Recitation 5

Thursday September 24, 2024

1 Recap: Rank, column/row spaces, and factorization

If A is an $m \times n$ matrix with **rank** r, then:

- The rank is the number of pivots ($\neq 0$ by definition.
- The rank is *also* the dimension of the column space $C(A)$: the number of independent columns.
- The rank is *also* the dimension of the row space $C(A^T)$: the number of independent columns. The ranks of A and A^T are the same!
- You can write $A = CR$ where the $m \times r$ matrix C consists of *any* basis for $C(A)$, and R is $r \times n$. (That is, the columns of A are made from the columns of C.). At the same time, the columns of R are a basis for $C(A^T)$. In fact, the rank r can also be defined as the *smallest* number for which such a factorization is possible.
- The dimension of the nullspace $N(A)$ is $n - r$. (This is sometimes called the **"rank–nullity theorem"**, though we haven't been using that name in lecture.)

One possible basis for $C(A)$ is to take the pivot columns of A (i.e. look in U to find where the pivots are, and take the corresponding columns of A). In the long run, we will choose a different basis using the "SVD" of A, but this choice is nice because it tells us the relationship between the dimension $C(A)$ and the number of pivots.

Four important cases are:

- $m = n = r$: square, invertible matrices. $Ax = b$ always has a unique solution.
- $m = r \leq n$: full row rank ("wide" matrices). The solution exists (but is not unique if $n > m$).
- $n = r \leq m$: full column rank ("tall" matrices). The solution is unique (but may not exist if $m > n$).
- $r < m, n$: rank deficient (also called "low rank" informally). $Ax = b$ may not be solvable, and if it is solvable the solution won't be unique. (On the other hand, a CR-like factorization gives a very "compressed" representation of A if $r \ll m, n$.)

2 Exercises

1. If A is an $m \times n$ matrix of rank r, what is the dimension of $N(A^T)$ (the "left" nullspace)?

2. Let A and B be $n \times n$ matrices. For any sets P and Q, denote $P + Q = \{p + q \mid p \in P, q \in Q\}$. Determine whether the following statements are true or false.

   (a) $C(A + B) = C(A) + C(B)$

   (b) $N(A + B) = N(A) + N(B)$.

   (c) If $x \in C(A)$ and $x \in C(B)$, then $x \in C(A + B)$.

   (d) If $x \in N(A)$ and $x \in N(B)$, then $x \in N(A + B)$.

   (e) If $N(A) \cap C(B) \neq \{\mathbf{0}\}$, then AB is not invertible.

   (f) If $N(A) \supseteq C(B)$ then $AB = 0$.

   (g) A square matrix A is *diagonal* if $a_{ij} = 0$ when $i \neq j$. The rank of a diagonal matrix is the number of nonzero entries.

   (h) If P is an $n \times n$ permutation matrix, then $N(P) = \mathbb{R}^n$.

3. If w_1, w_2, w_3 are independent vectors in $\mathbb{R}^3$, show that the differences

$$\begin{aligned} v_1 &= w_2 - w_3 \\ v_2 &= w_1 - w_3 \\ v_3 &= w_1 - w_2. \end{aligned}$$

   are *dependent*. Find the matrix A so that

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}.$$

   Which matrices above are singular?

4. True or False (give a good reason if true/counterexample or reason if false)

   (a) If the zero vector is in the column space of a matrix A, then the columns of A are linearly dependent.

   (b) If the columns of a matrix are dependent, so are the rows.

   (c) The column space of a 2×2 matrix is the same as its row space.

   (d) The column space of a 2×2 matrix has the same dimension as its row space.

   (e) The columns of a matrix are a basis for the column space.

   (f) A and A^T have the same number of pivots.

   (g) A and A^T have the same left nullspace.

   (h) If the row space equals the column space then $A^T = A$.

   (i) If $A^T = -A$, then the row space of A equals the column space.

3 Solutions

1. A^T has the same rank r and is $n \times m$, so its nullspace $N(A^T)$ must have dimension $\boxed{m-r}$.

2. (a) False. A counterexample is in $\mathbb{R}$ when $A = \begin{bmatrix}1\end{bmatrix}$ and $B = \begin{bmatrix}-1\end{bmatrix}$. Then $C(A) = C(B) = \mathbb{R}$ which makes $C(A) + C(B) = \mathbb{R}$. However, $A + B = \begin{bmatrix}0\end{bmatrix}$ whose column space is $\{0\}$.

