18.C06 Linear Algebra and Optimization
MIT Fall 2024

Recitation 1

Thursday September 10, 2024

Welcome to 18.C06. In this recitation, we will review some key points from Lecture 1 and do a few practice problems together. The remaining exercises are left to you for reference. Solutions will be released before next recitation.

A friendly reminder to sign up for Piazza which is our primary Q&A platform. Office hours are listed on Canvas.

1 Recap

Recall that the process of **Gaussian elimination** involves subtracting rows to turn a matrix A into an **upper triangular matrix** U. We also want to do the same row operations to the right-hand side b of an equation $Ax = b$ we want to solve; for hand calculations, it is convenient *augment* the matrix A with an additional column b. By doing the same row operations to both A and b, we arrive at an equivalent equation $Ux = c$ that is easy to solve by **backsubstitution** (solving for one variable at a time, working from the last row to the top row).

For example, suppose we are solving:

$$Ax = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & -1 \\ 3 & 11 & 6 \end{pmatrix} x = \begin{pmatrix} 9 \\ 1 \\ 35 \end{pmatrix} = b$$

We would perform the following elimination process.

$$\underbrace{\left[\begin{array}{rrr|r} \boxed{1} & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 6 & 35 \end{array}\right]}_{[\,A\,|\,b\,]} \xrightarrow{r_2 - r_1,\; r_3 - 3r_1} \left[\begin{array}{rrr|r} \boxed{1} & 3 & 1 & 9 \\ 0 & \boxed{-2} & -2 & -8 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow{r_3 + r_2} \underbrace{\left[\begin{array}{rrr|r} \boxed{1} & 3 & 1 & 9 \\ 0 & \boxed{-2} & -2 & -8 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]}_{[\,U\,|\,c\,]},$$

where we have indicated the row operations above the arrows (e.g. $r_3 - 3r_1$ means to subtract $3\times$ row 1 from row 3). The boxed values are known as the **pivots**. Now we do **backsubstitution** to solve $Ux = c$, working from the bottom up. The last row represents a single equation in a single unknown:

$$1x_3 = 0 \implies x_3 = 0.$$

Now that we know x_3, the second row gives:

$$-2x_2 - 2x_3 = -8 \implies -2x_2 - 0 = -8 \implies x_2 = 4.$$

Finally, now that we know x_2 and x_3, the first row gives:

$$1x_1 + 3x_2 + 1x_3 = 9 \implies x_1 + 12 + 0 = 9 \implies x_1 = -3.$$

If we get a zero in the pivot position (on the diagonal), we can't divide by it, but we can try to *swap* it with a subsequent row to get a non-zero pivot (this is just re-ordering the equations). If you can't do this—if *all* of the entries below the pivot are zero too, then the matrix is **singular** and $Ax = b$ might not have a solution (or otherwise has infinitely many solutions). We'll spend more time on this kind of thing later.

2 Exercises

1. In this exercise, we compute matrices associated to certain geometric transformations of vectors:
   (a) Find a 2×2 matrix such that when you multiply a 2-dimensional vector by it, the result is the *reflection* of the vector across the origin.
   (b) Find a 3×3 matrix such that when you multiply a 3-dimensional vector by it, it *swaps* the second and third coordinates of the vector.
   (c) If you have a 4×4 matrix A, which 4-dimensional vector x can you choose such that Ax is the second column of A?
2. A system of linear equations $Ax = b$ cannot have *exactly two* solutions. An easy way to see why: if two vectors x and $y \neq x$ are two solutions (i.e., $Ax = b$ and $Ay = b$), what is another solution? (Hint: $x + y$ is almost right.) (From Strang, section 2.2, problem 11.)
3. Use Gaussian elimination to convert the following matrix to upper-triangular form U:
$$A = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$
   What conditions on a, b, c guarantee that U has 3 nonzero pivots?
4. Consider the following *tridiagonal* matrix
$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}.$$
   Convert it to upper-triangular form U with Gaussian elimination. What do you notice about the pattern of nonzero entries in U, and the amount of work you had to perform compared to elimination on a generic 5×5 matrix?

3 Solutions

1. (a) A 2-dimensional vector $\begin{bmatrix} x \\ y \end{bmatrix}$ has a reflection across the origin

$$\begin{bmatrix} -x \\ -y \end{bmatrix} = x \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, the matrix representing this transformation is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) A 3-dimensional vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ must yield a result $\begin{bmatrix} x \\ z \\ y \end{bmatrix}$ which is

$$\begin{bmatrix} x \\ z \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Therefore, the matrix representing this transformation is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) Let's say $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then

$$\begin{aligned} Ax = \; & x_1 \cdot \big(A\text{'s } 1^{\text{st}} \text{ column}\big) + x_2 \cdot \big(A\text{'s } 2^{\text{nd}} \text{ column}\big) + \\ & x_3 \cdot \big(A\text{'s } 3^{\text{rd}} \text{ column}\big) + x_4 \cdot \big(A\text{'s } 4^{\text{th}} \text{ column}\big). \end{aligned}$$

Since we want Ax to be just the second column of A, it follows that $x_1 = x_3 = x_4 = 0$ and $x_2 = 1$. Therefore, $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

2. $Ax = b$ and $Ay = b$, so $A(x + y) = Ax + Ay = 2b$. [The key property that $A(x + y) = Ax + Ay$ a consequence of *linearity*, and is what makes matrix–vector multiplication a part of *linear* algebra.] But we want b on the right-hand side, so we can just divide both sides by 2: $A((x + y)/2) = b$, so $(x + y)/2$ is a solution. (Since $x \neq y$, this is a *new* solution, halfway between x and y.)

In fact, there are infinitely many solutions: anything on the line connecting x and y. Let $z = \alpha x + (1 - \alpha)y$ for any scalar α. Then z lies on the line connecting x and y, and in fact as α varies over all real numbers α the vector z traverses this entire line (check this on paper with your favorite vectors x and y in the plane!). Then z is another solution, again thanks to linearity:

$$Az = A(\alpha x + (1 - \alpha)y) = \alpha Ax + (1 - \alpha)Ay = \alpha b + (1 - \alpha)b = b.$$

3. We start by using Gaussian elimination to find U. Subtracting the first row from the second and third gives
$$\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix}.$$
Subtracting the second row from the third gives
$$\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} = U.$$
To guarantee U has 3 pivots, we need $a \neq 0$, $a \neq b$ and $b \neq c$.

4. We do Gaussian elimination to compute U:
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_2-2r_1} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 4 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_3-3r_2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$
$$\xrightarrow{r_4+2r_3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_5+r_4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = U.$$
We notice that U has nonzero entries only on its diagonal and one above each diagonal (this is called a "bidiagonal" matrix). And we only had to do *one* elimination step per column, and each elimination step only involved a small number of arithmetic operations since most of the entries in each row are zero. Because of this, Gaussian elimination on tridiagonal matrices scales *linearly* with the size n of the $n \times n$ matrix (as noted in class), compared to a generic matrix where the effort scales proportional to n^3.