Math Science 8000, Fall 2015
In Class Test I

NAME:

Directions: Show all work on the test to receive possible partial credit. Unsupported guesses will meet a red pen. You are allowed to use a calculator and an 8×11.5 formula sheet of your own construction.

1. Short Answer (20 points total, 2 points for each part)

a) What is $\Gamma(1/2)$?

$\sqrt{\pi}$

b) What is the mean of a binomial variable with 1000 trials and success probability 1/2?

5000

c) If a random variable has $E[X^2] = \infty$, what is $E[X^4]$?

∞ , $E(x^4) = [\underbrace{E(x^2)}_{\infty}]^2 + \underbrace{Var(x^2)}_{\geq 0}$

d) True or False: If X and Y are continuous random variables with $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for all real x, y, then X and Y are necessarily independent.

T

e) Which of the following distributions would be inappropriate probability model(s) for the number of bear attack deaths in Oconee county this year: Poisson, Geometric, Binomial?

✓ ✗ ✓

f) Which of the following distributions would be inappropriate probability model(s) for the proportion of people on welfare in the United States: Normal, Beta, Exponential.

g) If the joint cumulative distribution function of $X_1 \geq 0$ and $X_2 \geq 0$ is

$$F_{X_1,X_2}(x_1, x_2) = (1 - e^{-x_1})(1 - e^{-x_2})$$

for $x_1, x_2 > 0$ what is $F_{X_2}(x_2)$?

$$\lim_{x_1 \to \infty} F_{x_1,x_2}(x_1, x_2) = 1 - e^{-x_2}$$

odd function

h) IF Z is standard normal, what is $E[Z^7]$?

$$E(Z^7) = 0 \quad \text{since} \int_{-\infty}^{\infty} x^7 \cdot \frac{\exp(\frac{x^2}{2})}{2\pi} dx = 0$$

i) What is the variance of a Beta random variable with parameters $\alpha = 3$ and $\beta = 2$?

$$\frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

$$E(X) = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+1)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}$$

$$E(X^2) = \frac{\Gamma(\alpha+2)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$E(X^2) - [E(X)]^2 =$$

j) True or False: Exponential random variables are also Gamma random variables.

$$\lambda e^{-\lambda x} \qquad \Gamma(1, \lambda)$$

$$\frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\cancel{\alpha^3}+\alpha^2\beta+\cancel{\alpha^2}+\alpha\beta-\cancel{\alpha^3}-\cancel{\alpha^2\beta}-\cancel{\alpha^2}}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

2. (10 points) Sampson, an annoying neighborhood dog, is receiving death threats by telephone messages to his owners. The number of death threat messages that Sampson receives in any week has a Poisson distribution with mean 5. The FBI is called in to monitor the situation for a two week period. Let X_1 and X_2 denote the number of death threats in weeks 1 and 2, respectively, and suppose that X_1 and X_2 are independent.

$$\frac{\lambda^k}{k!}\cdot e^{-\lambda} \quad k=0.$$

a) What is mean number of death threats that Sampson receives in a week?

5.

b) What is $P(X_1 + X_2 = 0)$?

By indep.

$$P(X_1+X_2=0) = P(X_1=0 \cap X_2=0) = P(X_1=0)P(X_2=0)$$

$$= e^{-5}e^{-5} = e^{-10}$$

c) **Derive** the distribution of $X_1 + X_2$?

$$P(X_1+X_2=y) = \sum_{x_2=0}^{y} P(X_1=y-x_2 \mid X_2=x_2)P(X_2=x_2)$$

$$= \sum_{x_2=0}^{y} \frac{\lambda_1^{x_2}}{x_2!}e^{-\lambda} \cdot \frac{\lambda_2^{y-x_2}}{(y-x_2)!}e^{-\lambda}$$

$$= \sum_{x_2=0}^{y} \frac{\lambda_1^{x_2}\cdot\lambda_2^{y-x_2}}{(x_2)!(y-x_2)!}e^{-2\lambda}$$

$$= \frac{e^{-2\lambda}}{(\lambda_1+\lambda_2)^y}\sum_{x_2=0}^{y}\binom{y}{x_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x_2}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{y-x_2}$$

$$= (\lambda_1+\lambda_2)^{-y}\cdot e^{-(\lambda_1+\lambda_2)} = 10^{-y}e^{-10} \quad y=(0,1,2\cdots\cdots)$$

3. (10 points) Suppose that X has an exponential distribution with a mean of unity and set $Z = X - \lfloor X \rfloor$, where $\lfloor y \rfloor$ is the greatest integer less than or equal to y. What is the distribution of Z?

