MISC036: Some thoughts on leveraging MPSs for preparation of gauge-invariant states

Christopher White

April 21, 2020

tl;dr: We can turn an MPS into a quantum circuit by reinterpreting the MPS tensors as unitaries. If the MPS represents a state with a Gauss's law, we might be able to engineer those unitaries to be block-diagonal and to commute with certain operators related to the Gauss's law.

By (1) implementing the unitaries by continuous time evolution and (2) frequently measuring those commuting operators, we can reduce error in the prepared state.

The heart of the method is engineering the representation of the MPS virtual spaces (the "virtual registers") so that the unitaries commute with the appropriate operators; in some sense the rest is ancillary. I'm still not entirely clear on how to do this; I discuss in Sec. 3.3.

Contents

1 Model and setup

Suppose we want to prepare the ground state of the 1D Schwinger model on a quantum computer. The model is

$$\begin{aligned} H =& x\sum_j [\sigma_j^+ e^{i\theta_j}\sigma_{j+1}^- + h.c.] + \frac{1}{2}\mu\sum_j [1+(-1)^j\sigma_j^z] \\ &+ \sum_j (i\partial_{\theta_j} - \alpha)^2 \,. \end{aligned} \tag{1}$$

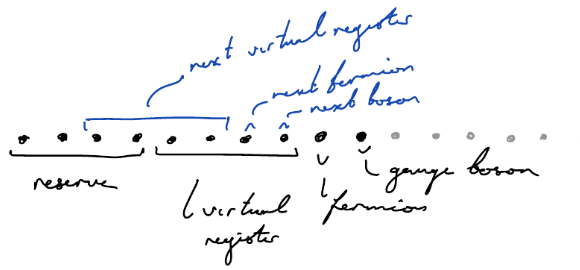

Figure 1: **Basic setup for MPS-based state preparation**: which qubits are playing which roles?

For concreteness, suppose further that $x, \mu \ll 1$ and $\alpha = \pi$, so the gauge field is $\pm\pi$ and we can represent the gauge boson with a single qubit. (The whole construction will generalize naturally to any finite gauge boson Hilbert space, but the exposition is a little more transparent if we have a single qubit.) The model commutes with Gauss's law operators

$$G_j = E_j - E_{j-1} + n_j$$

where $E_j = i\partial_{\theta_j}$.

Organize the qubits in the qubit model of the system in (what I think of) as the usual way, with the boson $i\partial_{\theta_n}$ to the right of the spin s_n. For MPS purposes, though, we'll want to group s_n and $i\partial_{\theta_n}$ into a single Hilbert space $\mathbb{C}^4 \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$.

Write the MPS representation for the ground state

$$|\Omega\rangle = \sum A^{s_1 n_1}_{\alpha_1} A^{s_2 n_2}_{\alpha_1\alpha_2} \dots A^{s_{N-1} n_{N-1}}_{\alpha_{N-1}\alpha_N} A^{s_N n_N}_{\alpha_N} \tag{2}$$

where the A are in appropriate canonical form. (Will discuss that later.)

2 General state preparation given an MPS

2.1 Waymarker states and preparation unitaries

(I haven't actually read this anywhere, but there's no way on God's green earth this is original. The new stuff starts in the next section.)

Work inductively, right-to-left: prepare s_N, l_N, then s_{N-1}, l_{N-1}, and so on.[1] Suppose we have prepared sites $l+1$

[1] Why right-to-left? "Historical reasons": I got started thinking about it that way, and never switched myself over.

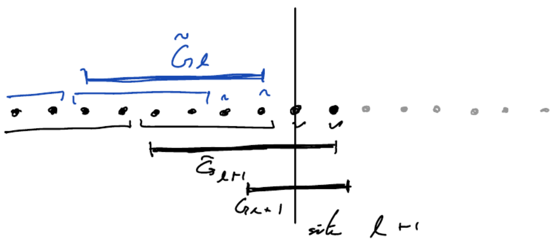

Figure 2: **Supports of the physical and half-virtual Gauss's laws** during the preparation of $|\Omega_l\rangle$ from $|\Omega_{l+1}\rangle$. Compare Fig. 1.

through N (i.e. $s_{l+1}, n_{l+1}, \dots s_N, l_N$). We have some state representing the partial MPS

$$|\Omega\rangle_{l+1} = A^{s_{l+1}n_{l+1}}_{\alpha_l \alpha_{l+1}} \dots A^{s_N n_N}_{\alpha_N} \tag{3}$$

with the dangling virtual leg $\alpha_l \in 1 \dots \chi$. What state? Represent that virtual Hilbert space on the "virtual register" qubits $(l - \lceil \lg \chi \rceil) \dots l$—i.e. reinterpret the partial MPS as a state on a Hilbert space of dimension $\chi 2^{N-l+1}$, then represent that Hilbert space on qubits $(l - \lceil \lg \chi \rceil) \dots N$.

