Computational Physics Homework Set #2

(Due 10/8, at noon)

1) In general relativity (GR), the orbit of a test particle around a source obeys the following differential equation (see lecture notes),

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + 3\frac{GM}{c^2}u^2, \tag{1}$$

where $u \equiv 1/r$ is the inverse radial position of the orbit, M is the source mass, h is the specific angular momentum of the orbit, G is Newton's constant, and c is the speed of light. Compared to the Newtonian result, Eq. (1) contains a relativistic correction (depending on c) that is of order (GM/c^2r) of the Newtonian, linear in u, term. This dimensionless number is characteristic of GR, since (GM/c^2r) is roughly the ratio of the Schwarzschild radius of the source ($2GM/c^2$,where GR gravity is very different from Newton's) to the characteristic size of the orbit r.

a) Write a code that, using 4^{th}-order explicit Runge-Kutta, solves Eq. (1).

b) Use your code for the specific case of Mercury orbiting around the Sun[1]. Switching off the relativistic term in Eq. (1), check that your code produces a closed orbit and compare it to the exact Newtonian result,

$$u = \frac{GM}{h^2}\Big(1 + e\ \cos(\phi)\Big), \tag{2}$$

where the eccentricity obeys $h^2/GM = a(1 - e^2)$ with a the semi-major axis of the orbit. Make a plot of this comparison for a few different choices of the timestep in your integrator. For initial conditions, you can start for example at perihelion where $du/d\phi = 0$.

c) Use your code for the relativistic case and compare the result you get for the perihelion precession with that of the famous perturbative result,

$$\Delta\phi^{\text{shift}} = \Delta\phi - 2\pi \simeq 6\pi\Big(\frac{GM}{hc}\Big)^2 \simeq 43\text{''}/\text{century} \tag{3}$$

where $\Delta\phi$ denotes the difference in angle between one perihelion and the next.

Hint: at the beginning, you may want to artificially increase the precession to make easier its detection by amplifying arbitrarily the last term in Eq. (1). Also, in order to avoid making the timestep very small (which makes it slow) to find the perihelion, you may want to fit a quadratic to the three points closest to perihelion to better find the angle at perihelion.

[1] For Mercury parameters see e.g. nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html. *In your writeup, please state clearly the information you use.*

2) Suppose the gravitational Newtonian force is changed to

$$F = \frac{GMm}{r^2} \times \left(\frac{r_0}{r}\right)^{\delta} \tag{4}$$

a) Find the equation of the orbit for this force law, writing it in a form analogous to Eq. (1).

b) Using your Runge-Kutta solver, compute the perihelion shift for $\delta = 0.05$ and $r_0 = h^2/GM$, where $h = r^2\dot{\phi}$ is the angular momentum per unit mass.

3*) The stellar structure equations for Newtonian gravity are,

$$\frac{dp}{dr} = -\frac{G\rho(r)m(r)}{r^2}, \tag{5}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(r), \tag{6}$$

where $\rho(r)$ is the mass density and p is the pressure. Equation (5) describes hydrostatic equilibrium, and receives relativistic corrections, leading to the Tolman, Oppenheimer & Volkov (TOV) equation,

$$\frac{dp}{dr} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left(1+\frac{p}{\epsilon}\right)\left(1+\frac{4\pi r^3 p}{mc^2}\right)\frac{1}{\left(1-\frac{2Gm}{c^2r}\right)}, \tag{7}$$

where we have used the special relativity (SR) relationship between mass (ρ) and energy (ϵ) density, $\epsilon = \rho c^2$. The first two corrections in parenthesis in Eq. (7) correct Newtonian hydrostatic equilibrium for SR effects, the last correction factor in Eq. (7) is due to general relativity (compare with Eq. (1)).

In order to solve the two coupled equations (6) and (7) for p and m as a function of radius r, one needs to provide and *equation of state* (EOS) that relates the pressure to the energy density, and appropriate boundary conditions,

$$p(0) = p_0 > 0, \qquad m(0) = 0, \qquad p(R) = 0, \qquad m(R) = M, \tag{8}$$

where R and M are the radius and mass of the star, to be found by integrating from $r = 0$ outwards until the pressure vanishes. Note that one has to supply the central pressure p_0 (which must be positive, of course).

a) Write a code that, using 4$^{\text{th}}$-order explicit Runge-Kutta, solves Eqs. (6-7).

b) A *neutron star* is a star supported by degeneracy pressure from neutrons (due to Pauli's exclusion principle). A simplified EOS in this case is given by

$$\hat{\epsilon} = 2.4216\,\hat{p}^{3/5} + 2.8663\,\hat{p}, \tag{9}$$

*This problem counts as extra credit for undergraduate students.

where the first term corresponds to the non-relativistic dependence, and the second mimics SR effects. Here variables with hats are dimensionless quantities, i.e. $x \equiv \epsilon_0\,\hat{x}$ where ϵ_0 is a characteristic energy density given by

$$\epsilon_0 \equiv \frac{m_n^4 c^5}{(3\pi^2\hbar)^3} = 5.346 \times 10^{36}\,\frac{\text{ergs}}{\text{cm}^3} = 3.0006 \times 10^{-3}\,\frac{M_\odot c^2}{\text{km}^3} \tag{10}$$

where m_n is the neutron mass and $\hbar = h/2\pi$ with h Planck's constant, and $M_\odot$ is the sun's mass.

Rewrite all the equations in terms of dimensionless variables, using ϵ_0 for energy densities and pressure as above, $M_\odot$ for masses (i.e. $\hat{m} = m/M_\odot$) and km for radii ($\hat{r} = r/\text{km}$), since neutrons stars have masses of order $M_\odot$ and radii of order ten km.

c) Using $\hat{p}_0 = 0.01$ make plots of $\hat{p}$ and $\hat{m}$ as a function of $\hat{r}$ for both the Newtonian solutions and the relativistic ones. Does the difference between Newtonian and relativistic make sense? Explain.

d) Solve the relativistic equations for star mass $\hat{M}$ and radius $\hat{R}$ as a function $\hat{p}_0$, starting from $\hat{p}_0 = 0.01$ and increasing it up to about $\hat{p}_0 = 0.05$. Plot $\hat{M}$ vs $\hat{R}$ (obtained parametrically from the different $\hat{p}_0$ values). You should see there is a maximum value for the mass, find it and the radius it corresponds to. This is a famous result from Oppenheimer & Volkov (1939).