Recitation 12

Tuesday Oct 22, 2024

1 Recap

1.1 Determinants

1.1.1 Algebraic View

The *determinant* of a square matrix $A \in \mathbb{R}^{n \times n}$ is given by

$$\det A = \sum_{\sigma} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)},$$

where the sum is over all permutations σ of the set $\{1, \ldots, n\}$, and $\mathrm{sign}(\sigma)$ denotes the parity (even or odd) of the permutation. While this is a closed formula, it is computationally impractical for large n, as it involves $n!$ terms, leading to $\sim n \times n!$ operations.
A more efficient way to compute the determinant is by transforming A to its upper triangular form U using Gaussian elimination, then taking the product of the pivot elements:

$$\det A = (\pm 1) \times (\text{product of pivots in } U),$$

where the sign is determined by the number of row swaps made during the elimination process. This method requires $\sim n^3$ operations, making it much more feasible for larger matrices (though an explicit determinant calculation is not needed that often in practice). The properties in 1.1.3 explains how row operations affect the determinant.

1.1.2 Geometric View

The determinant of a square matrix A is the signed scaling factor between $vol(S)$ and $vol(\phi_A(S))$, where $\phi_A(S) = \{Ax : x \in S\}$ is the region which ϕ_A maps S into.

1.1.3 Properties of Determinants

The following three axioms define the determinant:

1. **Normalization:** $\det(I) = 1$ for the identity matrix I.
2. **Swaps:** If you swap two rows or two columns of a matrix, the determinant flips sign.
3. **Linearity:** The determinant is linear in each row and each column. If a row or a column is scaled by a factor α, the determinant is scaled by α.

Many other properties can be derived from these three, for example:

4. **Repetition:** If a matrix has two identical rows or two identical columns, its determinant is zero.

5. **Row and column operations:** Subtracting a multiple of one row from another, or one column from another, does not change the determinant.

6. **Zeros:** A matrix with a row (or column) of zeros has a determinant of zero.

7. **Triangular matrices:** The determinant of a triangular matrix (upper or lower) is the product of its diagonal entries.

8. **Singularity:** A matrix is singular if and only if its determinant is zero.

9. **Multiplication:** The determinant of a product of matrices is the product of their determinants: $\det(AB) = \det(A)\det(B)$.

10. **Transpose:** The determinant of a matrix equals the determinant of its transpose: $\det(A) = \det(A^T)$.

1.2 Eigenvalues and Eigenvectors

1.2.1 Definition

Let A be a square $n \times n$ matrix. A nonzero vector $x \in \mathbb{R}^n$ is an *eigenvector* if and only if $Ax = \lambda x$ for some scalar λ. The scalars λ that satisfy this equation are referred to as *eigenvalues*.

1.2.2 Characteristic Polynomial

For a square matrix $A \in \mathbb{R}^{n\times n}$, its characteristic polynomial, denoted as $p(\lambda)$, is defined as:

$$p(\lambda) = \det(A - \lambda I) \text{ or } \det(\lambda I - A) \text{ (not all authors use the same sign)},$$

Observe that $\det(\lambda I - A)$ is a polynomial of degree n in λ. The roots to this polynomial, denoted as $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ (which may not all be distinct), are the eigenvalues of A. Therefore, a matrix $A \in \mathbb{R}^{n\times n}$ can have at most n distinct eigenvalues since they are the roots of a degree-n polynomial. Once you know an eigenvalue λ, corresponding eigenvectors are a basis for $N(A - \lambda I)$. (For the common case of non-repeated roots, this nullspace is 1d and you only need one eigenvector.)

2 Exercises

1. Using properties 1-3 in 1.1.3, prove property 6, which is that a matrix with a row (or column) of zeros has a determinant of zero.

2. Let A be the following 2×2 matrix:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Using your answer in part (b), find the eigenvalues and eigenvectors for the following matrices:

      i. $2A$

      ii. A^2

      iii. A^{-1}

      iv. $A + 4I$

   (c) Now suppose that A and B are $n \times n$ matrices with known eigenvalues and eigenvectors, and let c be a scalar. For each of the matrices cA, A^{2024}, A^{-1} (if A is invertible), $A + cI$, AB, and $A + B$, can you express the eigenvalues and eigenvectors in terms of those of A and B?

3. Describe as many eigenvalues and corresponding eigenvectors as you can (without doing any serious calculation) for the following matrices:

   a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

   b) A projection matrix P onto some subspace S of $\mathbb{R}^3$.

   c) The reflection matrix in $\mathbb{R}^2$ around the line $L : y = x$.

   d) The permutation matrix $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

   e) A rank one matrix uv^T (with specific vectors u, v in $\mathbb{R}^3$)

4. True, false, or neither (that is, sometimes true, sometimes false):

   a) If v, w are eigenvectors of A, then so is $v + w$ and cv for c any scalar $\neq 0$.

   b) If $v \in N(A)$ is not the zero vector, then v is an eigenvector of A.

