Critical dynamics in three-dimensional space

Mai Xuan Lý

Vietbak Technical Institute, Socialist Republic of Vietnam
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A new method is proposed for the direct computation, in three-dimensional space, of the dynamical exponent that characterizes the anomalous relaxation time of a thermodynamic system near a second-order phase transition point. The renormalized Green functions and the Ward identity are used to derive a relation between the vertex function and the dynamical exponent for a system with nonconserved order parameter and energy. The correctness of the dynamic scaling hypothesis in the three-dimensional case is confirmed by direct calculation. The obtained numerical result is also compared with the experimental data on critical ultrasound absorption.

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I. INTRODUCTION

A large number of papers have recently been published on the critical dynamics of slightly nonequilibrium systems undergoing a second-order phase transition. The main attention in these papers is given to the computation of the dynamical exponent that describes the character of the singularities of all the kinetic coefficients. In Ref. 1 the dynamical exponent is obtained with the aid of Wilson's ε-expansion procedure. The same problem has been solved[2] with the aid of the parquet-diagram summation method, used in conjunction with the ε-expansion procedure. A field-theoretic approach based on the Callan-Symanzik equation is proposed in Ref. 3. The dynamical problem can be reduced with the aid of the Fokker-Planck equation and path integration to a static problem in which the recursion method is directly applicable.[4] The discrete renormalization group, which has had great success in the static theory, finds further application in the dynamical theory.[5] Furthermore, many authors now apply numerical methods to the problem in question.

Although the approaches mentioned here are different, all of them are direct generalizations of the corresponding static approaches, and the ε-expansion is used in all the continuous-renormalization-group procedures. It would therefore be desirable to construct a dynamical theory directly in three-dimensional space, i.e., without the use of the ε-expansion method.

As is well known, even in the case of critical statics, the solution of the phase-transition problem directly in three-dimensional space requires special procedures.[6–8] This necessity is due to the loss of the invariance property that is inherent in the four-dimensional problem, and makes the analysis significantly easier. For example, in contrast to the four-dimensional case, here the invariant charge explicitly depends on the distance to the phase-transition line (see below).

It follows from the foregoing that the analysis of the dynamical problem directly in three-dimensional space is of interest in its own right not only in the computational, but also in the methodological, respect.

In the present paper, using the renormalized Green functions and the Ward identity, we derive directly in the three-dimensional case a relation between the vertex function and the dynamical exponent for a system with nonconserved order parameter and energy (other models require special treatment). The correctness of the dynamic scaling hypothesis is confirmed by direct calculation. We also carry out a comparison of the numerical result with the experimental data on critical ultrasonic absorption. Our approach is, in a sense, a generalization of the static approach proposed in Ref. 8 by Ginzburg.

Let us note that the method proposed by us here allows us to study the effect of the terms that are insignificant in $(4-\varepsilon)$-dimensional space (i.e., of the interaction of order higher than the fourth in the Ginzburg-Landau Hamiltonian) on the dynamics of a system in the critical state.

2. THE RENORMALIZED GREEN FUNCTIONS

We shall consider a system with nonconserved real order parameter and nonconserved energy. The dynamics of such a system is described by the following kinetic equation[1]:

$$\frac{\partial \varphi_\alpha(\mathbf{r},t)}{\partial t} = -\Gamma_0 \frac{\delta H[\varphi]}{\delta \varphi_\alpha} + \eta_\alpha(\mathbf{r},t), \tag{1}$$

where Γ_0 is the kinetic coefficient and the $\eta_\alpha(\mathbf{r},t)$ are extraneous random forces normalized by the condition

$$\langle \eta_\alpha(\mathbf{r},t)\,\eta_\beta(\mathbf{r}',t')\rangle = 2\Gamma_0\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{2}$$

The Ginzburg-Landau Hamiltonian figuring in (1) has the well-known form

$$H = \int d^3r \left\{ \frac{\varkappa_0^2}{2}\varphi^2(\mathbf{r}) + \frac{1}{2}[\nabla\varphi(\mathbf{r})]^2 + \frac{u}{8}[\varphi^2(\mathbf{r})]^2 \right\},$$
$$\varphi^2(\mathbf{r}) = \sum_{\alpha=1}^{n}\varphi_\alpha^2(\mathbf{r}), \quad [\nabla\varphi(\mathbf{r})]^2 = \sum_{\alpha=1}^{n}[\nabla\varphi_\alpha(\mathbf{r})]^2, \tag{3}$$

where n is the number of order-parameter components.

