Recitation 4

Thursday September 19,, 2024

1 Recap: Null spaces, rank, and elimination

The nullspace $N(A)$ corresponds to all solutions of $Ax = 0$. In class, we showed that the nullspace is preserved by invertible row operations (i.e., multiplying by an invertible matrix on the left) such as those in Gaussian elimination, so if U is the upper-triangular matrix after Gaussian elimination, then $N(A) = N(U)$.

Also, if A is an $m \times n$ matrix with **rank** r (the number of pivots), then the result of Gaussian elimination generically looks like:

$$A \longrightarrow U = \begin{pmatrix} U_r & F \\ m - r \text{ rows of 0's} & \cdots \end{pmatrix}$$

where U_r is an $r \times r$ upper-triangular matrix containing the pivots (the "pivot columns"), F is an $r \times (n - r)$ matrix (the "free columns"), and below them are $m - r$ rows of zeros. Occasionally, if you run into a column of 0's during elimination, your pivot columns might be interleaved with the free columns, but you can always permute the variables to put the pivot columns together (just remember to permute the variables back at the end).

The key idea is that, when you are solving $Ux = 0$, you can partition the unknowns x into "pivot variables" $p \in \mathbb{R}^r$ and "free variables" $f \in \mathbb{R}^{n-r}$ that multiply the corresponding columns of U:

$$x = \begin{pmatrix} p \\ f \end{pmatrix}$$

and you can **solve for the pivot variables in terms of the free variables**. $Ux = 0 \Longleftrightarrow U_r p + Ff = 0$, which means that $p = U_r^{-1}(-Ff)$: for any value of f, you can just solve for p by backsubstitution. Plug them back into x and you have a vector in the null space.

To get a basis for the null space, we just need to a basis for f, solve for the corresponding p, and plug it back to find our basis vectors x of $N(U) = N(A)$. Picking the Cartesian basis $f = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}$ is an easy choice, and the corresponding basis x of the nullspace is called the "special solutions."

Key fact: The dimension of $N(A)$ is $n-r$, equal to the number of free variables.

Key fact: a set of vectors $x_1, x_2, \ldots, x_n$ are linearly independent if and only if the matrix $X = \begin{pmatrix} x_1 & x_2 & \cdots \end{pmatrix}$ with those columns has $N(X) = \{\vec{0}\}$, i.e. rank n. That is, no nontrivial combination of the columns gives zero.

2 Exercises

1. A is a 1000×10 matrix. The nullspace $N(A)$ is at *least* __-dimensional and at *most* __-dimensional. What about $N(A^T)$?

2. $AB = 0$ (the zero matrix) for two matrices A and B (of compatible sizes) if the null space of ___ is $=$ / $\subseteq$ / $\supseteq$ (choose one) the column space of ___?

3. Using Gaussian elimination, find a basis for $N(A)$ where $A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 2 & 3 & 6 & -1 \\ 1 & 2 & 3 & -1 \end{pmatrix}$.

   How would your answer change if the problem had swapped the second and third columns of A?

4. In class, we considered $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & -3 \\ 1 & 6 & 7 & -7 \end{pmatrix}$, transformed it by Gaussian elimination to $A \xrightarrow{r_2 - r_1,\, r_3 - r_1} (\cdots) \xrightarrow{r_3 - 2r_2} U = \begin{pmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{2} & 2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and solved for the basis $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ of $N(A) = N(U)$ (the "special solutions", with pivot variables in blue and free variables in red).

   (a) The nullspace is solutions to $Ax = 0$. Suppose that we instead want to solve $Ax = b$ for $b = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$. Show that elimination transforms this to $Ux = c$ for $c =$__, and find a solution x_0 by setting the free variables to *zero* and solving for the pivot variables.

   (b) Your solution x_0 was just *one* possible solution to $Ax = b$ (also called a "particular solution"). With the help of the null space basis above, give a formula for *any*/*all* possible solutions x to $Ax = b$. Your formula should be equivalent to different choices for the free variables in the previous step; why?

   (c) Is the set of all possible solutions x to $Ax = b$ in this problem a *subspace* of $\mathbb{R}^4$? Why or why not?

