Markov Chain Convergence and Rubik's Cube Scrambling

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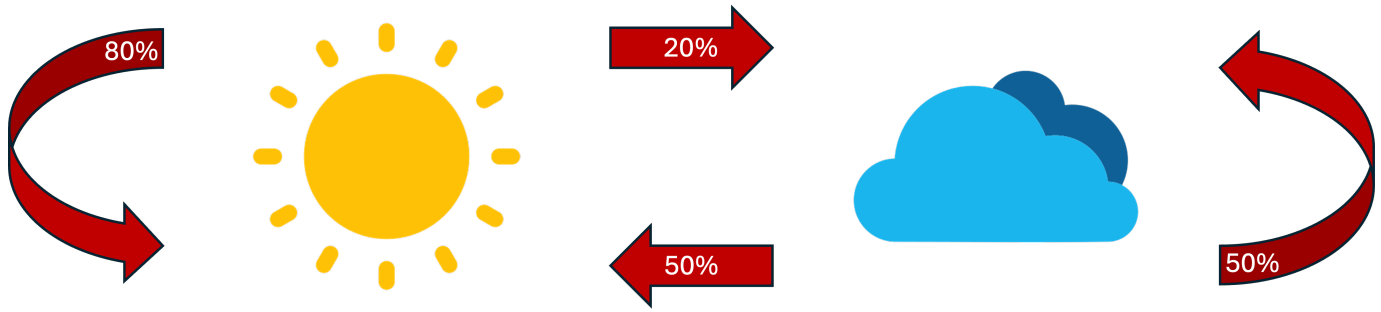
Joint work with Tomas Rokicki and Hillary Yang

What is the weather like?

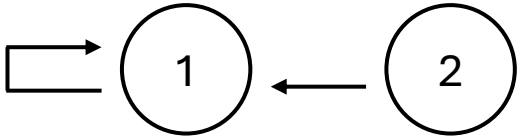


How fast does $P(X_n = \text{☀})$ converge to $P(X_\infty = \text{☀})$?

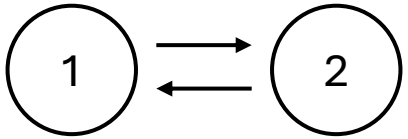
$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}, \quad \pi = \begin{pmatrix} 5/7 & 2/7 \end{pmatrix}, \quad \nu P^n \to \pi, \quad n \to \infty$$

When does a finite-state MC converge?

- Irreducibility: $\forall\, x, y,\ \exists\, m,$ s.t. $p^m(x, y) > 0$.
- Aperiodicity: $\forall\, x,\ \gcd\{m : p^m(x, x) > 0\} = 1$.
- An irreducible aperiodic finite-state Markov chain is positive recurrent and converges to a unique stationary distribution exponentially fast.

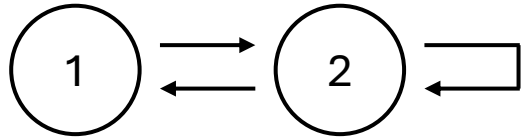


reducible



irreducible

periodic



aperiodic

How is convergence measured?

- Total variation distance:

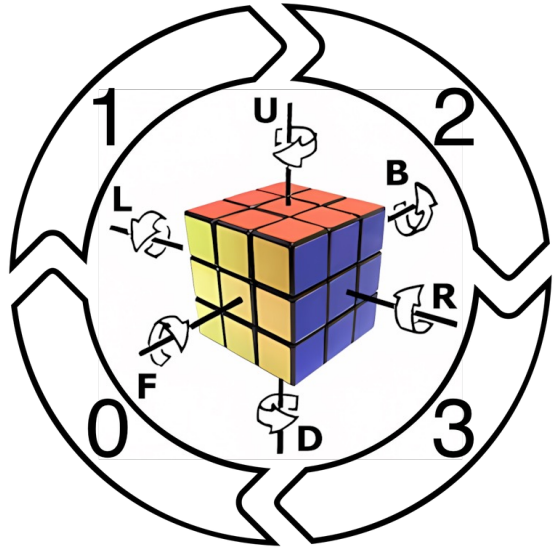
$$\begin{aligned}\mathrm{TV}(X_n, X_\infty) &= \sup_{0 \le h \le 1} |\mathbb{E}h(X_n) - \mathbb{E}h(X_\infty)| \\ &= (1/2) \sum_{x \in \mathcal{X}} |P(X_n = x) - P(X_\infty = x)|\end{aligned}$$

- Mixing time:

$$t_{\mathrm{mix}}(\epsilon) = \min\{n : \mathrm{TV}(X_n, X_\infty) < \epsilon\}, \quad t_{\mathrm{mix}} = t_{\mathrm{mix}}(1/4)$$

Rubik's Cube

- U:up, D:down, F:front, B:back, L:left, R:right;
- 1:90°, 2:180°, 3:270° (clockwise);
- U1, U2, U3, D1, D2, D3, F1, F2, F3,
  B1, B2, B3, L1, L2, L3, R1, R2, R3;
- $|\mathcal{X}| = 43,252,003,274,489,856,000$;



- Let f be a move sampled uniformly at random from the 18 options above. The scrambling process is $X_{n+1} = f_{n+1}(X_n)$ where f_{n+1}'s are iid copies of f.
- Exercise: i) it is irreducible, ii) it is aperiodic, iii) its equilibrium is uniform.

