ELU 501
Data science, graph theory and social network studies

Yannis Haralambous (IMT Atlantique)

April 16, 2018

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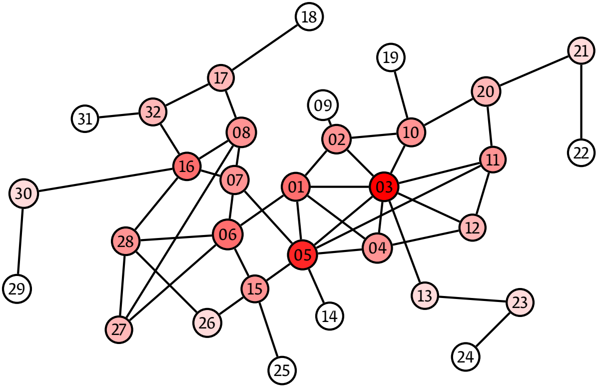
Part II

Lecture 2
Measures

Centrality

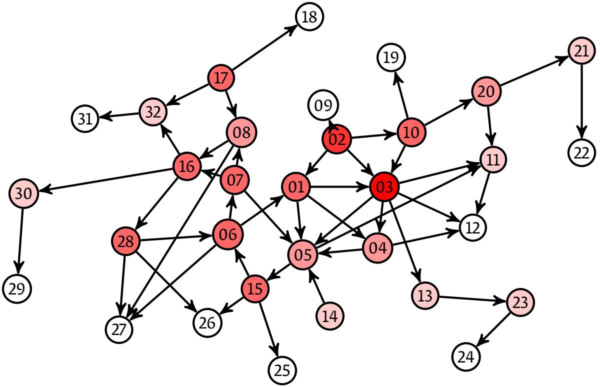
- In the following we will study graphs according to their metric properties.
- The first question we may ask is "which are the most important / most central vertices in a graph?"

Degree centrality



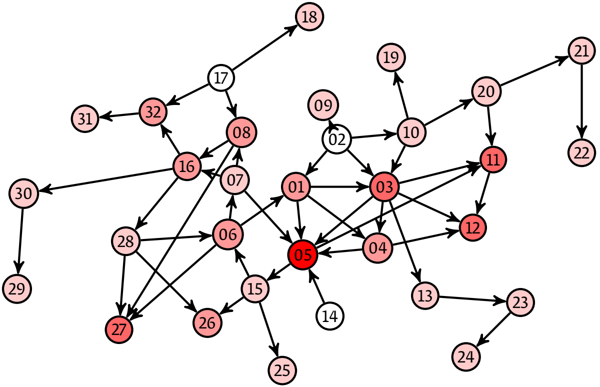
- Simplistic approach: *the most central vertex is the one of highest degree*: we call this measure, *degree centrality* ⟨*centralité par le degré*⟩.
- We calculate for each v_i the quantity $\frac{d(v_i)-d_{\min}}{d_{\max}-d_{\min}}$.
- In the figure, v_{14} has a very weak measure although it is a neighbor or v_5.

Outdegree centrality



- In the directed case we can consider *as central the vertex transmitting to a maximum of neighbors*: this is *outdegree centrality* ⟨*centralité par le degré sortant*⟩.
- We calculate for each v_i the quantity $\frac{d^+(v_i)-d^+_{\min}}{d^+_{\max}-d^+_{\min}}$.
- In the figure, v_{12} has a very weak measure although it is a neighbor or v_3.

In-degree centrality



- We can also consider *as central the vertex which receives from a maximum of neighbors*: this is *indegree centrality* ⟨*centralité par le degré entrant*⟩.
- We calculate for each v_i the quantity $\frac{d^-(v_i)-d^-_{\mathrm{min}}}{d^-_{\mathrm{max}}-d^-_{\mathrm{min}}}$.
- In the figure, v_{14} has a very weak measure although it is a neighbor or v_5.