It is not difficult to see that Eqs. (1)–(3) are not suitable for our purpose, since we want to compute the dynamical exponent on the basis of the renormalized Green functions and the Ward identity. Because of this, we shall use the results of Ref. 9, where it is shown that, in the classical limit, the description of the critical dynamics by Eqs. (1)–(3) is

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equivalent to the description with the aid of the following standard Lagrangian:

$$-L[\varphi]=\frac{1}{2}\sum_{n}\sum_{\mathbf{p}}\left(\frac{\omega_n \operatorname{sign}\omega_n}{\Gamma_0}+\varkappa_0^2+p^2\right)\varphi^2(\mathbf{p},\omega_n)$$
$$+\frac{u}{8}\sum_{n_1}\sum_{n_2}\sum_{n_3}\int d^3r(\varphi(\mathbf{r},\omega_{n_1})\varphi(\mathbf{r},\omega_{n_2}))$$
$$\times(\varphi(\mathbf{r},\omega_{n_3})\varphi(\mathbf{r},-\omega_{n_1}-\omega_{n_2}-\omega_{n_3})), \tag{4}$$

where ω_n takes on the discrete values

$$\omega_n=2\pi n,\quad n=0,\pm1,+2,\ldots, \tag{5}$$

and $\phi(\mathbf{p},\omega_n)$ and $\phi(\mathbf{r},\omega_n)$ are the order parameters in the momentum–frequency and frequency representations.

Let us define the Green functions:

$$\mathcal{G}_{\alpha\beta}(p,\omega_n)=\delta_{\alpha\beta}\mathcal{G}(p,\omega_n)=\langle\varphi_\alpha(\mathbf{p},\omega_n)\varphi_\beta(\mathbf{p},\omega_n)\rangle, \tag{6}$$

where the averaging is performed with the weight $\exp(L[\phi])$. The complete Green function is equal to

$$\mathcal{G}(p,\omega_n)=[\omega_n\operatorname{sign}\omega_n/\Gamma_0+\varkappa_0^2+p^2-\Sigma(p,\omega_n)]^{-1}, \tag{7}$$

where $\Sigma(p,\omega_n)$ is the self-energy part. Let us note that the diagrammatic procedure for computing the corrections to the Green function (7) is similar to the procedure expounded in Ref. 10 for the Matsubara function.

Let us now use the following idea, first stated by Polyakov: the dynamical properties of a system are determined by the retarded Green function $G^{(r)}(p,\omega)$, which is analytic in the upper complex half-plane, and possesses the following property[11]:

$$G^{(r)}(p,\omega)=\mathcal{G}(p,i\omega_n),\quad \omega_n>0. \tag{8}$$

To compute $G^{(r)}(p,\omega)$ or $\Sigma^{(r)}(p,\omega)$, which is given by the formula[3]

$$G^{(r)}(p,\omega)=[-i\omega/\Gamma_0+\varkappa_0^2+p^2-\Sigma^{(r)}(p,\omega)]^{-1},\quad \operatorname{Im}\omega>0, \tag{9}$$

we must continue the diagrams for $\Sigma(p,\omega_n)$ into the upper half-plane. In this case the separation of the most singular terms should be carried out only after the analytic continuation of the whole sum.

The dynamical exponent ρ is determined in our approach from the asymptotic form of the function $G^{(r)}(p,\omega)$ for $T=T_c$, $p=0$, and $\omega\to0$:

$$G^{(r)}(0,\omega)\,|_{T=T_c,\,\omega\to0}\sim1/\omega^{1-\rho}. \tag{10}$$

Thus, the analysis reduces to the study of the asymptotic behavior of the retarded Green function $G^{(r)}(p,\omega)$.

As is well known, the construction of the perturbation theory yields a series in the parameter $u/\varkappa_0$, and such an expansion is totally inapplicable when $u/\varkappa_0\gg1$. A way out of this situation is to renormalize u and $\varkappa_0$, a procedure which is similar to the renormalization of charge and mass in field theory. After such a procedure, the entire dependence on the large parameter $u/\varkappa_0$ becomes a dependence on the renormalized coupling constant u_R, which we shall call the invariant charge, and the renormalized chemical potential $\varkappa$.

As in the static theory, it is expedient to construct the theory, using the real, and not the bare, pole of the Green function as the zeroth approximation. But the situation is somewhat different in the dynamical case. It is well known that the most important contribution to the singularities of thermodynamic quantities is made by the small-momentum and low-frequency regions.[12,13] Therefore, in separating out the singularities of the static quantities, we set $\omega_n=0$, and use the pole of the Green function in the momentum (p) representation. Strictly speaking, such a procedure is equivalent to the neglect of the dynamical exponent. But in our case, being interested in the frequency dependence of the Green function, we shall consider the true pole of the function in the frequency representation at $p=0$.

