Recitation 3

Tuesday September 16,, 2024

1 Recap

1.1 Vector Space

A vector space V is a set of elements (e.g., vectors in $\mathbb{R}^n$, polynomials, and diagonal 2×2 matrices), and these elements are referred to as "vectors." Furthermore, a vector space is defined over a "field" F of scalars (e.g., real or complex numbers). Vector spaces must be closed under the following two operations:

1. **Vector Addition/Subtraction**: For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V, their sum $\mathbf{u} + \mathbf{v}$ and difference $\mathbf{u} - \mathbf{v}$ (sum of additive inverse) must also belong to the vector space V. (Hence $\mathbf{u} - \mathbf{u} = \mathbf{0}$ must also be in the vector space.)

2. **Scalar Multiplication**: For any vector $\mathbf{v}$ in V and any scalar c in F (e.g., a real or complex number), the product $c\mathbf{v}$ must also belong to the vector space. (Note that $(-1)\mathbf{v} = -\mathbf{v}$ turns addition into subtraction: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$. And $0\mathbf{u} = \mathbf{0}$.)

If $W \subseteq V$ is also a vector space with respect to the operations in V, then W is called a *subspace* of V.

Key Fact. If S_1 and S_2 both are vector space, then $S = S_1 \cap S_2$ is a subspace.

1.2 Column Space

The **column space** of an $m \times n$ matrix A, denoted $C(A)$, is the set of all **linear combinations** of columns of A, or the **span** of A.

- $C(A) = \{Ax \mid x \in \mathbb{R}^n\}$.
- $Ax = b$ has a solution if and only if $b \in C(A)$.
- If $m = n$, then A is invertible if and only if $C(A) = \mathbb{R}^n$.

1.3 Null Space

The **null space** of A, denoted $N(A)$, is the set of vectors x such that $Ax = 0$.

- $N(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$.
- If B is a square and invertible matrix, then $N(A) = N(BA)$.
- If A is $n \times n$, then $C(A) = \mathbb{R}^n$ is equivalent to $N(A) = \{\mathbf{0}\}$ which is also equivalent to A being invertible.

2 Exercises

1. Let A be a 3×3 matrix, and let x and b be 3-dimensional vectors such that $Ax = b$.
   (a) If A is invertible, can there be a non-trivial solution for $b = \mathbf{0}$?
   (b) Assume that when $b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$, there are infinitely many solutions. Is b in the column space of A? Is A invertible?
   (c) Assume that when $b = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T$, there is a unique solution x. Is b in the column space of A? Is A invertible?

2. If $A = uv^T$ where $u = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then
   (a) Write A explicitly in terms of its entries.
   (b) Describe the column space $C(A)$. (Not much calculation required!)
   (c) Describe the row space $C(A^T)$. (Not much calculation required!)
   (d) Describe the null space $N(A)$. (Not much calculation required! Not just $\{\mathbf{0}\}$!)

3. For each of the following, is V a real vector space?
   (a) V is the set of all n-dimensional vectors with positive entries, with usual vector operations.
   (b) V is the set of all n-dimensional vectors whose elements sum to 0, with usual vector operations.
   (c) V is the set of all $n \times n$ diagonal matrices, with usual matrix operations.
   (d) V is the set of all polynomials with degree up to d, with usual polynomial operations.
   (e) V is the set of all constant functions, i.e. $f(x) = c$ for some constant $c \in \mathbb{R}$, with usual real number operations.
   (f) V is the set of all single-variable polynomial whose value at 0 is 1, i.e. polynomial P with $P(0) = 1$, with usual polynomial operations.

4. True or False (give a good reason if true, and a counterexample or reason if false):
   (a) If the zero vector is in the column space of a matrix A, then the columns of A are linearly dependent.
   (b) The columns of a matrix are a basis for the column space.
   (c) Define the row space of matrix A as the span of the row vectors of A. If A is a square matrix, then the row space of A equals the column space.
   (d) The row space of A is equal to the column space of A^T.
   (e) If the row space of A equals the column space, then $A^T = A$.
   (f) A 4 by 4 permutation matrix has column space equal to $\mathbb{R}^4$.
   (g) Let $v \in N(A)$. If x is a solution to equation $Ax = b$, so is $x + v$.

3 Solutions

1. (a) If A is invertible, we can solve $Ax = \mathbf{0}$ by left-multiplying both sides by A^{-1}, which gives $x = \mathbf{0}$, that is a unique solution (the trivial solution only).