Eigenvector centrality

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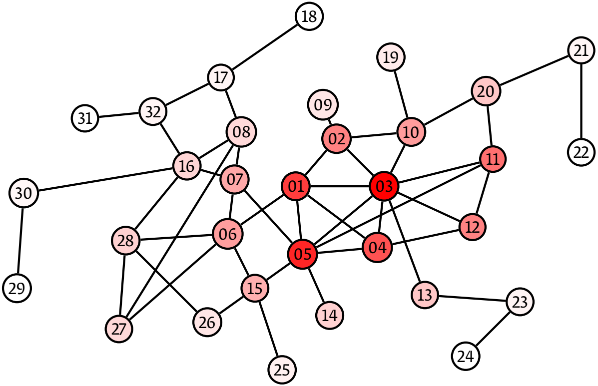


- More sophisticated: *consider as central the vertex having the most <u>central</u> neighbors*: we call it *eigenvector centrality* ⟨🇫🇷 *centralité par vecteur propre*⟩.
- Notice that v_{14} is a bit stronger than v_{25}, itself stronger than v_{22}.

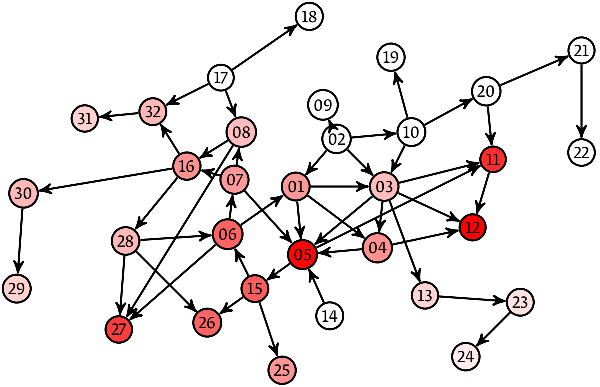
Eigenvector centrality

- But how do we calculate this kind of centrality, where we want the centrality of neighbors to be recursively taken into account?
  1. we attach value 1 to each vertex $x_i = 1$,
  2. we do a first calculation $x'_i = \sum_j a_{i,j} x_j$, i.e., $X' = A \cdot X$,
  3. after t steps we will have $X(t) = A^t \cdot X(0)$,
  4. like any vector, $X(0)$ can be written as a linear sum of eigenvectors of A: $X(0) = \sum_i c_i \vec{v_i}$,
  5. then, if κ_i is the ith eigenvalue (by decreasing order)

     $$X(t) = A^t \cdot \sum_i c_i \vec{v_i} = \sum_i c_i \kappa_i^t \vec{v_i} = \kappa_1^t \sum_i c_i \left(\frac{\kappa_i}{\kappa_1}\right)^t \vec{v_i},$$

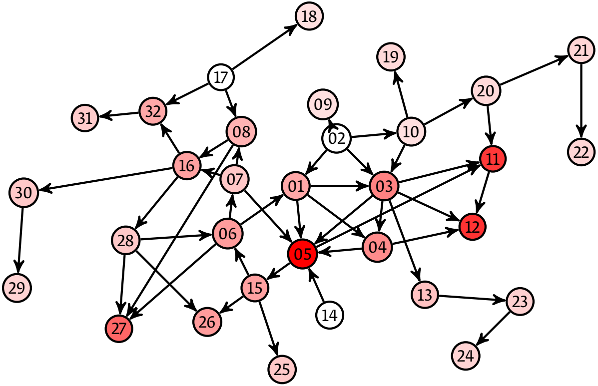
     (the second equality comes from the fact that if κ_i is an eigenvalue and v_i the corresponding eigenvector, then $(A - \kappa_i \mathrm{Id}) \cdot v_i = 0$ and hence $A v_i = \kappa_i v_i$ and $A^t v_i = \kappa_i^t v_i$),
  6. when $t \gg 0$, $X(t) \sim c_1 \kappa_1^t \vec{v_1}$ and hence centrality is proportional to the first eigenvector.
- We define eigenvector centrality of X as: $A \cdot X = \kappa_1 X$.