Main result

Rubik's Cube scrambling requires at least 26 random moves, i.e., $t_{\mathsf{mix}} \geq 26$.

Proof

distribution over 4.3×10^{19} states

distribution over $\{0, 1, 2, ..., 20\}$

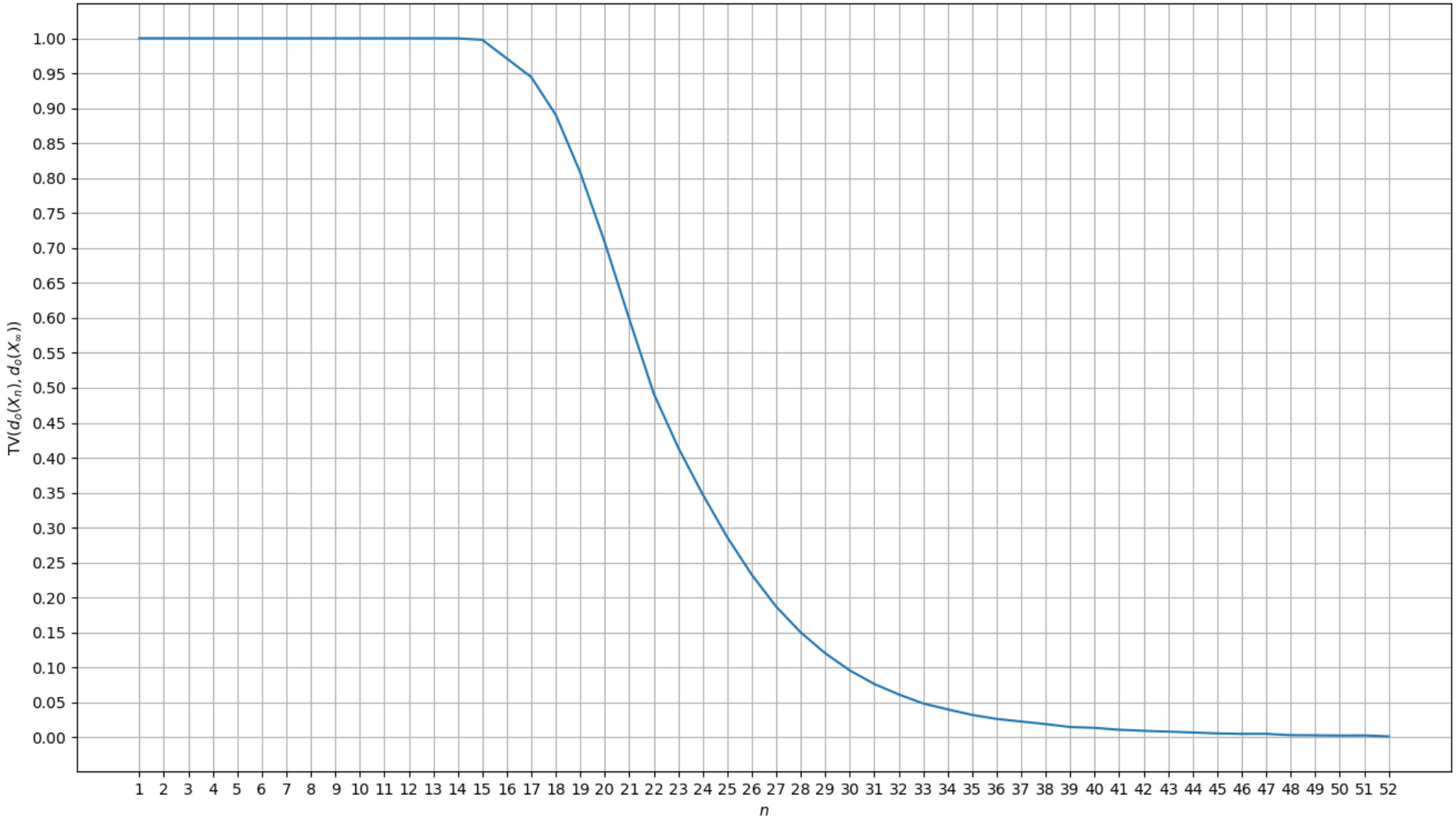
optimally solving 2M random cubes

$$\mathrm{TV}(X_{25}, X_\infty) \geq \mathrm{TV}(d_o(X_{25}), d_o(X_\infty))) = 0.286329 \pm 0.000697 > 0.25$$

