Deep Learning for Computer Vision

Lecture 6: The Perceptron, the XOR Challenge, Going Deep, Love for Feed Forward Networks, Jacobians, and Tensors

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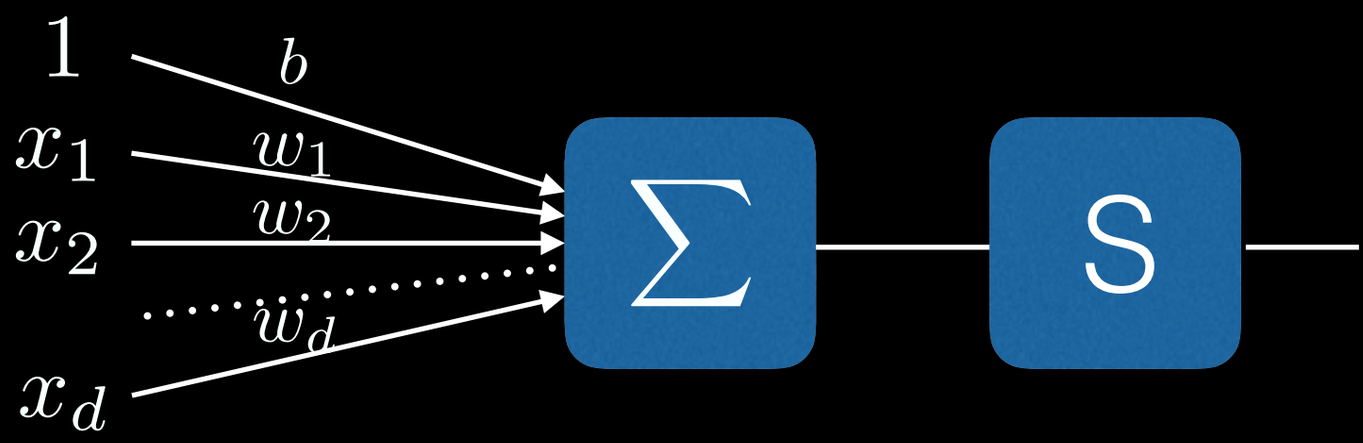
The Perceptron

[Rosenblatt 57]



$$P(y = 1|\mathbf{x}, \mathbf{w}, b)$$

$$f(\mathbf{x}; \mathbf{w}) = S(\mathbf{w}^T\mathbf{x} + b) = \frac{1}{1 + e^{-w_1x_1 - \ldots - w_dx_d + b}}$$

$$S(z) = \frac{1}{1 + e^{-z}}$$

Feedforward Networks

- Let $y = f^*(\mathbf{x})$ be some function we are trying to approximate
- This function could be assignment of an input to a category as in a classifier
- Let a feedforward network approximate this mapping $y = f(\mathbf{x}; \theta)$ by learning parameters θ

Feedforward Networks

- Feedforward networks have **NO** feedback
- These networks can be represented as directed acyclic graphs describing the composition of functions
- These networks are composed of functions represented as **“layers”** $f(\mathbf{x}) = l^3(l^2(l^1(\mathbf{x})))$
- The length of the chain of compositions gives the **“depth”** of the network

Feedforward Networks

- The functions defining the layers have been influenced by neuroscience
- Our training dictates the values to be produced output layer and the weights are chosen accordingly
- The weights for intermediate or **“hidden”** layers are learned and not specified directly
- You can think of the network as mapping the raw input space $\mathbf{x}$ to some transformed feature space $\phi(\mathbf{x})$ where the samples are ideally linearly separable

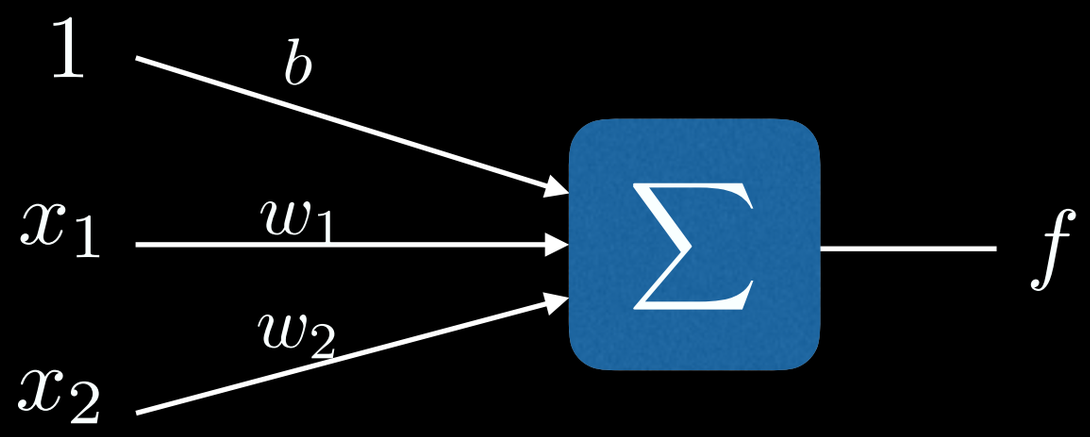
Feedforward Networks

- The goal of the feedforward network is then to learn this mapping $y = f(\mathbf{x}; \theta, \mathbf{w}) = \phi(\mathbf{x}; \theta)^T \mathbf{w}$

- If this mapping is simply the identity then our network is linear and this certainly not going to work for most computer vision problems.

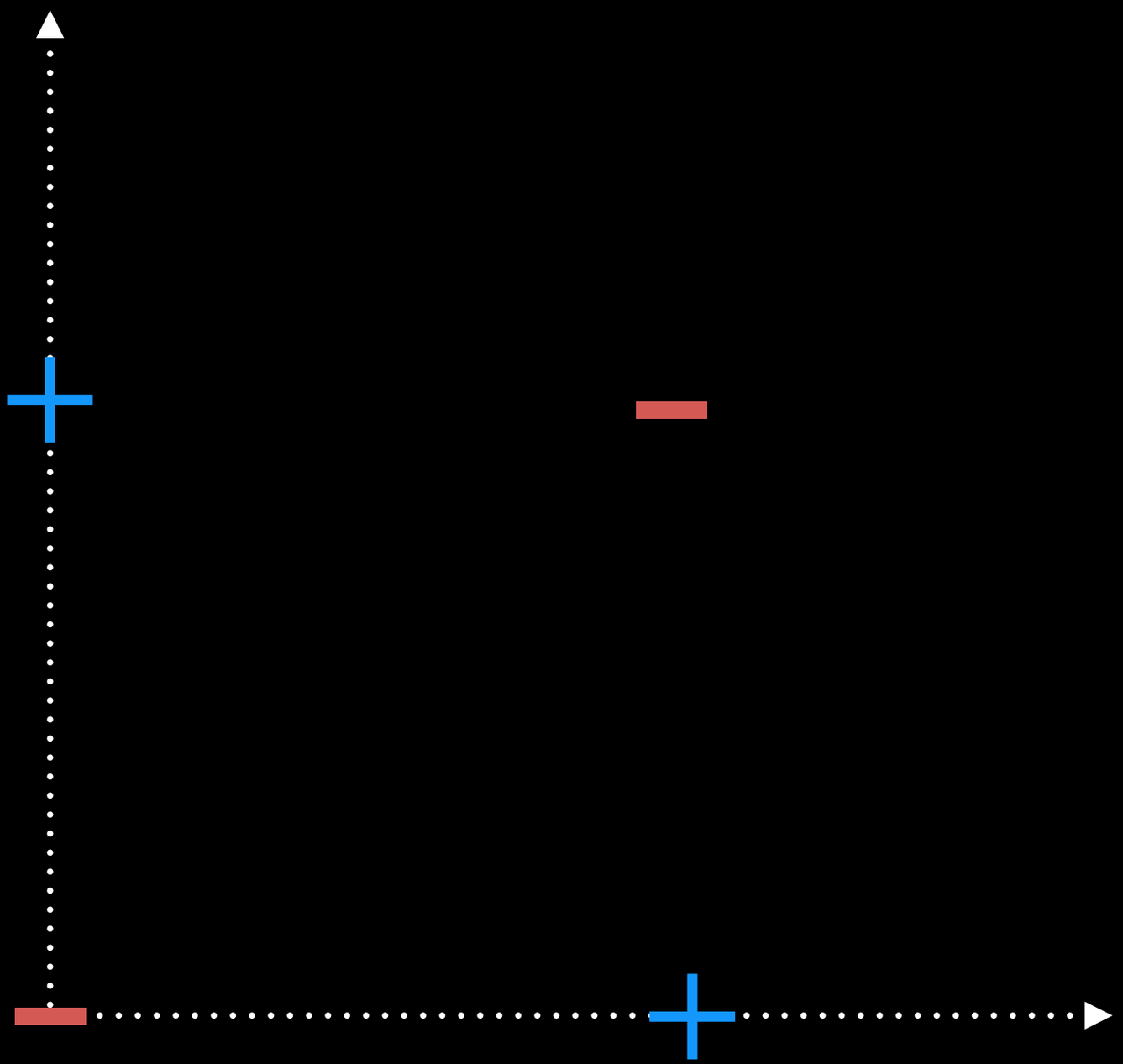
- Consider approximating something as simple as the XOR function with a linear network…

2D Linear Network

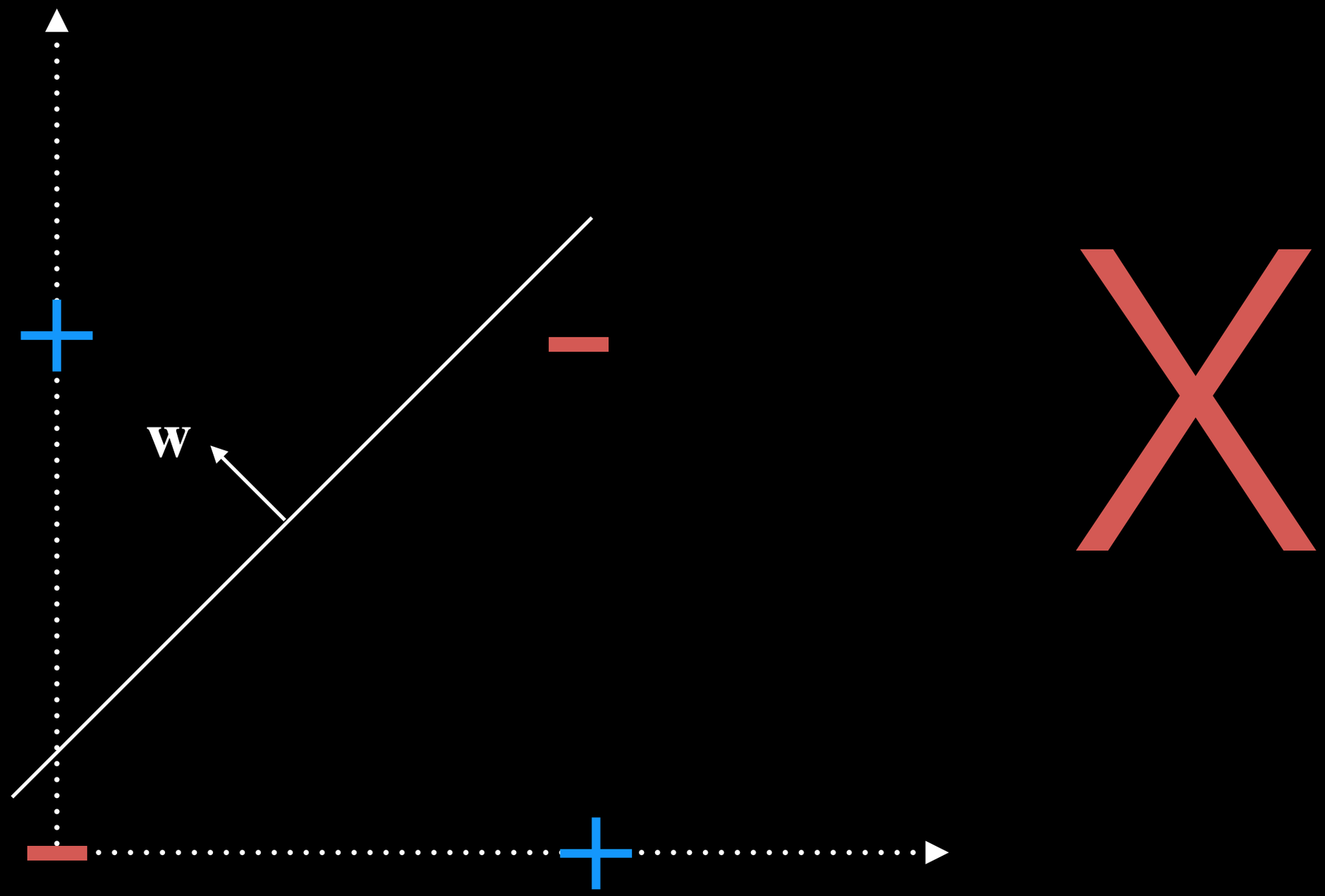


$$f(\mathbf{x}; \mathbf{w}, b) = \mathbf{w}^T \mathbf{x} + b$$

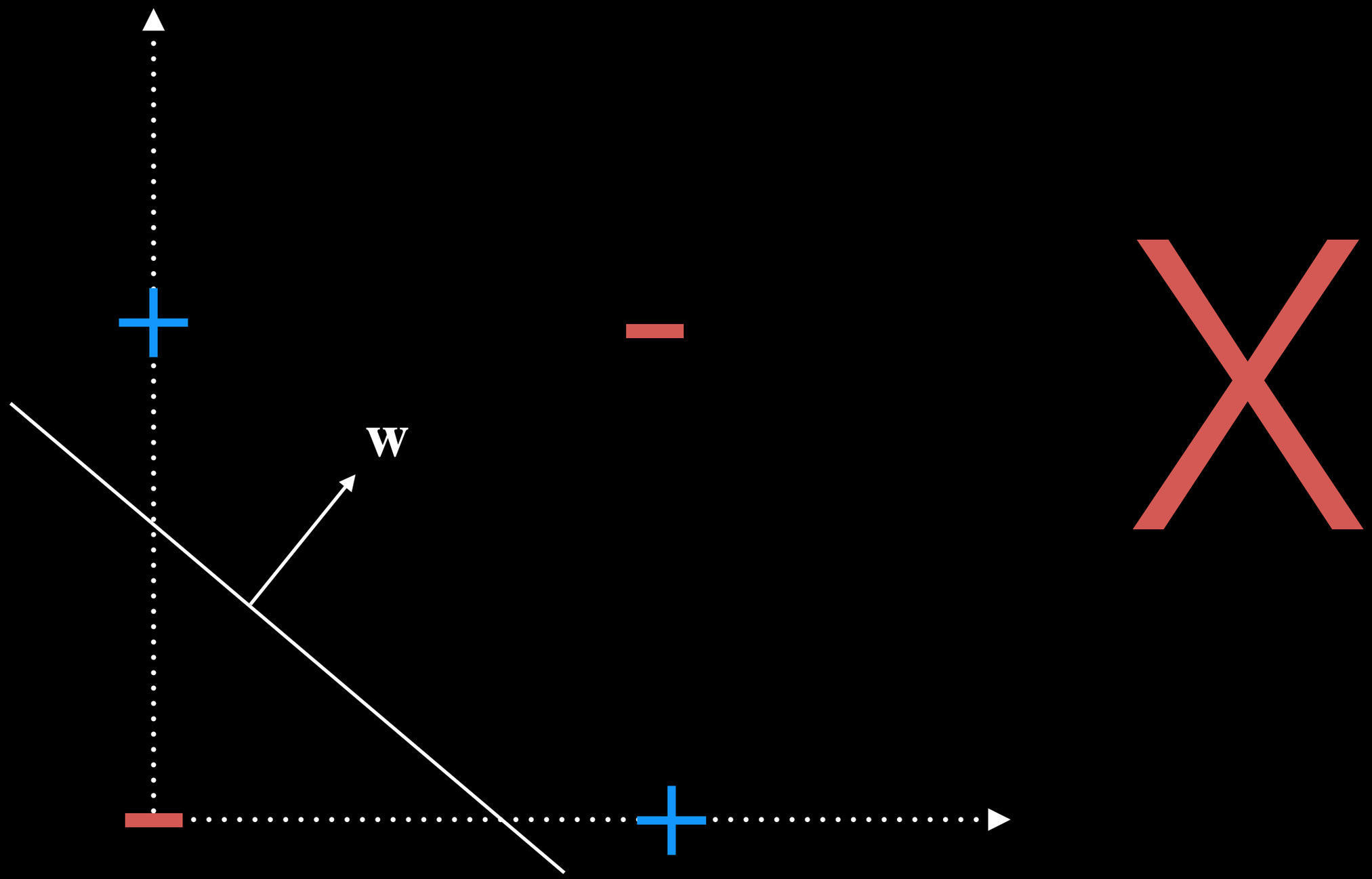
XOR



XOR



XOR



Let’s show this by trying fit a linear network to this data and choose the loss function to be mean squared error (MSE).

XOR with Linear Network

$$L(\mathbf{w}, b) = \frac{1}{4} \sum_{\mathbf{x} \in \mathbb{X}} (f^*(\mathbf{x}) - f(\mathbf{x}; \mathbf{w}, b))^2$$

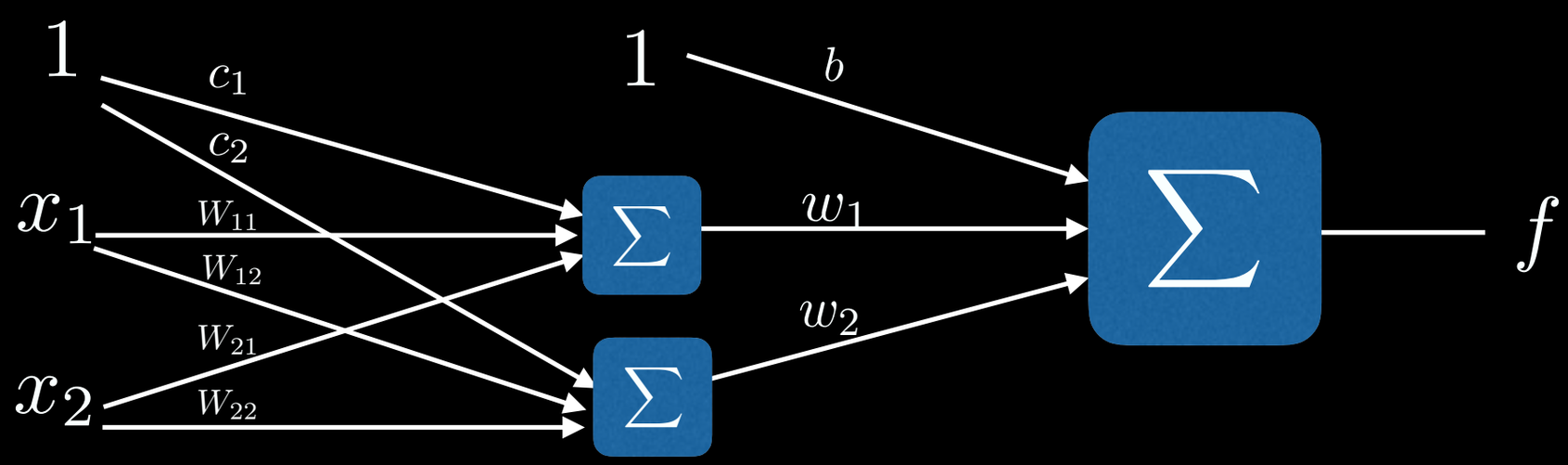
$$f(\mathbf{x}; \mathbf{w}, b) = \mathbf{w}^T \mathbf{x} + b$$

$$\hat{\mathbf{w}}, \hat{b} = \underset{\mathbf{w}, b}{\operatorname{argmax}}\ L(\mathbf{w}, b) = \mathbf{0}, \frac{1}{2}$$

There is no way to approximate the XOR function with this single layer linear network!

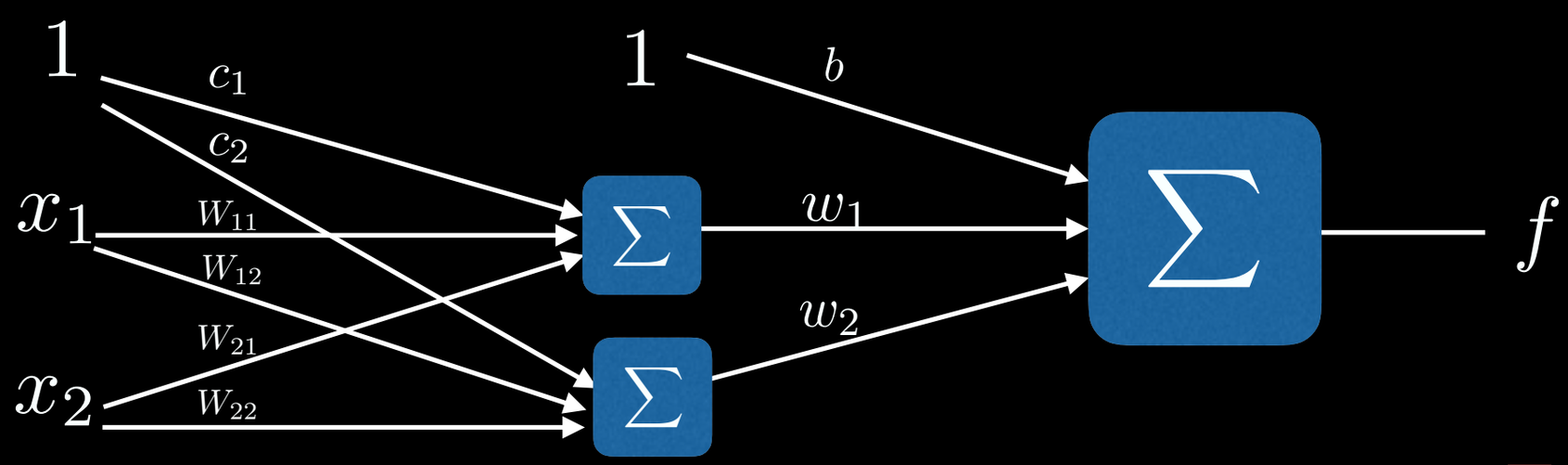
What if we introduced a hidden layer and chose this layer to be linear?

