Differential Equations

Jack David Carson - February 2, 2025

01 Introduction

Ordinary Differential Equations DEFINITION 1

A differential equation is an equation between specified derivative on an unknown function, its values, and known quantities and functions.

Ordinary DEs are classified according to their order. The order of a DE is defined as the largest positive integer, n, for which an nth derivative occurs in the equation.

First we can say that any variables we are differentiating with respect to are *independent variables*, and any variables we are differentiating are *dependent variables*. For instance in $\frac{\mathrm{d}y}{\mathrm{d}t}$, t is the ind. variable and y is the dep. variable. For some quick notation we can remind ourselevs that $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ and $y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and more generally $y^{(n)} = \frac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

A linear differential equation has a predictable form

$$\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}$$

where $\boldsymbol{a}(x)$ is a function of only the independent variable. Anything that *doesn't* look like this is a non-linear DE. In general linear DEs tend to have closed form solutions, while nonlinear ones often do not. In other words, for an nth order DE, $F\left(x, y, y', y'', \cdots, y^{(n)}\right) = 0$

We say a DE is *homogeneous* if $b(x) = 0$ and inhomogeneous for $b(x) \neq 0$ for the "constant term" $a_1(x) = b(x)$.

02 First Order Equations

All first order equations can be spsecified in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$. It is considered *seperable* if I can say $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$. We can generally solve these by nicely saying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$
$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

which will have a solution.

Similarly, all first order linear homogenous equations with constant coefficients in form $\frac{\mathrm{d}y}{\mathrm{d}x} + ky = 0$ can be expressed as

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -ky \\
\int \frac{1}{y}\,\mathrm{d}y &= -\int k\,\mathrm{d}x \\
\ln|y| + c_1 &= -kx + c_2 \\
|y| &= e^{-kx+c}
\end{aligned}$$

We can say then that $|y| = e^c e^{-kx}$ or $y = \pm C e^{-kx}$.

Theorem: Lets suppose we have two solutions $y_1(x), y_2(x)$ to a linear differential equation $y' + a(x)y = 0$. The linear combination $y = c_1 y_1(x) + c_2 y_2(x)$ is also a solution to $y' + a(x)y = 0$. This is true of all *linear* DE's. In the inhomogenous case where we have solutions y_1, y_2 to $y' + a(x)y = b_i(x)$, our sum $y_1(x) + y_2(x)$ is the solution to $y' + a(x)y = b_1(x) + b_2(x)$

03 First Order, Linear, Inhomogenous

A first order, linear, inhomogenous equation will be of the form $y' + a(x)y = b(x)$. Let's suppose we have a solution $y_{\text{hom}}(x)$ for the homogeneous equation. We have some mystery function $\varphi(x)$ such that $y = \varphi(x) y_{\text{hom}}(x)$ is a solution the inhomogenous DE. We call this method **Variation of Parameters**.

$xy' + 2xy = x^2$ EXAMPLE 1

In a more standard form we could divide by x to say $y' + 2y = x$. The hommogeneous version now is clearly $y' + 2y = 0$ such that $y_{\text{hom}}(x) = e^{-2x}$. We now introduce $\varphi(x)$ such that $y = \varphi(x)e^{-2x}$ and plug y into $y' + a(x)y = b(x)$ to get

$$\begin{aligned}
(\varphi(x)e^{-2x})' + 2\varphi(x)e^{-2x} &= x \\
\varphi'(x)e^{-2x} - 2e^{-2x}\varphi(x) + 2\varphi(x)e^{-2x} &= x \\
\varphi'(x)e^{-2x} &= x
\end{aligned}$$

or $\varphi'(x) = xe^{2x}$ and $\varphi(x) = \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x} + C$ such that our general solution is

$$y(x) = \frac{1}{2}x - \frac{1}{4} + Ce^{2x}$$