$z \in [0,1)$

$$P(Z=z) = \sum_{i=0}^{\infty} P(X = i+z)$$

$$= \sum_{i=0}^{\infty} \lambda e^{-\lambda(i+z)}$$

$$= \sum_{i=0}^{\infty} \lambda e^{-\lambda i} \cdot e^{-\lambda z}$$

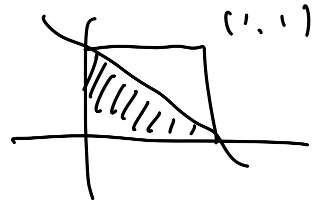
$$= \lambda e^{-\lambda z} \cdot \sum_{i=1}^{\infty} e^{-\lambda i}$$

$$= \lambda e^{-\lambda z} \cdot \left(1 + \frac{1}{e^{\lambda}} + \frac{1}{e^{2\lambda}} + \cdots +\right)$$

$$= \lambda e^{-\lambda z} \cdot \frac{1}{1 - \frac{1}{e^{\lambda}}}$$

$$= \frac{\lambda e^{-\lambda z} e^{\lambda}}{e^{\lambda}-1} = \lambda e^{-\lambda} \left(\frac{e^{\lambda}}{e^{\lambda}-1}\right)$$

truncated Exp(1)



4. (10 points) Suppose that the random pair (X, Y) has the joint probability density

$$f_{X,Y}(x, y) = cxy, \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 1$$

for come constant c.

a) What is c?

$$\int_0^1 \int_0^{1-x} cxy \, dy \, dx = \int_0^1 cx \left. \frac{y^2}{2} \right|_0^{1-x} dx$$

$$= \int_0^1 cx \cdot \frac{(1-x)^2}{2} dx = \int_0^1 \frac{c}{2} \cdot (x^3 - 2x^2 + x) \, dx$$

$$= \frac{c}{2} \cdot \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) = \frac{c}{2} \cdot \frac{3-8+6}{12} = 1$$

$$c = 24$$

b) What is the conditional density of X given that $Y = y$?

$$f_Y(y) = \int_0^{1-y} 24xy \, dx$$

$$= 12x^2 y \Big|_0^{1-y}$$

$$= 12(1-y)^2 y$$

$$f_{X|Y}(x|y) = \frac{24xy}{12(1-y)^2 y} = \frac{2x}{(1-y)^2}$$

$$x \in (0, 1-y)$$

c) What is $E[X|Y = y]$?

$$E(X|Y) = \int_0^{1-y} x \cdot \frac{2x}{(1-y)^2} dx$$

$$= \int_0^{1-y} \frac{2x^2}{(1-y)^2} dx = \frac{1}{(1-y)^2} \left. \frac{2}{3} x^3 \right|_0^{1-y} = \frac{(1-y)^3}{(1-y)^2} \cdot \frac{2}{3}$$

$$= \frac{2}{3}(1-y)$$

c) What is $\mathrm{Var}(X|Y = y)$?

$$E(X^2|Y) = \int_0^{1-Y} x^2 \cdot \frac{2x}{(1-y)^2} dx$$

$$= \frac{1}{(1-y)^2} \left. \frac{1}{2} x^4 \right|_0^{1-y} = \frac{1}{2}(1-y)^2$$

$$\mathrm{Var} = \frac{1}{2}(1-y)^2 - \frac{2}{3}(1-y) = \frac{y^2}{2} - \frac{5}{3}y - \frac{1}{6}$$

5. (10 points) Show that a geometric random variable X is memoryless in that

$$P(X > i + j | X > i) = P(X > j)$$

for all $i, j \in \{1, 2, \ldots\}$.