Linear Network with Hidden Layer



$$f(\mathbf{x}; W, \mathbf{c}, \mathbf{w}, b) = \mathbf{w}^T(W\mathbf{x} + \mathbf{c}) + b$$
$$= \tilde{\mathbf{w}}^T\mathbf{x} + \tilde{b}$$

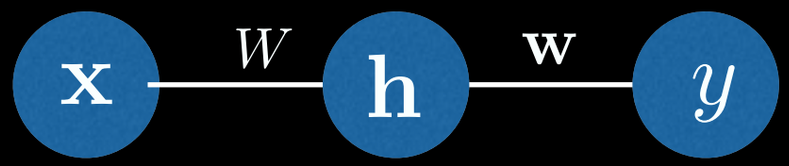
Linear Network with Hidden Layer



$$f(\mathbf{x}; W, \mathbf{c}, \mathbf{w}, b) = \mathbf{w}^T(W^T\mathbf{x} + \mathbf{c}) + b$$
$$= \tilde{\mathbf{w}}^T\mathbf{x} + \tilde{b}$$

Network notation is cumbersome. Let’s simplify.

“Deep” Network with Hidden Layer

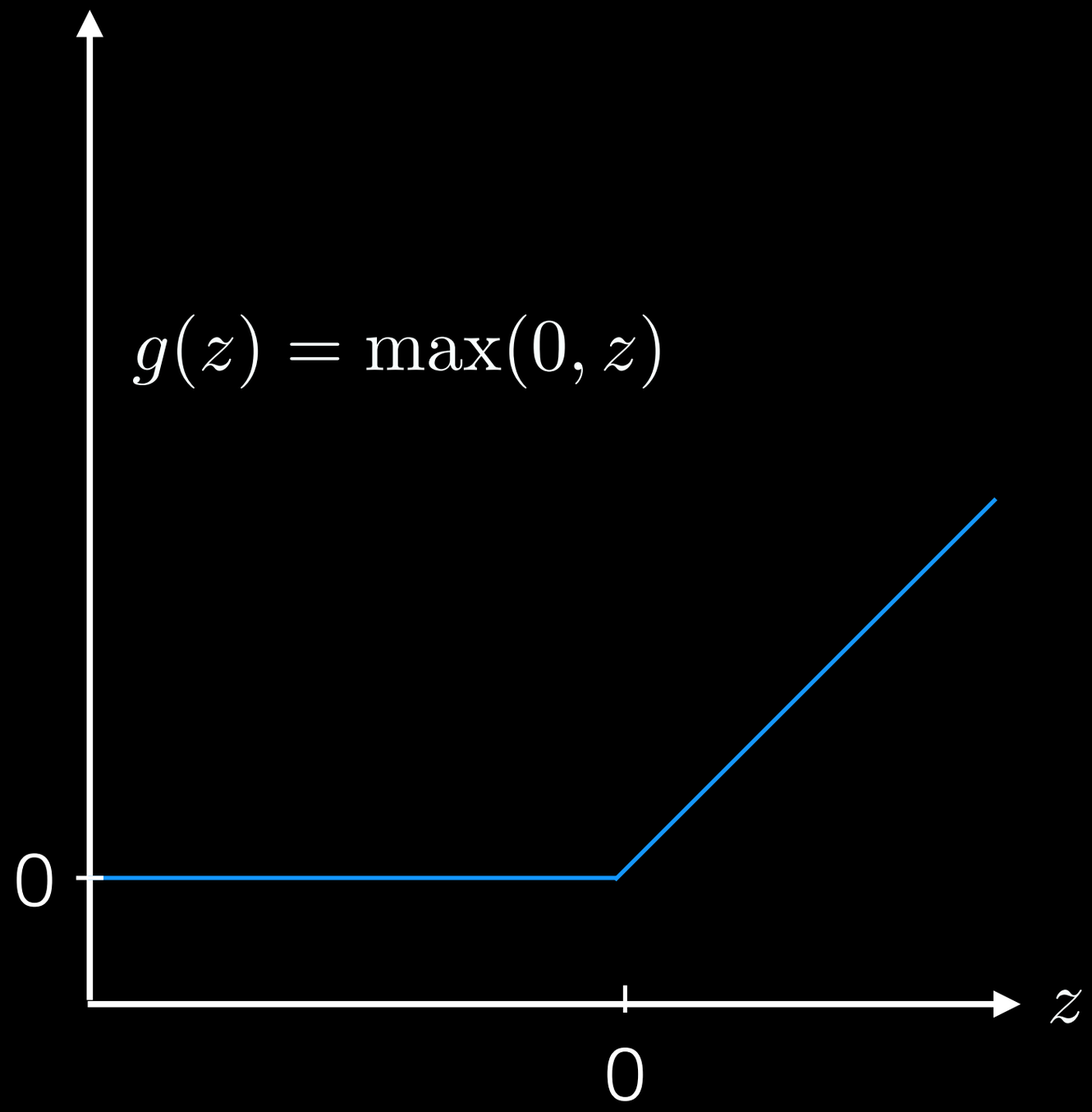


$$\mathbf{h} = g(W^T \mathbf{x} + \mathbf{c})$$

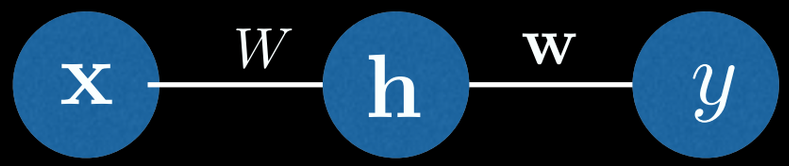
$$\begin{aligned} y &= \mathbf{w}^T \mathbf{h} + b \\ &= \mathbf{w}^T g(W^T \mathbf{x} + \mathbf{c}) + b \end{aligned}$$

Let’s choose $\mathbf{g}$ to be a non-linear function as linear did not work for us before!

Rectified Linear Unit ReLU



Hidden Layer with ReLU



$$\mathbf{h} = \max(\mathbf{0}, W^T \mathbf{x} + \mathbf{c})$$

$$\begin{aligned} y &= \mathbf{w}^T \mathbf{h} + b \\ &= \mathbf{w}^T \max(\mathbf{0}, W^T \mathbf{x} + \mathbf{c}) + b \end{aligned}$$

XOR and Simple Non-Linear Network

Let $W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\mathbf{c} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $b = 0$

Our XOR samples are $X = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

XOR and Simple Non-Linear Network

Then $W^T X + c = \begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{h} = \max(\mathbf{0}, W^T X + c) = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$y = \mathbf{w}^T \max(\mathbf{0}, W^T \mathbf{x} + \mathbf{c}) + b = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

XOR and Simple Non-Linear Network

Then $W^T X + c = \begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

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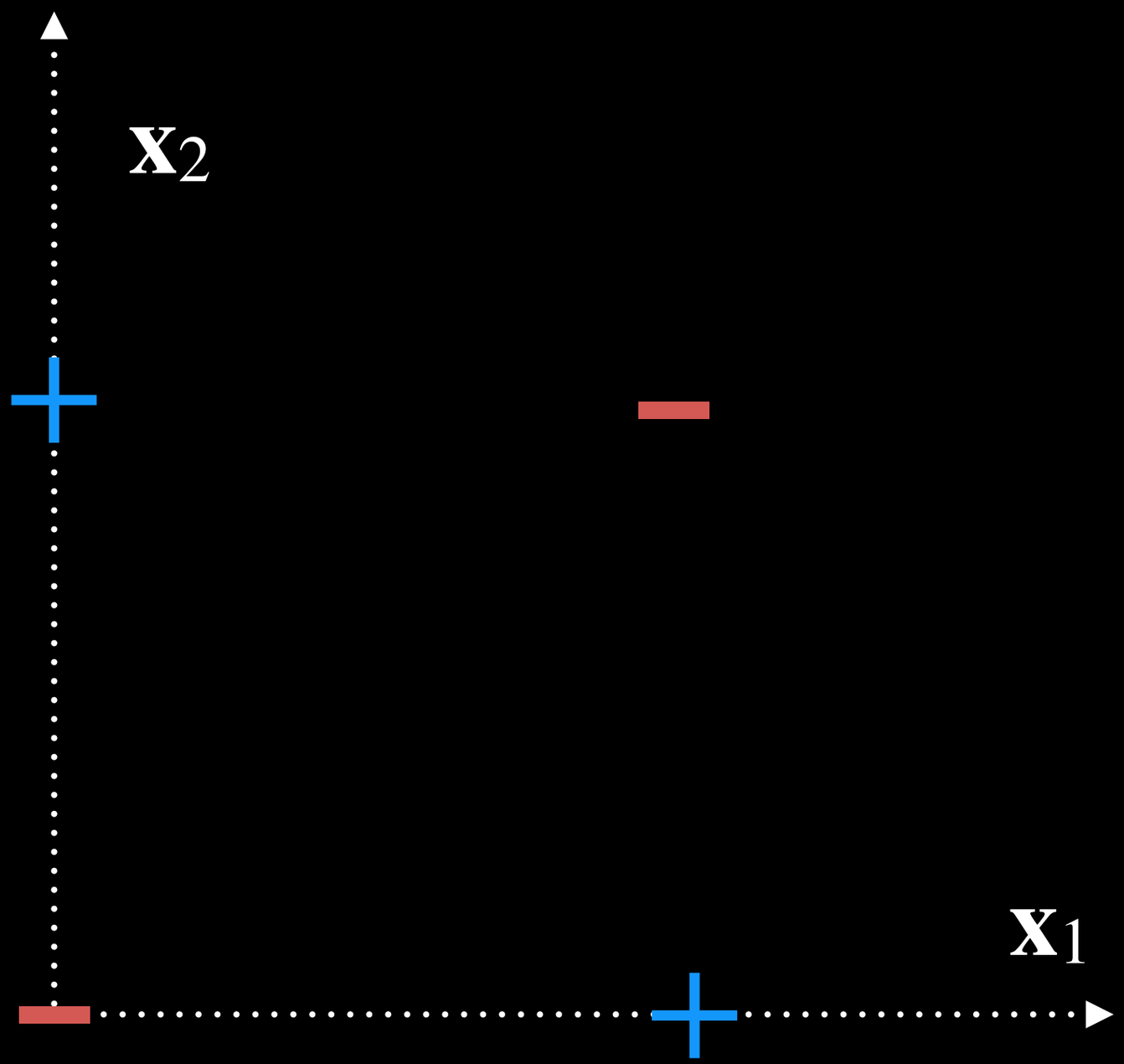
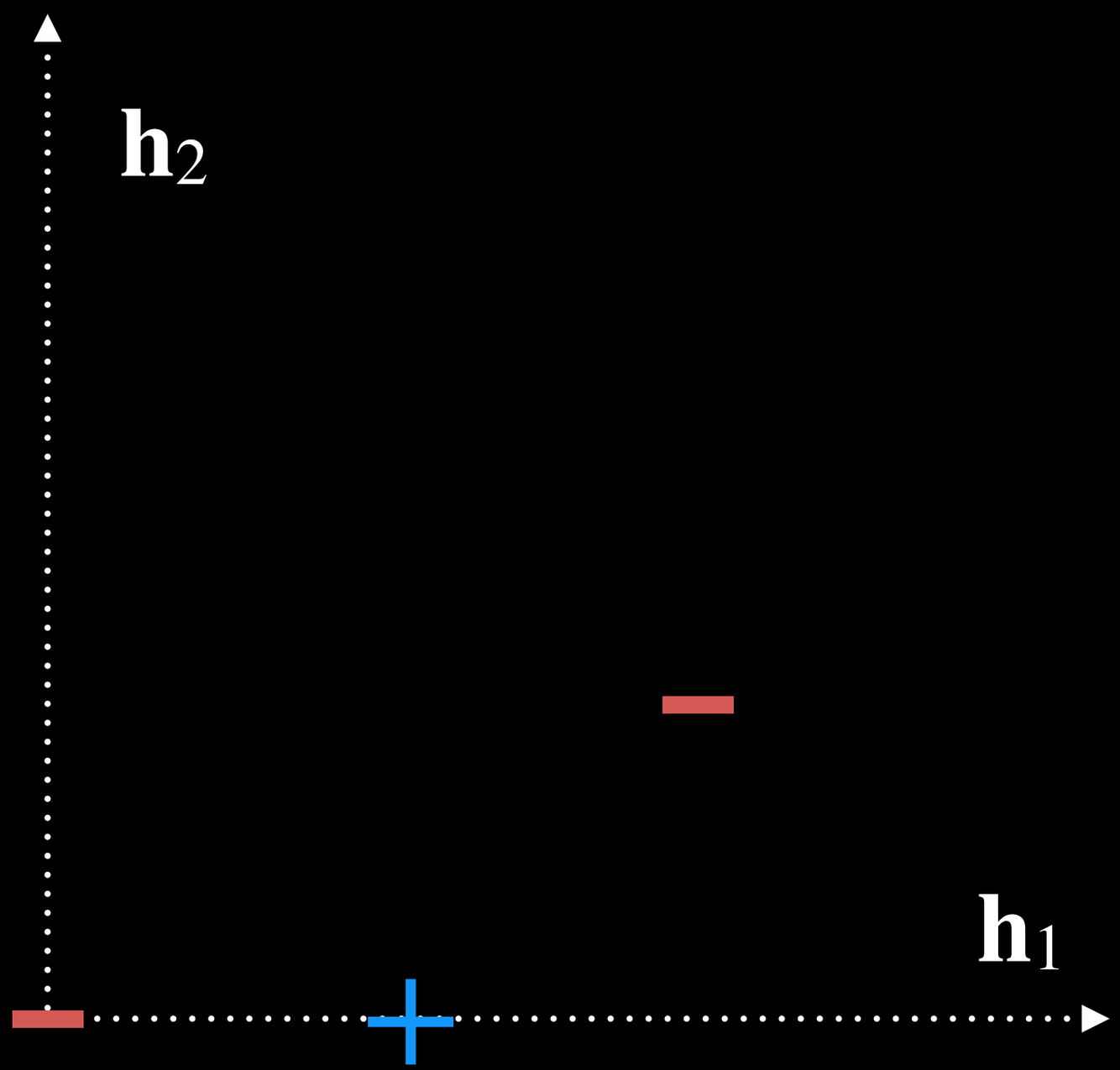
$$y = \mathbf{w}^T \max(\mathbf{0}, W^T \mathbf{x} + \mathbf{c}) + b = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

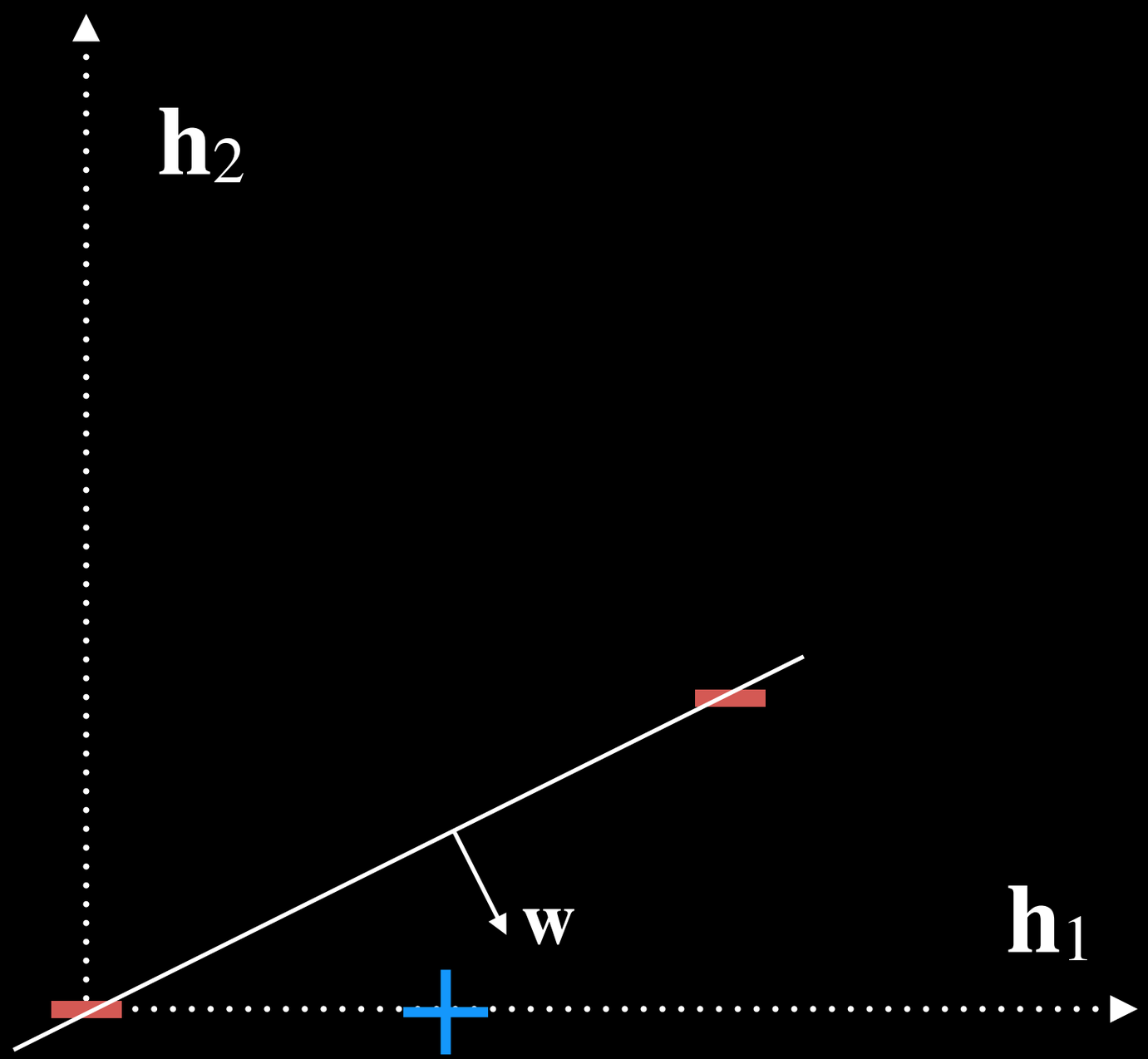
Input Feature Space



Transformed Feature Space: $\mathbf{h} = \phi(\mathbf{x})$



Transformed Feature Space: $\mathbf{h} = \phi(\mathbf{x})$



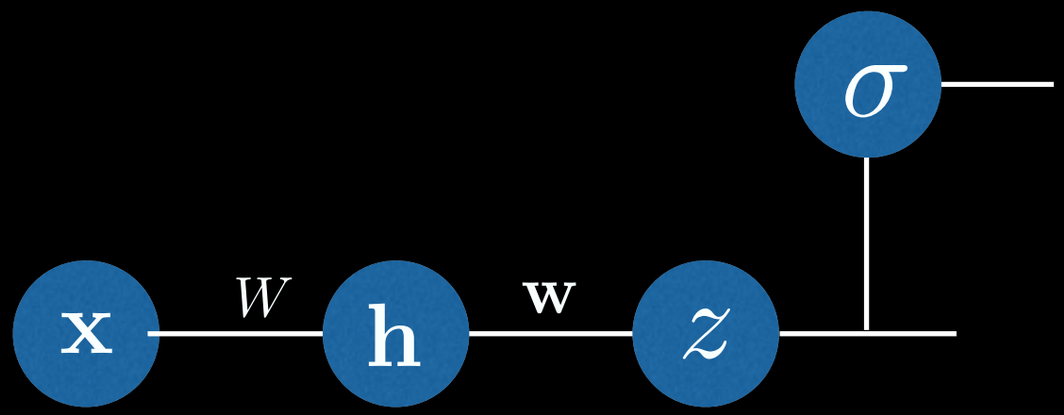
Non-linear mappings are f#$king awesome!

For this problem, we had a binary classifier with desired outputs of 0 and 1.

Now that we know it is possible, let's make this more principled. Recall our logistic regression from before!

Let’s use our logistic regression, but stick it on the top of our little non-linear network.

Simple Non-linear Network



$$\mathbf{h} = \max(\mathbf{0}, W^T\mathbf{x} + \mathbf{c})$$

$$z = \mathbf{w}^T\mathbf{h} + b$$

$$P(y = 1|\mathbf{x};\theta) = \sigma(z)$$

$$P(y = 0|\mathbf{x};\theta) = \sigma(-z)$$

$$\sigma(r) = \frac{1}{1 + e^{-r}}$$ Sigmoid

$$\theta = \{W, \mathbf{c}, \mathbf{w}, b\}$$

We need a cost (loss) function to optimize.

