Recitation 3

Thursday September 12, 2024

1 Recap

1.1 Matrix Multiplication

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix,. We can multiply these matrices to obtain $C = AB$, the operation is valid because the inner dimensions of A and B match. The resulting matrix C has dimensions $m \times p$. There are multiple ways to interpret this matrix multiplication.

1. **Entry-wise:** For each $1 \le i \le m$ and $1 \le j \le p$, we have

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

2. **Inner product:** C_{ij} is the inner product of the i^{th} row in A and the j^{th} column in B. In other words, let's say x_i is the i^{th} row of A and b_j is the j^{th} column of B. Then,

$$C = \begin{bmatrix} — & x_1 & — \\ — & x_2 & — \\ & \vdots & \\ — & x_m & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x_1 \bullet b_1 & x_1 \bullet b_2 & \cdots & x_1 \bullet b_p \\ x_2 \bullet b_1 & x_2 \bullet b_2 & \cdots & x_2 \bullet b_p \\ \vdots & \vdots & \ddots & \cdots \\ x_m \bullet b_1 & x_m \bullet b_2 & \cdots & x_m \bullet b_p \end{bmatrix}.$$

3. **Column-wise:** the j^{th} column of matrix C is a matrix-vector product of A and the j^{th} column of B. In other words, if b_j is the j^{th} column of B, then

$$C = A \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \dots & Ab_p \\ | & | & & | \end{bmatrix}.$$

4. **Outer product:** C is the sum of the product of i^{th} column of A and the i^{th} row of B – ranging from $i = 1$ to n. In other words, let's say a_i is the i^{th} column of A and y_j is the j^{th} row of B. Then,

$$C = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} — & y_1 & — \\ — & y_2 & — \\ & \vdots & \\ — & y_n & — \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} | \\ a_i \\ | \end{bmatrix} \begin{bmatrix} — & y_i & — \end{bmatrix}.$$

Each $a_i y_i$ is an $m \times p$ matrix itself.

1.2 Properties of Matrix Multiplication

1. Associative: $A(BC) = (AB)C$.
2. Distributive:
   - $A(B + C) = AB + AC$
   - $(A + B)C = AC + BC$
3. Non-commutative (in general): $AB \neq BA$.
4. Identity Matrix: I_n is an $n \times n$ square matrix with 1's on the diagonal and 0's elsewhere. For any $m \times n$ matrix A, $I_m A = A$; $AI_n = A$.

2 Exercises

1. Let A be the following 3×4 matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

   (a) Find a 3×3 matrix P such that when it is left multiplying A, the result, denoted as $B = PA$, is A after subtracting the second row from the third row, and then swapping the first and second rows.

   (b) Find a 4×4 matrix Q such that when it is right multiplying A, the result, denoted as $C = AQ$, is A after dividing the first column of A by two, and then adding the first column to the second and third columns.

   (c) If we perform the operations in (a) and then the operations in (b) on a A, would the result be the same as first performing operations in (b) then operations in (a)?

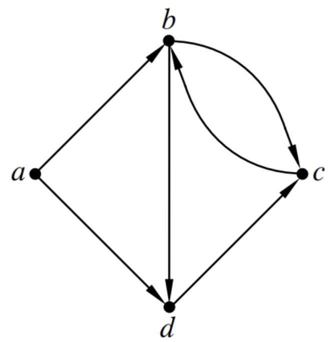
2. Let A and B be arbitrary $n \times n$ matrices. Indicate whether the following statements are True or False.

   (a) $AB = BA$.

   (b) If AB is a zero matrix (aka every entry is 0), then either A or B is a zero matrix.

   (c) If $AB = BA$, then either $A = I_n$ or $B = I_n$.

3. Counting Walks
   In this problem we will explore directed graphs where each edge points from one vertex, the “head", to another, the “tail". Now the only allowable walks will be ones that traverse the edges in the forwards direction. You can think of them as one-way streets.



(a) Write the adjacency matrix A for the graph. Note the edge from a to b is not the same as the edge from b to a.

(b) Are there walks of length 2 that start at node a and end at node b? If so, how many? What about from node a to c?

(c) Are there walks of length 3 that start at node a and end at b? If so, how many?

(d) How do you interpret $A + I_4$ from an aspect of the graph?

Hint: Adding I_4 is equivalent to adding 4 lines to the graph. But which lines are they?

(e) In class, we established that entries of A^2 represent the number of length-2 walks. What do the entries of $(A+I)^2$ represent? What can we tell if an entry is zero/non-zero?

