Homework 2 - Solving ODEs by Runge-Kutta Method

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Note: The zip folder include main c++ code in /comp/hw2/ and ROOT plot code in /root/macros. Code is compiled in g++ (clang) in Mac OSX.

1 Problem 1

Let $\frac{du}{d\phi}$ = v and $\frac{d^2u}{d\phi^2}$ = dv then the given equation of motion (1) becomes the system of equation:

$$\frac{dv}{d\phi} + u = \alpha + \beta u^2 \tag{1}$$

$$\frac{du}{d\phi} = v \tag{2}$$

with $\frac{GM}{h^2} = \alpha$ and $\frac{3GM}{c^2} = \beta$. This system of first-order ODE equation can be solved by Runge-Kutta method. Let $\vec{y} = (u, v)$ the system can be written as:

$$\frac{d\vec{y}}{d\phi} = \frac{d}{d\phi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \beta u^2 - u + \alpha \end{pmatrix} = \vec{f}(u, v, \phi) \tag{3}$$

Apply Runge-Kutta method with initial condition $\vec{y}(0) = \begin{pmatrix} a(1+e) \\ 0 \end{pmatrix}$ where a is the major axis and e is the eccentricity, thus u(0) is the inverse of the perihelion.

1.a Code implementation

Please see the attached file "comp/hw2/problem1.cc" for the code.

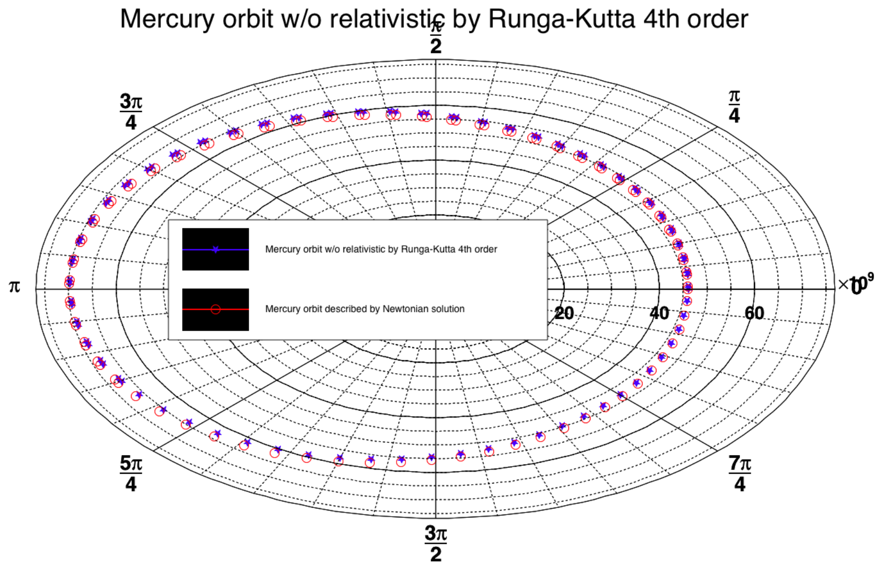

Figure 1: Close orbit produce by Runge-Kutta without relativistic term

1.b Mercury orbit around the Sun

The close orbit is shown by plotting the radius and ϕ in polar coordinate as in Figure (1). This this plot, the relativistic term β is turned off. The Runge-Kutta solution (blue) and Newtonian exact solution (red) are plotted together for different step sizes in Figure (2) (3) and (4) .It is obviously that, the more steps used, the more overlapping between the two solutions.

1.c The perihelion shift

The relativistic term β is now included. To find the shift in $\delta\phi$, the Runge-Kutta program runs for just pass one period with initial position as in section 1.b. The number of step was set to 10^7 for high precision and numerical values are set at high precision.

