Solutions for Homework 1

1. Base case: Let $n = 9$. Clearly, $3^9 < 8!$.

   Inductive step: Assume $3^k < (k-1)!$ where $k > 9$. Then:

$$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &\leq 3 \cdot (k-1)! \text{ (from the assumption)} \\ &< k \cdot (k-1)! \text{ (since, for } k > 9,\ k > 3) \\ &= k! \end{aligned}$$

   Hence, $3^n < (n-1)!$.

2. Let
$$f(x) = e^{-x} + x - 1.$$
   Now $f(0) = 0$, and by differentiating, we see that f is non-decreasing on $(0, 1]$ and non-increasing on $(-\infty, 0]$. It follows that f is non-negative, so
$$1 - x \leq e^{-x}$$
   Since both sides are non-negative, we can then raise both sides to the power y.

3. First we form the Langrangian $J(x, y, \lambda) = f(x, y) + \lambda(x^2 + y^2 - 1)$ where λ is the *Lagrange multiplier*. Then, setting the derivatives to zero, we get the following 3 equations:
$$\begin{aligned} \frac{\partial J}{\partial x} = 3 + 2\lambda x &= 0 \\ \frac{\partial J}{\partial y} = 6 + 2\lambda y &= 0 \\ \frac{\partial J}{\partial \lambda} = x^2 + y^2 - 1 &= 0 \end{aligned}$$
   which has the solutions:
$$\begin{aligned} (x_1, y_1, \lambda_1) &= \left(+\frac{\sqrt{5}}{5}, +\frac{2\sqrt{5}}{5}, -\frac{3\sqrt{5}}{2}\right) \\ (x_2, y_2, \lambda_2) &= \left(-\frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}, +\frac{3\sqrt{5}}{2}\right) \end{aligned}$$
   where using the 2nd derivative test we can show that (x_1, y_1, λ_1) is a maximum which occurs at $f(x_1, y_1) = 3\sqrt{5}$, while (x_2, y_2, λ_2) is a minimum which occurs at $f(x_2, y_2) = -3\sqrt{5}$.

4. (a) Note that in general, proving that $E[UV] = E[U]E[V]$ is not sufficient for showing that the two variables are independent. You must show that $f_{UV}(u, v) = f_U(u)f_V(v)$, i.e. that the joint pdf can be written as the product of the marginal pdf's. Thus, in this case, letting $U = X + Y, V = X - Y$, you would need to find the joint pdf of U, V and show that it factors. To find the pdf of two variables which are functions of two other random

variables, you can use the Jacobian (see any probability textbook), and indeed the pdf will factor. Another way to show independence is with characteristic functions.

However, a simpler solution comes from the fact that *jointly* Gaussian rv's have two special properties that could be exploited for this problem:

- Linear transformations of jointly Gaussian rv's are also jointly Gaussian.
- If the covariance of two *jointly* Gaussian rv's is zero, then they are independent. Note that it is *not* sufficient for the two variables to each be marginally Gaussian – they must be jointly Gaussian.

Since U, V is just a linear transformation of X, Y, they are indeed jointly Gaussian, and the covariance can be shown to be zero via:

$$\begin{aligned}
\mathrm{COV}(X-Y, X+Y) &= E\left[[(X-Y) - \underbrace{E[(X-Y)]}_{=0}][(X+Y) - \underbrace{E[(X+Y)]}_{=0}]\right] \\
&= E[(X-Y)(X+Y)] \\
&= E[X^2] - E[Y^2] \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
E[X^3 - Y^3 | X-Y] &= E[(X-Y)^3 | X-Y] + 3E[(X-Y)(XY) | X-Y] \\
&= (X-Y)^3 + \frac{3}{4}(X-Y)E[(X+Y)^2 - (X-Y)^2 | X-Y] \\
&= (X-Y)^3 + \frac{3}{4}(X-Y)\left(E[(X+Y)^2] - (X-Y)^2\right) \\
&= \frac{1}{4}(X-Y)^3 + \frac{3}{2}(X-Y)
\end{aligned}$$

where the last line comes from the fact that $E[(X+Y)^2] = 2$.

