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1

$$2\dddot{y} + 6\ddot{y} - 4\dot{y} + 8y = 16u$$

$$\dddot{y} + 3\ddot{y} - 2\dot{y} + 4y = 8u$$

Taking Laplace: $G(s) = \frac{Y(s)}{U(s)}$

$$G(s) = \frac{8}{s^3 + 3s^2 - 2s + 4}$$

Let $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$

$$\Longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -4x_1 + 2x_2 - 3x_3 + 8u \end{cases}$$

Using the same model used in 5.4.2, question 2:

$\Longrightarrow$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$, $C = [1, 0, 0]$

$D = 0$

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(2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \rightsquigarrow 1$$

$$y = \begin{bmatrix} 1 & -2 \end{bmatrix} x$$

Since all initial conditions are zero and $D = 0$:

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) \, d\tau$$

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s & 5 \\ -1 & s+2 \end{bmatrix}^{-1} \right\}$$

$$e^{At} = \mathcal{L}^{-1}\left\{ \frac{1}{s(s+2)+5} \begin{bmatrix} s+2 & -5 \\ 1 & s \end{bmatrix} \right\}$$

$$e^{At} = \begin{bmatrix} e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) & -\frac{5}{2}e^{-t}\sin(2t) \\ \frac{1}{2}e^{-t}\sin(2t) & e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t) \end{bmatrix}$$

$$y(t) = \int_0^t [1, -2] \begin{bmatrix} e^{-t-\tau}\cos(2(t-\tau)) + \frac{1}{2}e^{-t-\tau}\sin(2t-2\tau) & \ldots\ldots \\ \frac{1}{2}e^{-t-\tau}\sin(2(t-\tau)) & \ldots\ldots \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} (1) \, d\tau$$

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$$y(t) = \int_0^t -\frac{5}{2} e^{-t+\tau} \sin(2(t-\tau))\, d\tau$$

$$\boxed{y(t) = \frac{1}{2} e^{-t} \sin(2t) + e^{-t} \cos(2t) - 1}$$

Let $u = e^{-t+\tau}$ $\quad dv = \sin(2(t-\tau))\, d\tau$

$du = e^{-t+\tau}$ $\quad v = \frac{1}{2}\cos(2(t-\tau))$

$$\Rightarrow \frac{1}{2}\cos(2(t-\tau)) e^{-t+\tau}\Big]_0^t - \int_0^t \frac{1}{2}\cos(2(t-\tau)) e^{-t+\tau}\, d\tau$$

$$\frac{1}{2} - \frac{1}{2}\cos(2t) e^{-t} - \int_0^t \frac{1}{2}\cos(2(t-\tau)) e^{-t+\tau}\, d\tau$$

Let $u = e^{-t+\tau}$ $\quad dv = \frac{1}{2}\cos(2(t-\tau))\, d\tau$

$du = e^{-t+\tau}$ $\quad V = -\frac{1}{4}\sin(2(t-\tau))$

$$\Rightarrow \frac{1}{2} - \frac{1}{2}\cos(2t) e^{-t} - \left[-\frac{1}{4}\sin(2(t-\tau)) e^{-t+\tau}\Big]_0^t + \frac{1}{4}\int_0^t \sin(2(t-\tau)) e^{-t+\tau}\, d\tau \right]$$

$$\Rightarrow \frac{1}{2} e^{-t} \sin(2t) + e^{-t} \cos(2t) - 1$$

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(4) $$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} x \qquad D = 0$$

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau \qquad (u(\tau) = \delta(\tau))$$

$$G(s) = C(sI - A)^{-1} B + D$$

$$G(s) = C \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} B + 0$$

$$G(s) = \begin{bmatrix} 2 & 1 \end{bmatrix} \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{G(s) = \frac{3}{s+1}}$$

$$G(s) = \frac{Y(s)}{U(s)} \Rightarrow Y(s) = G(s) U(s)$$

$\mathcal{L}\{u(t)\} = 1$

$$\boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s+1} \times 1\right\} = 3e^{-t}}$$

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(9)

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{} = \overbrace{\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} u$$

$$y = \underbrace{[2 \quad 1]}_{C} X \underbrace{- u(t)}_{D}$$

a) Assuming all initial conditions are zero,

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = C \begin{bmatrix} s-1 & 3 \\ -4 & s+5 \end{bmatrix}^{-1} B + D$$

$$G(s) = [2 \quad 1] \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+5 & -3 \\ 4 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1$$

$$\boxed{G(s) = \frac{3}{s+1} - 1 = \frac{-s+2}{s+1}}$$

b) $y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = \mathcal{L}^{-1}\{G(s)\}$ since $U(s) = \mathcal{L}\{\delta(t)\} = 1$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} + \mathcal{L}^{-1}\{-1\}$$

$$\boxed{y(t) = 3e^{-t} - \delta(t)}$$

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c)

$$u(t) = 2 \text{ , so } \mathcal{L}\{u(t)\} = \mathcal{L}\{2\} = U(s) = \frac{2}{s}$$

