Math Science 8000, Fall 2018
In Class Test II

NAME:

**Directions:** Show all work on the test to receive possible partial credit. Unsupported guesses will meet a red pen. You are allowed to use a calculator and an $8 \times 11.5$ formula sheet of your own construction.

1. Short Answer (20 points total, 2 points for each part)

a) True or False: The characteristic function always exists (is finite).

b) True or False: The moment generating function always exists (is finite).

c) If a random variable $X \geq 0$ is known to take values in [0,100], what is the best type of transform to use?

d) True or False: An eigenvalue of a covariance matrix can be zero.

e) True or False: The sample variance $S^2$ is an unbiased estimate of the variance $\sigma^2$.

f) What is the mean of a $\chi$-squared random variable with 8 degrees of freedom?

g) Which is theoretically better: Conditional expectation prediction or linear prediction?

h) If $\varphi$ is a characteristic function, what is $\varphi''''(0)$ (assuming this quantity exists)?

i) True or False: The inverse of a covariance matrix always exists.

j) True or False: The sample mean and variance are independent when the data are independent and identically distributed and drawn from a normal distribution.

2. (15 points) Derive the moment generating function of a Poisson random variable $X$ with mean $\lambda > 0$. What is $E[X^3]$?

3. (10 points) A winter storm has induced ten car accidents over the next 15 miles of road. Assuming that all accidents occur uniformly over this stretch of road and independently of one and other, how far does a tow truck driver have to drive until she encounters the first accident?

4. (10 points) Suppose that $\mathbf{X} = (X_1, X_2, X_3)^T$ are jointly normally distributed with

$$\vec{\mu} = E[\mathbf{X}] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \mathbf{\Lambda} = \mathrm{Var}(\mathbf{X}) = \begin{pmatrix} 2.0 & 1.0 & 0.5 \\ 1.0 & 2.0 & 1.0 \\ 0.5 & 1.0 & 2.0 \end{pmatrix}.$$

What is the distribution of $X_1 + X_2$ given that $X_1 - X_3 = -2$?

5. (10 points) The number of hurricanes hitting the United States mainland in any calendar year has a Poisson distribution with mean $\lambda > 0$. Each hurricane causes a random amount of damage with mean $\mu_H$ and variance $\sigma_H^2$. Let $D$ be the total damage due to all hurricanes over the calendar year. What are $E[D]$ and $\mathrm{Var}(D)$? You may assume that damages by different hurricanes are independent.

6. (10 points) Show that if $X_i$ has the Cauchy $C(0,1)$ distribution for each $i \geq 1$, then so does $n^{-1}\sum_{i=1}^{n} X_i$ (you may assume mutual independence of the $X_i$'s).

7. (10 points) Suppose that $X_1, \ldots, X_n$ is an independent and identically distributed sample from a normal distribution with mean $\mu$ and variance $\sigma^2$.

a) (4 points) Show that $\bar{X} = n^{-1}(X_1 + \cdots + X_n)$ is an unbiased estimate of $\mu$.

b) (6 points) Show that

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

is an unbiased estimate of $\sigma^2$.

8. (15 points) A random variable $X \in \{0, 1, 2, \ldots\}$ is known to have probability generating function

$$g_X(t) = E[t^X] = \frac{1}{2}e^{-3(1-t)} + \frac{1}{6}(t + t^2 + t^3).$$

a) (7 points) What is $P[X = k]$ for $k \in \{0, 1, 2, \ldots\}$?

b) (8 points) What are $E[X]$ and $\mathrm{Var}(X)$?

Bonus (10 points) Bonus (10 points) Suppose that $X_1, X_2, \ldots$ are IID with common cumulative distribution function $F_X(\cdot)$. Independent of the $X_i$'s, there is a positive integer-valued random variable $N$ with generating function $g_N(\cdot)$. Setting

$$Y = \max\{X_1, X_2, \ldots, X_N\},$$

what is the cumulative distribution function of $Y$?

Bonus (1 point) Trivia. What is the nearest star to Earth other than the sun?