The initial point of (ϕ , u) is (0,1/perihelion) = (0, 2.17382199142713e-11). Three points nearest the maximum of u were fit into a quadratic u = $a\phi^2 + b\phi + c$ which first derivative was used to find the value of ϕ at maximum of u, i.e the perihelion. $\phi_{perihelion}$ found to be 6.283184785 Thus

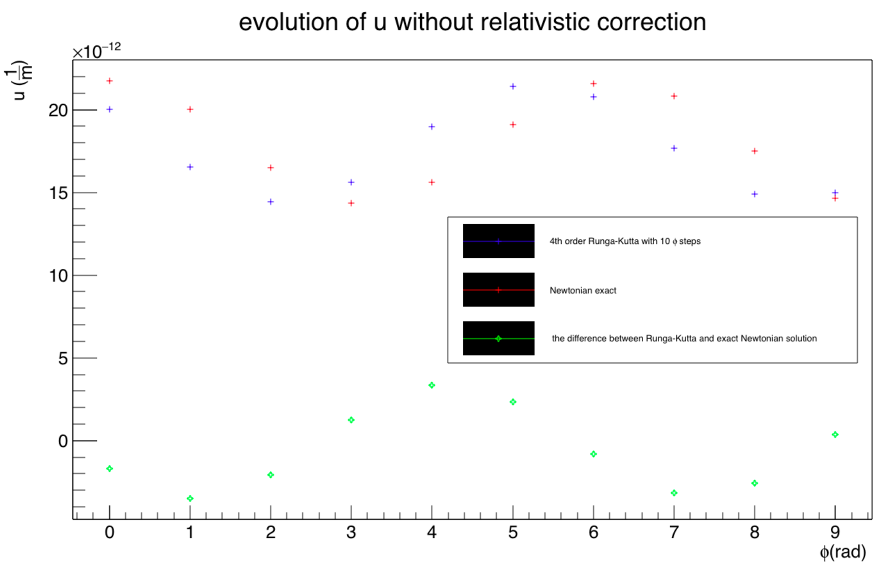

Figure 2: Comparing u with newtonian solution at 10 time steps

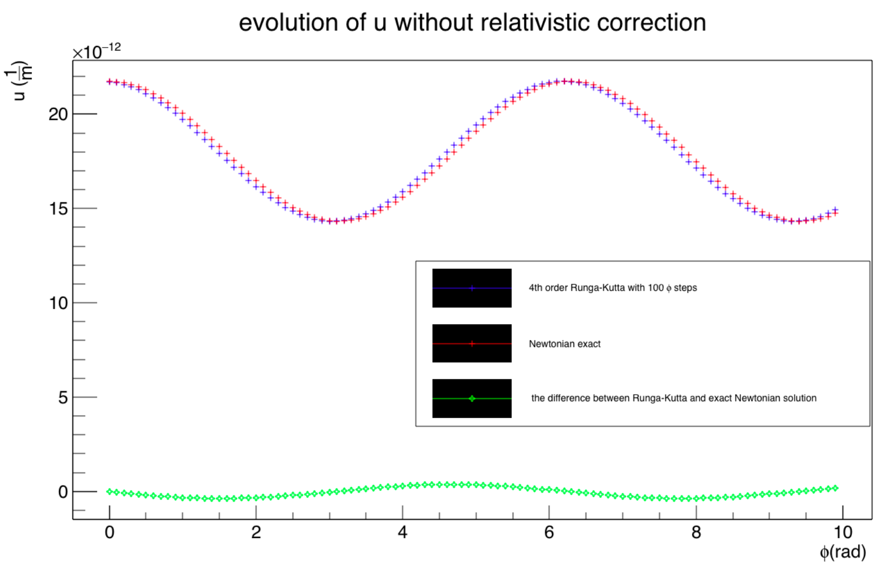

Figure 3: Comparing u with newtonian solution at 100 time steps

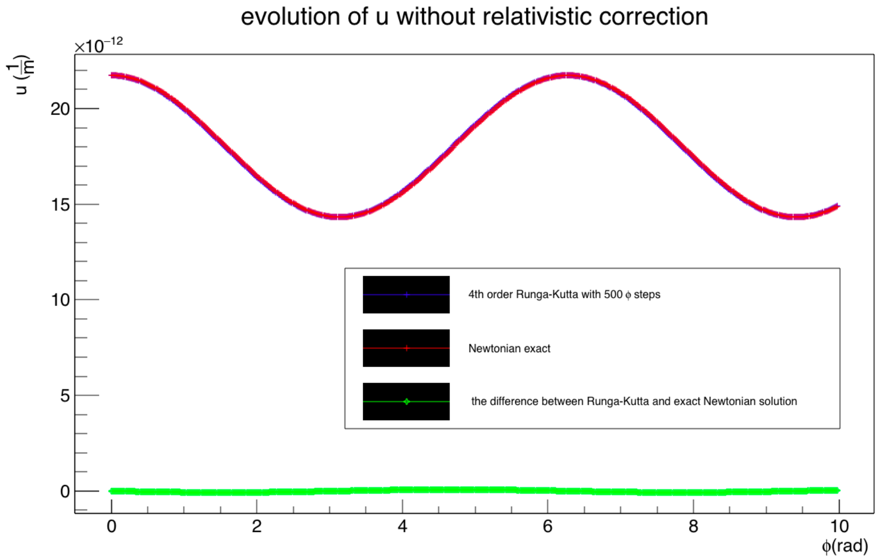

Figure 4: Comparing u with newtonian solution at 500 time steps

$$\Delta\phi_{shift} = (6.283184785 - 2\times\Pi).\frac{rad}{period}\times\frac{180}{\Pi}.\frac{degree}{rad}\times\frac{100}{0.241}.\frac{period}{century} \simeq 44.7\frac{arcsecond}{century} \quad (4)$$

This is approximately equal to the magnitude to the famous pertubative result. The result could be improved by increasing time steps or fitting with more points.

2 Problem 2

2.a Equation of orbit

The equation of motion of particle mass m subjected to central force f(r) in polar coordinate is:

$$m\ddot{\vec{r}} = f(r)\vec{e_r} \quad (5)$$

while taking derivative in polar coordinate give $\ddot{\vec{r}}$ = ($\ddot{r}$ - r$\dot{\phi}^2$)$\vec{e_r}$ + $(2\dot{r}\dot{\phi} + r\ddot{\phi})\vec{e_\phi}$ Thus the component-wise equation of motion are:

$$m(\ddot{r} - r\dot{\phi}^2) = f(r) \quad (6)$$