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{s(s+1)} - \frac{2}{s}\right\}$$

$$\frac{6}{s(s+1)} : \frac{A}{s} + \frac{B}{s+1}$$

$$A = G(s)(s)\Big|_{s=0} = 6$$

$$B = G(s)(s+1)\Big|_{s=-1} = -6$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{6}{s+1}\right\}$$

$$\boxed{y(t) = 4 - 6e^{-t}}$$

$$\boxed{\lim_{t \to \infty} y(t) = 4}$$

Also, $\boxed{\lim_{t \to \infty} y(t) = G(0)\,U(t) = 2 \times 2 = 4}$

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(2)

$$G(s) = \frac{2 + 6s^2 + 4s^3}{8 + 2s - 4s^2 + 2s^3}$$ => Choosing $D = 2$

=> $H(s) = G(s) - 2$

$$G(s) = \frac{-14 - 4s + 14s^2}{8 + 2s - 4s^2 + 2s^3} + 2$$

$$G(s) = \frac{-7 - 2s + 7s^2}{4 + s - 2s^2 + s^3} + 2$$

If $G(s) = \frac{c_0 + c_1 s + c_2 s^2}{a_0 + a_1 s + a_2 s^3 + a_3 s^3} + D$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [c_0 \ c_1 \ c_2] x + Du$$

So:

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= [-7 \ -2 \ 7] x + 2u \end{aligned}}$$

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1. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ -6 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = 2x_1 + 3x_2 + 2u$$

i) $$A = \begin{bmatrix} 7 & 12 \\ -6 & -10 \end{bmatrix}$$

$$|\lambda I - A| = 0 \implies \begin{vmatrix} \lambda - 7 & -12 \\ 6 & \lambda + 10 \end{vmatrix} = 0$$

$$\implies \lambda^2 + 3\lambda + 2 = 0 \implies \lambda = -1, -2$$

$$\boxed{\text{Since } Re(\lambda) < 0, \text{ the } 2\times 2 \text{ matrix is stable.}}$$

ii) $$G(s) = C(sI - A)^{-1} B + D$$

$$G(s) = C \begin{bmatrix} s-7 & -12 \\ 6 & s+10 \end{bmatrix}^{-1} B + D$$

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$$G(s) = [2 \quad 3] \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+10 & 12 \\ -6 & s-7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2$$

$$\boxed{G(s) = -\frac{1}{s+2} + 2 = \frac{2s+3}{s+2}}$$

With Mcmillan degree = degree denominator

$= \underline{1}$

iii) $u(t) = 2$

Note: For practice sake, this part was solved using a different method from the direct equation in the lecture

$\Rightarrow \dot{x}_1 = 7x_1 + 12x_2 + 2$

$\dot{x}_2 = -6x_1 - 10x_2 - 2$

Taking Laplace of $\dot{x}_1$ & $\dot{x}_2$:

$$sX_1(s) = 7X_1(s) + 12X_2(s) + \frac{2}{s}$$

$$sX_2(s) = -6X_1(s) - 10X_2(s) - \frac{2}{s}$$

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$$(s-7)X_1(s) - 12X_2(s) = \frac{2}{s} \quad \text{Eq. 1}$$

$$(s+10)X_2(s) + 6X_1(s) = -\frac{2}{s} \quad \text{Eq. 2}$$

From Eq. 1: $X_1(s) = \dfrac{12sX_2(s)+2}{s^2-7s}$

Substituting in Eq. 2 $\Rightarrow$ $X_2(s) = \dfrac{-2s+2}{s(s^2+3s+2)}$

Substituting in Eq. 1 $\Rightarrow$ $X_1(s) = \dfrac{2s-4}{s(s^2+3s+2)}$

$$X_1(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

A: $A = G(s)(s)\big|_{s=0} = -2$

B: $B = G(s)(s+1)\big|_{s=-1} = 6$

C: $C = G(s)(s+2)\big|_{s=-2} = -4$

$$X_1(t) = \mathcal{L}^{-1}\left\{-\frac{2}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-4}{s+2}\right\}$$

$$\boxed{X_1(t) = -4e^{-2t} + 6e^{-t} - 2}$$

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$$\boxed{\lim_{t\to\infty} x_1(t) = -2}$$

Using the same procedure,

$$X_2(s) = \frac{-2s+2}{s(s^2+3s+2)}$$

$$\Longrightarrow \boxed{x_2(t) = 3e^{-2t} - 4e^{-t} + 1}$$

$$\boxed{\lim_{t\to\infty} x_2(t) = 1}$$

Now $y = 2x_1 + 3x_2 + 4$ $\lim_{t\to\infty} y(t)$:

$y = 2(-2) + 3(1) + 4$

$\boxed{y(\infty) = 3}$

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(2) $\dot{x} = Ax + Bu$

$y = Cx$

$A = \begin{bmatrix} -4 & -2 \\ k & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $C = [2 \quad 2]$

i)

$|sI - A| = 0 \Rightarrow \begin{vmatrix} s+4 & 2 \\ -k & s-1 \end{vmatrix} = 0$

$\Rightarrow (s+4)(s-1) + 2k = 0$

$\Rightarrow s^2 + 3s + (2k-4) = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{25 - 8k}}{2}$

$x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{25-8k}$

So $\sqrt{25-8k} = 3 \Rightarrow k = 2$

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So k needs to be greater than 2

$k > 2$ to be stable

ii)

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = C\begin{bmatrix} s+4 & 2 \\ -k & s-1 \end{bmatrix}^{-1} B + D$$

$$G(s) = \begin{bmatrix} 2 & 2 \end{bmatrix} \frac{1}{s^2+3s-4+2k} \begin{bmatrix} s-1 & -2 \\ k & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0$$

$$G(s) = \frac{4s+2+2k}{(s-1)(s+4)+2k}$$

$$\lim_{t\to\infty} y(t) = G(0)u(t) = \frac{k+1}{k-2} \cdot 2 = 8$$

$\Rightarrow 2k+2 = 8k-16$

$\Rightarrow \boxed{k = 3}$