To prepare site l (that is, the state $A^{s_l n_l}_{\alpha_{l-1}\alpha_l} A^{s_{l+1}n_{l+1}}_{\alpha_l \alpha_{l+1}} \dots A^{s_N n_N}_{\alpha_N}$), apply the operator

$$U_l := A^{s_l n_l}_{\alpha_{l-1}\alpha_l} |\alpha_{l-1} s_l n_l\rangle \langle 00\alpha_l| \ . \tag{4}$$

(I'm assuming bonds $l-1$ and l both have bond dimension χ; if not, the virtual registers have different sizes and we'll need to append more or fewer $\langle 0|$ states in $\langle 0 \dots 0 \alpha_l|$.) Choosing A appropriately canonical, this is an isometry $\mathbb{C}^\chi \to \mathbb{C}^{4\chi}$; if we extend U_l to map $\langle 01\alpha_l|$ etc. to vectors perpendicular to $|\alpha_{l-1} s_l n_l\rangle$, $U^{(l)}$ is a unitary.

2.2 Preparation unitaries as continuous time evolution

Take a matrix log of the preparation unitary

$$U_l = e^{-2\pi i X_l} \tag{5}$$

—that is, eigendecompse

$$U_l = \sum_n e^{-2\pi i x_n} |n\rangle\langle n| \tag{6}$$

and construct

$$X_l = \sum_n x_n |n\rangle\langle n| \ . \tag{7}$$

Then $|\Omega_l\rangle$ is precisely the result of evolving $|\Omega_{l+1}\rangle$ by X_l for time $t = 2\pi$:

$$|\Omega_l\rangle = e^{-i(2\pi)X_l} |\Omega_{l+1}\rangle \ . \tag{8}$$

3 Gauss's law

3.1 Waymarker states and the half-virtual Gauss's law

Start with the waymarker state

$$|\Omega_{l+1}\rangle \equiv A^{s_{l+1}n_{l+1}}_{\alpha_l \alpha_{l+1}} \dots A^{s_N n_N}_{\alpha_N} |\alpha_l s_l n_l s_{l+1} n_{l+1} \dots\rangle \ . \tag{9}$$

as in the previous section. The Gauss's law operators $G_{l+1}, \dots$ are straightforwardly defined. To define the Gauss's law operator on $|\alpha_l s_l n_l\rangle$, recall that the indices α_l come with electric flux values $E(\alpha_l)$ (vide MISC034, on abelian gauge symmetries in matrix product states). So we can define an operator V_l,

$$V_l |\alpha_l\rangle = E(\alpha_l) |\alpha_l\rangle \ , \tag{10}$$

and the state

$$|\Omega_l\rangle := U_l |\Omega_{l+1}\rangle \tag{11}$$

is an eigenstate not only of $G_{l+1} \dots$ but also the half-virtual Gauss's law

$$\tilde{G}_l = (E_l - V_{l-1}) + n_l \ .$$

3.2 Preparation Hamiltonians and the half-virtual Gauss's law

We can arrange for the preparation Hamiltonian X_l to commute with $\tilde{G}_{l+1}$. Consider the two waymarker states

$$\begin{aligned} |\Omega_{l+1}\rangle &= A^{s_{l+1}n_{l+1}}_{\alpha_l \alpha_{l+1}} \dots A^{s_N n_N}_{\alpha_N} |0 \dots 0 \alpha_l s_{l+1} n_{l+1} \dots\rangle \\ |\Omega_l\rangle &= A^{s_l n_l}_{\alpha_{l-1}\alpha_l} \dots A^{s_N n_N}_{\alpha_N} |0 \dots 0 \alpha_{l-1} s_l n_l s_{l+1} n_{l+1} \dots\rangle \end{aligned} \tag{12}$$

and once again imagine constructing $|\Omega_l\rangle$ from $|\Omega_{l+1}\rangle$. By construction Ω_{l+1} must be an eigenstate of

- the virtual Gauss's law $\tilde{G}_{l+1} = (E_{l+1} - V_l) + n_l$,

while Ω_l must be an eigenstate of

- the physical Gauss's law G_{l+1} and
- the half-virtual Gauss's law $\tilde{G}_l$.