$$m(2\dot{r}\dot{\phi}+r\ddot{\phi})=0 \tag{7}$$

from equation 7 :

$$\frac{d}{dt}(r^2\dot{\phi})=0 \tag{8}$$

or $r^2\dot{\phi}$ = h = constant. Let u = $\frac{1}{r}$ then $\dot{r}=-\frac{1}{u^2}\dot{u}=-\frac{1}{u^2}\dot{\phi}\frac{du}{d\phi}=-h\frac{du}{d\phi}$. Differentiate this gives:

$$\ddot{r}=-h\frac{d}{dt}\frac{du}{d\phi}=-h\frac{d\phi}{dt}\frac{d}{d\phi}\frac{du}{d\phi}=-h\dot{\phi}\frac{d^2u}{d\phi^2}=-h^2u^2\frac{d^2u}{d\phi^2}$$

(9)

substituting r, $\dot{r}$ and $\dot{\phi}$ back to equation (6)

$$m(-h^2u^2\frac{d^2u}{d\phi^2}-\frac{1}{u}h^2u^4)=f(u^{-1}) \tag{10}$$

simplify, the orbital equation for a particle moving under a central for is

$$\frac{d^2u}{d\phi^2}+u=\frac{1}{mh^2u^2}f(u^{-1}) \tag{11}$$

plug in f=$\frac{GMn}{r^2}\times(\frac{r_o}{r})^\delta$ equation 11 becomes

$$\frac{d^2u}{d\phi^2}+u=\frac{GM}{h^2}(ur_o)^\delta \tag{12}$$

2.b Runge-Kutta method

As in problem 1, let $\frac{du}{d\phi}$ = v and $\frac{d^2u}{d\phi^2}$ = dv. Use Runge-Kutta to solve the system of ODE with initial condition $\vec{y}(0)=\begin{pmatrix} 1/perihelion \\ 0 \end{pmatrix}$, r$_o=h^2/GM$,$\delta$= 0.05.

$$\frac{d\vec{y}}{d\phi}=\frac{d}{d\phi}\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} v \\ -u+(\alpha)^{0.95}(u)^{0.05} \end{pmatrix}=\vec{f}(u,v,\phi) \tag{13}$$

with $\alpha=\frac{GM}{h^2}$ as in problem 1. The Runge-Kutta method code was written in general time dependent form and can be readily use to solve this system (see source code at