Common Loss Function for Binary Classifier

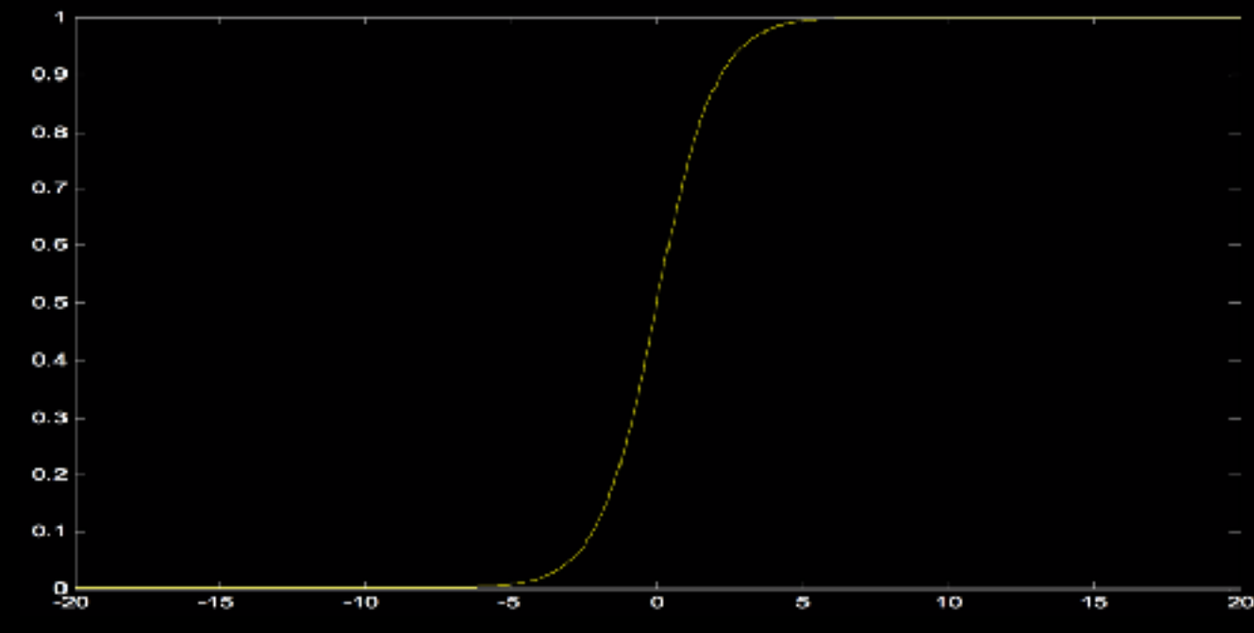
$$P(y|\mathbf{x};\theta) = \sigma((2y-1)z)$$

$$\sigma(r) = \frac{1}{1+e^{-r}} \qquad \text{Sigmoid}$$

$$\begin{aligned} L(\theta) &= -\log P(y|\mathbf{x};\theta) \\ &= -\log \sigma((2y-1)z) \\ &= \zeta((1-2y)z) \end{aligned}$$

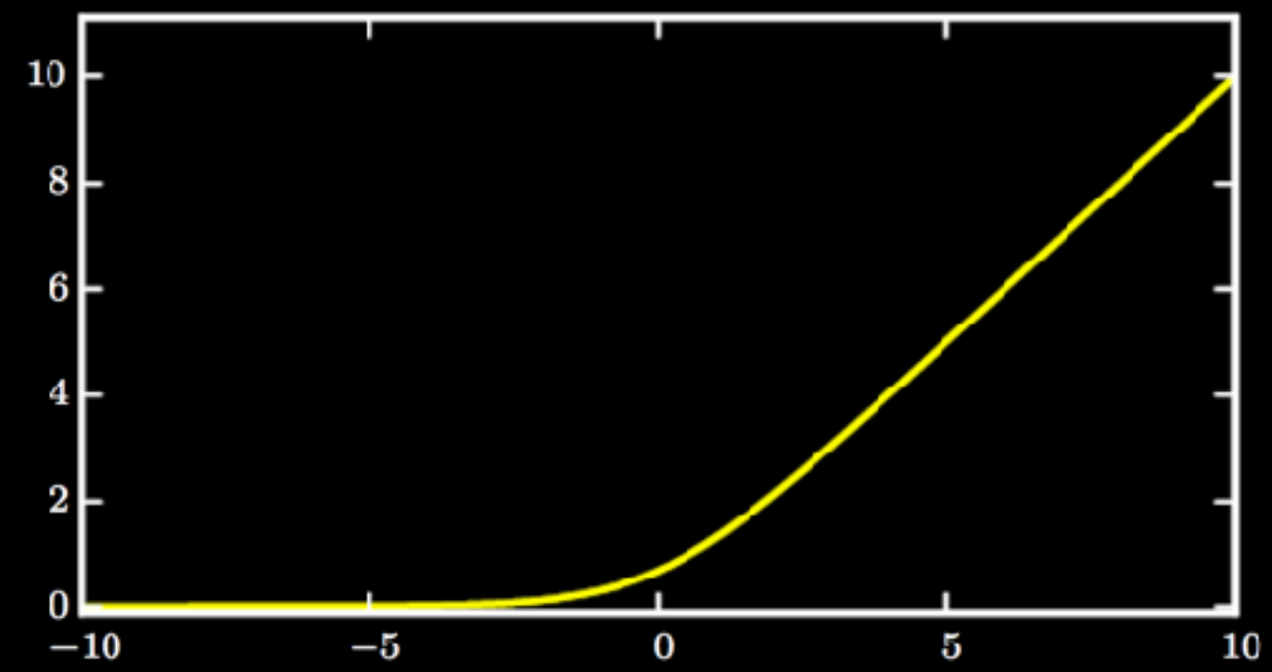
$$\zeta(r) = \log(1+e^{r}) \qquad \text{Softplus}$$

Sigmoid and Friend…Softplus



$$\sigma(r) = \frac{1}{1 + e^{-r}}$$

Sigmoid



$$\zeta(r) = \log(1 + e^{r})$$
$$\simeq \max(0, r)$$

Softplus

Don’t forget that for a collection of samples…

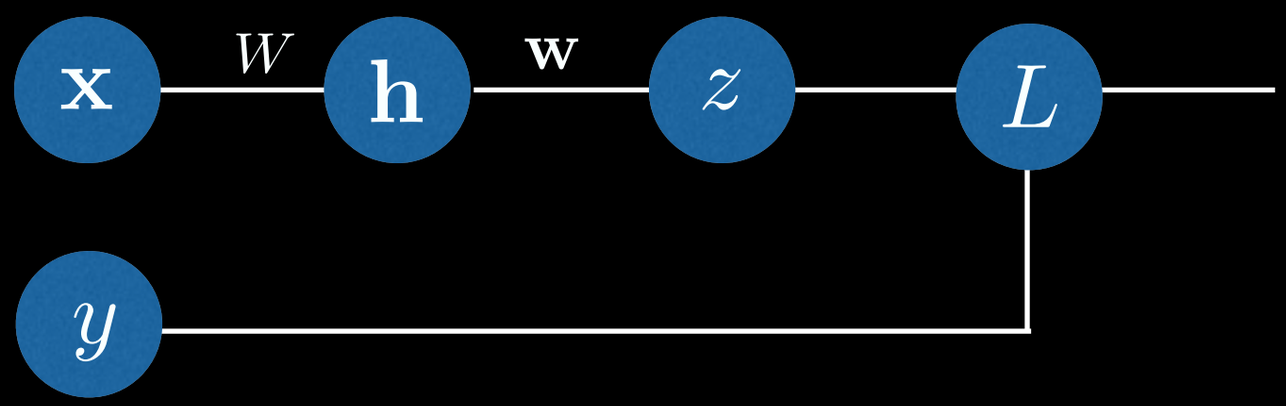
$$\operatorname*{argmax}_{\theta} P(\{y_i\}|\{\mathbf{x}_i\};\theta) = \operatorname*{argmax}_{\theta} \prod_{i=1}^{n} P(y_i|\mathbf{x}_i;\theta)$$

$$= \operatorname*{argmin}_{\theta} \; -\log \prod_{i=1}^{n} P(y_i|\mathbf{x}_i;\theta)$$

$$= \operatorname*{argmin}_{\theta} \sum_{i=1}^{n} -\log P(y_i|\mathbf{x}_i;\theta)$$

Common Loss Function for Binary Classifier

$$
\begin{aligned}
L(\theta) &= -\log P(y|\mathbf{x};\theta) \\
&= -\log \sigma((2y-1)z) \\
&= \zeta((1-2y)z) \qquad \text{Softplus}
\end{aligned}
$$



Common Loss Function for Binary Classifier

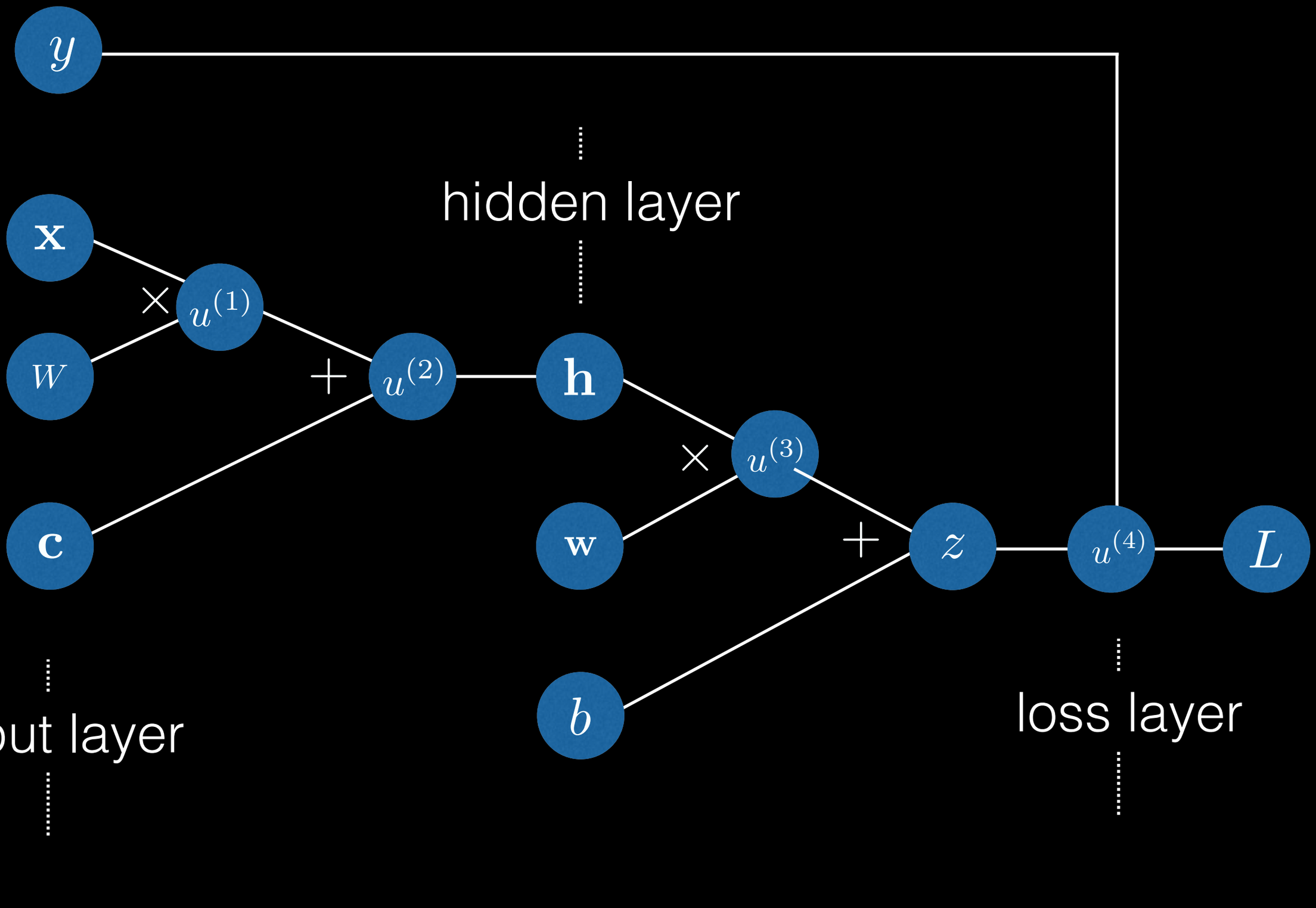
$$
\begin{aligned}
L(\theta) &= -\log P(y|\mathbf{x};\theta) \\
&= -\log \sigma((2y-1)z) \\
&= \zeta((1-2y)z)
\end{aligned}
$$

$\zeta(r) = \log(1+e^r)$ Softplus

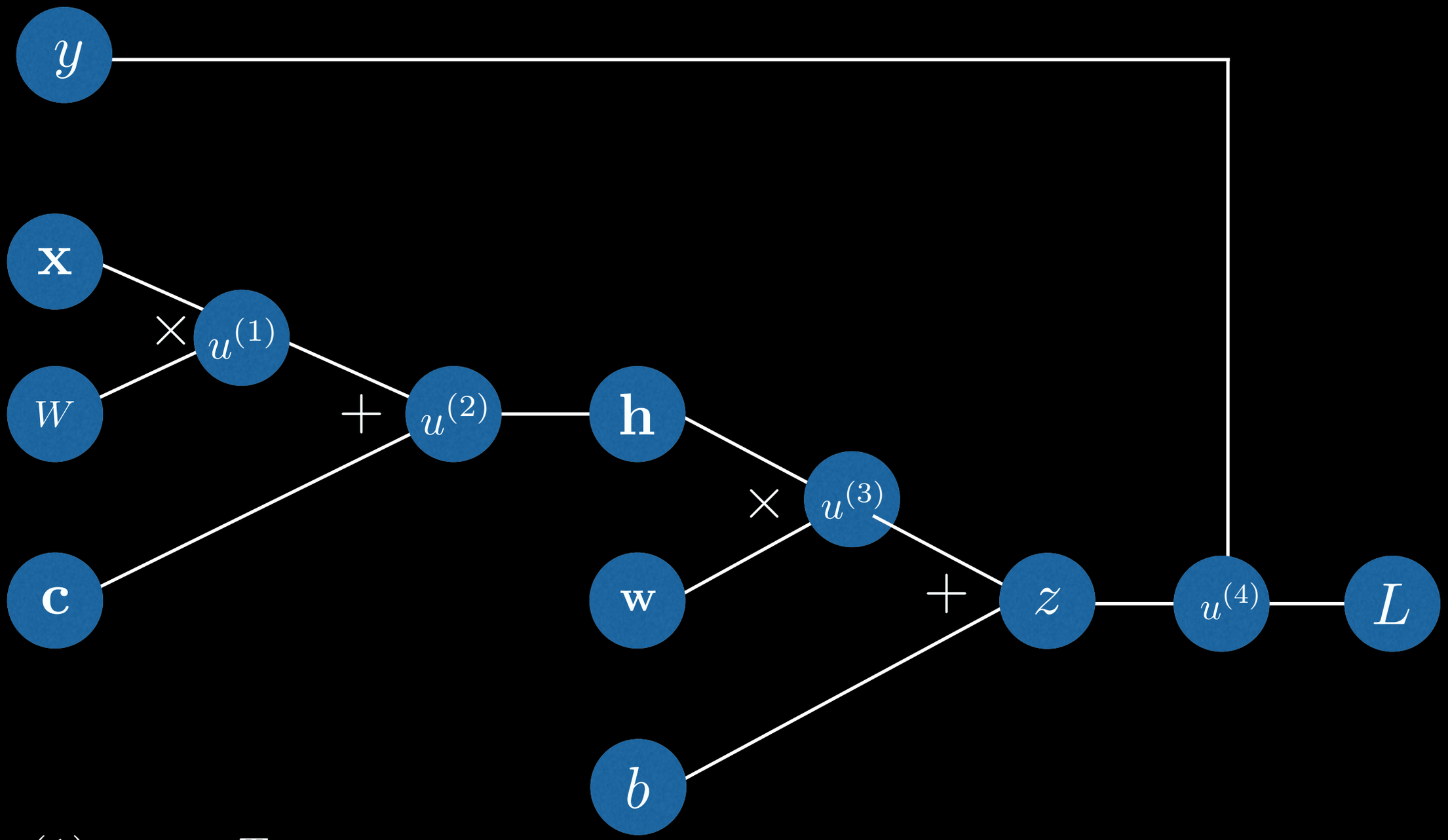
- Softplus has strong gradients when training sample is misclassified. It does not saturate.
- Softplus is differentiable everywhere.

Let’s make a more descriptive version of the computational graph for this network and then use backdrop to compute the gradient with respect to the network parameters.