Let at $p=0$ the retarded Green function $G^{(r)}(0,\omega)$ have a pole at $i\omega/\Gamma_0=\varkappa^2$ in the lower half-plane, $\varkappa^2$ being given by the root of the following equation:

$$-i\omega/\Gamma_0+\varkappa_0^2-\Sigma^{(r)}(0,\omega)=0. \tag{11}$$

Notice that $\varkappa^2$ is positive and must be real, since we are considering a continuous transition; otherwise we shall have an isolated point to deal with. With allowance for (11), the expression (9) can be written in the following form:

$$G^{(r)}(p,\omega)=[-i\omega/\Gamma_0+\varkappa^2+p^2-\Sigma^{(r)}(p,\omega)+\Sigma^{(r)}(0,-i\Gamma_0\varkappa^2)]^{-1}. \tag{12}$$

After the introduction of the notation

$$\Sigma_1^{(r)}(p,\omega)=\Sigma^{(r)}(p,\omega)-\Sigma^{(r)}(0,-i\Gamma_0\varkappa^2) \tag{13}$$

the function $G^{(r)}(p,\omega)$ assumes the form

$$G^{(r)}(p,\omega)=ZG_R^{(r)}(p,\omega), \tag{14}$$

where the renormalized retarded Green function is equal to

$$G_R^{(r)}(p,\omega)=[-i\omega/\Gamma_0+\varkappa^2+p^2-\Sigma_R^{(r)}(p,\omega)]^{-1}, \tag{15}$$

$$\Sigma_R^{(r)}(p,\omega)=Z\left[\Sigma_1^{(r)}(p,\omega)-\left(-\frac{i\omega}{\Gamma_0}+\varkappa^2+p^2\right)\frac{\partial\Sigma_1^{(r)}}{\partial\beta}\Big|_{\beta=-\varkappa^2-p^2}\right], \tag{16}$$

$$Z^{-1}=1-\frac{\partial\Sigma_1^{(r)}}{\partial\beta}\Big|_{\beta=-\varkappa^2-p^2}, \tag{17}$$

$$\beta=-i\omega/\Gamma_0. \tag{18}$$

We see that Z depends not only on $\varkappa^2$, but also on p. We can, neglecting the Fisher parameter η, set $p=0$ in the formula (17). Below we shall assume

$$Z^{-1}=1-\frac{\partial\Sigma_1^{(r)}(0,\omega)}{\partial\beta}\Big|_{\beta=-\varkappa^2}. \tag{19}$$

Let us emphasize once again that the approximation adopted here is, in fact, equivalent to the above-mentioned approximation in the static theory, where the frequency dependence is neglected (i.e., in which $\omega_n=0$). Since $G^{(r)}$ is a real function of the argument β (Ref. 3), Z will be a real function of the variable $\varkappa^2$. The renormalized Green function $\mathcal{G}_R(p,\omega_n)$ can easily be obtained from the renormalized retarded Green function $G_R^{(r)}(p,\omega)$ with the aid of the Lehmann representation[9,10]:

$$\mathcal{G}_R(p,\omega_n)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}G_R^{(r)}(p,x)\,dx}{x-i\omega_n}. \tag{20}$$

In the subsequent application of perturbation theory, we shall be in need of not the complete renormalized Green function, but its unperturbed value

$$\mathcal{G}_R^{(0)}(p, \omega_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} G_R^{(0,r)}(p, x)\, dx}{x - i\omega_n}, \tag{21}$$

which compares with the internal lines of the Matrsubara diagrams. It can be seen from (15) that

$$G_R^{(0,r)}(p, \omega) = [-i\omega/\Gamma_0 + \varkappa^2 + p^2]^{-1}, \tag{22}$$

and, consequently,

$$\operatorname{Im} G_R^{(0,r)}(p, \omega) = \frac{\omega/\Gamma_0}{(\omega/\Gamma_0)^2 + (\varkappa^2 + p^2)^2}. \tag{23}$$

The substitution of (23) into (21) yields

$$\mathcal{G}_R^{(0)}(p, \omega_n) = [\omega_n \operatorname{sign} \omega_n/\Gamma_0 + \varkappa^2 + p^2]^{-1}. \tag{24}$$

Notice that the function $\mathcal{G}_R^{(0)}(p, \omega_n)$ coincides with the temperature Green function chosen in Ref. 2. The formulas (21) and (23) will be used in the summation over the frequencies (see the Appendix).