3 Solutions

1. Assume that the matrix is M, if we multiply the zero row by zero, the matrix wouldn't change. Therefore, using property 3, the determinant is scaled by 0. More explicitly
$$\det(M) = 0 \cdot \det(M) \Rightarrow \det(M) = 0$$

2. (a) To find the eigenvalues and eigenvectors of A, first consider the characteristic polynomial:$\det(\lambda I - A) = 0$, where λ is the eigenvalue and I is the identity matrix. Solving this equation:
$$\begin{aligned}\det(\lambda I - A) &= \det\left(\begin{bmatrix}\lambda - 2 & 1 \\ 1 & \lambda - 2\end{bmatrix}\right) \\ &= (\lambda - 2)^2 - 1^2 = 0 \\ &\Rightarrow (\lambda - 1)(\lambda - 3) = 0\end{aligned}$$
Therefore, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. To find the corresponding eigenvectors, we solve the equation $Av = \lambda v$ for each eigenvalue. For $\lambda_1 = 1$, solving $(A - I)v_1 = 0$:
$$\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix} v_1 = 0$$
leads to many solutions, one of them is the eigenvector $v_1 = \begin{bmatrix}1 \\ 1\end{bmatrix}$.
Similarly, for $\lambda_2 = 3$, solving $(A - 3I)v_2 = 0$:
$$\begin{bmatrix}-1 & -1 \\ -1 & -1\end{bmatrix} v_2 = 0$$
This results in many solutions, one of them is the eigenvector $v_2 = \begin{bmatrix}1 \\ -1\end{bmatrix}$.
Therefore, the eigenvalues and eigenvectors of matrix A are:
$$\lambda_1 = 1, \quad v_1 = \begin{bmatrix}1 \\ 1\end{bmatrix} \quad \text{and} \quad \lambda_2 = 3, \quad v_2 = \begin{bmatrix}1 \\ -1\end{bmatrix}$$

   (b) For the following parts, assume that v and λ are corresponding eigenvalue and eigenvector for A

      i. For the matrix $2A$, note that
$$Av = \lambda v \Rightarrow 2Av = (2\lambda)v$$
The eigenvalues for $2A$ are double the values of those for A, i.e., $\lambda = 2$ and $\lambda = 6$, with the same eigenvectors v_1 and v_2 from part (a).

      ii. For the matrix A^2, we can multiply the eigenvalue equation by A:
$$A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$$
The eigenvalues for A^2 are the square of the eigenvalues for A, resulting in λ^2. Therefore, the eigenvalues are 1 and 9, with the same eigenvectors v_1 and v_2 from part (a).

iii. For the matrix A^{-1}, we multiply the eigenvalue equation for A by A^{-1}:

$$Av = \lambda v \Rightarrow A^{-1}(\lambda v) = v \Rightarrow A^{-1}v = \frac{1}{\lambda}v$$

The eigenvalues for A^{-1} are the reciprocals of the eigenvalues for A. Thus, the eigenvalues are 1 and $\frac{1}{3}$, with the same eigenvectors v_1 and v_2 from part (a).

iv. For the matrix $A + 4I$:

$$Av = \lambda v \Rightarrow (A + 4I)v = Av + 4v = (\lambda + 4)v$$

The eigenvalues for $A+4I$ are obtained by adding 4 to the eigenvalues for A. Thus, the eigenvalues are 5 and 7, with the same eigenvectors v_1 and v_2 from part (a).

(c) Assume that A has eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvector and $v_1, \dots, v_n$. Using the same proofs from part (b), we can show that

cA: the eigenvalues are $c\lambda_1, \dots, c\lambda_n$ the eigenvectors remain the same.

A^{2024}: the eigenvalues are $\lambda_1^{2024}, \lambda_2^{2024}, \dots, \lambda_n^{2024}$.

A^{-1}: (assuming A is invertible): the eigenvalues are $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$, and the eigenvectors remain the same.

$A + cI$: the eigenvalues are $\lambda_1 + c, \dots, \lambda_n + c$, and the eigenvectors remain the same as the eigenvectors of A.

A^T: Can't say anything about the eigenvectors in general, but the eigenvalues will be the same. The eigenvalues of A are the roots of $\det(A - \lambda I)$ and the determinant of a matrix and its transpose are equal, so $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$.

AB Can't say anything in general. If A and B happen to have an eigenvector, say v, in common, then it will be an eigenvector of AB (and the eigenvalues will multiply).

$A + B$ Can't say anything in general. Again, if A and B happen to have an eigenvector, say v, in common, then v will be an eigenvector of $A + B$, and the eigenvalues will add.

3. (a) For $A = I$, every vector is an eigenvector since $Ix = x$ for any x, with eigenvalue 1. For the upper-triangular matrix, an easy eigenvector is $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$, with eigenvalue 4. The other eigenvectors are more difficult to find.

(b) The matrix P fixes every vector in S, so all vectors in S are eigenvectors with eigenvalue 1. On the other hand, P maps every vector in $S^\perp$ to 0, making them eigenvectors with eigenvalue 0.

(c) Any vector along L, such as $\begin{bmatrix}1 & 1\end{bmatrix}^T$, is mapped to itself, making it an eigenvector with eigenvalue 1. Similarly, any vector w orthogonal to L is mapped to $-w$, making it an eigenvector with eigenvalue -1.

(d) The matrix M fixes the vector $v = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$, so v is an eigenvector with eigenvalue 1. The other eigenvalues and eigenvectors are more complex.

(e) Every nonzero vector in the nullspace is an eigenvector with eigenvalue 0. If q is not in the nullspace, then $(uv^T)q = (v^Tq)u$, meaning $(uv^T)q$ is always a scalar multiple of u. Thus, the only eigenvectors are scalar multiples of u, with eigenvalue v^Tu.

4. (a) The vector cv will always be an eigenvector, with the same eigenvalue as v. If v, w are eigenvectors with the *same* eigenvalue, then $A(v+w) = Av + Aw = \lambda v + \lambda w = \lambda(v+w)$, so $v+w$ is another eigenvector (as long as it's nonzero). If v, w have *different* eigenvalues, then $v+w$ is *not* an eigenvector.

   (b) True, since v is in $N(A)$, $Av = 0 = 0 \cdot v$, so v is an eigenvector with eigenvalue 0.