Entering eigenvector centrality



- We can generalize eigenvector centrality to directed graphs, but sometimes this is problematic.
- Here v_2 has zero entering centrality and therefore this also the case of v_{10}, etc., although these vertices have entering arrows...

Katz centrality

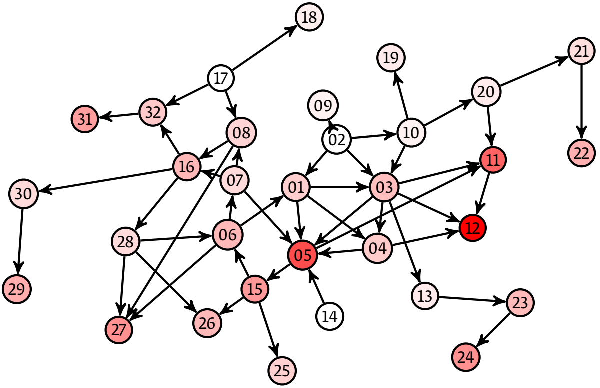


- Katz (1953) has improved entering eigenvector centrality, as we can see here only vertices without entering arrow have zero centrality.

Katz centrality

- Katz's idea: give a small amount of centrality to everyone and see how it evolves: $x'_i = \alpha \sum_j a_{i,j} x_j + \beta$.
- By setting $\beta = 1$ we get $X = (\mathrm{Id} - \alpha A)^{-1} \cdot 1$, *Katz centrality* ⟨*centralité de Katz*⟩.
- The choice of α is arbitrary, but we don't want the measure to diverge.
- It diverges when $\det(\mathrm{Id} - \alpha A) = 0$, i.e., $\alpha^{-1} = \kappa_*$, that is, for the first time for $\alpha = 1/\kappa_1$. So we set α slightly less than $1/\kappa_1$.
- In the example we took $\alpha = \frac{1}{4}$.
- To do calculations we will not invert A (complexity $O(n^3)$), but we will iterate $X' \leftarrow \alpha A \cdot X + \beta \cdot 1$.
- There is still a problem with Katz: if a vertex points to a million others, they will all get the same amount of centrality, while it would be better to disolve this centrality in the mass of entering arrows.

PageRank



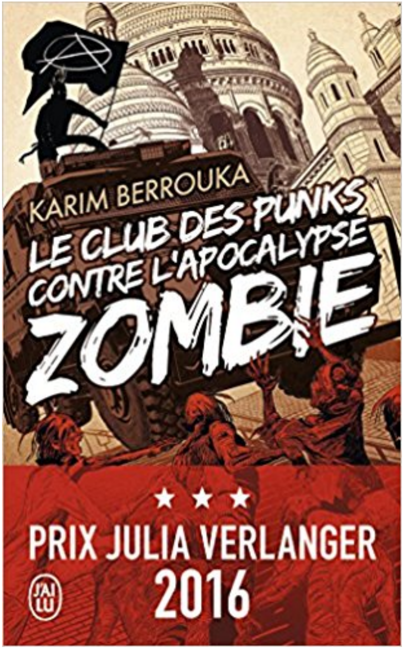
- A solution to this problem is given by the *PageRank* algorithm thanks to which Google is a major company today.
- Note that, contrarily to previous methods, isolated strings like $v_{13}v_{23}v_{24}$ ou $v_{30}v_{29}$) increase centrality. Note also that although v_3 and v_{12} both have 3 entering arrows, their centrality is quite different.

PageRank

- PageRank defines $x'_i = \alpha \sum_j a_{i,j} \frac{x_j}{d^+(v_j)} + \beta$, where $d^+(v_j)$ is the leaving degree of v_j.
- What happens when $d^+(v_j) = 0$? We replace it by $\max(d^+(v_j), 1)$.
- Let D be the diagonal matrix of the $\max(d^+(v_j), 1)$. Then we have $X = \alpha A \cdot D^{-1} \cdot X + \beta \cdot 1$ and hence for $\beta = 1$:
$$X = D \cdot (D - \alpha A)^{-1} \cdot 1.$$
- It is commonly agreed that the success of Google is not due to the relevance of the global set of results returned, but to their <u>order</u>.
- And behing the order you have PageRank.
- The same calculations as for Katz show that α must be less than 1. Google uses $\alpha = 0.85$.

