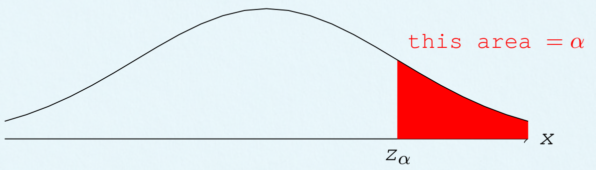
The cdf of $X \sim N(0,1)$

$$\Phi(x) = \int_{-\infty}^{x} f(u)du$$

satisfies

$$\Phi(-x) = 1 - \Phi(x) \quad \text{and} \quad \Phi^{-1}(p) = -\Phi^{-1}(1-p), \quad \text{for } 0 < p < 1.$$

Denote z_α as the solution to $1 - \Phi(z) = \alpha$



z_α is called the upper α critical point or the $100(1-\alpha)$th percentile.

Example

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ⓐ What is the probability that the current measurement is between $9\,\mathrm{mA}$ and $11\,\mathrm{mA}$?

ⓑ Determine the value for which the probability that a current measurement is below this value is 0.98.

Properties

Proposition (Basic properties)

1. $\mathrm{E}(X) = m$ *and* $\mathrm{Var}(X) = \sigma^2$
2. *If* $X \sim N(m,\sigma^2)$ *and* $Y = aX + b, a \neq 0$ *then*
$$Y \sim \mathrm{N}(am + b, a^2\sigma^2).$$
3. *If* $X_i \sim N(m_i,\sigma_i^2)$ *and are independent then*
$$\sum_{i=1}^{n} X_i \sim \mathrm{N}\left(\sum_{i=1}^{n} m_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

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17 Let $X_1 \sim N(2, 4)$ and $X_2 \sim N(-3, 5)$ be independent. Determine the distributions of the following random variables

Continuous Distributions | The Normal Distributions $N(m, \sigma^2)$

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(a) $X = X_1 + X_2$.

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Continuous Distributions | The Normal Distributions $N(m, \sigma^2)$

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(a) $X = X_1 + X_2$. (b) $Y = X_1 - X_2$.

(c) $Z = 3X_1 + 4X_2$.

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Example

18 Data from the National Oceanic and Atmospheric Administration indicate that the yearly precipitation in Los Angeles is a normal random variable with a mean of 12.08 inches and a standard deviation of 3.1 inches. Assume that the precipitation totals for the next 2 years are independent.

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(a) Find the probability that the total precipitation during the next 2 years will exceed 25 inches.

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(a) Find the probability that the total precipitation during the next 2 years will exceed 25 inches.

(b) Find the probability that next year's precipitation will exceed that of the following year by more than 3 inches.

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Continuous Distributions | The Normal Distributions $N(m, \sigma^2)$

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19 Consider three independent memory chips. Suppose that the lifetime of each memory chip has normal distribution with mean 300 hours and standard deviation 10 hours. Compute the probability that at least one of three chips lasts at least 290 hours.

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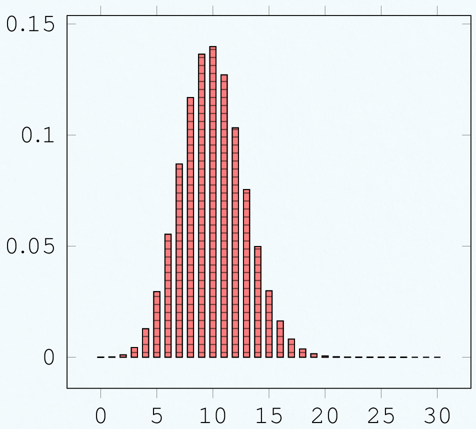
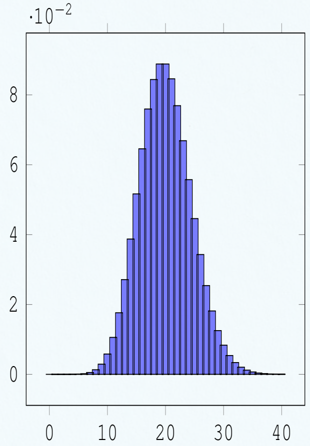
Normal Approximations

The binomial and Poisson distributions become more bell-shaped and symmetric as their mean value increase.





The Central Limit Theorem

Proposition (Central Limit Theorem)

Assume $X_1, X_2, \ldots$ *are i.i.d. (independently identically distributed) such that* $\sigma^2 = \mathrm{Var}(X) < \infty$. *Denote* $S_n = \sum_{i=1}^{n} X_i$ *and* $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. *Then*

$$\frac{S_n - nm}{\sigma\sqrt{n}} \simeq N(0,1).$$

and

$$\frac{\overline{X}_n - m}{\sigma/\sqrt{n}} \simeq N(0,1).$$

Normal Approximation to the Binomial Distribution

If X is a binomial random variable with parameters n and p then

$$Z = \frac{X - np}{\sqrt{npq}} \simeq \mathrm{N}(0,1).$$

The approximate is good if $np > 5$ and $n(1-p) > 5$

Continuity correction

$$P(X \le k) = P(X \le k + 0.5) \approx P\left(Z \le \frac{k + 0.5 - np}{\sqrt{npq}}\right)$$

and

$$P(X \ge k) = P(X \le k - 0.5) \approx P\left(Z \ge \frac{k - 0.5 - np}{\sqrt{npq}}\right)$$

Example

Suppose only 75% of all drivers in a certain state regularly wear a seat belt. A random sample of 500 drivers is selected. What is the probability that

1. Between 360 and 400 (inclusive) of the drivers in the sample regularly wear a seat belt?
2. Fewer than 400 of those in the sample regularly wear a seat belt?

Normal Approximation to the Poisson

If X is a Poisson random variable with $\mathsf{E}(X) = \lambda$ and $\mathsf{V}(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \simeq \mathsf{N}(0, 1)$$

- Continuity correction
- The approximation is good for $\lambda \geq 5$.

Example

Assume that the number of asbestos particles in a square meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a square meter of dust is analyzed, what is the probability that 950 or fewer particles are found?

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(20) Greater than 2.5 milligrams

Example

20 A producer of cigarettes claims that the mean nicotine content in its cigarettes is 2.4 milligrams with a standard deviation of 0.2 milligrams. Assuming these figures are correct, approximate the probability that the sample mean of 100 randomly chosen cigarettes is

1. Greater than 2.5 milligrams
2. Less than 2.25 milligrams