5. 5(a).The transition matrix is

$$p = \begin{bmatrix}
\frac{4}{9} & \frac{5}{9} & 0 & 0 & 0 & 0 \\
0 & \frac{4}{8} & \frac{4}{8} & 0 & 0 & 0 \\
0 & 0 & \frac{4}{7} & \frac{3}{7} & 0 & 0 \\
0 & 0 & 0 & \frac{4}{6} & \frac{2}{6} & 0 \\
0 & 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

5(b). Not irreducible; cannot get from any state to any other state(for example, from 0 to 5).

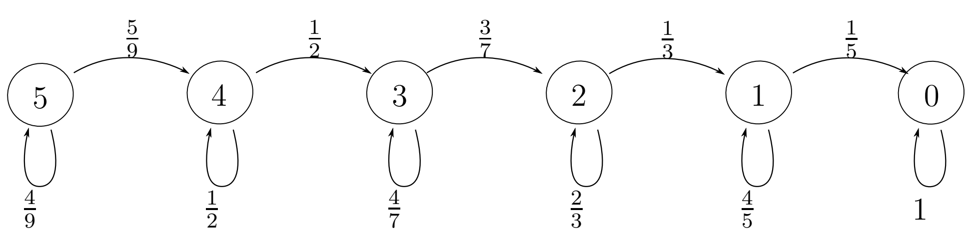

Figure 1: Solution to Problem 5(a).

6. Let T be the number of times you take ECE 5311, where $T \in \{1, 2, 3, 4\}$. The pmf of T is then given by the following table, using that the fact that the outcome for each taking of ECE 5311 is an i.i.d. Bernoulli random variable with parameter p:

outcome	T	probability
P	1	p
FP	2	$(1-p)p$
FFP	3	$(1-p)^2 p$
FFFP $\cup$ FFFF	4	$(1-p)^3 p + (1-p)^4$

Then,

$$\begin{aligned} E[T] &= 1p + 2(1-p)p + 3(1-p)^2 p + 4[(1-p)^3 p + (1-p)^4] \\ &= 4 - 6p + 4p^2 - p^3 \end{aligned}$$

7.

Let D be the event that she picks the double-headed coin, N be the event that she picks the normal coin, and Z be the event that she picks the double-tailed coin. Let H_{L_i} (resp. H_{U_i}) be the event that the lower (resp. upper) face of the coin on the ith toss is a head.

a)

$$\begin{aligned} P(H_{L_1}) &= P(H_{L_1} \mid D)P(D) + P(H_{L_1} \mid N)P(N) + P(H_{L_1} \mid Z)P(Z) \\ &= (1)(\frac{2}{5}) + (\frac{1}{2})(\frac{2}{5}) + (0)(\frac{1}{5}) \\ &= \frac{3}{5} \end{aligned}$$

b)

$$\begin{aligned} P(H_{L_1} \mid H_{U_1}) &= \frac{P(H_{L_1} \cap H_{U_1})}{P(H_{U_1})} \\ &= \frac{2}{5}/\frac{3}{5} \\ &= \frac{2}{3} \end{aligned}$$

c)

$$\begin{aligned} P(H_{L_2} \mid H_{U_1}) &= \frac{P(H_{L_2} \cap H_{U_1})}{P(H_{U_1})} \\ &= \frac{P(H_{L_2} \cap H_{U_1} \mid D)P(D) + P(H_{L_2} \cap H_{U_1} \mid N)P(N) + P(H_{L_2} \cap H_{U_1} \mid Z)P(Z)}{P(H_{U_1})} \\ &= \frac{(1)(\frac{2}{5}) + (\frac{1}{4})(\frac{2}{5}) + (0)(\frac{1}{5})}{\frac{3}{5}} \\ &= \frac{5}{6} \end{aligned}$$

d)

$$\begin{aligned} P(H_{L_2} \mid H_{U_1} \cap H_{U_2}) &= \frac{P(H_{L_2} \cap H_{U_1} \cap H_{U_2})}{P(H_{U_1} \cap H_{U_2})} \\ &= \frac{(1)(\frac{2}{5}) + (0)(\frac{2}{5}) + (0)(\frac{1}{5})}{(1)(\frac{2}{5}) + (\frac{1}{4})(\frac{2}{5}) + (0)(\frac{1}{5})} \\ &= \frac{4}{5} \end{aligned}$$