PageRank and random walks

- Another interpretation of PageRank is centrality can be obtained by *random walks* ⟨🇫🇷 *marche aléatoire*⟩ :
  A random walk is a random path on a graph where at each moment t the probability of visiting any neighboring vertex is $1/d$ where d is the degree of the current vertex.
- To random walk we add another operation called *teleport* (*Energy Mr Sulu!*): on each vertex, there is
  1. an α probability that the next move will be some other (not necessarily neighboring) vertex, uniformily chosen among all vertices of the graph,
  2. and a $1-\alpha$ probability that the standard random walk is pursued.
- Compare this with zombies (or tourists) randomly walking through Paris.


KARIM BERROUKA
LE CLUB DES PUNKS CONTRE L'APOCALYPSE
ZOMBIE
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PRIX JULIA VERLANGER
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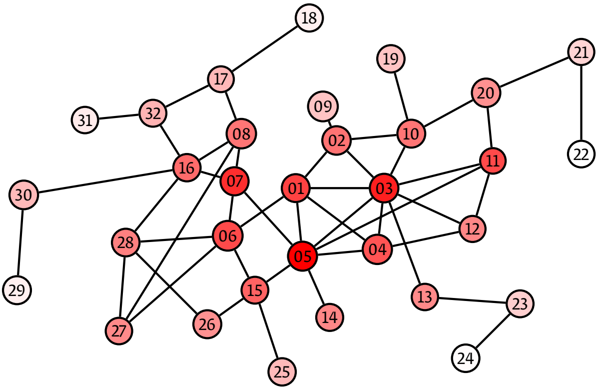
Closeness centrality

- Let us return to more geometric considerations.
- We can calculate the *mean geodesic distance* ⟨*distance géodésique moyenne*⟩ of a vertex v_i out of all other vertices: $\ell_i = \frac{1}{n}\sum_j d(v_i, v_j)$.
- More we are central, more this quantity diminuishes.
- We define *closeness centrality* ⟨*centralité de proximité*⟩ by

$$C_i = \frac{1}{\ell_i} = \frac{n}{\sum_j d(v_i, v_j)}.$$

Closeness centrality



Closeness centrality

- We have calculated closeness centrality for actors of database `imdb.com` (where edges mean: actors have played in the same movie) and the most central acror is Christopher Lee, with $C = 0,4143$ and the less central actor is Leia Zanganeh with $C = 0,1154$.
- There is almost half a million actors between those two and nevertheless the two values are quite close.
- Additional problem: in an unconnected graph, all C are zero...
- And if we limit ourselves to components, we get greater values for small components.

Closeness centrality

- The solution to this problem is to use an harmonic mean:
$$C'_i = \frac{1}{n-1}\sum_{j\neq i}\frac{1}{d(v_i, v_j)}.$$
- The good properties are: if v_i and v_j belong to distinct components, the fraction is zero.
- Close to each other vertices count more than far away ones.
- We also define the *mean geodesic distance* ⟨🇫🇷 *distance géodésique moyenne*⟩
$$\ell = \frac{1}{n}\sum_i \ell_i = \frac{1}{n^2}\sum_{i,j} d(v_i, v_j)$$
- and the *harmonic mean geodesic distance* ⟨🇫🇷 *distance géodésique harmonique moyenne*⟩
$$\ell' = \frac{n}{\sum_i C'_i} = n(n-1)\frac{1}{\sum_{i\neq j}\frac{1}{d(v_i,v_j)}}.$$

Betweenness centrality

- In a network where travel ideas, opinions, merchandises, IP packets between close to everyone may not be the most relevant centrality characteristic.
- Idea: let us consider paths going through a vertex.
- We want to quantify the geodesic paths traversing a given vertex.
- we define *betweenness centrality* ⟨🇫🇷 *centralité de synexité*⟩ as $x_i := \sum_{j,k} \frac{n_{i,j,k}}{g_{j,k}}$ where:
  - $n_{i,j,k}$ is the number of geodesic paths between v_j and v_k going through v_i,
  - $g_{j,k}$ is the number of geodesic paths between v_j et v_k.