Forward Propagation

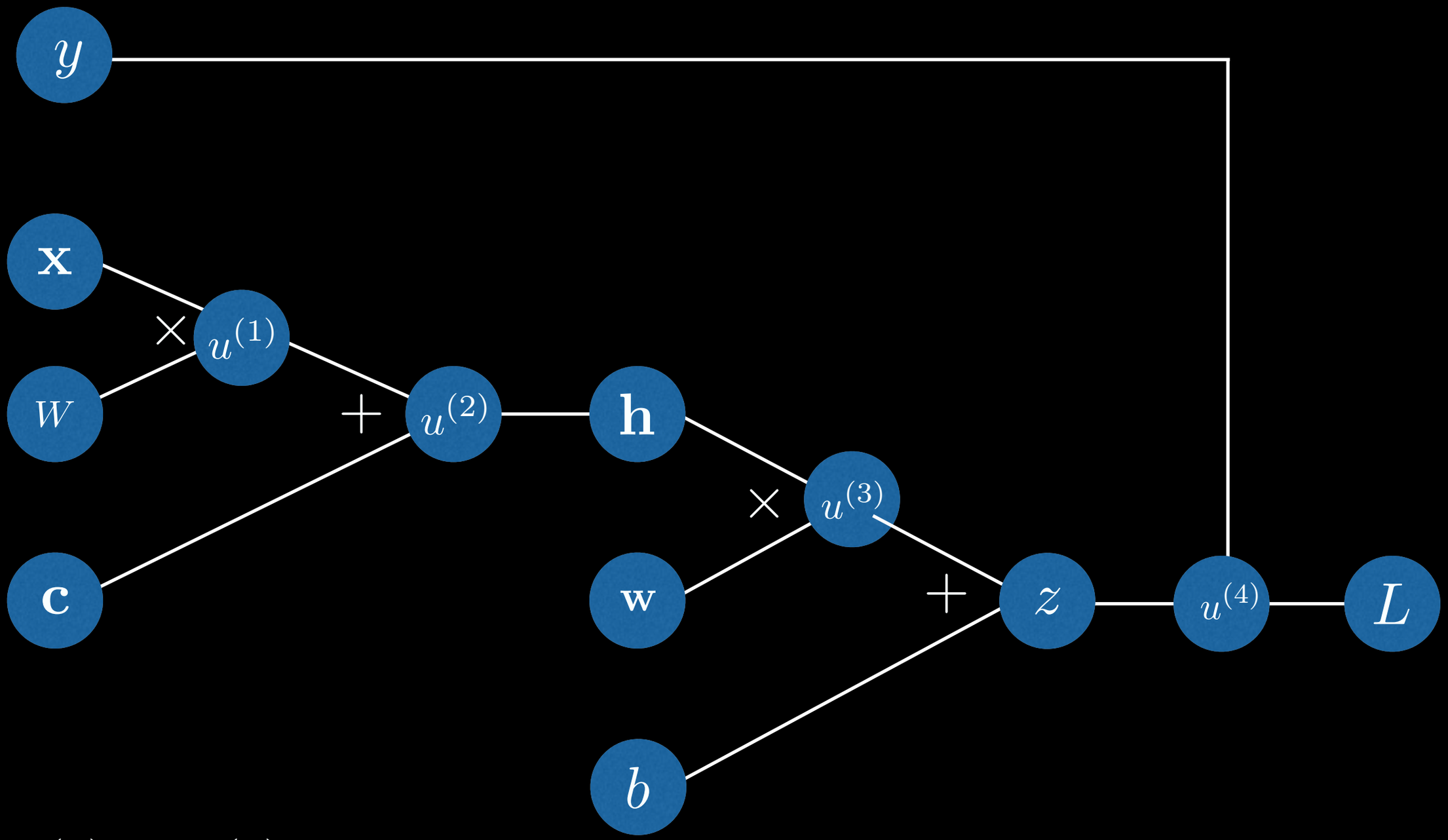


Forward Propagation



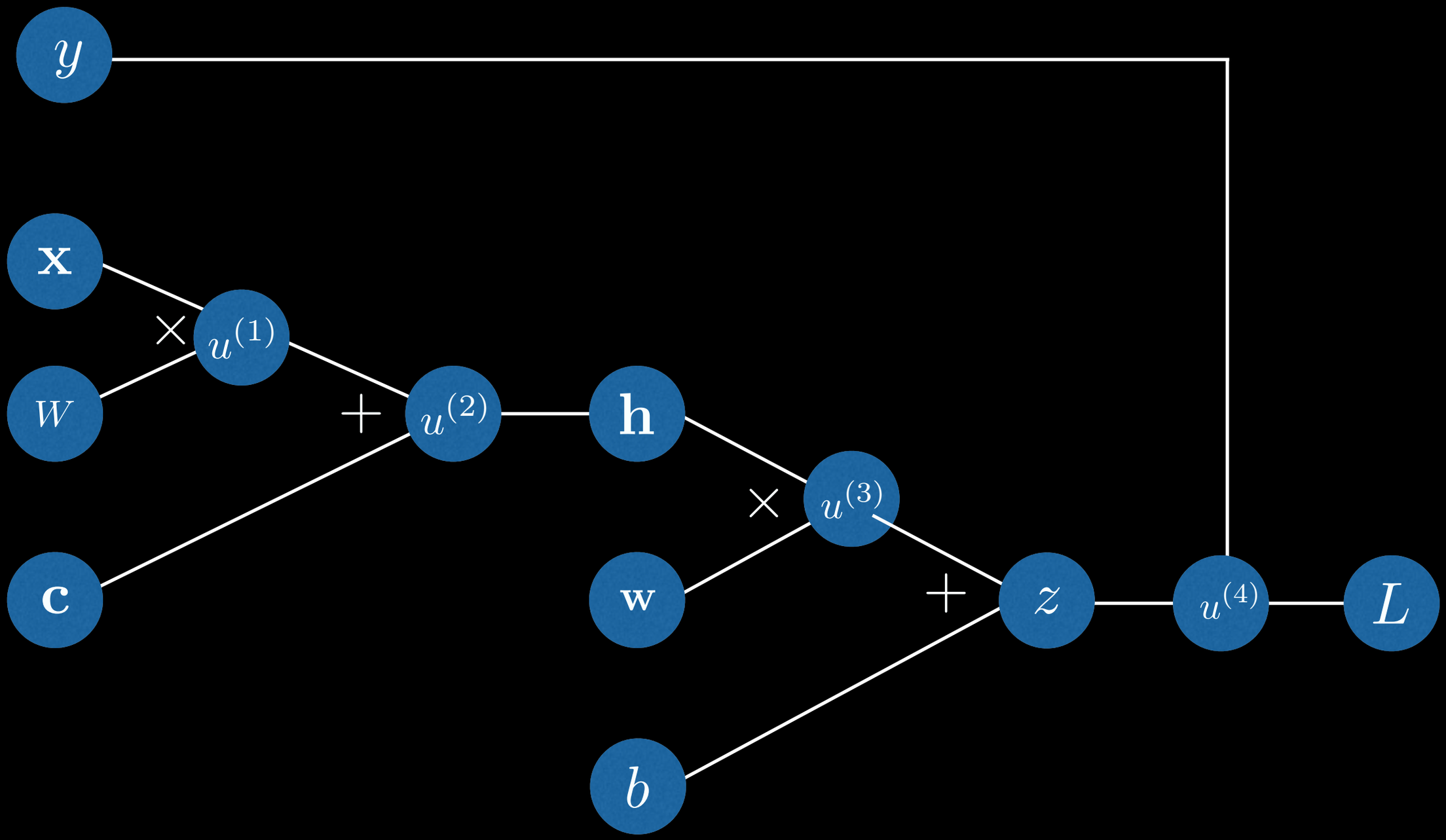
$$u^{(1)} = W^T \mathbf{x}$$

Forward Propagation



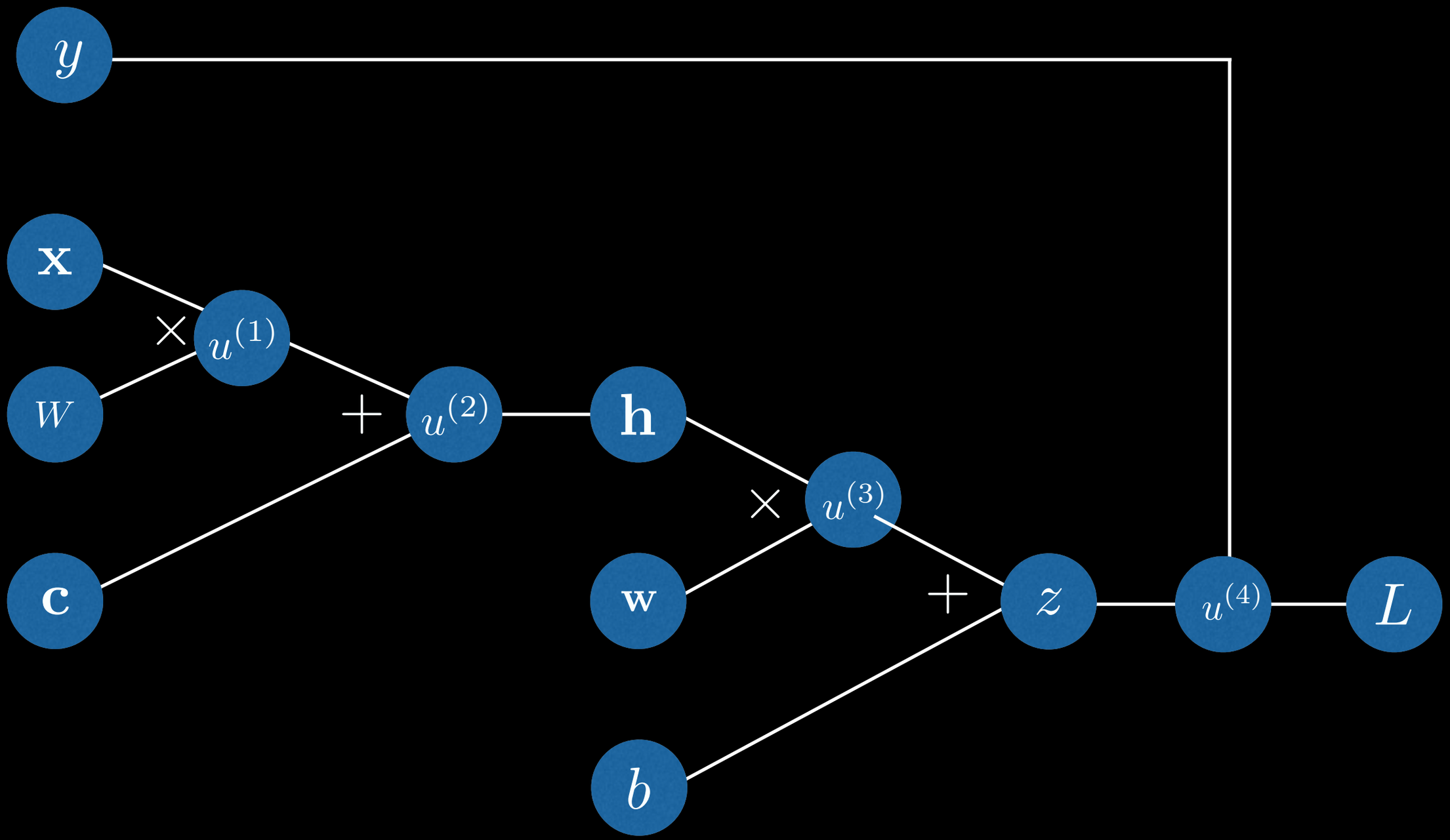
$$u^{(2)} = u^{(1)} + \mathbf{c}$$

Forward Propagation



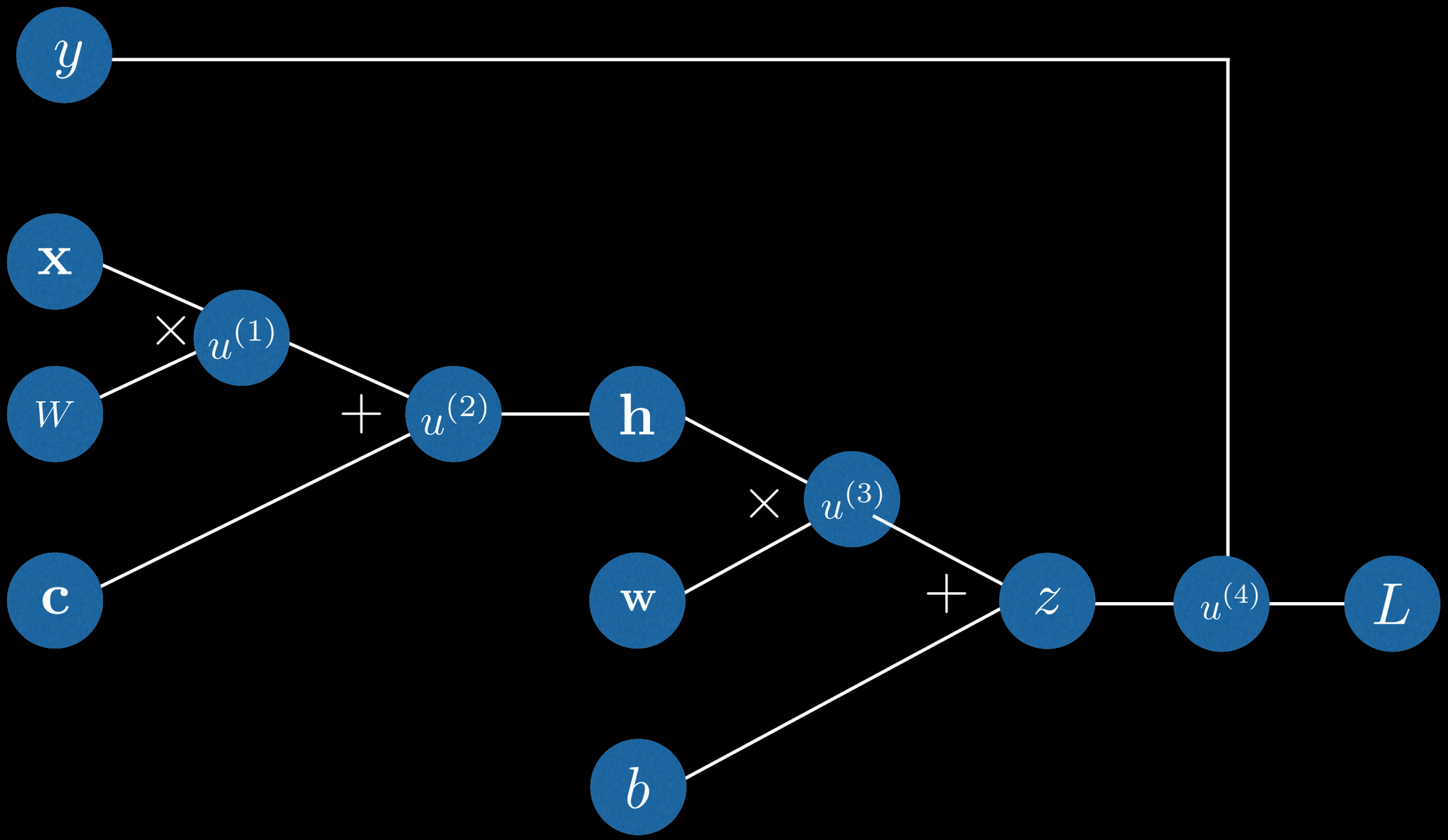
$$\mathbf{h} = \max(\mathbf{0}, \mathbf{u^{(2)}})$$

Forward Propagation



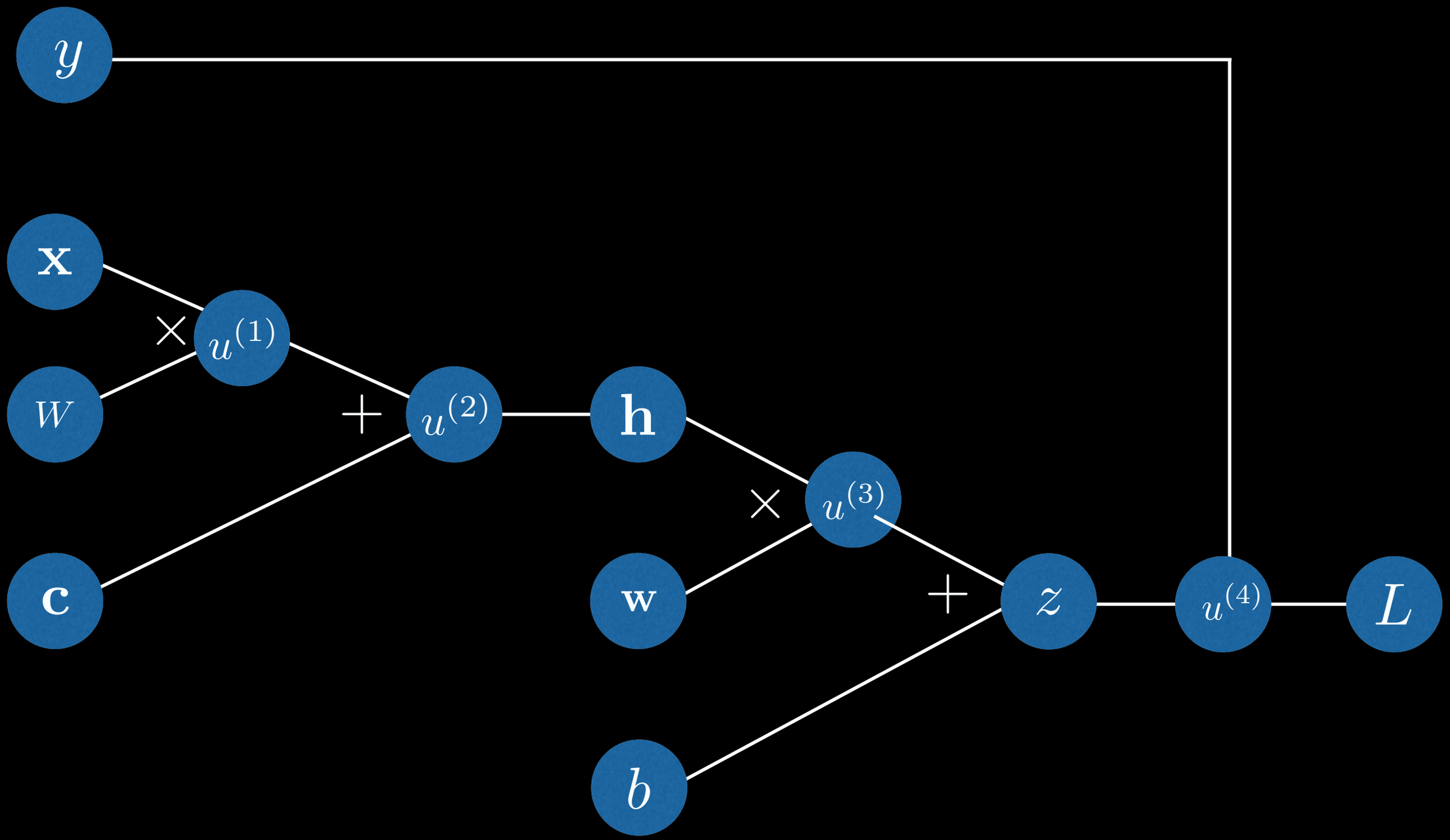
$$u^{(3)} = \mathbf{w}^T \mathbf{h}$$

Forward Propagation



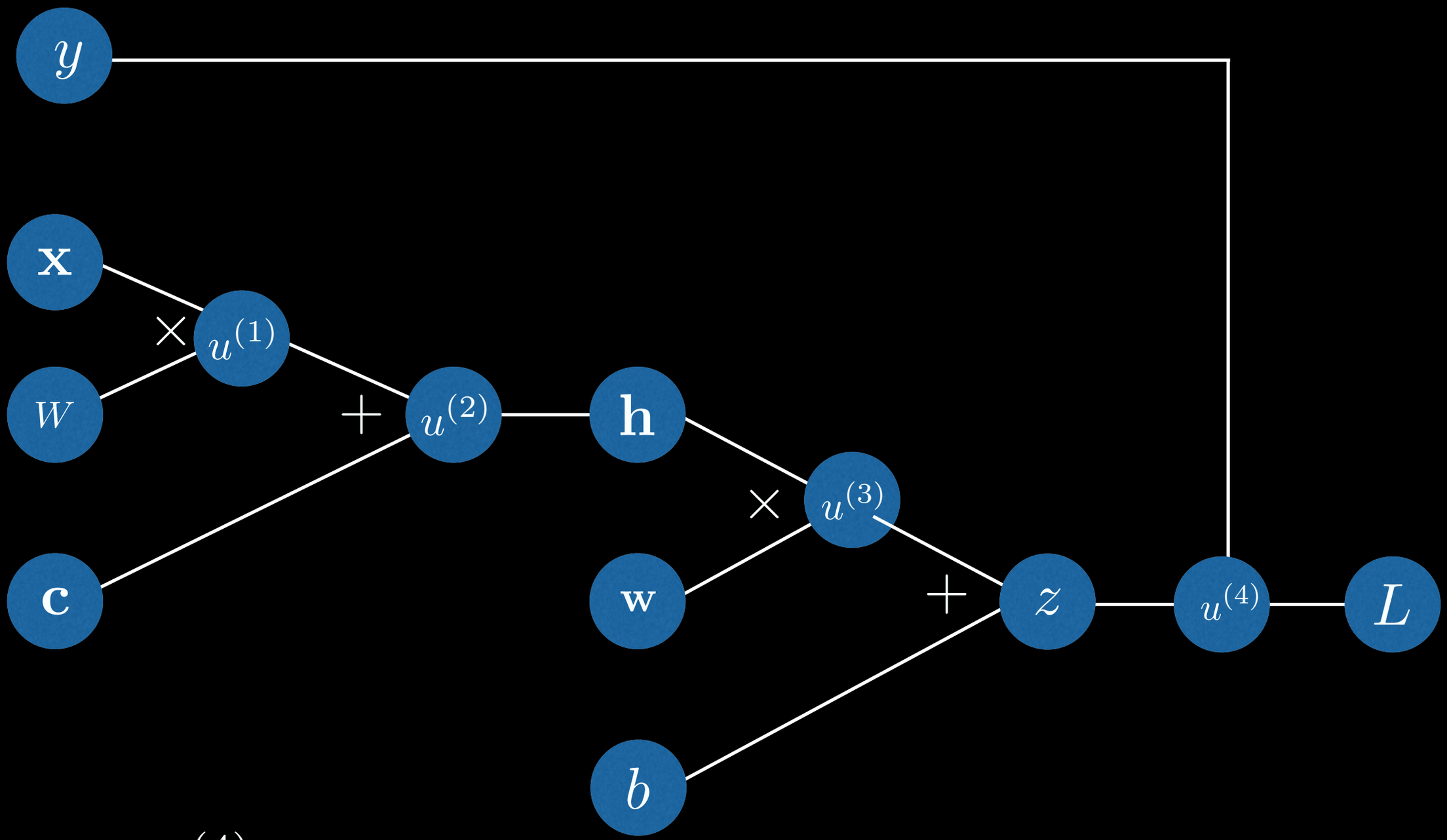
$$z = u^{(3)} + b$$

Forward Propagation



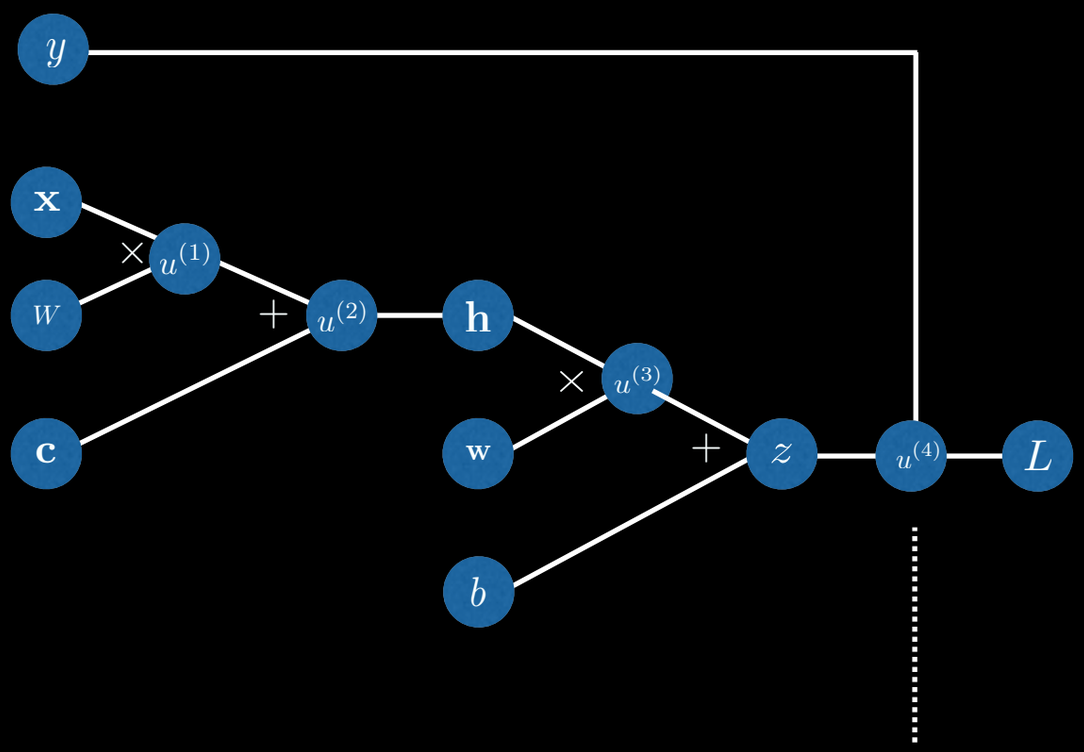
$$u^{(4)} = (1 - 2y)z$$

Forward Propagation



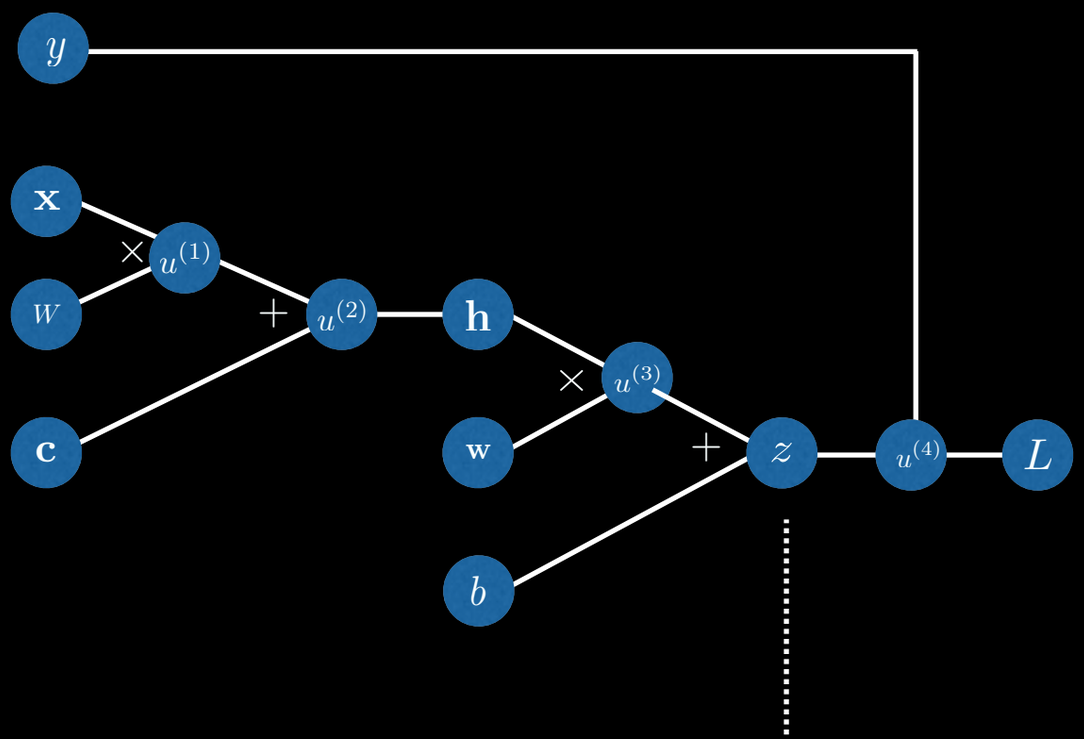
$$L = \zeta(u^{(4)})$$

Back-Propagation



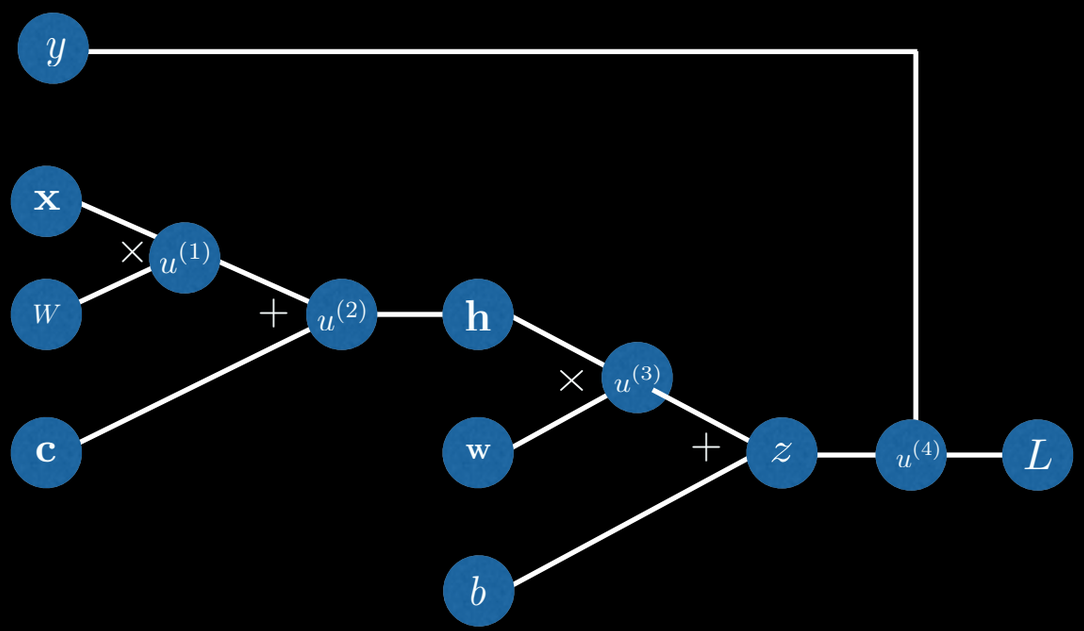
$$\frac{\partial L}{\partial u^{(4)}} = \sigma(u^{(4)})$$