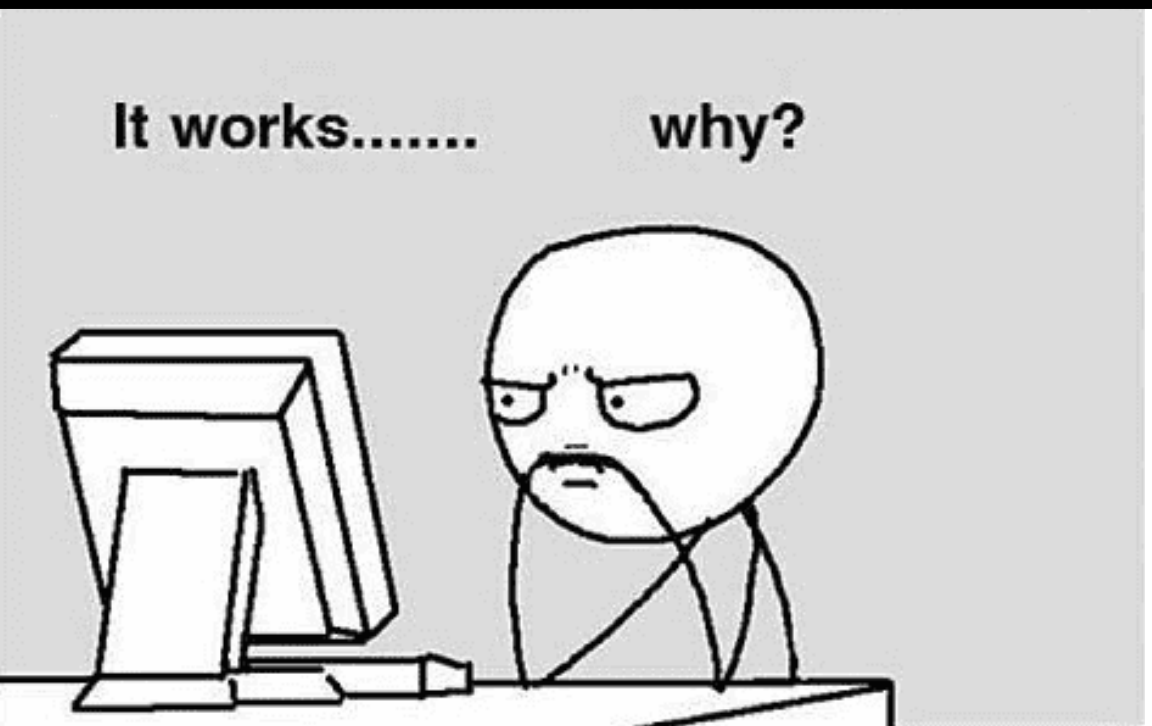
exercise

distance to the solved state

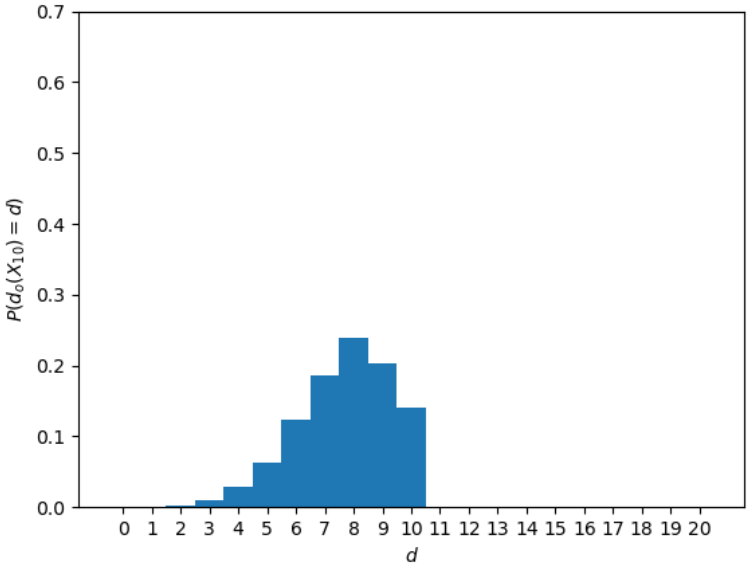
bootstrap

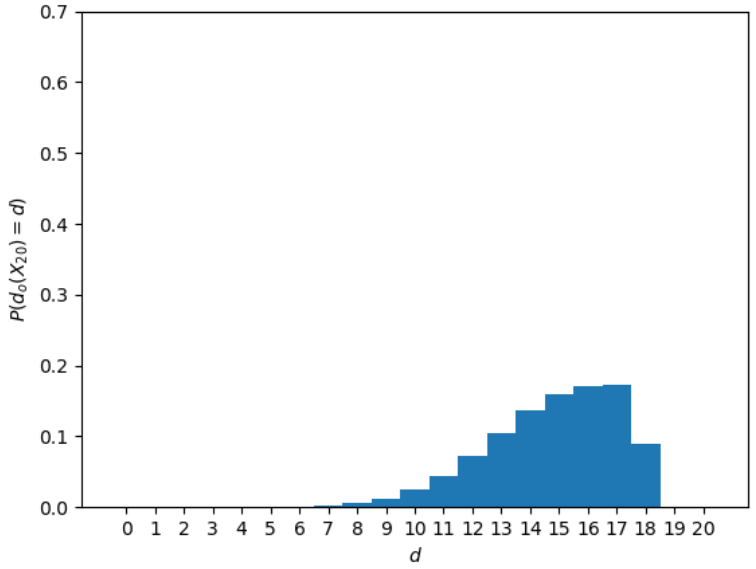

1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
TV($d_o(X_n), d_o(X_\infty)$)
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52
n