"/comp/hw2/problem2.cc"). Result: The points nearest to perihelion were found to be
6.283184785 , 2.17382199142695e-11
6.283185414 , 2.17382199142697e-11
6.283186043 , 2.17382199142684e-11
6.283184156 , 2.17382199142679e-11
These three points were fit into a quadratic $u = a\phi^2 + b\phi + c$ which first derivative was used to find the value of ϕ at the perihelion. The quadratic is:

$$-0.176908\phi^2 + 2.28125\phi + -5.1804 \tag{14}$$

and thus

$$\delta\phi_{shift} = -\frac{b}{2a} - 2\Pi = 0.164354(\text{rad/period}) \tag{15}$$

The result verifies that the orbit is less stable when the potential is not proportional to $1/r^2$

3 Problem3

3.a Code implementation

Please see attached files "/comp/hw2/probllem3.cc", "problem3newtonian.cc" and "problem3d.cc". The Runge-Kutta code was written for general time dependent case, so only the right hand side of ODEs have to be changed.

3.b Dimensionless equation

After substitution: $\epsilon = \epsilon_o\hat{\epsilon}, p = \epsilon_o\hat{p}, m = \hat{m}M_\odot, r = \hat{r}km$ into the given equation and necessary algebra, the dimensionless equations are:

$$\frac{d\hat{m}}{d\hat{r}} = 4\Pi\left(\frac{km^3\epsilon_o}{M_\odot c^2}\right)\hat{r}^2(2.4216\hat{p}^{3/5} + 2.8663\hat{p}) \tag{16}$$

$$\frac{d\hat{p}}{d\hat{r}} = -\frac{GM_\odot}{c^2km}\frac{(2.4216\hat{p}^{3/5} + 2.8663\hat{p})}{\hat{r}^2}\left(1 + \frac{\hat{p}}{2.4216\hat{p}^{3/5}\hat{} + 2.8663\hat{p}}\right)$$

$$\times\left(\hat{m} + 4\Pi\left(\frac{km^3\epsilon_o}{M_\odot c^2}\right)\frac{\hat{r}^3\hat{p}}{\hat{m}}\right)\left(\frac{1}{1-\frac{2GM_\odot}{c^2km}\frac{\hat{m}}{\hat{r}}}\right) \tag{17}$$

where $\frac{km^3\epsilon_o}{M_\odot c^2}$ and $\frac{GM_\odot}{c^2km}$ are dimensionless constants with value respectively $3.0006.10^{-3}$ and 1.477. The system of equation is now ready to be solved by Runge-Kutta method.