Back-Propagation



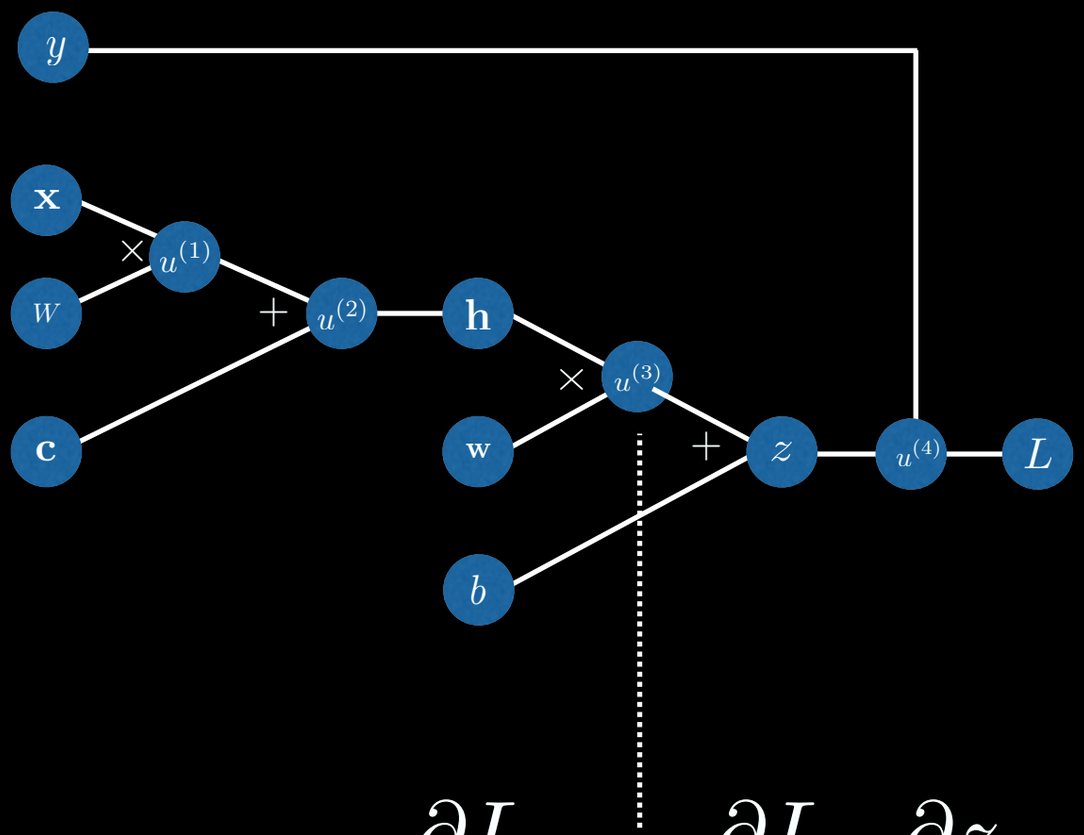
$$\frac{\partial L}{\partial z} = \frac{\partial L}{\partial u^{(4)}} \frac{\partial u^{(4)}}{\partial z} = \sigma(u^{(4)})(1-2y)$$

Back-Propagation



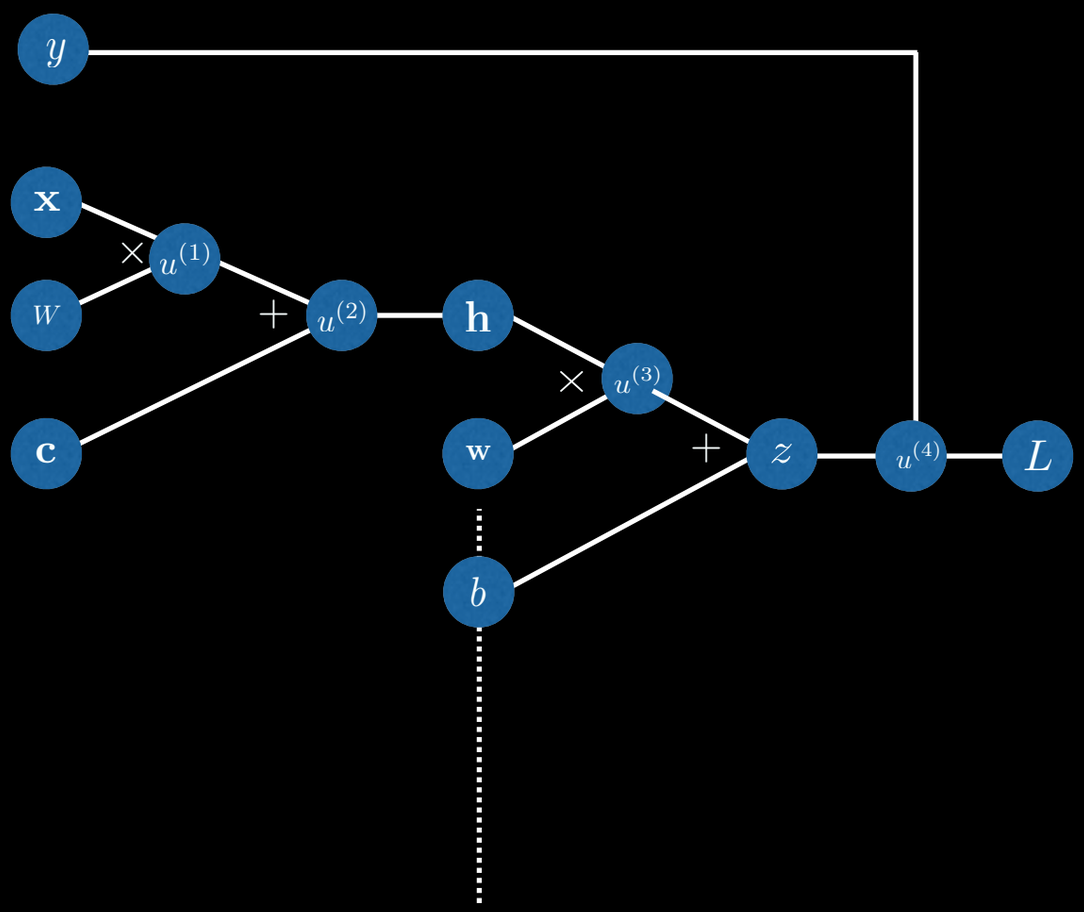
$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial b} = \sigma(u^{(4)})(1-2y)$$

Back-Propagation



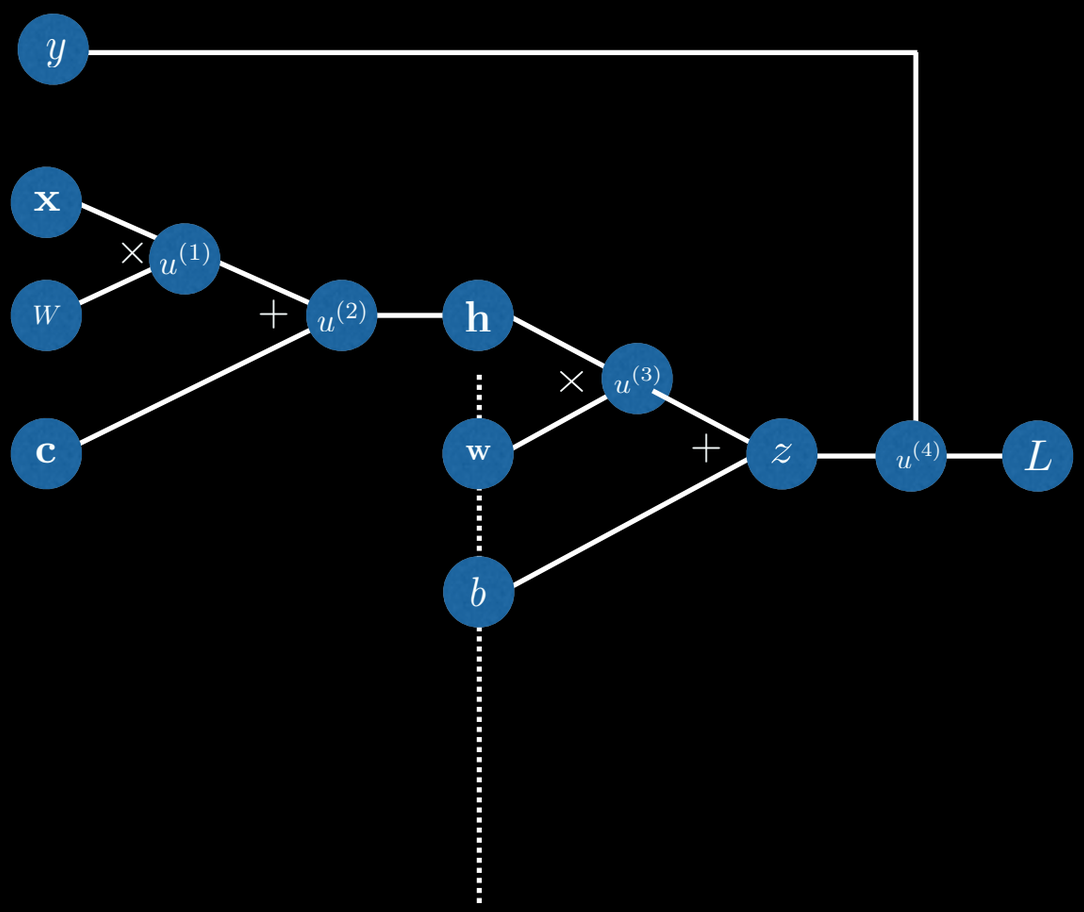
$$\frac{\partial L}{\partial u^{(3)}} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial u^{(3)}} = \sigma(u^{(4)})(1-2y)(1)$$

Back-Propagation



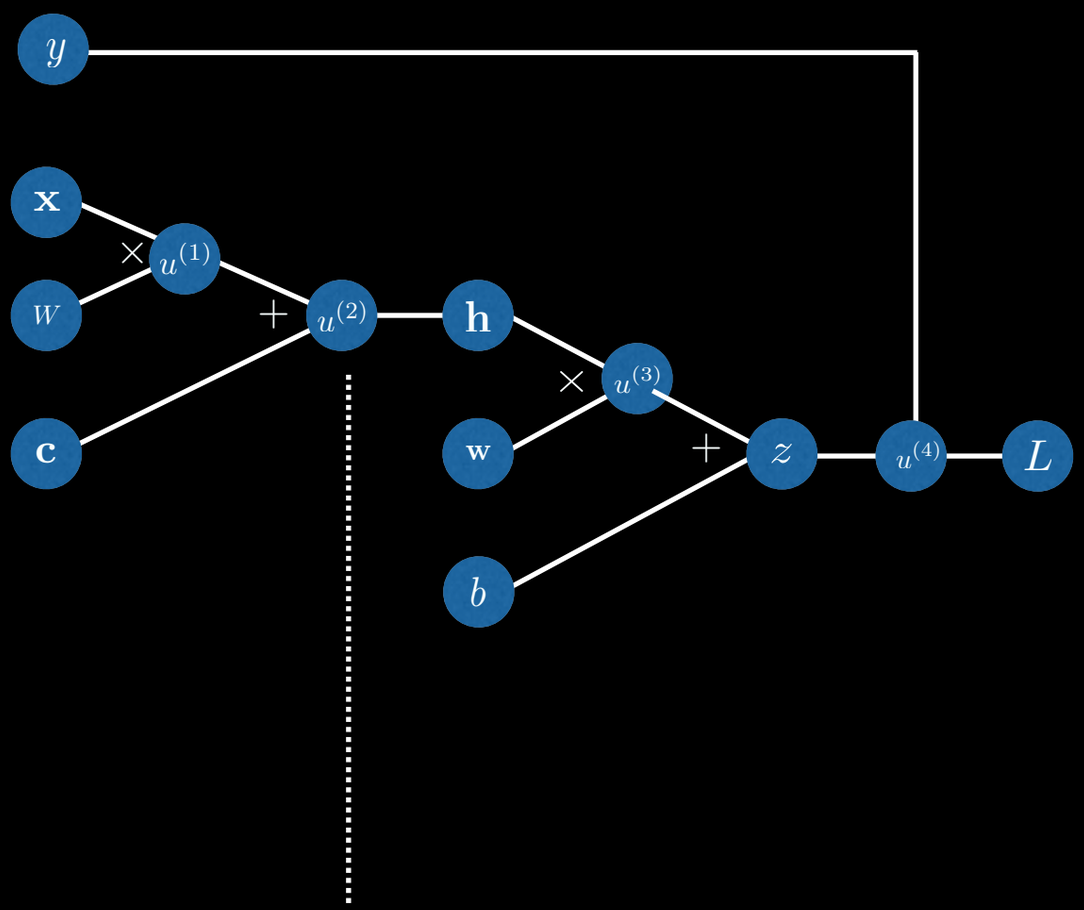
$$\nabla_{\mathbf{w}} L = \frac{\partial L}{\partial u^{(3)}} \nabla_{\mathbf{w}} u^{(3)} = \sigma(u^{(4)})(1-2y)\mathbf{h}$$

Back-Propagation



$$\nabla_{\mathbf{h}} L = \frac{\partial L}{\partial u^{(3)}} \nabla_{\mathbf{h}} u^{(3)} = \sigma(u^{(4)})(1 - 2y)\mathbf{w}$$

Back-Propagation



Here we have a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

$\mathbb{R}^2 \leftarrow \mathbb{R}^2$

$\mathbf{h} = \max(\mathbf{0}, \mathbf{u^{(2)}})$

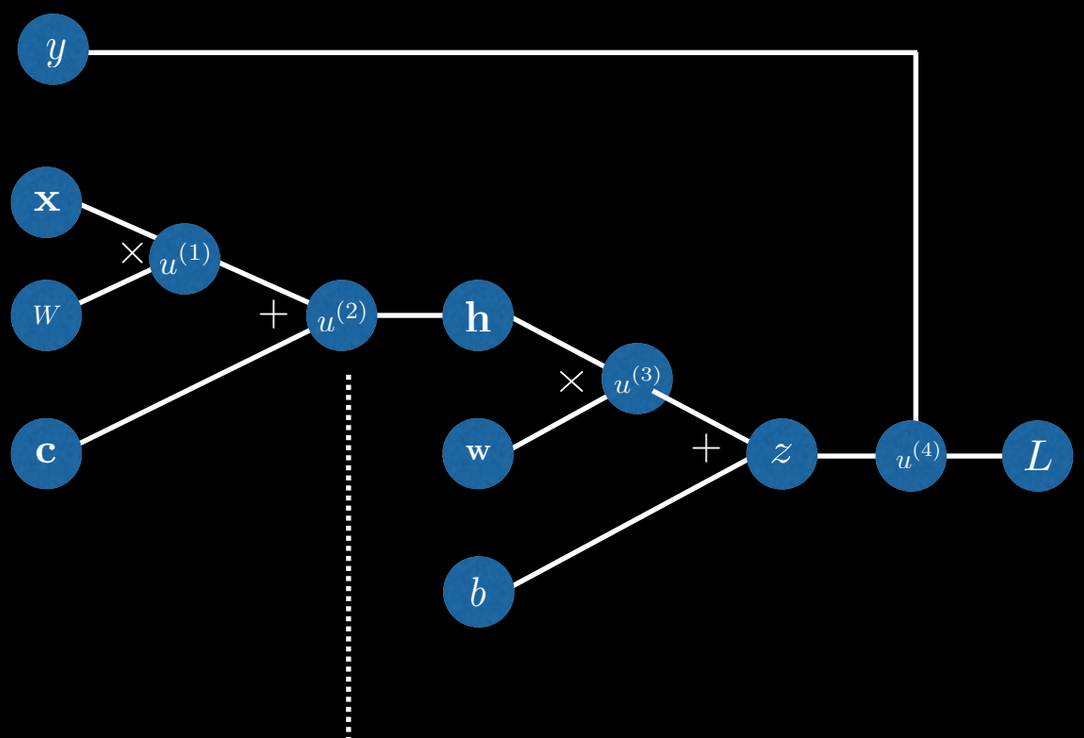
Jacobians and more…

$$\mathbb{R}^n \leftarrow \mathbb{R}^m \qquad \mathbb{R} \leftarrow \mathbb{R}^n \qquad \Longrightarrow \quad \nabla_{\mathbf{x}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)^T \nabla_{\mathbf{y}} z$$

$$\mathbf{y} = g(\mathbf{x}) \qquad z = f(\mathbf{y})$$

where the Jacobian $\dfrac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix}_{n \times m}$

Back-Propagation



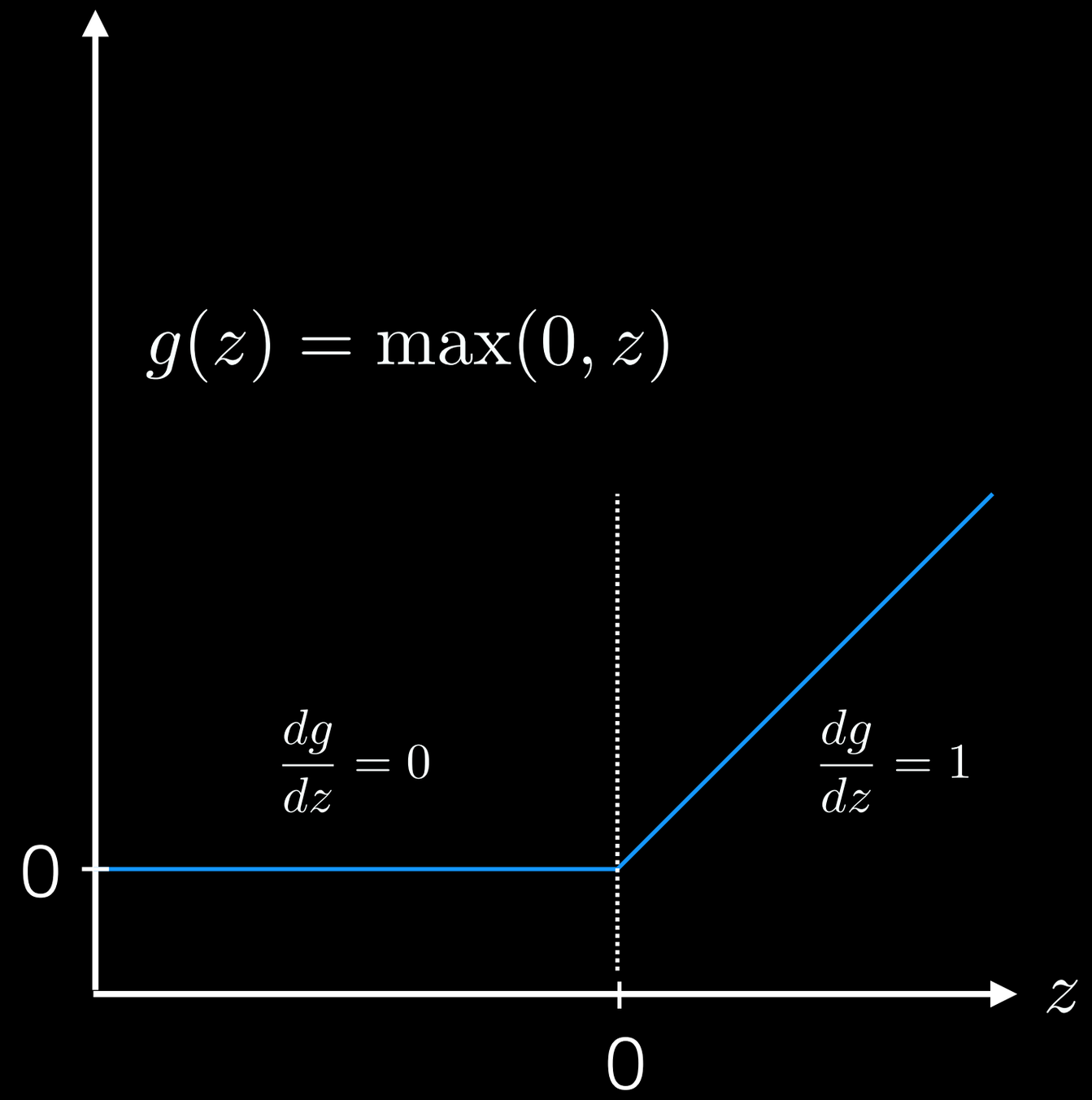
Here we have a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbb{R}^2 \leftarrow \mathbb{R}^2$$

$$\mathbf{h} = \max(\mathbf{0}, \mathbf{u^{(2)}})$$

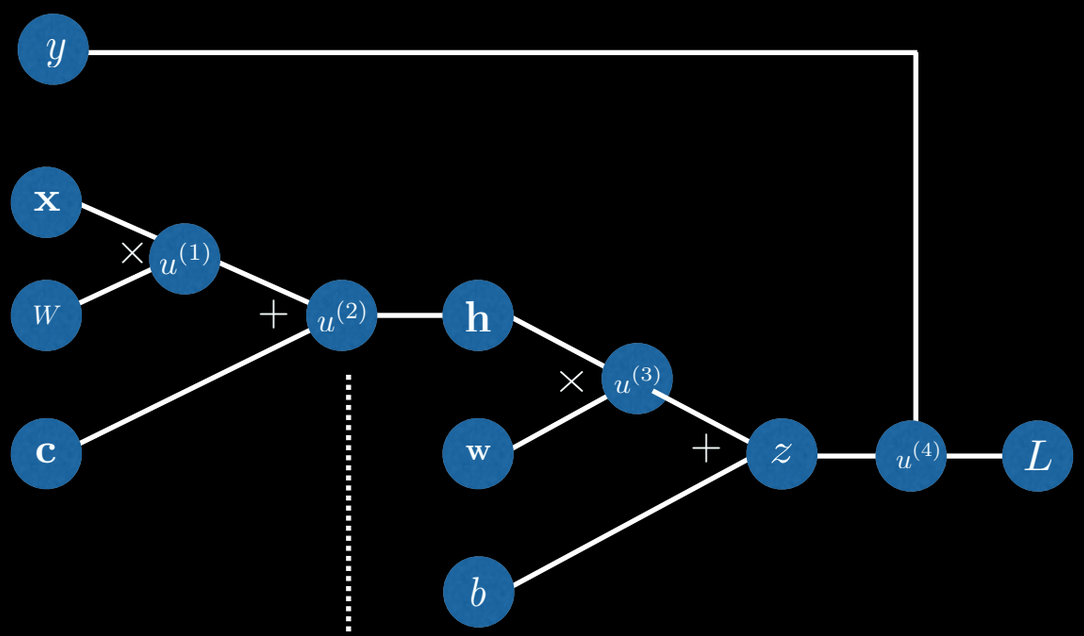
$$\frac{\partial \mathbf{h}}{\partial \mathbf{u^{(2)}}} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1^{(2)}} & \frac{\partial h_1}{\partial u_2^{(2)}} \\ \frac{\partial h_2}{\partial u_1^{(2)}} & \frac{\partial h_2}{\partial u_2^{(2)}} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1^{(2)}} & 0 \\ 0 & \frac{\partial h_2}{\partial u_2^{(2)}} \end{bmatrix}$$

