MISC034: Abelian gauge symmetries in matrix product states

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1 Model

Consider the Schwinger model

$$\begin{aligned} H = x \sum_j [\sigma^+_j e^{i\theta_j} \sigma^-_{j+1} + h.c.] + \frac{1}{2}\mu \sum_j [1 + (-1)^j \sigma^z_j] \\ + \sum_j (i\partial_{\theta_j} - \alpha)^2 \end{aligned} \quad . \tag{1}$$

This acts on a 1D chain with a spin-1/2 $\mathbb{C}^2$ σ^z_j on each site and an $O(2)$ rotor θ_j on each link; label each link by the site to its left. The operator $(i\partial_{\theta_j})$ is an electric flux.

The model has two symmetries: a physical $U(1)$ generated by

$$N = \sum_j n_j \tag{2}$$

defining $n_j = \frac{1}{2}(1 + \sigma^z_j)$ and an onsite gauge symmetry

$$G_j = i\partial_{\theta_{j-1}} - i\partial_{\theta_j} + n_j \ , \quad G_1 = -i\partial_{\theta_1} + n_1 \ . \tag{3}$$

Work in superselection sector given by

$$G_j \left|\psi\right\rangle = g_j \left|\psi\right\rangle \tag{4}$$

for some set of integers g_j.

2 Organizing the Hilbert space for matrix product states

Take the computational basis states to be simultaneous eigenstates of each spin σ^z_j (with eigenvalue s_j) and each flux $i\partial_{\theta_j}$ (with eigenvalue E_j). Group each $O(2)$ rotor (gauge boson) with the site to the left: that is, think of these computational basis states as

$$\begin{aligned} &\left|s_1 E_1 s_2 E_2 s_3 E_3 \dots\right\rangle \\ &= \left|(s_1 E_1)(s_2 E_2)(s_3 E_3) \dots\right\rangle \ \in \ (\mathbb{C}^2 \otimes \mathbb{C}^\infty)^L \end{aligned} \tag{5}$$

where I write L for system size.

3 Blocks of the MPS tensors

Schmidt decompose the system at bond l:

$$\left|\psi\right\rangle = \sum \left|u^{(l)}_\alpha\right\rangle s_\alpha \left|v^{(l+1)}_\alpha\right\rangle \ . \tag{6}$$

The reduced density matrix on sites $1 \dots l$ is

$$\rho^{(l)} = \sum \left|u^{(l)}_\alpha\right\rangle\!\left\langle u^{(l)}_\alpha\right| s^2_\alpha \ . \tag{7}$$

The Schmidt vectors $\left|u^{(l)}_\alpha\right\rangle$ must be eigenstates of $N^{(l)} = \sum_{j \le l} n_j$ with eigenvalues $N^{(l)}_\alpha$

$$N^{(l)} \left|u^{(l)}_\alpha\right\rangle = N^{(l)}_\alpha \left|u^{(l)}_\alpha\right\rangle \tag{8}$$

(The easiest way to see this is to note that N generates a tensor product unitary

$$e^{iN\varphi} = \prod_j [e^{in_j\varphi}] = e^{iN^{(l)}\varphi} \prod_{j>l} [e^{in_j\varphi}] \ ; \tag{9}$$

the product on $j > l$ disappears when we take the partial trace.)

The $\left|u^{(l)}_\alpha\right\rangle$ must also be eigenvalues of all the G_j and consequently of their sum:

$$\sum_{j \le l} G_j = N^{(l)} - i\partial_{\theta_l} \ ; \tag{10}$$

since they are eigenvalues of $N^{(l)}$, they must also be eigenvalues of $i\partial_{\theta_l}$

$$i\partial_{\theta_l} \left|u^{(l)}_\alpha\right\rangle = E^{(l)}_\alpha \left|u^{(l)}_\alpha\right\rangle \ . \tag{11}$$

Now Schmidt decompose at bond $l+1$

$$\left|\psi\right\rangle = \sum \left|u^{(l+1)}_\alpha\right\rangle s_\alpha \left|v^{(l+2)}_\alpha\right\rangle \ . \tag{12}$$

Once again

$$i\partial_{\theta_{l+1}} \left|u^{(l+1)}_\alpha\right\rangle = E^{(l+1)}_\alpha \left|u^{(l+1)}_\alpha\right\rangle \ . \tag{13}$$

But we can also write

$$\left|u^{(l+1)}_\alpha\right\rangle = \left|u^{(l)}_\beta\right\rangle \left|sE\right\rangle M^{sE}_{\beta\alpha} \ ; \tag{14}$$

M is precisely an MPS tensor in the appropriate gauge. Since $\left|u^{(l)}_\beta\right\rangle$ is an eigenvalue of $i\partial_{\theta_l}$ with eigenvalue $E^{(l)}_\beta$,

$$g_j \left|u^{(l+1)}_\alpha\right\rangle = G_j \left|u^{(l+1)}_\alpha\right\rangle \tag{15}$$

becomes

$$g_j \left|u^{(l)}_\beta\right\rangle \left|sE\right\rangle M^{sE}_{\beta\alpha} = (E^{(l)}_\beta - E + s) \left|u^{(l)}_\beta\right\rangle \left|sE\right\rangle M^{sE}_{\beta\alpha} \tag{16}$$

for

$$M^{sE}_{\beta\alpha} \neq 0 \quad \text{iff} \quad (E^{(l)}_\beta - E + s - g_j) = 0 \tag{17}$$