$$l(x, y \mid \mu, \sigma^2, \text{Tree}) \propto \prod_{i=1}^{I} \prod_{j=1}^{n_i} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(y_{ij}-\mu_i)^2}{2\sigma^2}\right)$$

$$\propto (\sigma^2)^{-\frac{N}{2}} \exp\left(-\frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2}{2\sigma^2}\right)$$

$$p(\mu_i \mid \theta_i, a) \propto (\sigma^2)^{-\frac{I}{2}} \exp\left(-\frac{\sum_{i=1}^{I} a(\mu_i-\theta_i)^2}{2\sigma^2}\right)$$

$$p(\sigma^2 \mid \alpha, \beta) \propto (\sigma^2)^{-\alpha-1} \exp\left(-\frac{\beta}{\sigma^2}\right)$$

$$p(\mu_i \mid \text{others}) \propto \exp\left(-\frac{\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2 + a(\mu_i-\theta_i)^2}{2\sigma^2}\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\left[(n_i+a)\mu_i^2 - 2\left(\sum_{j=1}^{n_i} y_{ij} + a\theta_i\right)\right]\right)$$

$$\sim N\left(\frac{\sum_{j=1}^{n_i} y_{ij} + a\theta_i}{n_i+a}, \quad \sigma^2 / n_i + a\right)$$

$$p(\sigma^2 \mid \alpha, \beta) \propto (\sigma^2)^{-\frac{N}{2}-\frac{I}{2}-\alpha-1} \exp\left(-\frac{1}{\sigma^2}\left(\beta + \sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2 + \sum_{i=1}^{I} a(\mu_i-\theta_i)^2\right)\right)$$

$$\sim IG\left(\frac{N}{2}+\frac{I}{2}+\alpha, \ \beta + \sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2 + \sum_{i=1}^{I} a(\mu_i-\theta_i)^2\right)$$

$\theta_i = 0$ if scaled.

Joint posterior: $f(\text{all} \mid x, y) \cdot p(\text{tree})$

$$\propto (\sigma^2)^{-\frac{N}{2}} \exp\left(-\frac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}(y_{ij}-\mu_i)^2}{2\sigma^2}\right)$$

$$(\sigma^2)^{-\frac{2}{2}} \exp\left(-\frac{\sum_{i=1}^{2} a\mu_i^2}{2\sigma^2}\right)(\sigma^2)^{-\alpha-1}\exp\left(-\frac{\beta}{\sigma^2}\right)$$

$$= (\sigma^2)^{-(\frac{N}{2}+\frac{2}{2}+\alpha)-1} \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^2\left((n_i+a)\mu_i^2 - 2\mu_i\left(\sum_{j=1}^{n_i} y_{ij}\right) + \sum_{j=1}^{n_i} y_{ij}^2 \right)\right)\right)\exp\left(-\frac{\beta}{\sigma^2}\right)$$

$$=(\sigma^2)^{-(\frac{N}{2}+\frac{2}{2}+\alpha)-1}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^2 \left(\sum_{j=1}^{n_i}(y_{ij})^2\right)\exp\left(-\frac{\beta}{\sigma^2}\right)\right.$$

$$\exp\left(-\frac{1}{2\sigma^2}(n_i+a)\left[\left(\mu_i - \frac{\sum_{j=1}^{n_i} y_{ij}}{n_i+a}\right)^2 - \left(\frac{\sum_{j=1}^{n_i} y_{ij}+a\theta_i}{n_i+a}\right)^2\right)\right)$$

$$=(\sigma^2)^{-(\frac{N}{2}+\frac{2}{2}+\alpha)-1}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^2\left(\sum_{j=1}^{n_i}y_{ij}^2 - \frac{(\sum_{j=1}^{n_i}y_{ij})^2}{n_i+a}\right)\right)$$

$$\prod_{i=1}^{n}\exp\left(-\frac{n_i+a}{2\sigma^2}\left(\mu_i-\frac{\sum_{j=1}^{n_i}y_{ij}}{n_i+a}\right)^2\right)$$

$$f(\text{tree}\mid x,y) = \int_{\sigma^2}\int_\mu f(\text{tree}\mid x,y)\,d\mu\,d\sigma^2$$

$$\propto \int (\sigma^2)^{-(\frac{N}{2}+\frac{2}{2}+\alpha)-1}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^2\left(\sum_{j=1}^{n_i}y_{ij}^2 - \frac{(\sum_{j=1}^{n_i}y_{ij})^2}{n_i+a}\right)\right)$$

$$\cdot\prod_{i=1}^2\left[\sqrt{2\pi\frac{\sigma^2}{n_i+a}}\right]\exp\left(-\frac{\beta}{\sigma^2}\right)d\sigma^2$$

$$\propto\int(\sigma^2)^{-(\frac{N}{2}+\alpha)-1}\cdot\prod_{i=1}^n(n_i+a)^{-\frac{1}{2}}\,d\sigma^2 \quad \cdot\exp\left(-\frac{\beta}{\sigma^2}\right)$$

$$\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^2\left(\sum_{j=1}^{n_i}y_{ij}^2 - \frac{(\sum_{j=1}^{n_i}y_{ij}+a\theta_i)^2}{n_i+a}\right)\right)d\sigma^2$$

-

$$= \frac{\Gamma\left(\frac{N}{2}+\alpha\right)}{\left[\beta + \frac{\sum_{i=1}^{J}\left(\sum_{j=1}^{n_i} y_{ij}^2 - \left(\sum_{j=1}^{n_i} y_{ij}\right)^2 / (n_i+a)\right)}{2}\right]^{\frac{N}{2}+\alpha}} \prod_{i=1}^{J}(n_i+a)^{-\frac{1}{2}}$$

$$\propto \left[\beta + \frac{\sum_{i=1}^{J}\left(\sum_{j=1}^{n_i} y_{ij}^2 - \left(\sum_{j=1}^{n_i} y_{ij}\right)^2 / (n_i+a)\right)}{2}\right]^{-\left(\frac{N}{2}+\alpha\right)}$$

$$\prod_{i=1}^{J}(n_i+a)^{-\frac{1}{2}}$$

$$\alpha = \frac{\nu}{2} \qquad \beta = \nu\lambda/2$$

$$\Rightarrow \lambda = \beta/\alpha \qquad \nu = 2\alpha$$