Suppose that Ω_l is *also* an eigenstate of the half-virtual Gauss's law $\tilde{G}_{l+1}$ with the correct eigenvalue, i.e.

$$\begin{aligned} \tilde{G}_{l+1} |\Omega_{l+1}\rangle &= \lambda |\Omega_{l+1}\rangle \\ \tilde{G}_{l+1} |\Omega_l\rangle &= \lambda |\Omega_l\rangle \end{aligned} \tag{13}$$

for the same λ. (We can arrange this by choosing the representation of the virtual registers: see Sec. 3.3.) Then U_l (hence X_l) commutes with $\tilde{G}_{l+1}$.

3.3 Engineering the virtual registers

We need

1. $\tilde{G}_l, \tilde{G}_{l+1}$, and G_{l+1} to have the same eigenspaces, i.e.

$$0 = [\tilde{G}_{l+1}, G_{l+1}] \tag{14a}$$
$$= [\tilde{G}_l, \tilde{G}_{l+1}] \tag{14b}$$
$$= [\tilde{G}_l, G_{l+1}] \ , \tag{14c}$$

and

2. $|\Omega_l\rangle, |\Omega_{l+1}\rangle$ to be in the same block of the resulting common eigendecomposition.

(You might feel the impulse to additionally require $0 = [\tilde{G}_l, G_l]$—I certainly did when I was writing this up—but that's covered by (14a) with $l \leftarrow l - 1$.)

3.4 Eigenspaces/commutation relations

Recall that

$$\tilde{G}_l = V_l - E_l + s_l \tag{15a}$$
$$\tilde{G}_{l+1} = V_{l+1} - E_{l+1} + s_l \tag{15b}$$
$$G_{l+1} = E_l - E_{l+1} + s_l \ , \tag{15c}$$

so we desire

$$0 = [\tilde{G}_{l+1}, G_{l+1}] \quad = \quad [V_{l+1}, E_l] \tag{16a}$$
$$= [\tilde{G}_l, \tilde{G}_{l+1}] \quad = \quad [V_l - E_l + s_l, V_{l+1}] \tag{16b}$$
$$= [\tilde{G}_l, G_{l+1}] \quad \text{[automatically satisfied]}. \tag{16c}$$

We can arrange for all of these to be satisfied by taking all the V_j diagonal in the computational basis (i.e. the $\{s_j, E_j\}$ eigenbasis).

3.5 Eigenvalues

We need not only that the operators have compatible eigenbasis, but also that the waymarker states $|\Omega_l\rangle, |\Omega_{l+1}\rangle$ are eigenstates.

By construction we have

$$\tilde{G}_{l+1} |\Omega_{l+1}\rangle = g_{l+1} |\Omega_{l+1}\rangle \tag{17a}$$
$$\tilde{G}_l |\Omega_l\rangle = g_l |\Omega_l\rangle \tag{17b}$$

where g_l, g_{l+1} are the desired superselection sectors. We want in addition

$$\tilde{G}_{l+1} |\Omega_l\rangle = g_{l+1} |\Omega_l\rangle \tag{18a}$$
$$\tilde{G}_l |\Omega_{l+1}\rangle = g_l |\Omega_{l+1}\rangle \ . \tag{18b}$$

To get the first, arrange

$$V_{l+1} |\alpha_l s_l E_l\rangle = E_l \ , \tag{19}$$

which can be achieved by

$$V_{l+1} = E_l \ . \tag{20}$$

In other words—Since $|\alpha_l s_l E_l\rangle$ is roughly $|\alpha_{l+1}\rangle$, this means that we need to store sector information in the rightmost bit of the virtual register, since that sector information is exactly the same as the boson quantum number that will end up on that bit. (This argument looks like it relies on the fact that gauge theories aren't real in 1D, i.e. virtual sector q.n. is exactly boson q.n. Does this only work in 1D?)

The second is

$$\begin{aligned}\tilde{G}_l |\Omega_{l+1}\rangle &= [V_l - E_l + s_l] A^{s_{l+1}E_{l+1}}_{\alpha_{l+1}\alpha_{l+2}} \dots |\alpha_{l+1} s_{l+1} E_{l+1} \dots\rangle \\ &= [V_l - E_l + s_l] |\alpha_{l+1}\rangle A^{s_{l+1}E_{l+1}}_{\alpha_{l+1}\alpha_{l+2}} \dots |s_{l+1} E_{l+1} \dots\rangle \\ &= [E_{l-1} - E_l + s_l] |\alpha_{l+1}\rangle A^{s_{l+1}E_{l+1}}_{\alpha_{l+1}\alpha_{l+2}} \dots |s_{l+1} E_{l+1} \dots\rangle\end{aligned} \tag{21}$$

by (20).