Betweenness centrality

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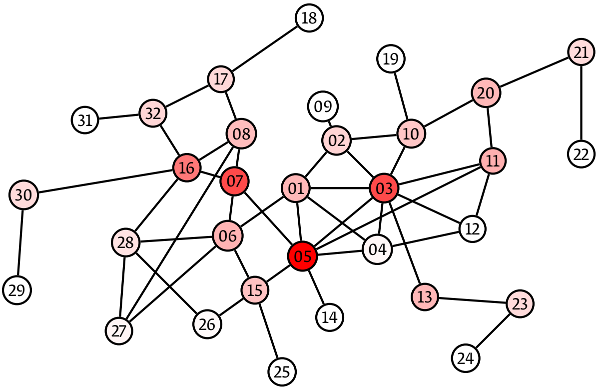
- Betweenness centrality is completely different than other centralities since it is totally independent of degree.
- Example: if $G = G_1 - v_i - G_2$ where G_1 and G_2 are very populated components, then v_i will have a high betweenness centrality even though its degree is only 2.
- Such a vertex is called a *broker* ⟨*courtier*⟩.

Betweenness centrality



Betweenness centrality

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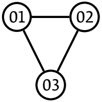
- Betweenness centrality values are distributed as follows:
- Maximal value is n^2 (think of the center of a star graph).
- Minimal value is $2n-1$ (think of the leaf of a linear graph, counting also a zero-length path).
- In IMDB, the actor of greatest betweenness centrality is Fernando Rey, he played both with US and European actors, in movies and TV, with $x = 7.47 \cdot 10^8$. The second one is again Christopher Lee with $x = 6.46 \cdot 10^8$, a difference of 14%.
- BC is much more stable than CC.
- We can also define *normalized betweenness centrality* ⟨*centralité de synexité normalisée*⟩, with values between 0 and 1:

$$x_i = \frac{1}{n^2} \sum_{j,k} \frac{n_{i,j,k}}{g_{j,k}}.$$

Transitivity

- In algebra, a relation $*$ is transitive when $a * b, b * c \Rightarrow a * c$.
- In social networks this can be interpreted as "the friend of my friend is my friend".
- A complete subgraph K_3 is called a *triad* ⟨*triade*⟩.



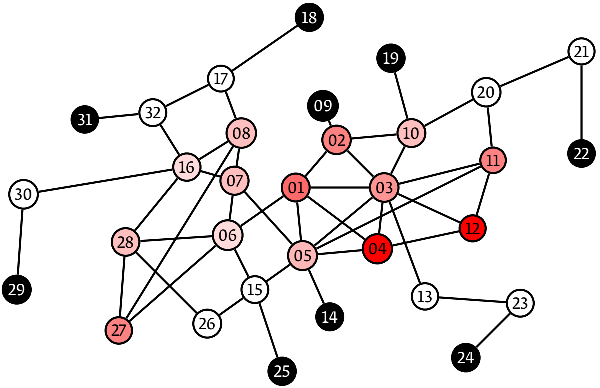
- We define the *global clustering coefficient* ⟨*coefficient de clustering global*⟩ of a graph as the ratio

$$C = \frac{\#\text{triads}}{\#\text{paths of length } 2}.$$

- For a given vertex v_i of degree ≥ 2, we define the *local clustering coefficient* ⟨*coefficient de clustering local*⟩ :

$$C_i = \frac{\#\text{connected neighbors of } v_i}{\#\text{pairs of neighbors of } v_i}.$$

Transitivity



Black vertices are of degree 1 and hence C_i cannot be defined. Note that v_4 and v_{12} are the "most transitive" vertices of the graph.