3 Solutions

1. The rank of A can be at *most* 10, since after 10 columns you can't have any more pivots. So $N(A)$ is at least $10 - 10 = 0$ dimensional, and could be at most 10-dimensional (if the matrix A consisted of all zeros, i.e. if the rank is zero). Conversely, A^T is a 10×1000 with a rank of at most 10, so $N(A^T)$ is at least $1000 - 10 = 990$ dimensional (for rank 10), and could be at most 1000-dimensional (if the rank is zero).

2. $AB = 0$ if $N(A) \supseteq C(B)$: if all the "outputs" of B (its column space) lie within the nullspace of A, so that A sends them to zero. Equivalently, AB consists of A multiplied by each column of B, so you get the zero matrix if each column of B (hence its column space) lies in $N(A)$.

3. (a) Elimination gives

$$A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 2 & 3 & 6 & -1 \\ 1 & 2 & 3 & -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1; r_3 - r_1} \begin{pmatrix} \boxed{1} & 1 & 3 & 0 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} \boxed{1} & 1 & 3 & 0 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U\,,$$

where the pivots are boxed. If we set $Ux = 0$ and solve for the pivot variables p in terms of the free variables f, we get the system:

$$\underbrace{\begin{pmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \end{pmatrix}}_{U_r} p = -\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}_{F} f\,.$$

following the notation from class. To get all possible solutions, we pick a basis for f, and as usual it is convenient to pick the Cartesian basis. Setting $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and solving for p by backsubstitution gives $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$. Setting $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and solving for p by backsubstitution gives $p = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Plugging these into x gives our basis for the nullspace:

$$N(A) = N(U) = \text{span of } \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

(You can easily plug these back in to check that $Ax = 0$.)

(b) If the problem had swapped the second and third columns of A, then we can see from the elimination steps that we would have encountered a zero in the 2nd column's pivot position and would have had to skip to the third column—so, we would have gotten the same U but with the second and third columns swapped. The pivot columns would have now been the first and *third* columns. But we would have done the *same* process to solve for the pivot variables from the free variables, just with slightly more annoying bookkeeping. And we would have gotten the same nullspace vectors, but with the second and third components swapped:

$$\text{new } N(A) = \text{span of } \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

4. In class, we considered $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & -3 \\ 1 & 6 & 7 & -7 \end{pmatrix}$, transformed it by Gaussian elimination to $A \xrightarrow{r_2 - r_1,\, r_3 - r_1} (\cdots) \xrightarrow{r_3 - 2r_2} U = \begin{pmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{2} & 2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and solved for the basis $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ of $N(A) = N(U)$ (the "special solutions", with pivot variables in blue and free variables in red).

   (a) If we represent the elimination steps by a matrix E multiplying on the left, so that $EA = U$, then what we will do to $Ax = b$ is to multiply both sides on the left by E, transforming it to $EAx = Eb$, or $Ux = c$ where $c = Eb$: we do the same elimination steps to b as we did to A. (This is what we *always* do to solve $Ax = b$, usually "augmenting" A with b to help us keep track if we are doing it by hand. Doing these elimination steps gives:

$$b = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \xrightarrow{r_2 - r_1,\, r_3 - r_1} \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = c$$

   If we now set the free variables to zero, we get an equation $Ux = c$ of the form:

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{2} & 2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

   The last row is all zeros on the left, but that is all right because the last row is also zero on the right. (This means that $b \in C(A)$.). The fact that the free variables are set to zero means that we can ignore the free columns, and just get a triangular system

$$\begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

   that we can easily solve by backsubstitution to find $p_2 = 1$ and $p_1 = 2$. Hence, our "particular" solution is

$$x_0 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(b) We can get more solutions by adding anything in the nullspace: any linear combination of our “special” solutions can be added to x_0, so the general form of the solution is:

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + f_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + f_2 \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

for any scalars $f_1, f_2 \in \mathbb{R}$. But adding these terms together shows that f_1 and f_2 exactly corresponds to solutions with those choices for the free variables!

(c) It is not a subspace, because it does not contain the origin: $x = 0$ not one of the solutions. (It is what is sometimes called an “affine hyperplane”.)