In renormalizing the fourth-order vertex, we shall set

$$\Gamma_{\alpha\beta\mu\nu}(\omega_i = 0, \mathbf{p}_i = 0) = \gamma_R I_{\alpha\beta\mu\nu},$$
$$I_{\alpha\beta\mu\nu} = \delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}. \tag{25}$$

We can introduce the invariant charge

$$u_R = Z^2\gamma_R, \tag{26}$$

and expand all the quantities of interest to us in powers of it. For example, in second-order perturbation theory, the renormalization of the diagram in Fig. 1 yields the following correction to the renormalized mass operator:

$$\delta\Sigma_R^{(r)} = \frac{n+2}{2} u_R^2 \left\{ \Psi(p, \omega) - \Psi(0, -i\Gamma_0\varkappa^2) - \left(-\frac{i\omega}{\Gamma_0} + \varkappa^2 + p^2\right) \frac{\partial \Psi}{\partial \beta}\Big|_{\beta = -\varkappa^2 - p^2} \right\} \tag{27}$$

$$\Psi(p, \omega) = \mathscr{C}\left\{ \sum_l \sum_m \int\int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \mathcal{G}_R^{(0)}(p_1, \omega_l)\mathcal{G}_R^{(0)}(p_2, \omega_m)\mathcal{G}_R^{(0)} \times (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}, \omega_l + \omega_m + \omega_n) \right\}, \tag{28}$$

where $\mathscr{C}$ denotes the operation of analytic continuation into the upper half-plane. An expression of the type (27) corresponds to the subtraction procedure in field theory.[14]

3. THE WARD IDENTITY AND THE COMPUTATIONAL SCHEME FOR THE EXPONENT ρ

Let $\varkappa_{0c}$ be the value of $\varkappa_0$, i.e., the value at which the phase transition occurs. Let us set

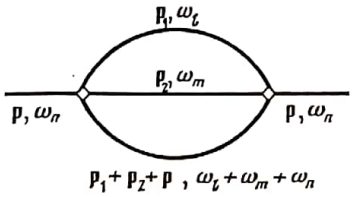

FIG. 1.

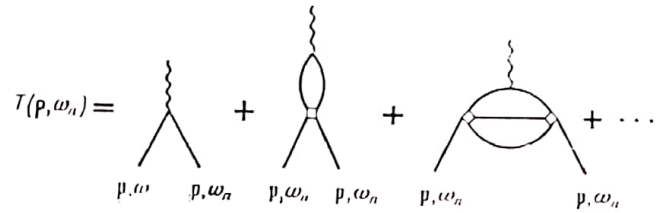

FIG. 2.

$$\xi = \varkappa_0^2 - \varkappa_{0c}^2. \tag{29}$$

Then the Ward identity for the Green function has the following form[12]:

$$\partial \mathcal{G}^{-1}(p, \omega_n)/\partial\xi = T(p, \omega_n), \tag{30}$$

where $T(p, \omega_n)$ is the sum of diagrams, some which are shown in Fig. 2 (a wavy line denotes differentiation).

Setting

$$T(0, 0) = t_R, \tag{31}$$

we can renormalize the function $T(p, \omega_n)$, i.e., expand it in a power series in u_R and t_R. Below we shall work not with $T(p, \omega_n)$, but with its analytic continuation into the upper half-plane, which we denote by $T^{(r)}(p, \omega)$. Then the formula (30) goes over into

$$\partial [G^{(r)}(p, \omega)]^{-1}/\partial\xi = T^{(r)}(p, \omega). \tag{32}$$

Let us renormalize for $T^{(r)}$ the diagrams in Fig. 2 for example. This yields

$$T^{(r)}(p, \omega) = t_R + \frac{3(n+2)}{2} t_R u_R^2 \mathscr{C}\left\{ \sum_l \sum_m \int\int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \times [\mathcal{G}_R^{(0)}(p_1, \omega_l)]^2 \mathcal{G}_R^{(0)}(p_2, \omega_m)[\mathcal{G}_R^{(0)}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}, \omega_l + \omega_m + \omega_n) - \mathcal{G}_R^{(0)}(\mathbf{p}_1 + \mathbf{p}_2, \omega_l + \omega_m)] \right\}. \tag{33}$$

The second diagram in Fig. 2 gives zero when the subtraction procedure is carried out. It should be emphasized that the subtraction must be carried out here in order to ensure the correctness of (31) in any order of perturbation theory.

From (33) and the analysis of more complicated diagrams we find that

$$T^{(r)}(p, \omega) = t_R \alpha(p^2/\varkappa^2, -i\omega/\Gamma_0\varkappa^2). \tag{34}$$

The dependence of α on the ratios $p^2/\varkappa^2$ and $-i\omega/\Gamma_0\varkappa^2$ is a consequance of the fact that, as will be shown below, $u_R \sim \varkappa$. The function α will play a special role below.