I think **this means that if we *start* with a state in the right superselection sectors, we can *end up* in a state with the right superselection sectors**.

4 Quantum Zeno and error resistance

We want to do the time evolution

$$|\Omega_l\rangle = e^{-2\pi i X} |\Omega_{l+1}\rangle \ , \tag{22}$$

but our real quantum computer is noisy: what we actually get is

$$\mathcal{T} \exp \left[-i \int_0^{2\pi} dt' \ (X + Z(t)) \right] |\Omega_{l+1}\rangle \tag{23}$$

where $Z(t)$ is the noise operator. (For the moment I'm only considering coherent errors; we'd deal with decoherence by writing some number of Lindblad operators Ξ_j.)

Write

$$Z = \xi\,\Xi + \upsilon\Upsilon,$$

where the operators Ξ, Υ are normalized in some appropriate sense (Frobenius? Talk to Paraj?), ξ, υ are scalar magnitudes with dimensions frequency, and

$$[\tilde{G}_{l+1}, \Xi] = 0 \ . \tag{24}$$

(To make this well-defined, we want to choose Ξ, Υ in the way that maximizes ξ—that'll give sensible error estimates later.) The error is (at leading order) something like

$$\epsilon \sim 2\pi \sqrt{\xi^2 + \upsilon^2} \ . \tag{25}$$

4.1 Staying on the right path

Suppose (!) $\upsilon \gg \xi$, so

$$\varepsilon \sim 2\pi\upsilon \tag{26}$$

Since X_l commutes with $\tilde{G}_{l+1}$, we can freely intersperse projective measurements of $\tilde{G}_{l+1}$ with the desired time evolution: if $\tilde{\mathfrak{G}}_{l+1}$ is the stochastic operator for that projective measurement, then

$$|\Omega_l\rangle = \prod \left[\tilde{\mathfrak{G}}_{l+1} e^{-i\delta t X} \right] |\Omega_{l+1}\rangle \tag{27}$$

But if we do this on our noisy computer, we buy ourselves something:

$$\prod \left[\tilde{\mathfrak{G}}_{l+1} e^{-i\delta t (X + \xi\,\Xi(t) + \upsilon\Upsilon(t))} \right] |\Omega_{l+1}\rangle \tag{28}$$

has error like

$$\varepsilon \sim 2\pi\,\delta t \upsilon^2 \tag{29}$$

4.2 Errors in when we stop

What about the remaining error like $2\pi\xi$, including "overrotation" (component of $\Xi \propto X$)? If we're somewhere in the general vicinity, projectively measuring G_{l+1} and $\tilde{G}_l$ will get us into the right superselection sector, at least, with high probability.

Also: any MPS is (IIRC) the ground state of some "stabilizer Hamiltonians"—a Hamiltonian

$$H = \sum_j h_j, \tag{30}$$

with the h_j local and

$$[h_j, h_k] = 0. \tag{31}$$

Projectively measuring these guys could help.

5 Questions

- Quantitatively, how much does this buy us? Two subsidiary questions:

  1. Suppose the noise is isotropic (i.e. Z Haar random). Then the amount of good the ongoing measurement of Sec. 4.2 does is determined by how finely $\tilde{G}_{l+1}$ dices the Hilbert space. Once we know that it should be a pretty straightforward analytical calculation, which we can then verify with numerics—but we need to know about the eigenspaces of $\tilde{G}_{l+1}$.
  2. Suppose the noise is *not* isotropic, and let's pick some reasonable noise model. (Indrakshi—you've been looking at cold-atom setups, right?) What's the resulting error? Can we re-engineer the virtual registers (and possibly the physical degrees of freedom) to arrange for the large-magnitude errors to be corrected by measuring $\tilde{G}_{l+1}$

- I talk about "projectively measuring $\tilde{G}_{l+1}$". What does $\tilde{G}_{l+1}$ look like in terms of the qubit operators?

- 2D: extension to PEPS?

- 2D $\mathbb{Z}_2$ gauge theory (bosonized fermions)? That'd be a cool application.