reasoning for interpolated actions

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Problem setting: two native actions $\{(p_1, F_1), (p_2, F_2)\}$ assuming that $p_1 < p_2, \quad F_1 > F_2$, threshold F_{th}, we choose action 1 with probability q and action 2 with probability $1-q$. The mapping between fidelity and bins $n(F) = \frac{1}{\Gamma}\ln\frac{F-1/4}{F_{th}-1/4}$

Case 1: Not distinguishing between the states
The probability for the events:

$$P_q(choice = 1; success) = qp_1 \quad P_q(choice = 2; success) = (1-q)p_2$$

$$P_q(success) = qp_1 + (1-q)p_2$$

We say successful transmission under the first action produces ρ_1 and the success for the second action produces ρ_2. The resulting state would then be

$$\rho_q = \rho_1 P_q(choice = 1|success) + \rho_2 P_q(choice = 2|success) = \frac{qp_1\rho_1 + (1-q)p_2\rho_2}{qp_1 + (1-q)p_2}$$

Since $F(a\rho_1 + b\rho_2, |\phi^+\rangle\langle\phi^+|) = aF(\rho_1, |\phi^+\rangle\langle\phi^+|) + bF(\rho_2, |\phi^+\rangle\langle\phi^+|)$, the fidelity of the state can be written as

$$F_q = F(\rho_q, |\phi^+\rangle\langle\phi^+|) = \frac{qp_1F_1 + (1-q)p_2F_2}{qp_1 + (1-q)p_2}$$

With the depolarization model, the corresponding quantum operation is

$$\mathcal{E}(\rho) = e^{-\Gamma n}\rho + (1-e^{-\Gamma n})\frac{I}{4}$$

We can verify that $F(\mathcal{E}(\rho), |\phi^+\rangle\langle\phi^+|) = e^{-\Gamma n}(F_\rho - \frac{1}{4}) + \frac{1}{4}$ Thus by the time of waiting until n, the state will evolve as

$$\rho_q \to \mathcal{E}(\rho_q)$$

and the fidelity of the state:

$$F_q \to e^{-\Gamma n}(F_q - \frac{1}{4}) + \frac{1}{4}$$

The state can be viewed as a normal state that starts depolarizing from F_q

The $F_q - p_q$ relation is

$$F_q(p_q) = \frac{1}{p_q}\frac{p_1p_2(F_2 - F_1)}{p_1 - p_2} + \frac{p_1F_1 - p_2F_2}{p_1 - p_2}$$

See the graph in Figure 1
How to conduct in reality:

1. sample from $X \sim Bernoulli(q)$, conduct action 1 if $X = 0$, conduct action 2 if $X = 1$
2. if the link is established, assign it's bin as $n_q = n(F_q) = \frac{1}{\Gamma}\ln\frac{F_q-1/4}{F_{th}-1/4}$

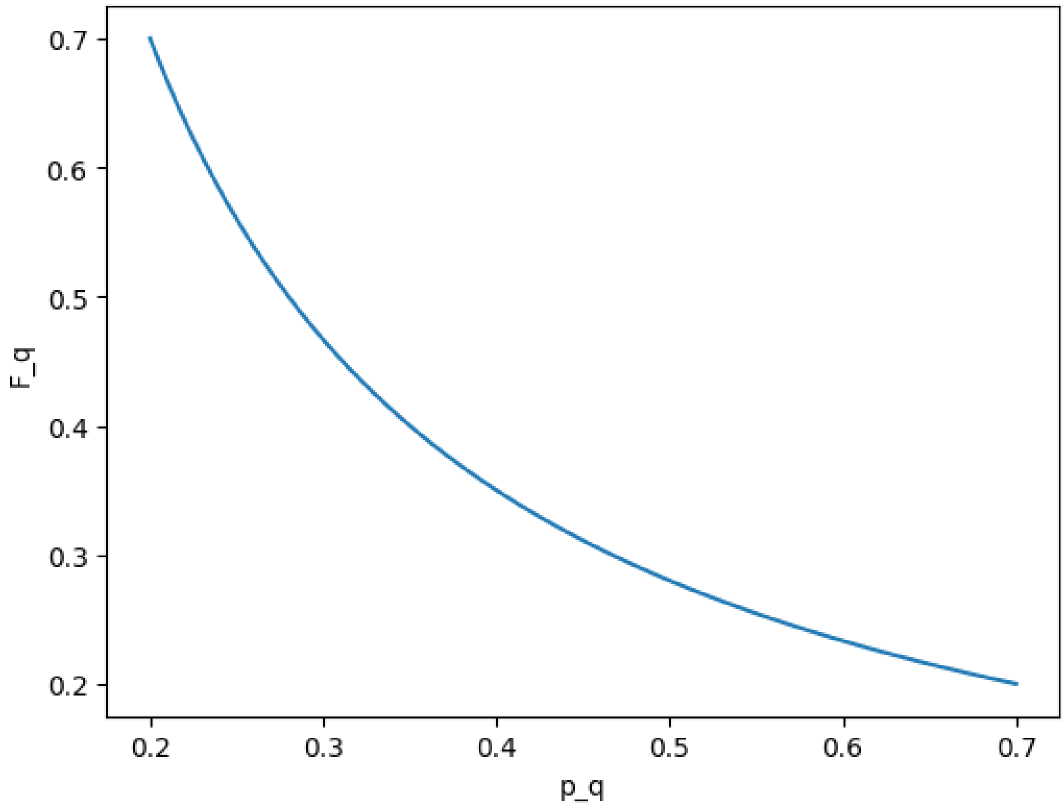

Figure 1: $F_q - p_q$

Case 2: Distinguishing between the states

Here because we assign different bins, a possible comparable concept is the average number of bins:

$$\bar{n} = n(F_1)P_q(choice = 1|success) + n(F_2)P_q(choice = 2|success) = \frac{qp_1 n(F_1) + (1-q)p_2 n(F_2)}{qp_1 + (1-q)p_2}$$

Which is smaller than the bin n_q in case 1 due to concavity:

$$\begin{aligned} \bar{n} = n(F_1)P_q(choice = 1|success) + n(F_2)P_q(choice = 2|success) \\ \leq n(F_1 P_q(choice = 1|success) + F_2 P_q(choice = 2|success)) = n_q \end{aligned} \quad (1)$$

Thus it seems like Case 1 gives better(and more consistent) performance.

How to conduct in reality:

1. sample from $X \sim Bernoulli(q)$, conduct action 1 if $X = 0$, conduct action 2 if $X = 1$
2. if the link is established, and the action is 1, assign bin $n(F_1) = \frac{1}{\Gamma} \ln \frac{F_1 - 1/4}{F_{th} - 1/4}$; if the link is established, and the action is 2. assign bin $n(F_2) = \frac{1}{\Gamma} \ln \frac{F_2 - 1/4}{F_{th} - 1/4}$