Solving (32) near $i\omega/\Gamma_0 = \varkappa^2$ and $p = 0$, we have

$$\partial\varkappa^2/\partial\xi = Z t_R(\varkappa)\alpha(0, -1). \tag{35}$$

It immediately follows from (32) and (35) that

$$Z\frac{\partial [G^{(r)}(0, \omega)]^{-1}}{\partial \varkappa^2} = \frac{\alpha(0, -i\omega/\Gamma_0\varkappa^2)}{\alpha(0, -1)}. \tag{36}$$

Let us introduce the function

$$g = \varkappa^2 G_R^{(r)}(0, \omega) = (\varkappa^2/Z) G^{(r)}(0, \omega), \tag{37}$$

which depends only on $-i\omega/\Gamma_0\varkappa^2$. Let us differentiate the last relation with respect to $\ln \varkappa^2$:

$$\frac{\partial \ln Z}{\partial \ln \varkappa^2}=\rho=1-\frac{\partial \ln g}{\partial \ln \varkappa^2}-\frac{\alpha(0,-i\omega/\Gamma_0\varkappa^2)}{\alpha(0,-1)}g, \tag{38}$$

where ρ is simply a separation parameter for the variables, since the left-hand side depends only on $\varkappa$, while the right-hand side depends only on $-i\omega/\Gamma_0\varkappa^2$. From the left-hand side of (38) we have

$$Z\sim\varkappa^{2\rho}; \tag{39}$$

from the right-hand side,

$$1+\frac{\partial \ln g(x)}{\partial \ln x}-\frac{\alpha(0,x)}{\alpha(0,-1)}\sigma(x)=\rho, \tag{40}$$

$$x=-i\omega/\Gamma_0\varkappa^2. \tag{41}$$

Let us first consider the expression (40) for $|x|\gg 1$. It is easy to show that the product $g(x)\alpha(x)\to 0$ as $|x|\to\infty$. Then

$$g(x)=x^{-1+\rho},\quad G_R^{(r)}(0,\omega)=\varkappa^{-2\rho}(-i\omega/\Gamma_0)^{-1+\rho},$$
$$G^{(r)}(0,\omega)=ZG_R^{(r)}(0,\omega)\sim 1/\omega^{1-\rho}. \tag{42}$$

This is the only possible solution, since the function $G^{(r)}(0,\omega)$ cannot depend on $\varkappa$ when $\omega/\Gamma_0\gg\varkappa^2$. Comparing (10) and (12), we see the ρ is the exponent of interest to us.

In order to find the equation with the aid of which we can determine ρ, we must consider (40) for small x. Since for $\omega/\Gamma_0\sim\varkappa^2$ we find that up to terms of the order of ρ

$$T^{(r)}(0,\omega)=t_R,\quad G_R^{(r)}(0,\omega)=(-i\omega/\Gamma_0+\varkappa^2)^{-1},$$

it is evident that the following relations are valid for $|x|\sim 1$:

$$|\alpha(0,x)-1|\infty\rho,\quad |g(x)-1/(x+1)|\sim\rho. \tag{43}$$

The last expressions make it possible for us to linearize (40). Using the condition that $g(x)$ is finite at $|x|\to 0$, we find from (40) that

$$g(0)=(1-\rho)\alpha(0,-1). \tag{44}$$

Further, let us set

$$g(x)=\frac{g(0)}{x+1}\frac{1}{1-\zeta(x)},\quad \zeta(0)=0. \tag{45}$$

Substituting (45) into (40), and linearizing the equation, we obtain

$$x(x+1)\zeta'(x)-\zeta(x)=\alpha(0,x)+\rho x-1. \tag{46}$$

This equation has two singular points: $x=0$ and $x=1$. Its solution is regular at these points only when

$$\rho=-\alpha'(0,0). \tag{47}$$

The last equation allows us to compute ρ directly in three-dimensional space in terms of the vertex function.

4. THE INVARIANT CHARGE u_R

Let us now proceed to determine u_R. For this purpose, let us again use the Ward identity. Let us compute the quantity $\partial\gamma_R/\partial\xi$. In lowest-order perturbation theory we have

$$\frac{\partial\gamma_R}{\partial\xi}=t_R\gamma_R^2(n+8)\sum_m\int\frac{d^3p}{(2\pi)^3}\mathcal{G}^3(p,\omega_m) \tag{48}$$

(see Fig. 3). From (26) and (35) we obtain

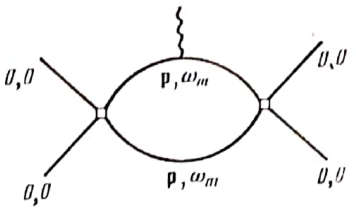

FIG. 3.

$$\alpha(0,-1)\frac{\partial u_R}{\partial\varkappa^2}-\frac{2}{Z}u_R\alpha(0,-1)\frac{\partial Z}{\partial\varkappa^2}$$
$$=(n+8)u_R^2\sum_m\int\frac{d^3p}{(2\pi)^3}\mathcal{G}_R^3(p,\omega_m). \tag{49}$$

