STAT206B Homework #1

Due: 01/20 (Th) - Tentative.

1. Let $X \sim \text{Exp}(\lambda)$, where $\text{E}(X) = 1/\lambda$. What is the pmf (probability mass function) of $Y = \lfloor X \rfloor$ (the floor of X)? Do you recognize it as a distribution that you have studied in the past?

2. Let X_1 and X_2 be two independent random variables such that $X_i \sim \text{Ga}(a_i, b)$ for any $a_1, a_2, b > 0$. Define $Y = X_1/(X_1 + X_2)$ and $Z = (X_1 + X_2)$.
   (a) Find the joint pdf for Y and Z and show that these two random variables are independent.
   (b) Find the marginal pdf of Z. Do you recognize this pdf as belonging to some family that you know?
   (c) Find the marginal pdf of Y. Do you recognize this pdf as belonging to some family that you know?
   (d) Compute $\text{E}(Y^k)$ for any $k > 0$.
   (e) What does this result imply if $a_i = b = 1$?

3. Consider three independent random variables X_1, X_2 and X_3 such that $X_i \overset{indep}{\sim} \text{Gamma}(a_i, b)$, $i = 1, 2, 3$. Let
$$\boldsymbol{Y} = (Y_1, Y_2, Y_3) = \left(\frac{X_1}{X_1 + X_2 + X_3}, \frac{X_2}{X_1 + X_2 + X_3}, \frac{X_3}{X_1 + X_2 + X_3} \right).$$
   (a) Show that $\boldsymbol{Y} \sim \text{Dirichlet}(a_1, a_2, a_3)$, a Dirichlet distribution.
   (b) How can this result be used to generate random variables according to a Dirichlet distribution? Write a simple function in `R` or `Matlab` (your choice) that takes as inputs n, the number of trivariate vectors to be generated, and $\boldsymbol{a} = (a_1, a_2, a_3)$ and generates a matrix of size $n \times 3$ whose rows correspond to independent samples from a Dirichlet distribution with parameter (a_1, a_2, a_3).
   Use each of $\boldsymbol{a} = (0.01, 0.01, 0.01)$, $(100, 100, 100)$, and $(3, 5, 10)$ and comment how the density of $\boldsymbol{Y}$ changes over $\boldsymbol{a}$.

4. Y follows an inverse Gamma distribution with shape parameter a and scale parameter b ($Y \sim \text{IG}(a, b)$) if $Y = 1/X$ with $X \sim \text{Gamma}(a, b)$ (assume the Gamma distribution is parameterized such that $\text{E}(X) = ab$).
   (a) Find the density of Y.
   (b) Compute $\text{E}(Y^k)$. Do you need to impose any constrain on the problem for this expectation to exists?
   (c) Compare $\text{E}(Y^k)$ to $1/\text{E}(X^k)$ (hint: look at the ratio of the two quantities).

5. Y follows a log normal distribution with parameters μ and σ^2 (denotes as $Y \sim \text{Log-N}(\mu, \sigma^2)$ if $Y = \exp(X)$ where $X \sim \text{N}(\mu, \sigma^2)$).
   (a) Find the density of Y.

(b) Compute the mean and the variance of Y.

6. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ with $X \sim \mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ and set $\boldsymbol{Z}_1 = (X_1, \ldots, X_q)$ and $\boldsymbol{Z}_2 = (X_{q+1}, \ldots, , X_p)$ with $1 < q < p$. Show that

$$\boldsymbol{Z}_1 \mid \boldsymbol{Z}_2 \sim \mathrm{N}_q\left(\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{Z}_2 - \boldsymbol{\mu}_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right),$$

where $\boldsymbol{\mu}_k$ and $\Sigma_{k\ell}$ denote the blocks of $\boldsymbol{\mu}$ and Σ where the rows correspond to the variables in $\boldsymbol{Z}_k$ and the columns to the variables in $\boldsymbol{Z}_\ell$.