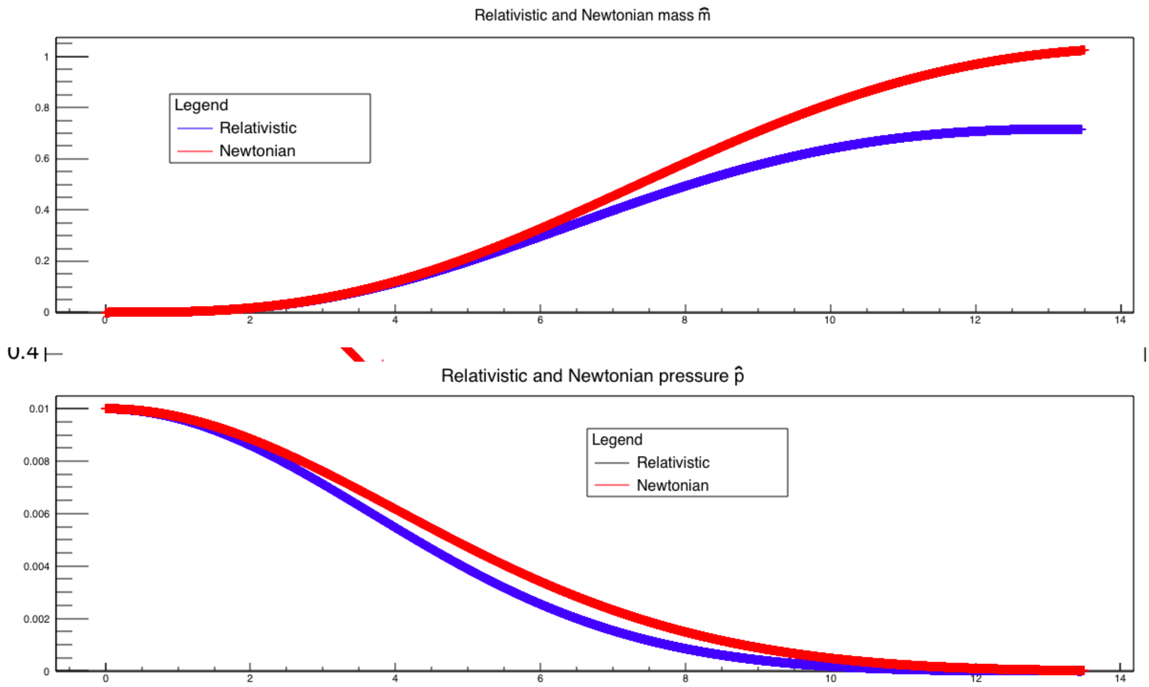

Figure 5: Comparing the pressure and mass of Newtonian and relativistic solution. Unit of mass are in solar mass, unit of radius in km and unit of pressure are in ϵ_o

3.c Plot of $\hat{p}$ and $\hat{m}$ as a function of $\hat{r}$

Referring to Figure 5, the Newtonian mass increases faster than the relativistic one. That means a higher average density, consistent with larger pressure shown in the second graph.

All the corrections in TOV equations are greater than 1 and strengthen gravitational interaction, consequently put a stronger constraint on maximum mass within a radius (to not form a black hole) , as shown on Figure 5

3.d Finding for $\hat{M}$ and $\hat{R}$

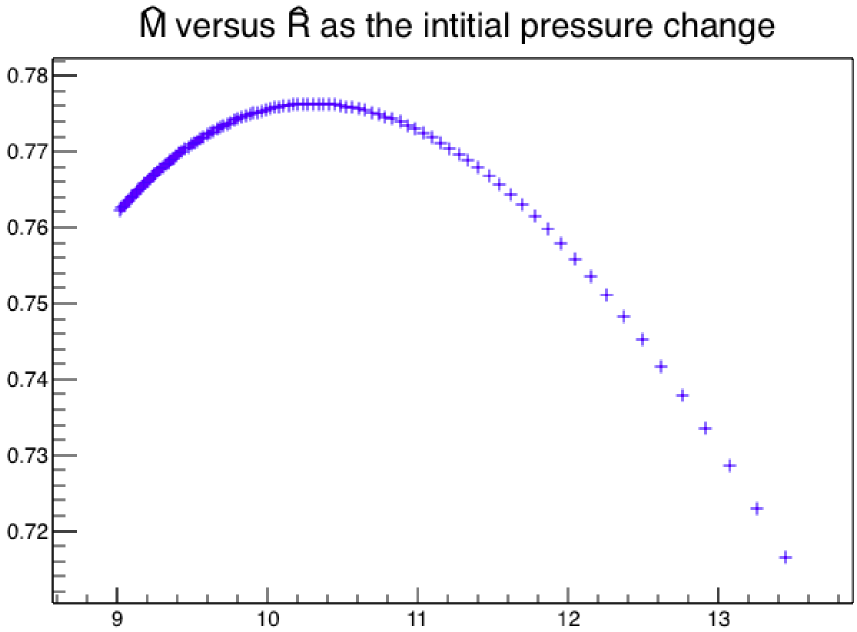

Figure 6: $\hat{M}$ and $\hat{R}$ are solved for 100 points of initial pressure between 0.01 and 1.1. From left to right, the initial pressure decreases

The $\hat{M}$-$\hat{R}$ plot is shown in Figure 6. Clearly, there is a maximum for the mass. To find out, as in problem 1 and 2, three nearest points to the maximum mas was fitted to a quadratic and the maximum was found by setting the first derivative of the quadratic equal to zero.

The result is $\hat{M}_{max}$=0.776339 solar mass and $\hat{R}$=10.3064km. These are close to famous result from Oppenheimer and Volkow 1939.

References

[1] Fowles and Cassiday, *Analytical Mechanics*

[2] Roman Scoccimarro *Lecture notes*