Homework 1: Due at start of class on 26-Jan.

Please complete all seven problems.

1. Using induction, prove that $3^n < (n-1)!$ for every $n \in \{9, 10, 11, \ldots\}$.

2. Prove that $(1-x)^y \leq e^{-xy}$ for all $x < 1$ and $y > 0$.

3. Using the method of Lagrange multipliers, find both the maximum and minimum values of the function $f(x, y) = 3x + 6y$ on the circle $x^2 + y^2 = 1$.

4. Let X and Y be i.i.d. random variables distributed as $X, Y \sim \mathcal{N}(0, 1)$.

   (a) Show that $X + Y$ and $X - Y$ are independent.

   (b) Determine $E[X^3 - Y^3 | X - Y]$.
   Hint: Use your result from part (a). And, depending on how you approach the problem, it may help to use the fact that $XY = \frac{1}{4}[(X+Y)^2 - (X-Y)^2]$.

5. An urn initially contains five black balls and four white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; if the ball is white it is put back in the urn, otherwise it is left out. Let $X_i \in \{0, 1, 2, 3, 4, 5\}$ be the number of black balls remaining in the urn after i draws from the urn.

   (a) Draw the state transition diagram for the Markov chain X_i, and provide the state transition matrix.

   (b) Is this Markov chain irreducible?

6. Your academic advisor tells you that you *must* pass ECE 5311 in order to graduate. But, he/she makes a deal: if you *fail* ECE 5311 four times, he/she will have mercy and let you graduate anyway. Assume that you will pass any given semester of ECE 5311 with probability p, and that you forget everything you learned each time you re-take ECE 5311 (so that subsequent re-takings are independent). What is the expected number of times you will have to take ECE 5311 in order to graduate?

7. Gertrude has five fair coins in her pocket: two are double-headed, one is double-tailed, and two are normal.

   (a) She closes her eyes, chooses a coin at random, and tosses it. What is the probability that the lower face is a head?

   (b) She opens her eyes, and sees that the upper face is a head. What is the probability that the lower face is a head?

   (c) She closes her eyes, and tosses the same coin again. What is the probability that the lower face is a head?

   (d) She opens her eyes, and sees that the upper face is a head. What is the probability that the lower face is a head?