(f) Suppose we have a gigantic graph G and we want to check if there exists any walk of length *at most* ℓ that goes from node u to v. How do we do that?

3 Solutions

1. (a) To find the matrix P that achieves the specified transformation, we can break it down into two steps and find the corresponding elementary matrices:

   (1) Subtract the second row from the third row of matrix A:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ -4 & 3 & 1 & 0 \end{bmatrix}$$

   (2) Swap the first and second rows of the resulting matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ -4 & 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 & 1 \\ 2 & 1 & 0 & 0 \\ -4 & 3 & 1 & 0 \end{bmatrix}$$

   So, the matrix P that accomplishes the desired transformation is:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

   Note that the order of the operations is important. When we apply the row operations represented by E_1, followed by those described by E_2, the resulting matrix representing these consecutive operations is denoted as $P = E_2E_1$. This choice is made because it results in the product PA being equivalent to $E_2(E_1A)$.

   (b) Similar to part (a) to how we can use right multiplication to apply column operations. First, we can break it down into two steps and find the corresponding elementary matrices. Note that these matrices are easy to find if you think about them in the column-wise product form.

   (1) Divide the first column of A by two:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 0 & -1 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

   (2) Add the first column to the second and third columns

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

   Now we want to apply both operations. When we apply the column operations represented by F_1, followed by those described by F_2, the resulting matrix representing these consecutive operations is denoted as $(AF_1)F_2 = A(F_1F_2)$. Therefore, the matrix Q that accomplishes the desired transformation is:

$$Q = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) The first operation is achieved by left-multiplying the matrix by P and the second operation is achieved by right-multiplying the matrix by Q. Thus, the result of applying row operations followed by column operations is $(PA)Q$, while applying column operations followed by row operations results in $P(AQ)$.

Because matrix multiplication is associative, the order of operations doesn't affect the final result.

2. (a) False. Matrix multiplication is non-commutative.

(b) False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(c) False. Counterexample: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So we have $AB = BA =$ zero matrix.

3. (a) The adjacency matrix A must have dimension 4×4 as we have 4 nodes. We can designate the first/second/third/forth rows and columns to be associated with nodes $a/b/c/d$ respectively.

Each entry is 0 or 1 – indicating whether there is a directed edge pointing from the associated row to column. For instance, we have $A_{2,1} = 0$ since there is no directed edge from node b to a. On the other hand, we have $A_{4,3} = 1$ since there is a directed edge from node d to c.

The full adjacency matrix is $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(b) We first compute $A^2 = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

Since $(A^2)_{1,2} = 0$, there is no length-2 walks of node a to b. Conversely, $(A^2)_{1,3} = 2$ means there are 2 walks of length 2 from node a to c.

(c) We can further compute $A^3 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

The number of walks of length 3 from node a to b is $(A^3)_{1,2} = 2$.

(d) Adding I_4 changes entries in A's main diagonal from 0 to 1. This makes $A_{1,1} = 1$ which means there exists a directed edge that goes out of a and into a. In other word, we can add a *self-loop* at node a. The same argument applies for the remaining three nodes.

Therefore, adding I_4 is equivalent to adding 1 self-loop at each of the 4 nodes.

(e) With the same reasoning, we can argue that entries of $(A + I)^2$ represent the the number of length-2 walks **which allow the use of self-loop(s)**. Using self-loops mean we increase the number of steps without actually moving. In other words, $(A + I)^2$ represent the the number walks of length *at most* 2.

However, we make a crucial note that such number of walks includes permutation of self-loops which means one walk might be counted more than once. For instance, going from a to b can be done by either 1) $a \to a \to b$, or 2) $a \to b \to b$.

For this reason, we cannot *exactly* count the number of walks of length at most ℓ by $(A+I)^\ell$. But we can pinpoint whether or not there *exists* a walk of length at most ℓ by comparing entries of $(A+I)^\ell$ to 0. If an entry is 0, there is no walk. If it is non-zero, there must exist at least one walk.

(f) The answer was already given above but here is a summary.

i. Build an adjacency matrix A.

ii. Add to A by an identity matrix I with proper size. So now we have $A+I$.

iii. Compute $(A+I)^l$.

iv. Find an entry of $(A+I)^\ell$ that corresponds to row u and column v. If it is zero, there is no walk of length at most ℓ. Otherwise if it is non-zero, at least 1 walk exists.