7. Show that if $X \sim \mathrm{Exp}(\beta)$, then

   (a) $Y = X^{1/\gamma}$ has a Weibull distribution with parameters γ and β with $\gamma > 0$ a constant.

   (b) $Y = (2X/\beta)^{1/2}$ has the Rayleigh distribution.

   For both parts, derive the form of the pdf, verify that is a pdf, and calculate the mean and the variance.

8. Let $Y \mid X \sim \mathrm{Poisson}(X)$ and let $X \sim \mathrm{Exp}(\lambda)$. What is the marginal distribution of Y?

9. (Robert) If $y \sim \mathrm{Binomial}(n, \theta)$ and $x \sim \mathrm{Binomial}(m, \theta)$, and $\theta \sim \mathrm{Beta}(\alpha, \beta)$. Find the predictive distribution of y given x.

10. (Robert) Give the posterior and the marginal distributions in the following cases:

    (a) $x \mid \sigma^2 \sim \mathrm{N}(0, \sigma^2)$ and $1/\sigma^2 \sim \mathrm{Gamma}(1, 2)$.

    (b) $x \mid p \sim \text{Negative-Binomial}(10, p)$ and $p \sim \mathrm{Beta}(1/2, 1/2)$.

11. Assume that an observation, $x_1, \ldots, x_n$ are iid from $\mathrm{N}(\theta, \sigma^2)$, where μ and σ^2 are unknown. Consider $\tilde{\pi}(\theta, \sigma^2) \propto 1/\sigma^2$ (not a probability density, i.e., improper, Jeffreys prior).

    (a) Find the joint posterior distribution.

    (b) Find the posterior distributions $\pi(\theta \mid \bar{x}, s^2, \sigma^2)$ and $\pi(\sigma^2 \mid \bar{x}, s^2)$.

    (c) Find the marginal posterior distribution of θ, $\pi(\theta \mid \bar{x}, s^2)$.

12. Consider $\boldsymbol{x}_i \mid \boldsymbol{\theta}, \Sigma \overset{iid}{\sim} \mathrm{N}_p(\boldsymbol{\theta}, \Sigma)$, $i = 1, \ldots, n$, where $\mathrm{N}_p(\boldsymbol{\theta}, \Sigma)$ represents the $p-$dimensional normal distribution with mean vector $\boldsymbol{\theta} \in \mathbb{R}^p$ and covariance matrix Σ ($p \times p$ positive definite matrix). Suppose $\boldsymbol{\theta}$ and Σ are unknown. Consider the following conjugate prior distributions;

$$\boldsymbol{\theta} \mid \Sigma \sim \mathrm{N}_p(\boldsymbol{\mu}, 1/n_0\Sigma), \text{ and } \Sigma^{-1} \sim \mathrm{Wishart}_p(\alpha, W).$$

The Wishart distribution is described in Robert Exercise #3.21. Note that if $\Sigma^{-1} \sim \mathrm{Wishart}_p(\alpha, W)$, $\Sigma \sim \text{inverse-Wishart}_p(\alpha, W^{-1})$.

fact: Given n observations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of $\mathrm{N}_p(\boldsymbol{\theta}, \Sigma)$, a sufficient statistic is

$$\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i, \text{ and } S = \sum_{i=1}^{n} (\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^t.$$

Tip: Read Robert §4.4.2 (and/or BDA §3.6).

(a) Find an expression of the joint posterior distribution as follows;

$$\pi(\boldsymbol{\theta}, \Sigma^{-1} \mid \bar{\boldsymbol{x}}, S) = \pi_1(\boldsymbol{\theta} \mid \Sigma, \bar{\boldsymbol{x}}, S)\pi_2(\Sigma^{-1} \mid \bar{\boldsymbol{x}}, S).$$

Also, identify $\pi_1(\boldsymbol{\theta} \mid \Sigma, \bar{\boldsymbol{x}}, S)$ and $\pi_2(\Sigma^{-1} \mid \bar{\boldsymbol{x}}, S)$

(b) Is the prior conjugate? Explain.