Homework 2: Due at start of class on 02-Feb.

Please complete all nine problems.

1. Let X, Y, Z be jointly defined continuous random variables. Please state whether each of the following statements is true or false, and provide an explanation.
   (a) If the conditional pdf of X, Y given Z factors as $f_{X,Y|Z}(x, y|z) = f_{X|Z}(x|z)f_{Y|Z}(y|z)$, then X and Y are independent.
   (b) If $E[XY] = E[X]E[Y]$, then X and Y are independent.
   (c) If X and Y are i.i.d., then $E[XY] = E[X]E[Y]$.
   (d) If X and Y are jointly Gaussian and $E[XY] = E[X]E[Y]$, then X and Y are independent.
   (e) $VAR(X + Y) = VAR(X) + VAR(Y) + 2 \cdot COV(X, Y)$.

2. For the random vector
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho \\ \rho & \sigma_y^2 \end{bmatrix}\right)$$
   calculate each of the following quantities:
   (a) $E[(X - Y)^2]$
   (b) $E[5X^2 + 2]$
   (c) $E[X|Y]$

3. C&T 2.1

4. C&T 2.4

5. C&T 2.7. You may omit part (b).

6. C&T 2.8

7. C&T 2.12. You may omit part (f).

8. Let X_1, X_2 be i.i.d. random variables taking values in $\{0, 1\}$ with $\Pr(X_1 = 1) = \Pr(X_2 = 1) = \frac{1}{2}$. For $n \geq 3$, define X_n as follows:
$$X_n = \begin{cases} 0, & \text{if } X_{n-1} \neq X_{n-2} \\ 1, & \text{if } X_{n-1} = X_{n-2} \end{cases}$$
   (a) Compute $H(X_n)$ for $n \geq 3$.
   (b) Compute $H(X_n|X_1, X_2)$ for $n \geq 3$.

9. Let $X, Y \sim p(x, y)$ be jointly distributed discrete random variables.
   (a) Show that **one** of the following is true (where $\cdot$ is multiplication):
      i. $H(X|X \cdot Y) \geq H(X, Y) - H(X \cdot Y)$
      ii. $H(X|X \cdot Y) \leq H(X, Y) - H(X \cdot Y)$

      Please provide a proof and justify all steps.
   (b) Find the conditions for equality in part (a).