   (b) Consider one solution x such that $Ax = b$. Recall that Ax can be written as a linear combination of the columns of A, which implies that $b \in C(A)$. A is *not* invertible. Now, consider two distinct solutions u and v of $Ax = b$. This gives $Au = Av = b$. Therefore, $A(u - v) = \mathbf{0}$, which implies $u - v \in N(A)$. In addition, $u - v$ is not a zero vector since u and v are distinct. Thus, $N(A)$ is non-trivial, implying that A is not invertible.

   (c) If $Ax = b$ has a unique solution x_0, then $N(A) = \{\mathbf{0}\}$ because if y were a non-trivial vector in the null space of A, then

   $$A(x_0 + y) = Ax_0 + Ay = b + \mathbf{0} = b,$$

   which means that $x_0 + y$ is another solution for $Ax = 0$ that is different from y, which is impossible. Therefore, $N(A) = \{\mathbf{0}\}$, indicating that A is invertible.

2. (a) $A = uv^T = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} = \begin{pmatrix} 2u & u \end{pmatrix} = \begin{pmatrix} v^T \\ 2v^T \\ -v^T \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{pmatrix}$.

   (b) Every column of A is a multiple of u, so $C(A)$ is just the 1d subspace of $\mathbb{R}^3$ spanned by u (the line parallel to u through the origin).

   (c) Every column of A^T (from the rows of A) is a multiple of v, so $C(A^T)$ is just the 1d subspace of $\mathbb{R}^2$ spanned by v (the line parallel to v through the origin).

   (d) Since $A = \begin{pmatrix} 2u & u \end{pmatrix}$, we can get zero by subtracting twice the second column from the first, or any multiple thereof. That is, $A \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. A little thought tells us that multiples of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are the *only* way to get zero from the columns of A. So, $N(A)$ is the 1d subspace of $\mathbb{R}^2$ spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (i.e. the line through the origin parallel to this vector).

3. (a) No, it's not possible. Consider the vector $v = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$. All entries of v are positive, so v belongs to V. However, $(-1) \cdot v = \begin{bmatrix} -1 & -1 & \dots & -1 \end{bmatrix}^T$ has negative entries, making it *not* in V.

   (b) Yes, we can. If a vector v has elements that sum to 0, then λv also has elements that sum to $\lambda \cdot 0 = 0$. Moreover, if we have another vector u whose elements also sum to 0, then $u + v$ has elements that sum to u's sum + v's sum, which is $0 + 0 = 0$.

   (c) Yes, it is. The sum of two diagonal matrices is still a diagonal matrix, and the scalar multiple of a diagonal matrix is also a diagonal matrix.

   (d) Yes, they are. The sum of two polynomials with degrees up to d also has a degree at most d, and scalar multiplication does not change the degree.

(e) Yes, they are. For any two constant functions $f(x) = c_1$ and $g(x) = c_2$, and any real numbers a and b, $af(x) = ac_1$ is a constant function, and $f(x) + g(x) = c_1 + c_2$ is a constant function.

(f) No, it's not. If $c = 1$ and we have a polynomial P such that $P(0) = 1$, then a scalar multiple of it $Q = 5P$ has $Q(0) = 5P(0) = 5$, which violates the scalar multiplication condition. Furthermore, if we have two polynomials P and Q with $P(0) = Q(0) = 1$, then $(P + Q)(0) = P(0) + Q(0) = 1 + 1 = 2$, which violates the vector addition condition as well.

4. (a) False; $A = I$ is a counterexample. The zero vector is in the column span of every matrix, because the zero vector is in every subspace.

(b) False. The columns will always span the column space, but they may not be linearly independent. A counterexample is any matrix with a column of all 0's, or any matrix with more columns than rows.

(c) False. A counter example is $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ where the column space is the multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and the row space are the multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which are different spaces.

(d) True. The set of rows of A is identical to the set of columns of A^T.

(e) False. Counter example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Both the column and row space are equal to $\mathbb{R}^2$, but $A \neq A^T$.

(f) True. Any permutation matrix is invertible which implies that its span is $\mathbb{R}^4$.

(g) True. If $v \in N(A)$, then $Av = \mathbf{0}$. As x is a solution to $Ax = b$, we have $Ax = b$. This gives us $A(x + v) = Ax + Av = b + \mathbf{0} = b$ which means $x + v$ is also a solutions.