Special Pythagorean triplet

Quoc-Tin Phan

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Problem. A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2$$

For example, $32 + 42 = 9 + 16 = 25 = 52$.

There exists exactly one Pythagorean triplet for which $a + b + c = 1000$. Find the product abc.

Solution.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (a+b)^2 - 2ab &= c^2 \end{aligned}$$

Since $a + b = 10^3 - c$, we have:

$$\begin{aligned} (10^3 - c)^2 - 2ab &= c^2 \\ ab + 10^2c &= 5 * 10^5 \\ ab &= 5 * 10^5 - 10^2c \end{aligned}$$

Given
$\begin{cases} a + b = 10^3 - c \\ ab \quad = 5 * 10^5 - 10^2c \end{cases}$
we can apply Viète theorem, a, b are solution of:

$$x^2 + (c - 10^3)x + (5 * 10^5 - 10^3c) = 0$$

which produces
$\begin{cases} a \ = \frac{10^3 - c - \sqrt{(c-10^3)^2 - 4*(5*10^5 - 10^3c)}}{2} \\ b = \frac{10^3 - c + \sqrt{(c-10^3)^2 - 4*(5*10^5 - 10^3c)}}{2} \end{cases}$
Since there exists only once solution, searching c in $[1, 999]$ gives us:

$$\mathbf{a = 200, b = 375, c = 425}$$