Linear Mappings

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01 Algebra

Vector Space DEFINITION 1

A vector space is any collection of objects that can be added together and multiplied by a scalar, and the resulting object is still a vector.

This property can be specified by considering two vector spaces V, W and a mapping $\Phi : V \to W$ that preserves the structure of the vector space if and only if

$$\Phi(\boldsymbol{x} + \boldsymbol{y}) = \Phi(\boldsymbol{x}) + \Phi(\boldsymbol{y})$$
$$\Phi(\lambda \boldsymbol{x}) = \lambda \Phi(\boldsymbol{x})$$

for *all* $\boldsymbol{x}, \boldsymbol{y} \in V$ and $\lambda \in \mathbb{R}$. From this we can define a linear mapping

Linear Mapping DEFINITION 2

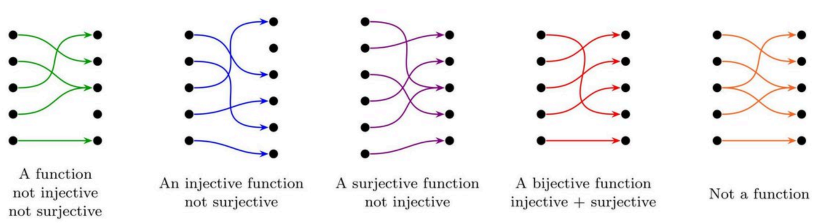
A linear mapping, also known as a *vector space homomorphism* or *linear transform* $\Phi : V \to W$ is defined for

$$\forall \boldsymbol{x}, \boldsymbol{y} \in V \ \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda \boldsymbol{x} + \psi \boldsymbol{y}) = \lambda \Phi(\boldsymbol{x}) + \psi \Phi(\boldsymbol{y})$$

which essentialy reads for all vectors $\boldsymbol{x}, \boldsymbol{y}$ in V and all λ, ψ in $\mathbb{R}$ the linear mapping Φ satisfies the scalar multiplication and additivity constraints.

These definitions are useful because we can define certain special mappings.

- **Injective** if $\forall \boldsymbol{x}, \boldsymbol{y} \in V : \Phi(\boldsymbol{x}) = \Phi(\boldsymbol{y}) \Rightarrow \boldsymbol{x} = \boldsymbol{y}$
- **Surjective** if $\Phi(V) = W$
- **Bijective** if Φ is both injective and surjective



This specifies that if Φ is surjective, then every element in W can be “reached” from $\Phi(V)$. For a bijective Φ, we can furthermore define an inverse mapping $\Psi : W \to V$ such that $\Psi \circ \Phi(\boldsymbol{x}) = \boldsymbol{x}$.

Then, we can also introduce some special cases of linear mappings
- **Isomorphism:** $\Phi : V \to W$ is linear and bijective
- **Endomorphism:** $\Phi : V \to V$ is linear
- **Automorphism:** $\Phi : V \to V$ linear and bijective

02 Matrix Representation

Based on this we can state the theorem that *finite-dimensional vector spaces V and W are ismorphic if and only if* $\dim(V) = \dim(W)$. From hence we can define a matrix from first principles as representing an n-dimensional vector space isomorphic to $\mathbb{R}^n$. Specifically, we consider a basis $\{\boldsymbol{b}_1, ..., \boldsymbol{b}_n\}$ of an n -dimensional vector space V isomorphic to $\mathbb{R}^n$. As we are familiar with in linear algebra, this basis represents a linear transformation on a coordinate system.

Remark: An **ordered** basis is a special form of basis in which the orer of vectors matters. Our canonical linear algebra intuition relies on this constraint, since matrix $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] \neq [\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1]$. However it is rarely necessary for understanding vector spaces. We specify ordered basis $B = (\boldsymbol{b}_1, ..., \boldsymbol{b}_n)$ and unordered basis $\mathcal{B} = \{\boldsymbol{b}_1, ...\boldsymbol{b}_n\}$

Transformation Matrix Definition 3

Consider vector spaces V, W with ordered bases $B = (\boldsymbol{b}_1, ..., \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, ..., \boldsymbol{c}_n)$, scalars $\boldsymbol{\alpha} \in \mathbb{R}^n$ and a linear mapping $\Phi : V \to W$ such that

$$\Phi(\boldsymbol{b}_j) = \alpha_{1j}\boldsymbol{c}_1 + \cdots + \alpha_{mj}\boldsymbol{c}_m = \sum_{i=1}^{m} \alpha_{ij}\boldsymbol{c}_i$$

The result of this in matrix form will be the *transformation matrix* corresponding to Φ (with respect to the chosen bases B and C) which we call $\boldsymbol{A}_{\boldsymbol{\Phi}}$ such that for $\boldsymbol{y} = \Phi(\boldsymbol{x}) \in W$,

$$\boldsymbol{A}_{\boldsymbol{\Phi}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$$

And just like that we've gone from algebra to linear algebra.

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