Jacobian Matrix

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If we consider a function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and vector $\boldsymbol{x} = [x_1, ..., x_n]^\mathsf{T} \in \mathbb{R}^n$, the coresponding vector of function values is $\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^m$. It is like a vector of functions $[f_1, ..., f_m]^\mathsf{T}$ for $f_i : \mathbb{R}^n \to \mathbb{R}$. We can say then that

$$\frac{\partial \boldsymbol{f}}{\partial x_i} = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_m}{\partial x_i} \end{pmatrix} \in \mathbb{R}^m$$

which implies

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} &= \begin{pmatrix} \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_m(\boldsymbol{x})}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n}. \end{aligned}$$

Jacobian DEFINITION 0.0.1

The Jacbian is defined as the collection of all first-order partial derivatives of $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$.

$$\boldsymbol{J} = \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}}$$

where $J(i, j) = \frac{\partial f_i}{\partial x_j}$