   (b) False. A counterexample is in $\mathbb{R}$ when $A = \begin{bmatrix}1\end{bmatrix}$ and $B = \begin{bmatrix}-1\end{bmatrix}$. Then $N(A) = N(B) = \{0\}$ which makes $N(A) + N(B) = \{0\}$. However, $A + B = \begin{bmatrix}0\end{bmatrix}$ whose null space is $\mathbb{R}$.

   (c) False. A counterexample is in $\mathbb{R}$ when $A = \begin{bmatrix}1\end{bmatrix}$, $B = \begin{bmatrix}-1\end{bmatrix}$, and $x = 1$. We see that $C(A) = C(B) = \mathbb{R}$ so $x \in C(A)$ and $x \in C(B)$. However, $A + B = \begin{bmatrix}0\end{bmatrix}$ whose column space does not contain $x = 1$.

   (d) True. If $x \in N(A)$ and $x \in N(B)$, then $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. Then we have $(A+B)x = Ax + bx = \mathbf{0} + \mathbf{0} = \mathbf{0}$ which means $x \in N(A+B)$.

   (e) True. If $N(A) \cap C(B) \neq \{\mathbf{0}\}$, then there must exists a non-zero vector $v \in N(A) \cap C(B)$. This means $v \in N(A)$ and $v \in C(B)$. So $Av = \mathbf{0}$ and there must exist another vector u which $Bu = v$. Note that if $u = \mathbf{0}$, then $v = \mathbf{0}$ which is contradictory; therefore $u \neq \mathbf{0}$. So we have $(AB)u = A(Bu) = Av = \mathbf{0}$ and $u \neq \mathbf{0}$. This implies $N(AB) \neq \{\mathbf{0}\}$ which means AB is not invertible.

   (f) True. For any vector x we have $Bx \in C(B) \subseteq N(A)$, and thus $ABx = 0$, which implies that AB is the zero matrix.

   (g) True. As long as they are nonzero, the column (or row) vectors are linearly independent.

   (h) False. If P is a permutation matrix, then $N(P) = \{0\}$, since $Px = 0$ implies $x = 0$.

3. To show that v_1, v_2, v_3 are dependent, we need to find a linear relation that they satisfy. Playing around, you can see that

$$v_1 - v_2 + v_3 = (w_2 - w_3) - (w_1 - w_3) + (w_1 - w_2) = 0.$$

   The matrix A is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

   The matrix $(v_1\ v_2\ v_3)$ is singular, and so is A (if A weren't singular, then it would be impossible for $(v_1\ v_2\ v_3)$ to be singular).

4. True or False (give a good reason if true/counterexample or reason if false)

   (a) If the zero vector is in the column space of a matrix A, then the columns of A are linearly dependent.
   Solution: False; $A = I$ is a counterexample. The zero vector is in the column span of every matrix, because the zero vector is in every subspace.

(b) If the columns of a matrix are dependent, so are the rows.
Solution: False. A counterexample is any matrix with more columns than rows, but full row rank, e.g.
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(c) The column space of a 2×2 matrix is the same as its row space.
Solution: False. Consider
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$
Then $C(A) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$, but $R(A) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$.

(d) The column space of a 2×2 matrix has the same dimension as its row space.
Solution: True. The dimensions of both spaces are the rank of A.

(e) The columns of a matrix are a basis for the column space.
Solution: False. The columns will always span the column space, but they may not be linearly independent. A counterexample is any matrix with a column of all 0's, or any matrix with more columns than rows.

(f) A and A^T have the same number of pivots.
Solution: True. The number of (nonzero) pivots is the rank of A, which is equal to the rank of A^T.

(g) A and A^T have the same left nullspace.
Solution: False. The left nullspace of A is $N(A^T)$. The left nullspace of A^T is $N(A)$. These are usually not equal; for example, if A is 2×3, then $N(A)$ is a subspace of $\mathbb{R}^3$ and $N(A^T)$ is a subspace of $\mathbb{R}^2$.

(h) If the row space equals the column space then $A^T = A$.
Solution: False. A counterexample is any invertible matrix which is not symmetric, like
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$
.

(i) If $A^T = -A$, then the row space of A equals the column space.
Solution: True. $C(A) = C(-A) = C(A^T) = R(A)$.