- embedded 2x2x2 pocket cube
- 3,674,160 states
- mixing time 19

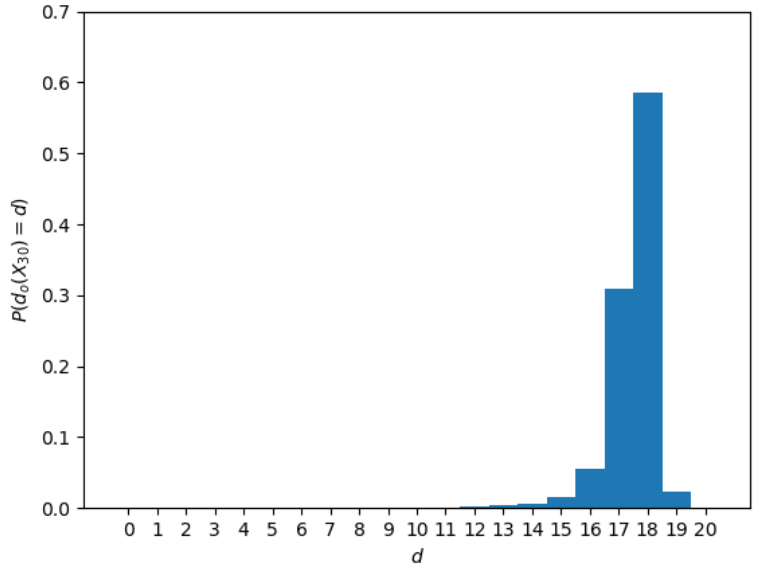


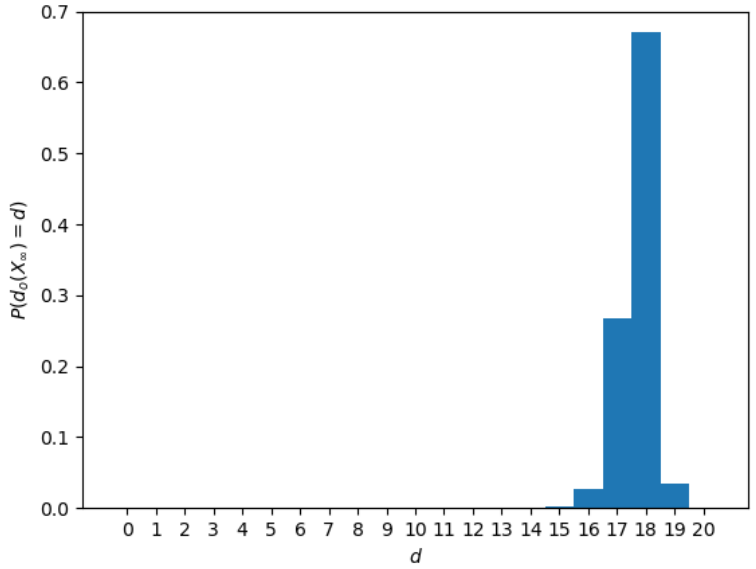
- distance to the solved state
- 21 states
- mixing time 26





moving instead of mixing





Thank You

https://arxiv.org/abs/2410.20630

Rubik's Cube Scrambling Requires at Least 26 Random Moves