Computational Physics Homework Set #4

(Due 11/19, at noon)

1) Consider the following linear PDE (with a, b constants and n integer),

$$\frac{\partial f}{\partial t} + b\, x\, \frac{\partial f}{\partial x} = a\, x^n. \tag{1}$$

a) Solve Eq. (1) by the method of characteristics. Express your solution in terms of the initial condition $f(x, t = 0) = f_0(x)$.

b) Plot your solution as a function of x at different times $t = 0.1, 2$, for $f_0(x) = \exp -(x-1)^2$, $b = n = 1$. Consider two cases, $a = 0$ and $a = 0.05$. Explain what you see in the plots.

2) Consider the Poisson equation,

$$\nabla^2 \Phi = 1 \tag{2}$$

in an irregular two-dimensional geometry defined by the equations,

$$0 \leq x \leq 1.5, \qquad 0 \leq y \leq 2, \qquad y \geq 1.5 - 2x, \qquad y \leq 2.75 - 1.5x \tag{3}$$

with boundary conditions $\Phi = 0$.

a) Write a code that solves this using Gauss-Seidel relaxation in a cartesian grid.

b) Solve the problem for a grid of size 100×100. Plot the error at step n as a function of n defined as

$$\text{error}(n) \equiv \text{Max}\ |\Phi^{(n)}(x,y) - \Phi^{(n-1)}(x,y)|, \tag{4}$$

from $n = 1$ until the solution converged to a reasonably small error.

c) Make a *contour plot* of $\Phi(x, y)$ for your solution.

d) Repeat a)-c) for the SOR method (choose the parameter $1 < w < 2$). Compare the convergence rate against Gauss-Seidel.

3) Consider the traffic flow problem, which obeys the continuity equation for density of vehicles $\rho(x,t)$,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, v) = 0, \tag{5}$$

where $v(x,t) = v_{\max}(1 - \rho/\rho_{\max})$ is the velocity field, with $v_{\max}$ the speed limit, and $\rho_{\max}$ the density corresponding to vehicles filling space bumper to bumper.

a) Explain why such a dependence of v on ρ makes physical sense. Calculate the wave speed $c(\rho)$ and compare with $v(\rho)$, explain their difference.

b) Impose periodic boundary conditions (as in a race track) and a square wave initial condition,

$$\rho(x,0) = \rho_{\max}, \qquad -L/4 < x < 0, \tag{6}$$

and zero otherwise, where L is the size of the track, $-L/2 \le x \le L/2$.

c) Solve the PDE by evolving ρ using the methods FTCS and Lax. Figure out reasonable values for $v_{\max}$, $\rho_{\max}$, L, and time of evolution (e.g. so that cars go around at least once). Make a choice of resolution of your spatial grid, and then fix your time step appropriately. Explain the logic behind your choices.

d) Make a 3D plot showing the density ρ as a function of space and time (or 2D plots with $\rho(x)$ for different times). Compare the results of the two methods and explain the main features of the solution.

4*) Consider the KdV equation in one dimension,

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} + \alpha\frac{\partial^3 f}{\partial x^3} = 0, \tag{7}$$

where $-L \le x \le L$ and periodic boundary conditions are imposed, the initial condition corresponds to a small amplitude cosine wave, $f(x, t=0) = (1/8)\cos(\pi x/L)$.

a) Solve the KdV equation using the Galerkin method, as discussed in class. In doing so, you convert the PDE into a system of coupled ODE's which can be solved by your Runge-Kutta solver. Use plane waves as modes to expand your solution, including up to $M = 20$ modes, and evolving from $t = 0$ to $t = 200$. Use $\alpha = 1$ and $L = 40$ for your numerical solution, and the standard inner product.

b) Make plots of $f(x,t)$ as a function of x for times $t = 0, 20, 40, 80, 120, 200$.

c) Calculate and make a plot of the power spectrum for the timesteps in b).

d) Explain what you see in the plots in b) and c). What would happen if we decrease α?

*This problem counts as extra credit for undergraduate students.