Rectified Linear Unit ReLU



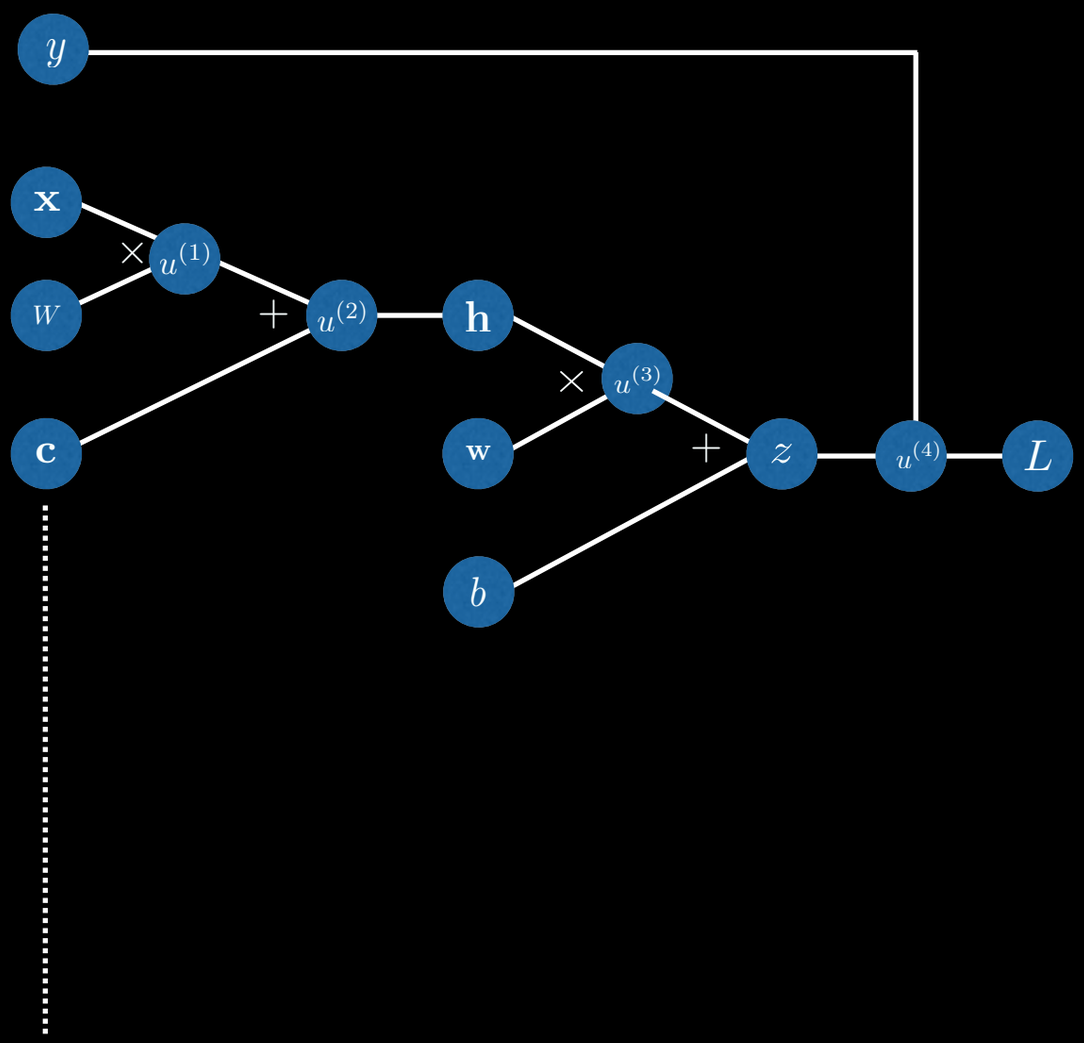
Let $f = \text{“}\frac{dg}{dz}\text{”}$

Back-Propagation



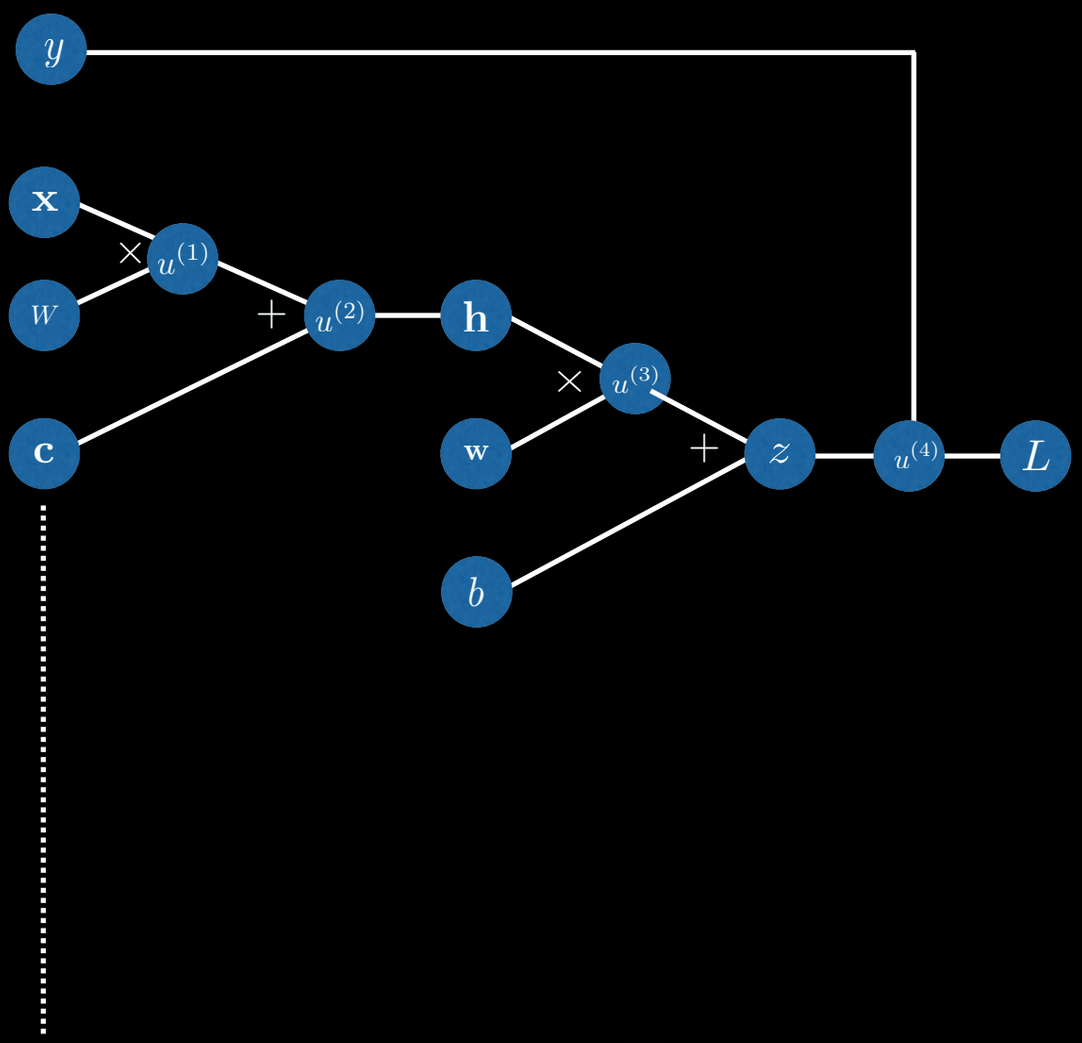
$$\nabla_{\mathbf{u^{(2)}}} L = \left(\frac{\partial \mathbf{h}}{\partial \mathbf{u^{(2)}}} \right)^T \nabla_{\mathbf{h}} L = \sigma(u^{(4)})(1 - 2y)\mathbf{f}(\mathbf{u^{(2)}}) \odot \mathbf{w}$$

Back-Propagation



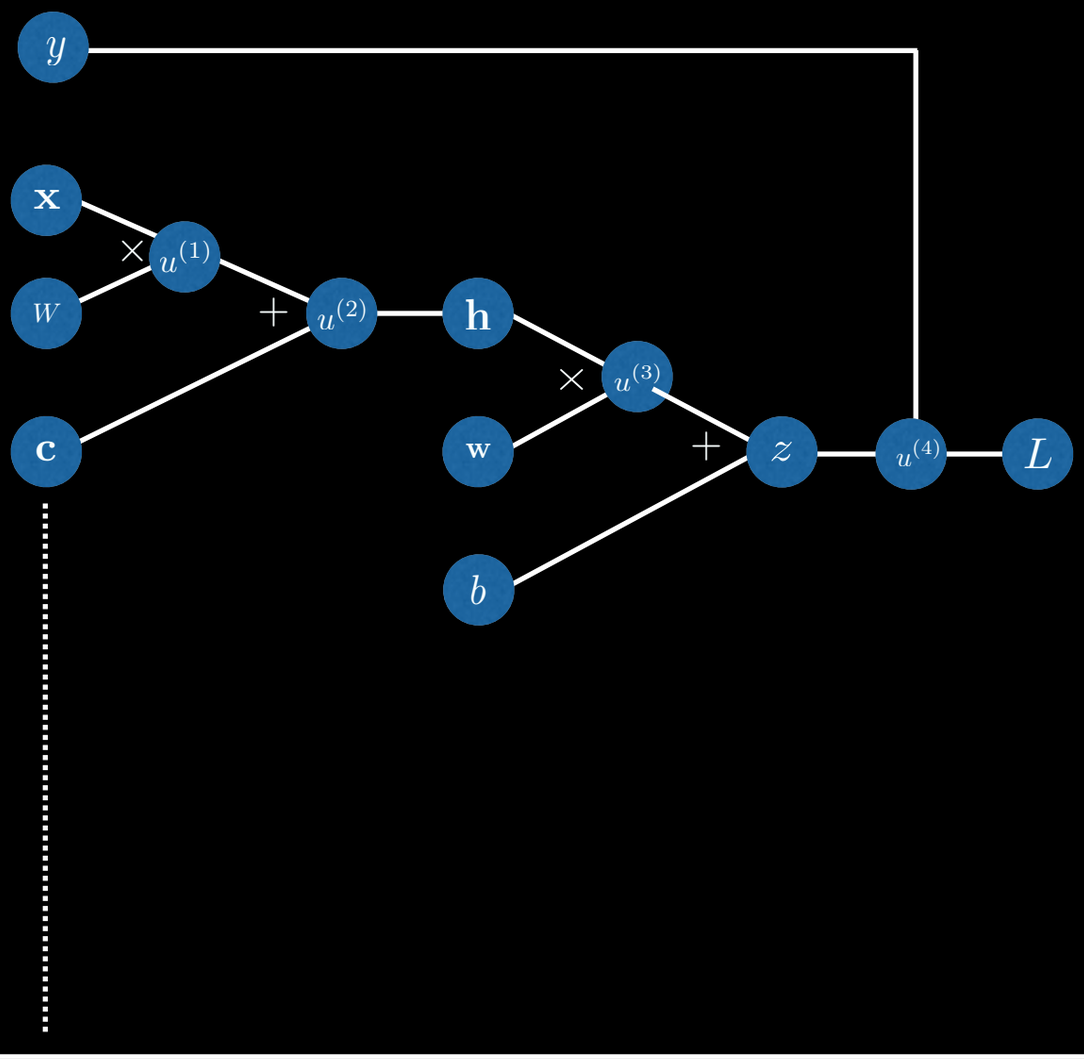
$$\nabla_{\mathbf{c}} L = \left(\frac{\partial \mathbf{u^{(2)}}}{\partial \mathbf{c}} \right)^T \nabla_{\mathbf{u^{(2)}}} L$$

Back-Propagation



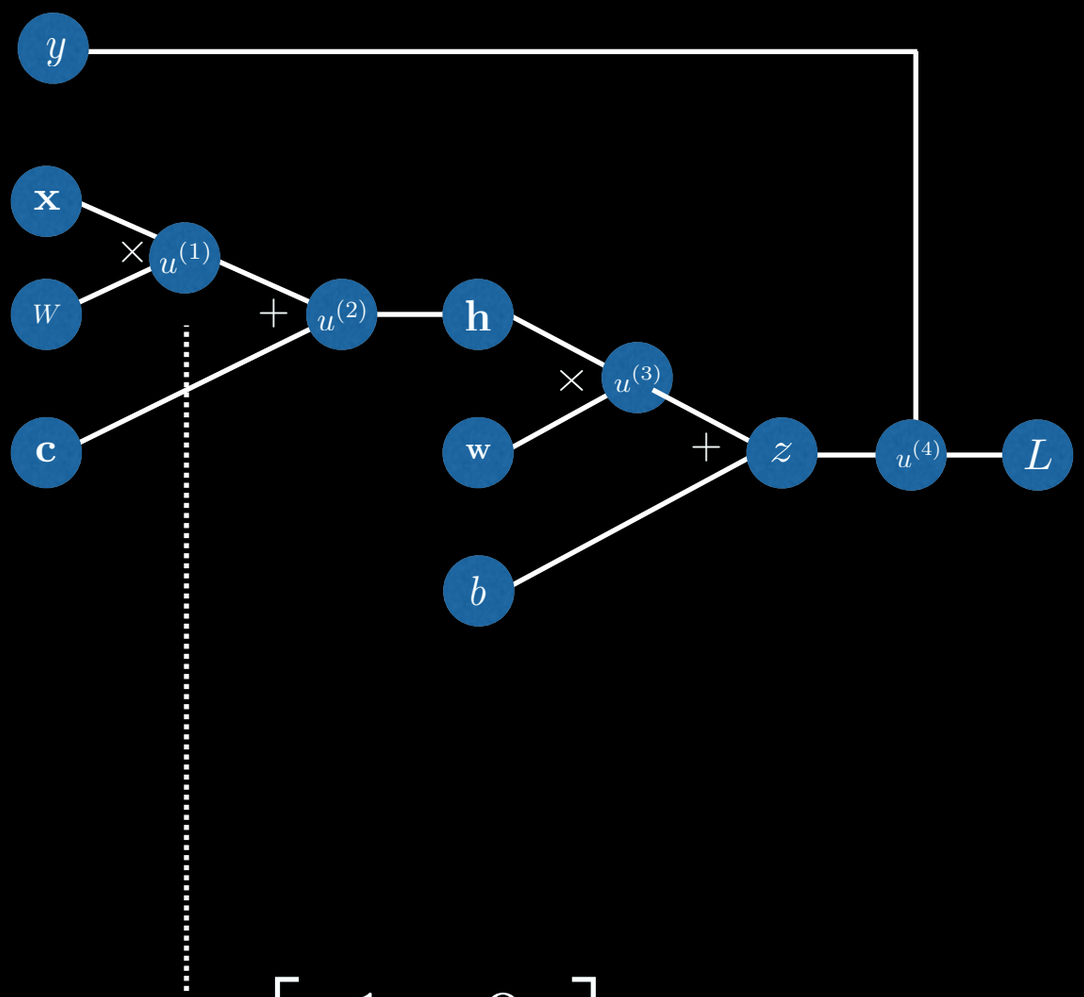
$$\nabla_{\mathbf{c}} L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T \nabla_{\mathbf{u^{(2)}}} L$$

Back-Propagation



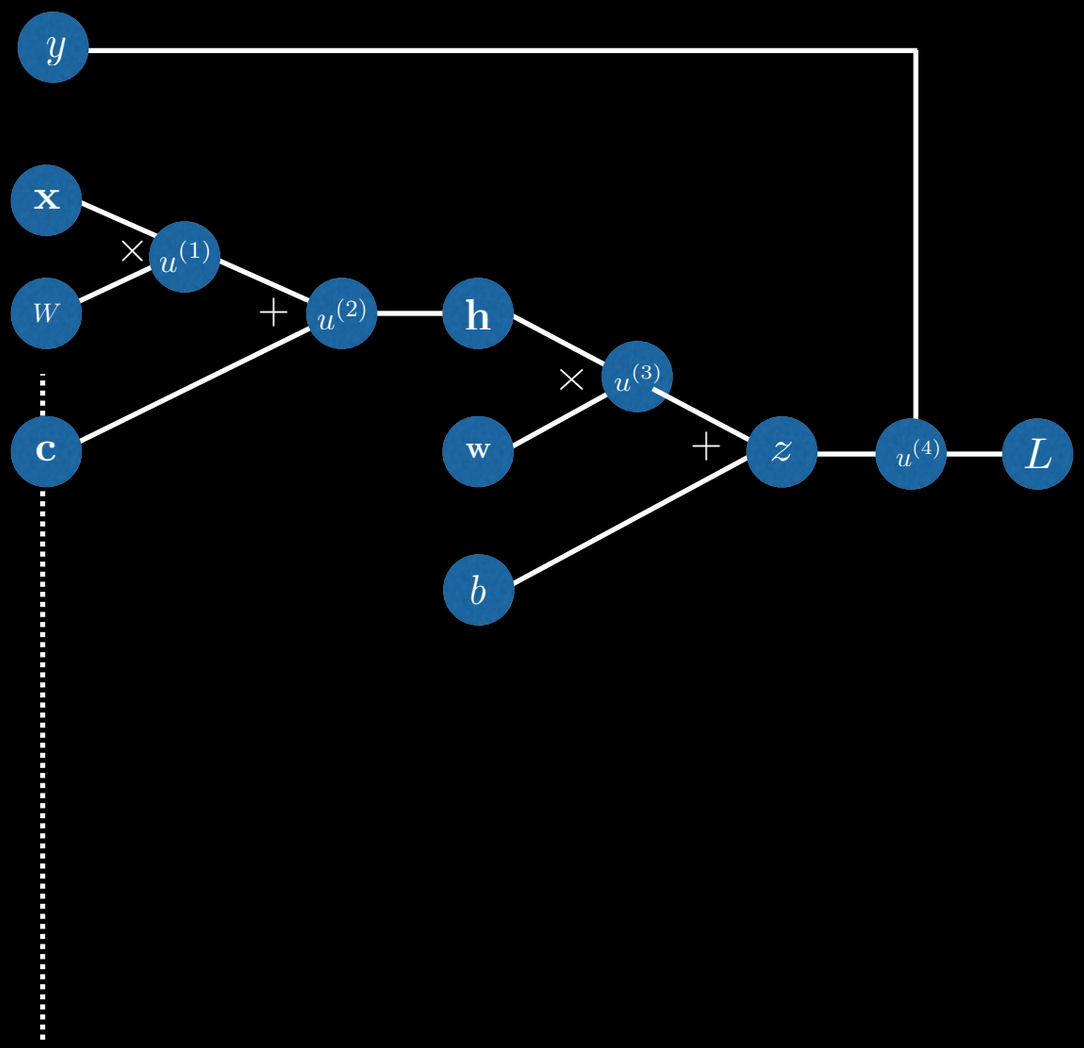
$$\nabla_{\mathbf{c}} L = \sigma(u^{(4)})(1-2y)\mathbf{f}(\mathbf{u^{(2)}}) \odot \mathbf{w}$$