$$P(X = x) = (1-p)^{x-1} p$$

$$P(X > i+j \mid X > i) = \frac{P(X > i+j \cap X > i)}{P(X > i)}$$

$$P(X > k) = (1-p)^k p + (1-p)^{k+1} p + \cdots$$

$$= \frac{(1-p)^k p}{1-(1-p)} = (1-p)^k$$

$$P(X > i+j \cap X > i) = P(X > i+j).$$

$$\Rightarrow P(X > i+j \mid X > i) = \frac{(1-p)^{i+j}}{(1-p)^i} = (1-p)^j$$

$$(1-p)^j = P(X > j)$$

Q.E.D.

6. (10 points) Suppose that the random pair (X, Y) is uniformly distributed over the unit dart board region governed by $x^2 + y^2 \leq 1$. What is the marginal density of X?

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \, dy$$

$$= \frac{2}{\pi} \sqrt{1-x^2} \qquad x \in \left(-\sqrt{1-y^2}, \sqrt{1-y^2}\right)$$

7. (15 points) Suppose that X_1 and X_2 are two independent random variables, each having the joint density

$$f(x) = \frac{e^{-x/2}}{2}, \quad x > 0.$$

If $Y_1 = X_1$ and $Y_2 = X_1 + X_2$, what is the joint density function of Y_1 and Y_2?

By indep:

$$f(x_1, x_2) = \frac{1}{4} e^{-\frac{x_1+x_2}{2}}$$

$Y_1 = X_1 \quad Y_2 = X_1 + X_2 \qquad Y_1 \in (0,\infty) \; Y_2 \in (0,\infty)$

$$\begin{cases} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \end{cases} \qquad |J| = \begin{vmatrix} \frac{\partial x_1}{\partial Y_1} & \frac{\partial x_1}{\partial Y_2} \\ \frac{\partial x_2}{\partial Y_1} & \frac{\partial x_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(Y_1, Y_2 - Y_1)|J|$$

$$= \frac{1}{4} e^{-y_2} \qquad Y_1 \in (0,\infty),\; Y_2 \in (0,\infty)$$

8. (15 points) Let X and Y be jointly distributed variables such that $Y|X = x$ has a Poisson distribution with mean x. Suppose also that the density function of X is

$$f_X(x) = 3x^2, \quad 0 \le x \le 1.$$

Compute $E[Y]$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X, Y)$.

$$Y|X \sim \frac{x^y}{y!} e^{-x}$$

$$x \sim 3x^2$$

$$E(Y) = EE(Y|x) = E(x) = \int_0^1 x \cdot 3x^2 \, dx$$

$$= \frac{3}{4}$$

$$\mathrm{Var}(Y) = E(\mathrm{Var}(Y|x)) + \mathrm{Var}(E(Y|x)).$$

$$= E(x) + \mathrm{Var}(x)$$

$$E(x^2) = \int_0^1 x^2 \cdot 3x^2 \, dx = \frac{3}{5}$$

$$\mathrm{Var}(x) = \frac{3}{5} - \frac{9}{16} = \frac{48 - 45}{80} = \frac{3}{80}$$

is 0 since x is given

$$\mathrm{Cov}(X, Y) = E\,\mathrm{Cov}(x, Y | x) + \mathrm{Cov}(E(x|x), E(Y|X)).$$

$$= \mathrm{Cov}(x, x)$$

$$= \mathrm{Var}(x)$$

$$= 3/80,$$

Bonus (10 points) A miner is trapper in a mine containing three doors. The first door leads to a tunnel that takes her to safety after two hours of travel. The second door leads to a tunnel that returns her to the mine after three hours. The third door leads to a tunnel that return her to the mine after five hours. Assuming that at all door decision times, the miner is equally likely to take any of the three doors, what is the expected length of time the it takes until the miner reaches safety?

Assume the Sum is S.

$$S = \frac{2}{3} + \frac{5+S}{3} + \frac{3+S}{3}$$

$$3S = 10 + 2S$$

$$S = 10.$$

Karakoram

Bonus. (Trivia, 1 point) What is the second tallest mountain on the planet?