Since

$$\partial Z/\partial\varkappa^2\sim\rho,\quad |\alpha(0,-1)-1|\sim\rho,$$

we find in second-order perturbation theory that

$$\frac{\partial u_R}{\partial\varkappa^2}=(n+8)u_R^2\sum_m\int\frac{d^3p}{(2\pi)^3}[\mathcal{G}_R^{(0)}(p,\omega_m)]^3. \tag{50}$$

We must, in summing over the frequencies in the last formula, make the classical approximation (see Sec. A of the Appendix). Finally we have

$$\frac{\partial u_R}{\partial\varkappa^2}=(n+8)u_R^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{(p^2+\varkappa^2)^3}. \tag{51}$$

This equation coincides with the well-known equation for the invariant charge in the static case.[8] Therefore, the dynamical value coincides with the static value

$$u_R=16\pi\varkappa/(n+8). \tag{52}$$

The fact that the invariant charge is the same in both cases, once more confirms the correctness of the dynamic scaling hypothesis.[15]

5. COMPUTATION OF THE EXPONENT ρ

Let us find the exponent ρ, using the formula (47). We shall, in computing α, limit outselves to the lowest order in u_R. From (33) and (34) we have

$$\alpha\left(\frac{p^2}{\varkappa^2},-\frac{i\omega}{\Gamma_0\varkappa^2}\right)$$
$$=1+\frac{3(n+2)}{2}u_R^2\mathcal{C}\left\{\sum_l\sum_m\int\int\frac{d^3p_1d^3p_2}{(2\pi)^6}[\mathcal{G}_R^{(0)}(p_1,\omega_l)]^2\right.$$
$$\times\mathcal{G}_R^{(0)}(p_2,\omega_m)[\mathcal{G}_R^{(0)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p},\omega_l+\omega_m+\omega_n)$$
$$\left.-\mathcal{G}_R^{(0)}(\mathbf{p}_1+\mathbf{p}_2,\omega_l+\omega_m)]\right\}. \tag{53}$$

The details of the summation over the ω_l and ω_m are contained in Sec. B of the Appendix. In the classical approximation the function α is split into a static and a dynamic part:

$$\alpha=\alpha_s+\alpha_d, \tag{54}$$

where

$$\alpha_s=1+\frac{3(n+2)}{2}u_R^2\int\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\frac{1}{(p_1^2+\varkappa^2)^2(p_2^2+\varkappa^2)}$$

$$\times\left[\frac{1}{(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p})^2+\varkappa^2}-\frac{1}{(\mathbf{p}_1+\mathbf{p}_2)^2+\varkappa^2}\right], \quad (55)$$

$$\alpha_d=-\frac{3(n+2)}{2}u_R^2\mathscr{C}\left\{\int\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\frac{-\omega_n\,\mathrm{sign}\,\omega_n/\Gamma_0}{(p_1^2+\varkappa^2)(p_2^2+\varkappa^2)}\right.$$

$$\times[(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p})^2+\varkappa^2]^{-1}$$

$$\times[p_1^2+p_2^2+(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p})^2+3\varkappa^2+\omega_n\,\mathrm{sign}\,\omega_n/\Gamma_0]^{-1}[(p_1^2+\varkappa^2)^{-1}$$

$$\left.+(p_1^2+p_2^2+(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p})^2+3\varkappa^2+\omega_n\,\mathrm{sign}\,\omega_n/\Gamma_0)^{-1}]\right\}. \quad (56)$$

With allowance for the expression (8), the analytic continuation in the last formula can be carried out without difficulty, i.e., we need to make the substitution $i\omega_n\to\omega$, where $\omega_n>0$, $\mathrm{Im}\,\omega>0$. Going over to new dimensionless integration variables, $p_1\to p_1/\varkappa$ and $p_2\to p_2/\varkappa$, and taking (52) into account, we have

$$\alpha_d\left(\frac{p^2}{\varkappa^2},-\frac{i\omega}{\Gamma_0\varkappa^2}\right)=\frac{3(n+2)(16\pi)^2}{2(n+8)^2}\int\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\frac{i\omega}{\Gamma_0\varkappa^2}$$

$$\times[(p_1^2+1)(p_2^2+1)]^{-1}[(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}/\varkappa)^2+1]^{-1}$$

$$\times[p_1^2+p_2^2+(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}/\varkappa)^2+3-i\omega/\Gamma_0\varkappa^2]^{-1}$$

$$\times[(p_1^2+1)^{-1}+(p_1^2+p_2^2+(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}/\varkappa)^2+3-i\omega/\Gamma_0\varkappa^2)^{-1}]. \quad (57)$$

We see that the function α_d vanishes at $\omega=0$, leaving only the static part (55). This verifies the correctness of our calculations, since the dynamical theory should yield the results of the static theory on passing to the limit $\omega\to 0$. It can be seen from (57) that the function α, like the self-energy part $\Sigma^{(r)}$, can have singularities only in the lower half-plane.