Back-Propagation



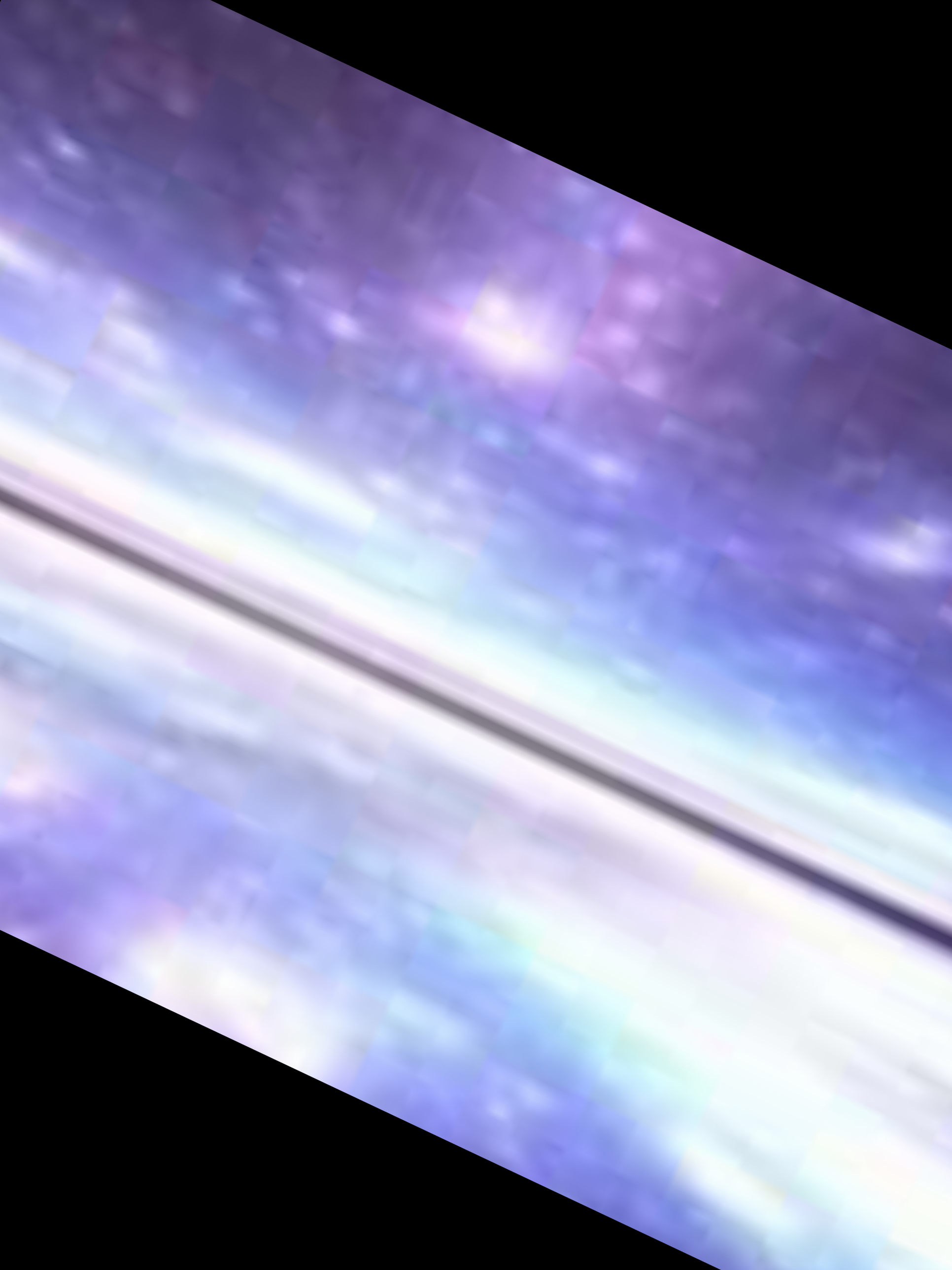
$$\nabla_{\mathbf{u^{(1)}}} L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \nabla_{\mathbf{u^{(2)}}} L = \sigma(u^{(4)})(1-2y)\mathbf{f}(\mathbf{u^{(2)}}) \odot \mathbf{w}$$

Back-Propagation



Here is mapping from $\mathbb{R}^{2\times 2} \rightarrow \mathbb{R}^2$

$$u^{(1)} = W^T \mathbf{x}$$



Tensors! …WTF?

- In physics tensors have specific roles and rules.
- In ML we usually think of tensors as a generalization of vectors and matrices.
- So we use tensors to organize data into some grid form. A **vector** has 1 index, a **matrix** has 2 indices, and **tensors** can be used when more indices are needed.

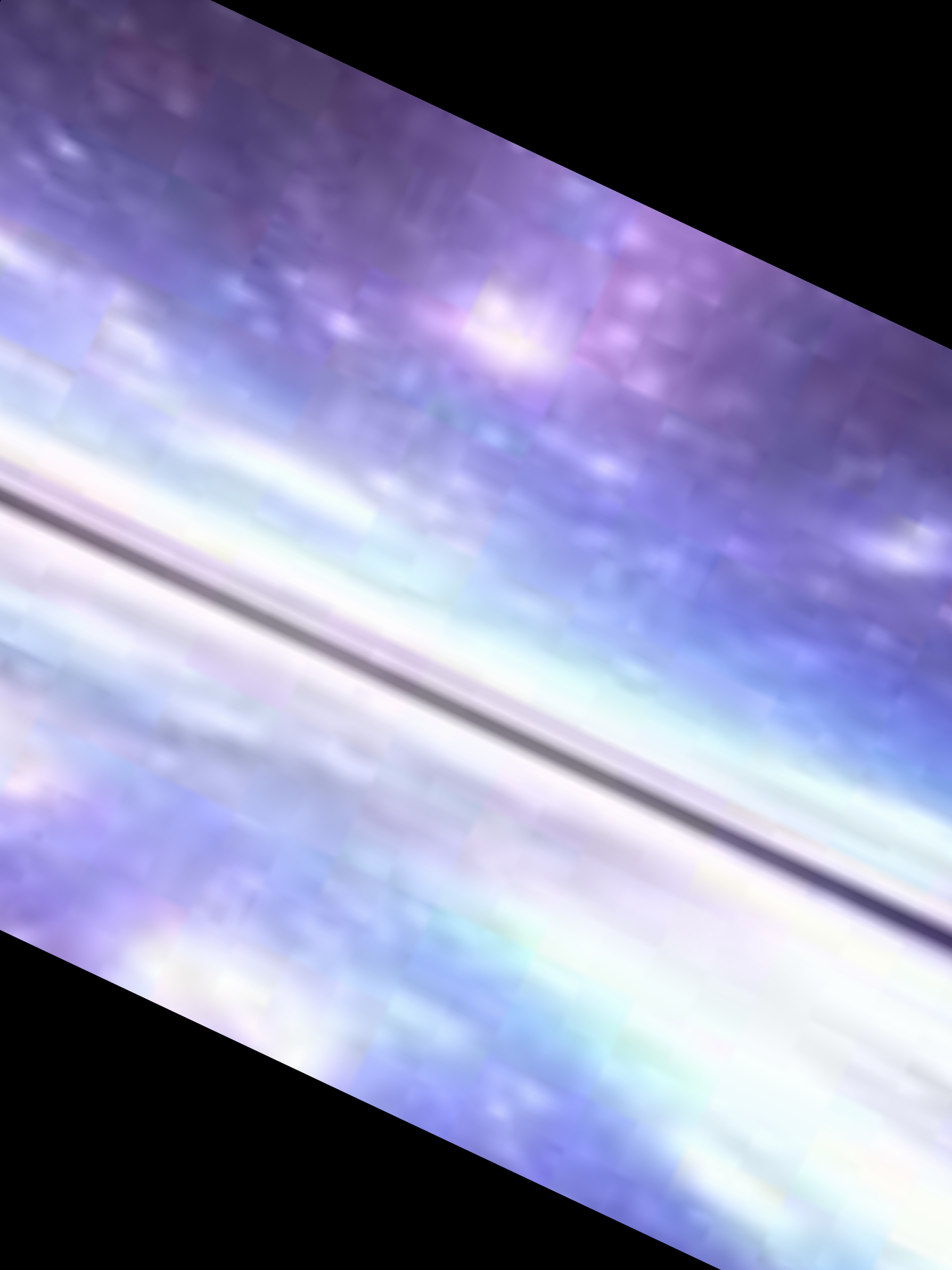
Tensors

- Let **X** be a tensor.
- Even though **X** may have any number of indices we can think of this tuple of indices as simply $\mathbf{i}$
- So an element of our tensor can be written $\mathbf{X_i}$
- And the gradient of z wrt to tensor **X** is $\nabla_{\mathbf{X}} z$

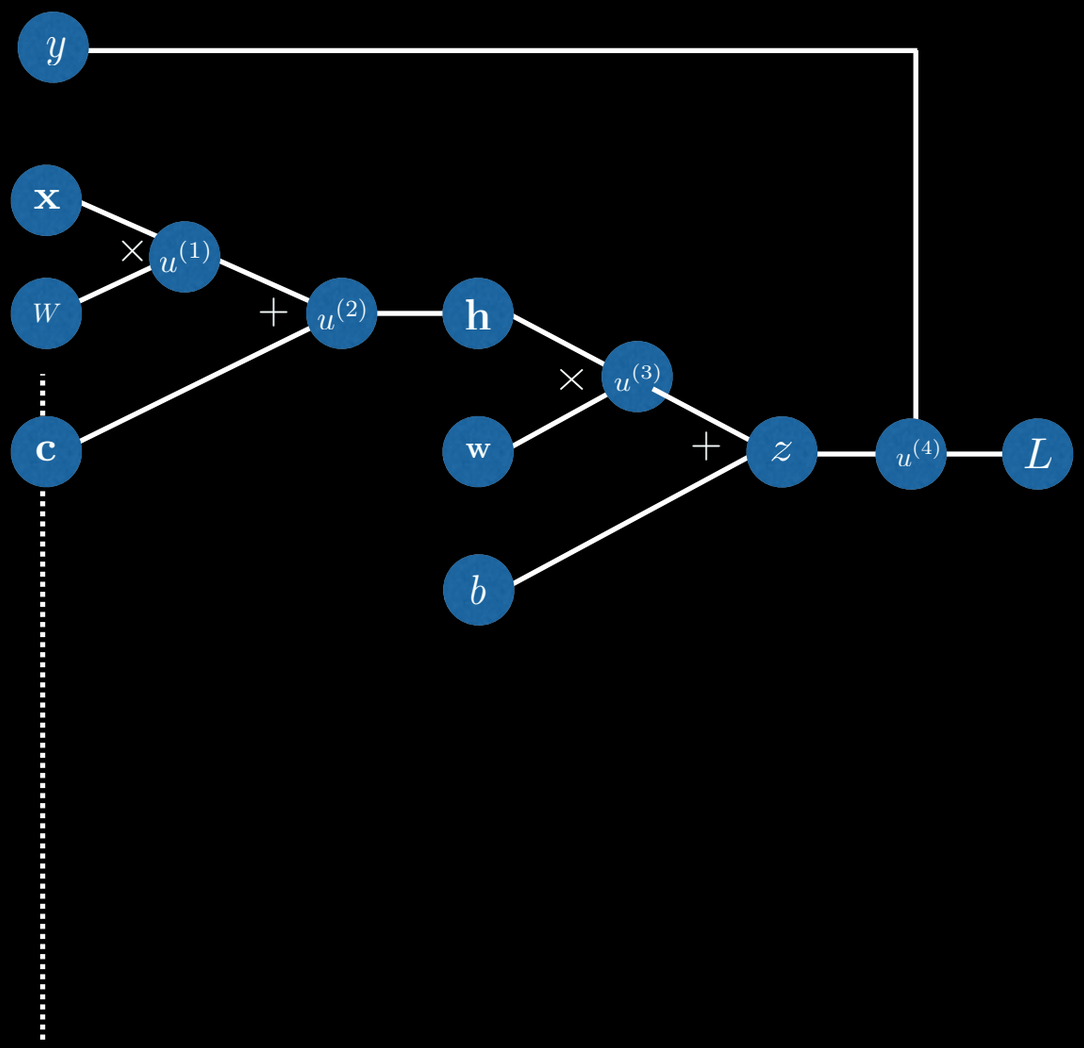
Tensors

- So if $\mathbf{Y} = g(\mathbf{X})$ and $z = f(\mathbf{Y})$ then

$$\nabla_{\mathbf{X}} z = \sum_j (\nabla_{\mathbf{X}} \mathbf{Y}_j) \frac{\partial z}{\partial \mathbf{Y}_j}$$



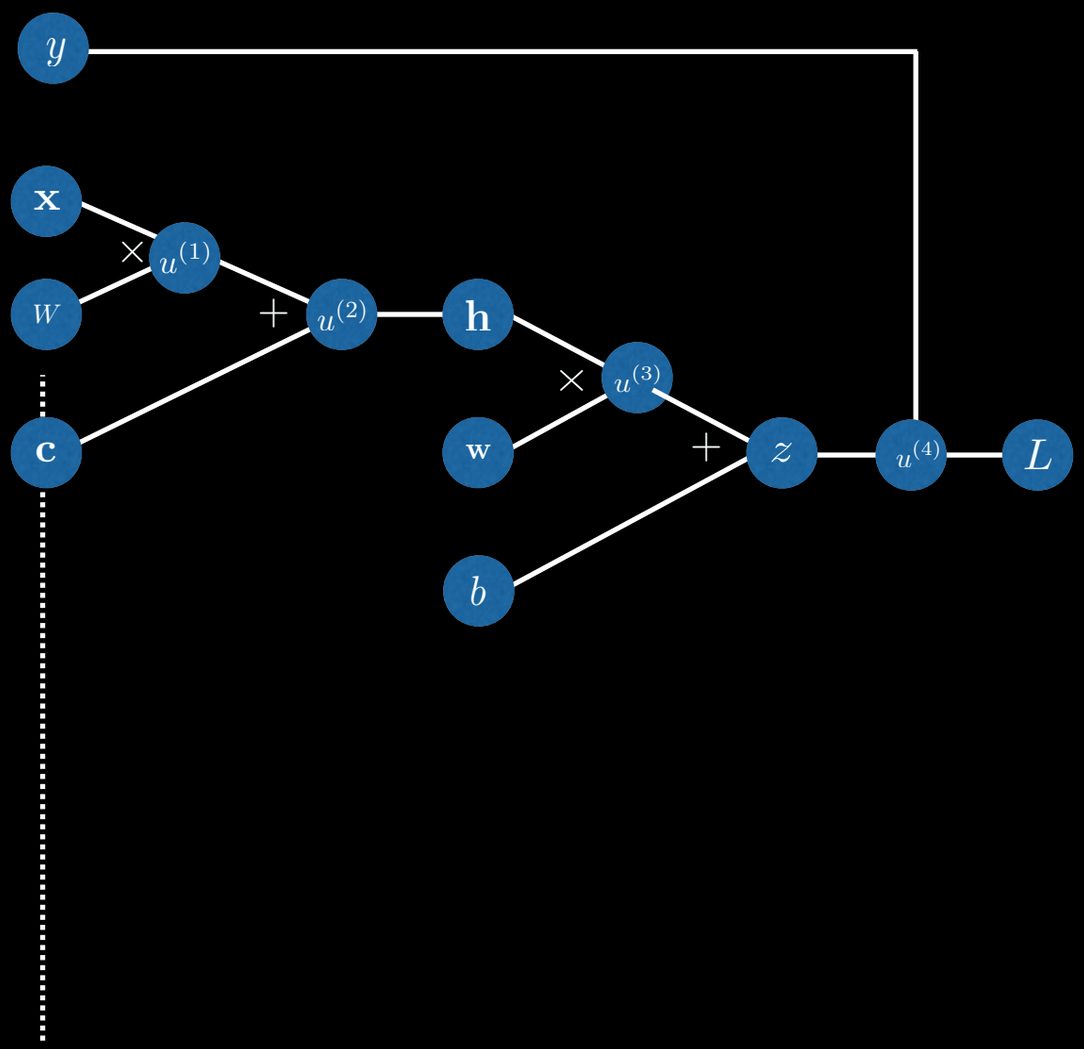
Back-Propagation



Here is mapping from $\mathbb{R}^{2\times 2} \rightarrow \mathbb{R}^2$

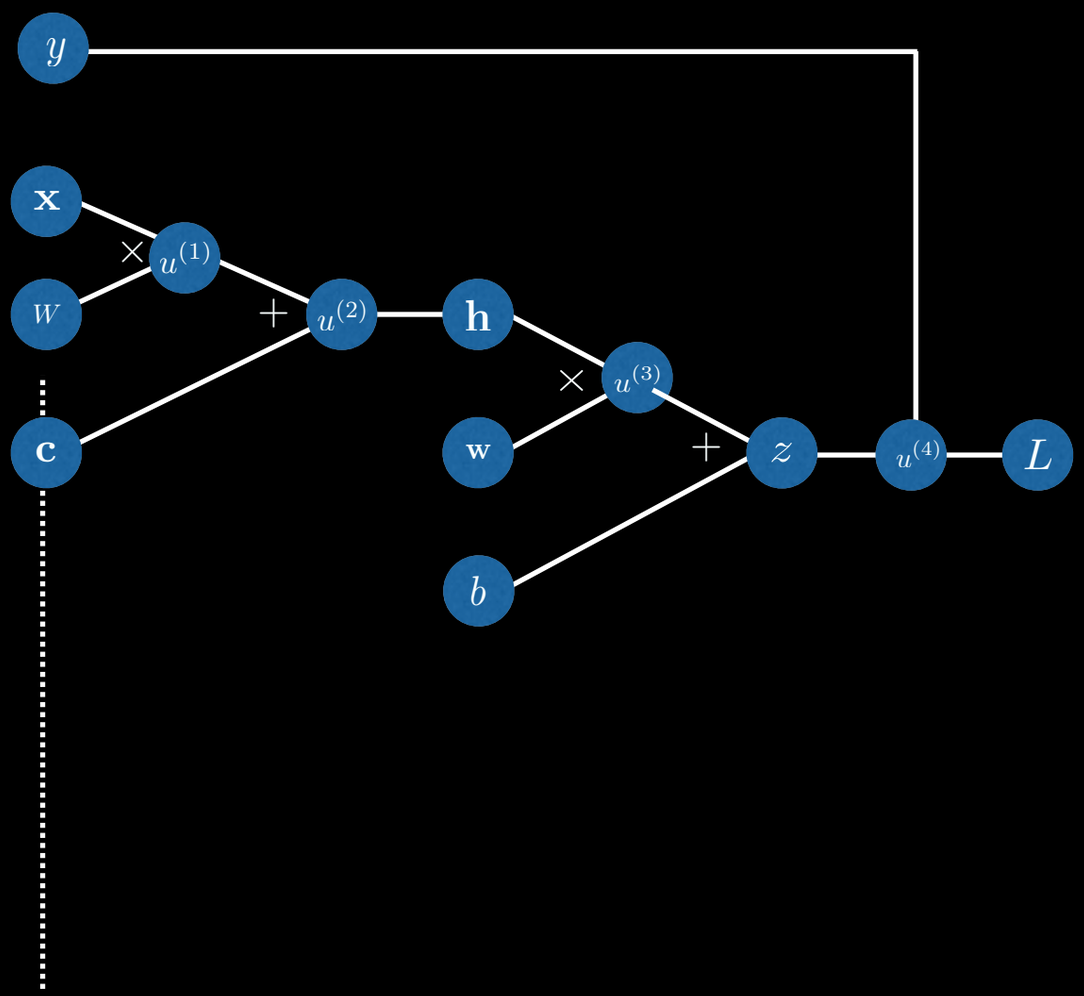
$$u^{(1)} = W^T \mathbf{x}$$

Back-Propagation



$$\nabla_W L = \sum_j \left(\nabla_W u_j^{(1)} \right) \left(\nabla_{\mathbf{u}^{(1)}} L \right)_j$$

