Tutorial 9: The Assignment 4

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COMP210
Discrete Structure

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Table of Contents

Review of Algorithm & Counting methods

Pseudo Code of Algorithms

- Algorithms are recipes to solve problems
  - Finite, precise
- For, while, if … then …, if … else if … then …
- Recursive algorithms
  - A routine that calls itself (with a reduced input)

Review of Algorithm & Counting methods cont.

Algorithmic Complexity

- Measures the # of basic operations
  - A function of input size
- Asymptotic notation (Big-O, Big-Ω , Big-Θ, small-o, small-ω)
  - Definitions
  - Finding the dominating terms
  - Write functions in forms of the asymptotic notations and compare their complexity

Review of Algorithm & Counting methods cont.

Big-O definition

DEF: Let f , g be functions with domain $\mathbf{R}_{\geq 0}$ or $\mathbf{N}$ and codomain $\mathbf{R}$. If there are constants C and k such

$$\forall\ x > k,\ |f(x)| \leq C \cdot |g(x)|$$

then we write:

$$f(x) = O(g(x))$$

- Big-Θ: f(x) = O(g(x)) & g(x) = O(f(x))

Review of Algorithm & Counting methods cont.

Rule of thumbs

- First, for input size n, determine the # of basic operations as f(n)
- Find the dominating term in f(n)
- The following functions are in growing order of complexity

$$\frac{1}{x}, \ln x, x, x^e, e^x, x^x$$

Review of Algorithm & Counting methods cont.

Counting methods

- Multiplication principle
  - Count in stages
- Addition principle
  - Divide the original set into disjoint sets
- Inclusion-exclusion principle
  - Generalization of the addition principle to overlapping sets
- Pigeon hole principle
  - Given N pigeon, k holes, at least one hole contains $\lceil N/k \rceil$ pigeons
  - Can also solve the inverse problem, how big N needs to be such that for k holes, at least one hole contains $\lceil N/k \rceil$ pigeons

Outline

Problem 1

Write an algorithm that reverses a string $s_1, \ldots, s_n$.
Example: If the sequence is AMY BRUNO ELIE, the reversed sequence is ELIE BRUNO AMY.

Problem 1 cont.

Algorithm 1 Reverse string s

Input: String s, where s ends with EOL

$i \leftarrow 1$, $word\_cnt \leftarrow 0$ {Parsing}
while $s(i) \neq EOL$ **do**
    while $s(i) = ''\textvisiblespace''$ **do**
        $i \leftarrow i + 1$
    end while
    if $s(i) \neq EOL$ **then**
        $word\_cnt \leftarrow word\_cnt + 1$, $char\_cnt \leftarrow 1$
        while $s(i) \neq EOL$ **and** $s(i) = ''\textvisiblespace''$ **do**
            $word[word\_cnt][char\_cnt] \leftarrow s(i)$
            $char\_cnt \leftarrow char\_cnt + 1$, $i \leftarrow i + 1$
        end while
    end if
end while
for $i \leftarrow word\_cnt$ **to** 1 **do** {Output}
    print $word[i]$
end for

Outline

Common Growth Functions

Table: Common Growth Functions (Table 4.3.3)

Theta Form	Name
$\Theta(1)$	Constant
$\Theta(\lg \lg n)$	Log log
$\Theta(\lg n)$	Log
$\Theta(n)$	Linear
$\Theta(n \lg n)$	n log n
$\Theta(n_2)$	Quadratic
$\Theta(n_3)$	Cubic
$\Theta(n_k)$	Polynomial
$\Theta(c_n)$	Exponential
$\Theta(n!)$	Factorial

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Then

$$f(n) = \frac{(n+1)(n+3)}{n+1} = \Theta(n) \quad \square$$

Outline

Problem 5.1

Express in theta notation the number of times the statement x = x + 1 is executed.

```
for i = 1 to n
    for j = 1 to n
        x = x + 1;
```

Problem 5.1 cont.

The basic operation runs 1 times. The for loops of j, runs n times, and the outer for loops of i runs n times. So based on the multiplication principle. Then total number is $1 \times n \times n = n^2$.

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$$f(n) = n^2 = \Theta(n^2) \quad \square$$

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&\textit{for all } n \geq \lg \frac{c}{k}, \\
&\qquad \lg\left(n^k + c\right) \leq \lg(2n^k) \\
&\qquad\qquad = k \lg n + \lg 2 \\
&\qquad\qquad \leq C_1 \lg n, \ \textit{where } C_1 = k + 1, \ n \geq 2 \\
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3. Problems cont.
   - Problem 10
   - Problem 12.2
   - Problem 14.1

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Two dice are rolled, one blue and one red. How many outcomes give the sum of 2 or the sum 12?

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8	2	6
	3	5
	4	4
	5	3
	6	2

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Table: Outcomes of dice

Sum	Blue	Red
2	1	1
12	6	6
8	2	6
	3	5
	4	4
	5	3
	6	2

1 outcome gives the sum of 2;
1 outcome gives the sum of 12.

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Problem 12.2

For integers from 5 to 200, inclusive. How many do not contain the digit 0?

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By Addition principle, the total is 167. $\square$

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Definition

A relation R on a set X is symmetric if ∀ x,y ∈ X, if (x,y) ∈ R, then (y,x) ∈ R.{Definition 3.3.9}

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Definition

A relation R on a set X is antisymmetric if $\forall$ x,y $\in$ X, if (x,y) $\in$ R and (y,x) $\in$ R then x=y. {Definition 3.3.12}

Problem 14.1 cont.

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$\Rightarrow$ n$\times$n by Multiplication principle

So, there are n^2 summetric and antisymmetric relations on an n-element set.

Q & A

Questions about the problems?