Naturally, we can obtain the explicit $-i\omega/\Gamma_0\varkappa^2$ dependence for $\alpha(0,-i\omega/\Gamma_0\varkappa^2)$, and then, using the formula (47), compute ρ. But let us, for the purpose of simplifying the calculations, differentiate right away with respect to the variable $-i\omega/\Gamma_0\varkappa^2$ the expression under the integral sign in (57) and then set $-i\omega/\Gamma_0\varkappa^2=0$. As a result, we have

$$\rho=\frac{3(n+2)}{2(n+8)^2}(16\pi)^2\int\int\frac{d^3p_1d^3p_2}{(2\pi)^6}[(p_1^2+1)(p_2^2+1)]^{-1}$$

$$\times[(\mathbf{p}_1+\mathbf{p}_2)^2+1]^{-1}\times[p_1^2+p_2^2+(\mathbf{p}_1+\mathbf{p}_2)^2+3]^{-1}$$

$$\times[(p_1^2+1)^{-1}+(p_1^2+p_2^2+(\mathbf{p}_1+\mathbf{p}_2)^2+3)^{-1}]. \quad (58)$$

After a tedious computation of the integrals, we obtain

$$\rho=\frac{4}{3\pi}\left\{\sqrt{3}\pi\left[1+4\left(\left(1+\frac{\sqrt{3}}{2}\right)^{1/2}-\left(1-\frac{\sqrt{3}}{2}\right)^{1/2}\right)\right]\right.$$

$$-7\pi-\frac{2(12+7\sqrt{3})}{(7+4\sqrt{3})^{1/2}}$$

$$\left.\times\mathrm{arctg}\,(7+4\sqrt{3})^{-1/2}+\frac{2(12-7\sqrt{3})}{(7-4\sqrt{3})^{1/2}}\mathrm{arctg}\,(7-4\sqrt{3})^{-1/2}\right\}\frac{n+2}{(n+8)^2}. \quad (59)$$

If we express ρ in terms of the Fisher parameter

$$\eta=8(n+2)/27(n+8)^2$$

found in Ref. 8, and roughly estimate all the terms in (59), then we find

$$\rho\approx 0.82\eta. \quad (60)$$

Knowing ρ, we can find the often used exponent z,[1] which is given by the following formula:

$$z=(2-\eta)/(1-\rho)=2+c\eta.$$

From (60) we easily find that $c\approx 0.64$. Let us recall that the ε-expansion method in second-order perturbation theory yields[1] $c\approx 0.73$, i.e., a value not very different from our value.

6. COMPARISON WITH EXPERIMENT

Suzuki[16] has measured the coefficient of ultrasound absorption near the structural phase transition point for the $KMnF_3$ crystal. In the temperature region $T_c<T<T_\Delta$, where $T_c=186.2$ K and $T_\Delta=187.9$ K, the absorption coefficient $\alpha_s\propto t^{-1.13}$, with $t=(T-T_c)/T_c$. On the other hand, it follows from theory that $\alpha_s\propto t^{-x}$, where the exponent $x=\nu z$, ν being the correlation length exponent. Further, according to Sokolov,[17] the $KMnF_3$ crystal can be considered to be an Ising system ($n=1$) in the above-indicated temperature range. Taking the values $\alpha={}^8/_{729}$ and $\nu=0.6$ from Ref. 8, and setting $c\approx 0.64$, we obtain $x\approx 1.20$. This value of the exponent x is quite close to the experimental value. Let us note that the ε-expansion method yields $x\approx 1.17$ and $x\approx 1.25$ respectively in second- and third-order perturbation theories,[17] i.e., values that are also close to 1.13. Of course, it is difficult to draw any definite conclusion on the basis of these numerical x values only, since there are quite large uncertainties in both the experimental and the theoretical values. Nevertheless, it may be hoped that the method proposed in the present paper gives a reasonable result.

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APPENDIX

A. Let us, in summing over the frequencies in the formula (50) of the main text, use the Lehmann representation (21):

$$\sum_{m=-\infty}^{\infty}[\mathscr{G}_R^{(0)}(p,\omega_m)]^3=\frac{1}{\pi^3}\iiint dx_1\,dx_2\,dx_3\,\mathrm{Im}\,G_R^{(0,r)}(p,x_1)$$

$$\times\mathrm{Im}\,G_R^{(0,r)}(p,x_2)\,\mathrm{Im}\,G_R^{(0,r)}(p,x_3)S, \quad (A.1)$$

$$S\equiv\sum_{m=-\infty}^{\infty}[(x_1-i\omega_m)(x_2-i\omega_m)(x_3-i\omega_m)]^{-1}.$$