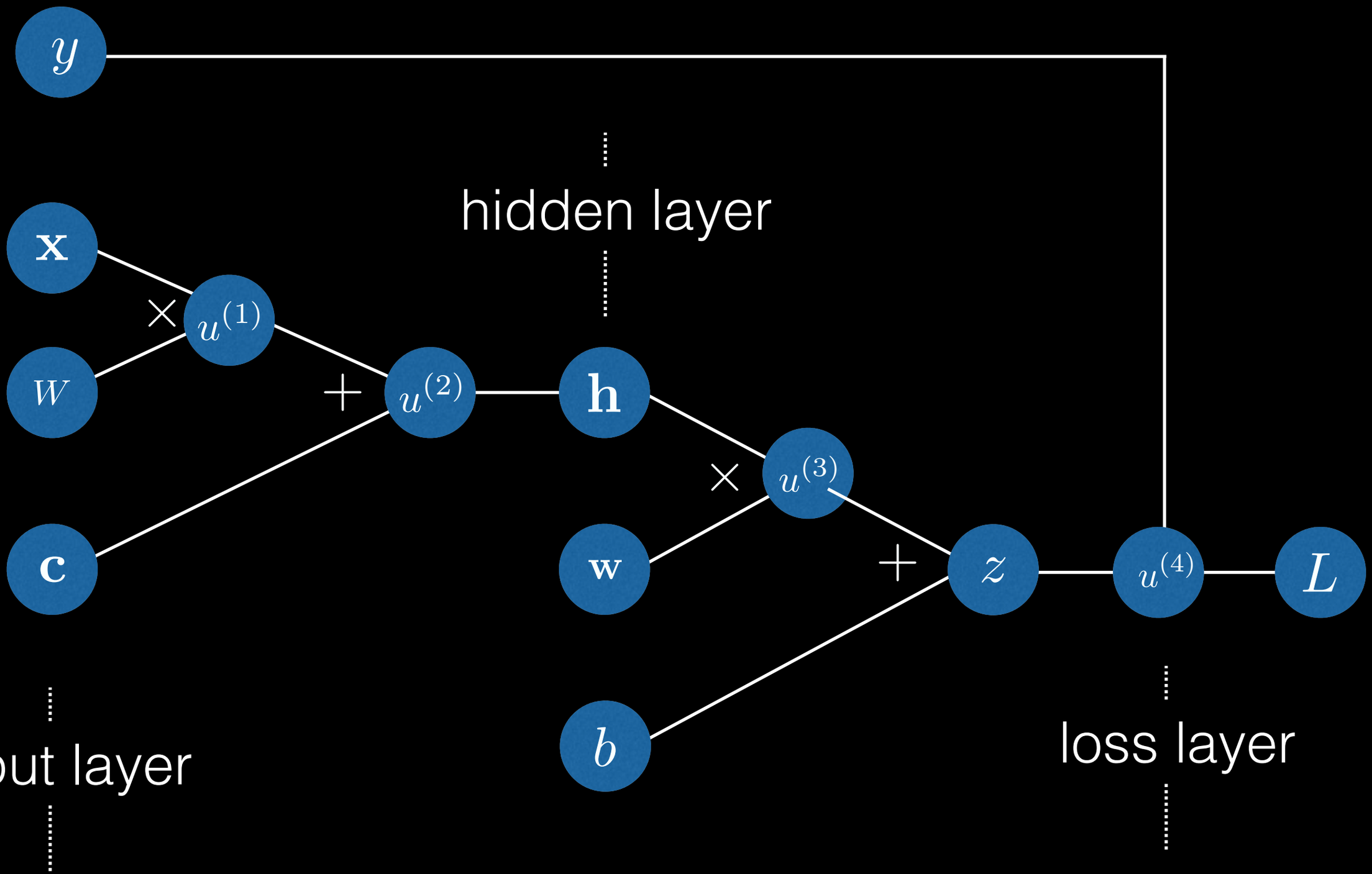
Back-Propagation



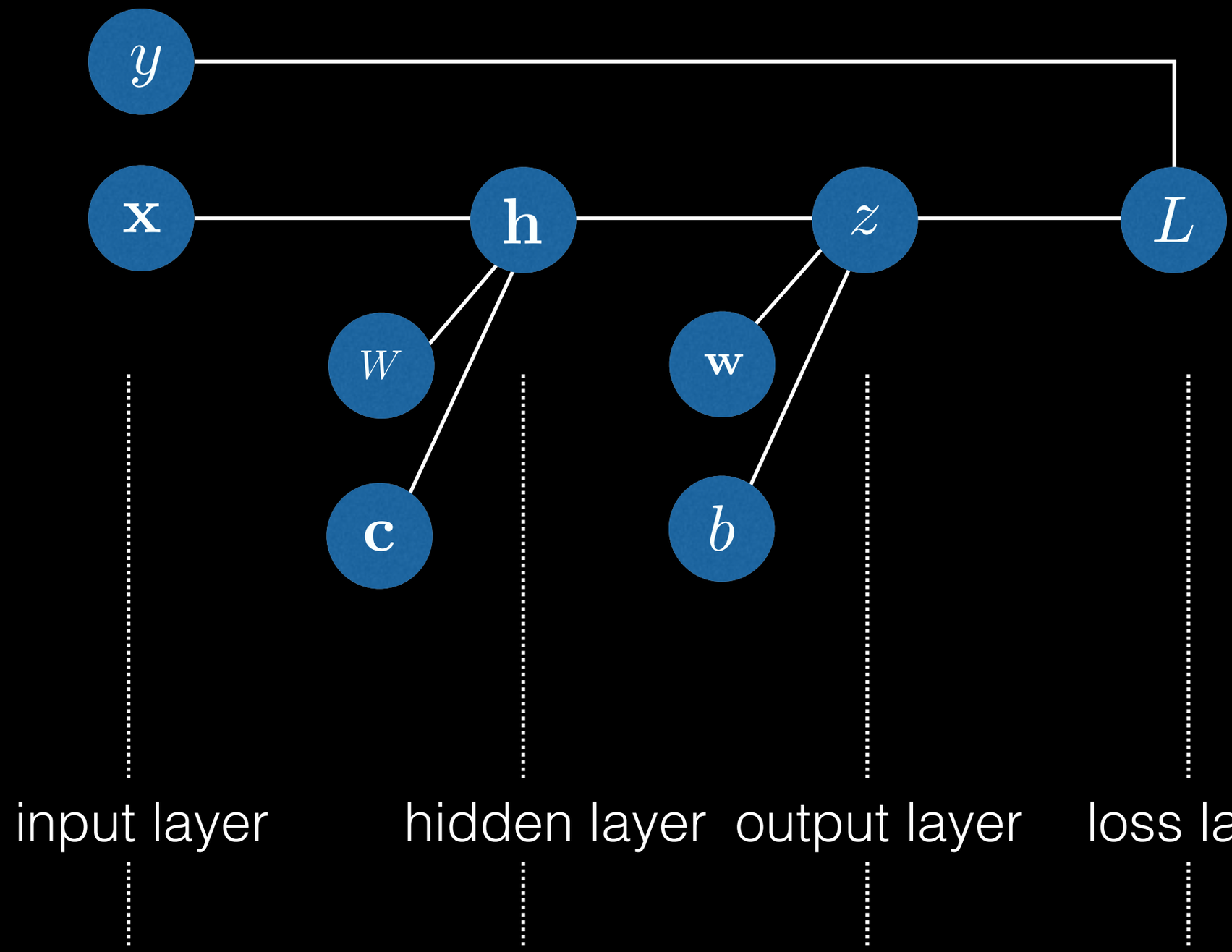
$$\nabla_W L = \sigma(u^{(4)})(1-2y)\mathbf{x}\,(\mathbf{f}(\mathbf{u^{(2)}}) \odot \mathbf{w})^T$$

So let’s collapse this diagram into functional layers.

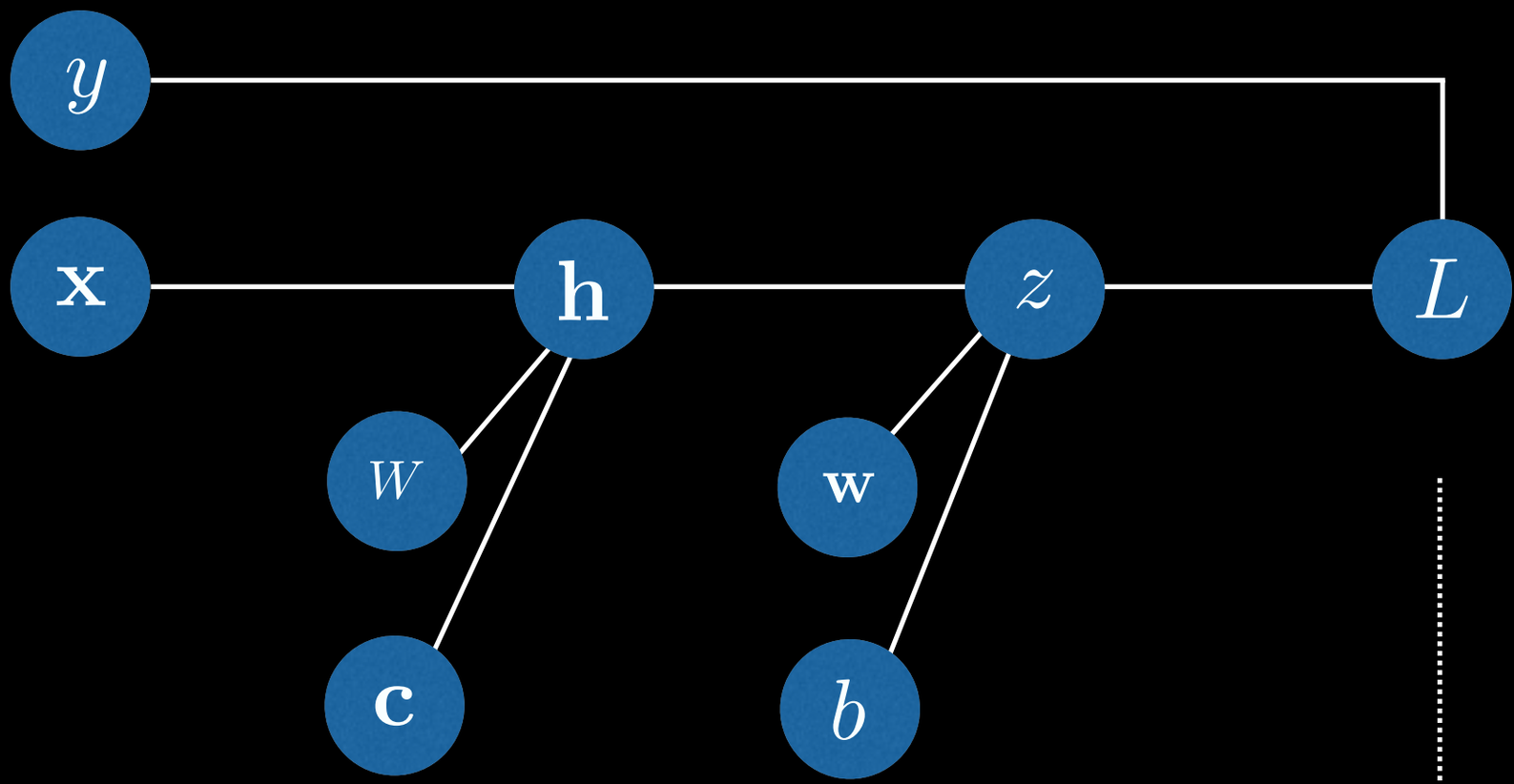
Simple MLP


y
hidden layer
$\mathbf{x}$
$\times$
$u^{(1)}$
W
$+$
$u^{(2)}$
$\mathbf{h}$
$\times$
$u^{(3)}$
$\mathbf{c}$
$\mathbf{w}$
$+$
z
$u^{(4)}$
L
b
input layer
loss layer

Simple MLP


y
$\mathbf{x}$
$\mathbf{h}$
z
L
W
$\mathbf{c}$
$\mathbf{w}$
b
input layer
hidden layer
output layer
loss layer

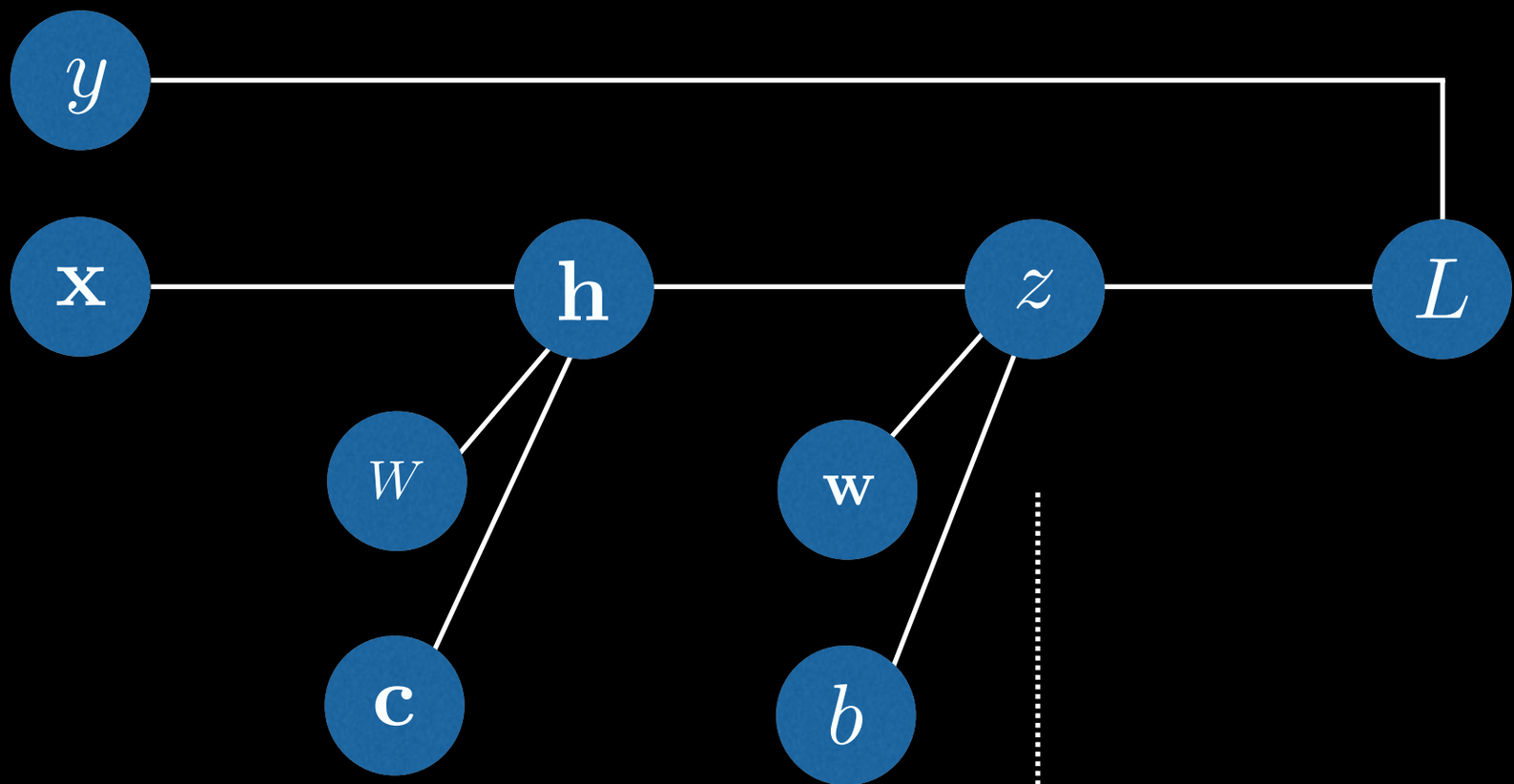
Simple MLP



$$\frac{\partial L}{\partial z} = \sigma((1-2y)z)(1-2y)$$

loss layer

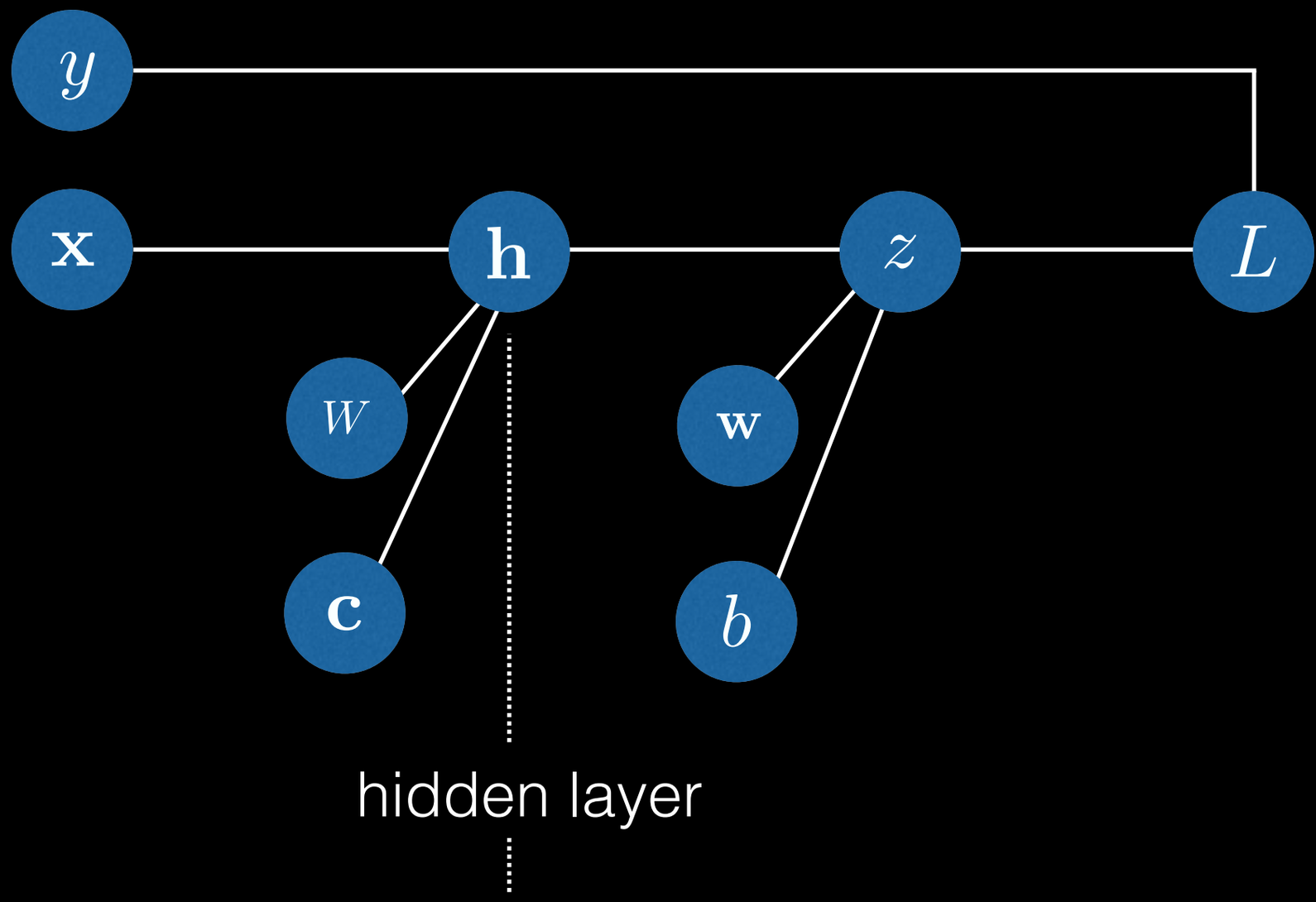
Simple MLP



$$\nabla_{\mathbf{h}} L = \frac{\partial L}{\partial z}\mathbf{w}$$

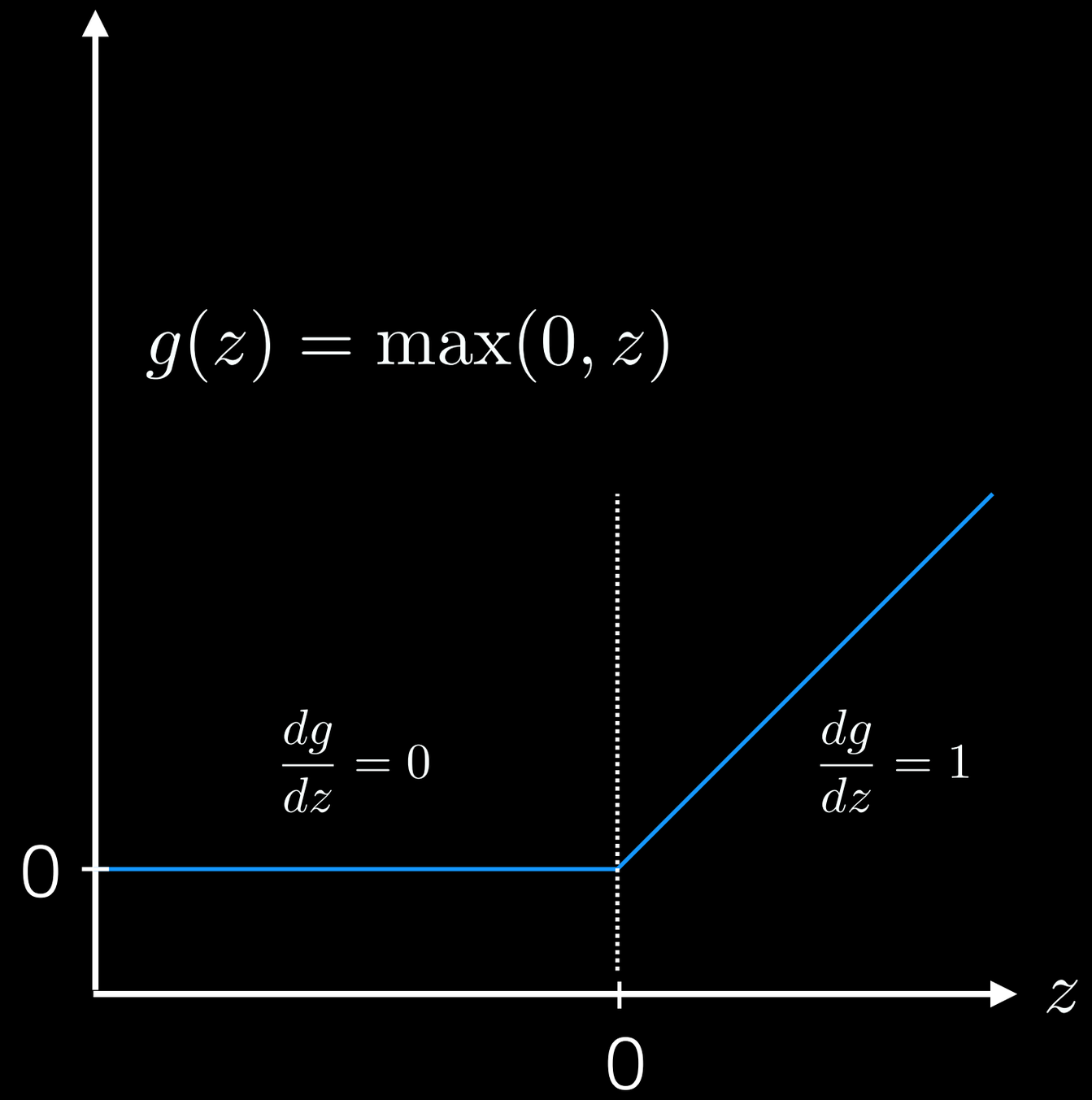
$$\boxed{\frac{\partial L}{\partial b} = \frac{\partial L}{\partial z} \qquad \nabla_{\mathbf{w}} L = \frac{\partial L}{\partial z}\mathbf{h}}$$

Simple MLP



$$\nabla_{\mathbf{c}} L = \mathbf{f} \odot \nabla_{\mathbf{h}} L \qquad \nabla_W L = \mathbf{x} \left(\mathbf{f} \odot \nabla_{\mathbf{h}} L \right)^T$$

Rectified Linear Unit ReLU



Let $f = \text{“}\frac{dg}{dz}\text{”}$