The summation over m in (A.1) can be carried out with the aid of the following formula[18]:

$$\sum_{n=-\infty}^{\infty}f(n)=-\pi\sum_{k=1}^{m}\mathrm{Res}[f(z)\,\mathrm{ctg}\,\pi z]_{z=a_k}, \quad (A.2)$$

where $a_1, \ldots, a_m$ are the poles of the function $f(z)$, which do not coincide with any of the points $z = 0, 1, 2, \ldots$, and $f(z)$ itself satisfies all the necessary conditions (see Ref. 18). A simple calculation of the residues yields

$$S = \frac{1}{2}\left[\frac{1}{(x_2-x_1)(x_3-x_1)}\operatorname{cth}\frac{x_1}{2} + \frac{1}{(x_1-x_2)(x_3-x_2)}\operatorname{cth}\frac{x_2}{2} + \frac{1}{(x_1-x_3)(x_2-x_3)}\operatorname{cth}\frac{x_3}{2}\right]. \quad (A.3)$$

The passage to the classical limit in (A.3) is accomplished by means of the substitution

$$\operatorname{cth}(x_i/2) \to 2/x_i, \quad i=1, 2, 3. \quad (A.4)$$

Evidently, this substitution is equivalent to the replacement of the Planck function by the Rayleigh–Jeans distribution.[3,11] Taking (A.3) and (A.4) into account, we obtain

$$\sum_{m=-\infty}^{\infty} [\mathcal{G}_R^{(0)}(p, \omega_m)]^3 = \frac{1}{\pi^3}\left[\int_{-\infty}^{\infty}\frac{dx}{x}\operatorname{Im} G_R^{(0,r)}(p, x)\right]^3. \quad (A.5)$$

After substituting the formula (23) into this expression, and performing the trivial integration over x, we arrive at the required result (51).

B. In summing over the ω_m and ω_l in the formula (53) we again use the Lehmann representation (21):

$$\sum_l\sum_m [\mathcal{G}_R^{(0)}(p_1, \omega_l)]^2 \mathcal{G}_R^{(0)}(p_2, \omega_m) \times \mathcal{G}_R^{(0)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}, \omega_l+\omega_m+\omega_n)$$

$$= \frac{1}{\pi^4}\int\sum_{i=1}^{4} dx_i \operatorname{Im} G_R^{(0,r)}(p_1, x_1)\operatorname{Im} G_R^{(0,r)}(p_1, x_2) \times \operatorname{Im} G_R^{(0,r)}(p_2, x_3)\operatorname{Im} G_R^{(0,r)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}, x_4) \times \sum_l\sum_m \{(x_1-i\omega_l)(x_2-i\omega_l)(x_3-i\omega_m) \times [x_4-i(\omega_l+\omega_m+\omega_n)]\}^{-1}. \quad (A.6)$$

The summation over m with allowance for (A.2) yields

$$\sum_{m=-\infty}^{\infty}\{(x_3-i\omega_m)[x_4-i(\omega_l+\omega_m+\omega_n)]\}^{-1} = [2(x_4-x_3-i\omega_l-i\omega_n)]^{-1} \times\left(\operatorname{cth}\frac{x_3}{2} - \operatorname{cth}\frac{x_4-i\omega_l-i\omega_n}{2}\right). \quad (A.7)$$

We can make here right away a classical approximation of the type (A.4); all the frequencies in the argument of the hyperbolic tangent function are then set equal to zero.[2] Thus, in the approximation in question

$$\sum_{m=-\infty}^{\infty}\{(x_3-i\omega_m)[x_4-i(\omega_l+\omega_m+\omega_n)]\}^{-1} = \frac{x_4-x_3}{x_3x_4(x_4-x_3-i\omega_l-i\omega_n)}. \quad (A.8)$$

Substituting (A.8) into (A.6), and carrying out a similar summation over l, we have

$$\sum_l{}'\sum_m [\mathcal{G}_R^{(0)}(p_1, \omega_l)]^2 \mathcal{G}_R^{(0)}(p_2, \omega_m) \times \mathcal{G}_R^{(0)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}, \omega_l+\omega_m+\omega_m) = \frac{1}{\pi^4}\int\prod_{i=1}^{4}\frac{dx_i}{x_i}\left\{1+\frac{i\omega_n}{x_4-x_3-x_1-i\omega_n}\left[1-\frac{x_1}{x_4-x_3-x_2-i\omega_n}\right]\right\} \times \operatorname{Im} G_R^{(0,r)}(p_1, x_1)\operatorname{Im} G_R^{(0,r)}(p_1, x_2) \times \operatorname{Im} G_R^{(0,r)}(p_2, x_3)\operatorname{Im} G_R^{(0,r)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}, x_4). \quad (A.9)$$

The remaining integration over x_i is easily performed, and we obtain as a result the formulas (54)–(56) of the main text.

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