Math Science 8000, Fall, 2015
Final Exam

NAME:

Directions: Show all work on the test to receive possible partial credit. Unsupported guesses will meet a red pen. You are allowed to use a calculator and an 8×11.5 formula sheet of your own construction.

1. (10 points) Suppose that a box contains six green balls and four red balls and that two of the balls are drawn at random from the box without replacement.

a) What is the chance the first draw is green?

a) What is the chance the second draw is green?

2. (10 points) The number of meteorites that hit the Earth each year has a Poisson distribution with mean $\lambda > 0$. The chance that any of these meteorites weighs more than a pound is $p \in (0, 1)$. Derive the distribution of the number of meteorites that hit the Earth in a year that weigh more than a pound. You may assume that the weights of the individual meteorites are independent from one and other.

3. (10 points) The probability density function of a Beta random variable X has the form

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}, \qquad 0 < x < 1,$$

where α and β are positive parameters. Derive an expression for the mean of this distribution.

4. (10 points) The joint density of X and Y is

$$f_{X,Y}(x,y) = c(x^2 - y^2)e^{-x}, \qquad x > 0, \qquad -x \le y \le x$$

Find the constant c. What is the conditional distribution of Y given that $X = x$?

5. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed normal random variables with mean zero and unit variance. Set

$$\bar{X}_k = \frac{X_1 + \ldots + X_{k-1}}{k-1}, \quad 2 \le k \le n.$$

Derive the distribution of

$$\sum_{k=2}^{n} \frac{k-1}{k} (X_k - \bar{X}_k)^2?$$

6. (10 points) Suppose that X and Y are nonnegative random variables with the density

$$\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \quad x > 0.$$

Here, $\alpha > 0$ and $\lambda > 0$ are parameters. What is the joint distribution of $X + Y$ and $X/(X + Y)$?

7. (10 points) Suppose that $X_i, i = 1, \ldots, n$, are mutually independent binomial random variables and that X_i has n_i trials and success probability $p \in (0, 1)$. What is the distribution of $\sum_{i=1}^{n} X_i$?

8. (10 points) Suppose that the random triple $\mathbf{X} = (X_1, X_2, X_3)'$ has a joint normal distribution with mean vector and covariance matrix

$$E[\mathbf{X}] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \quad \mathrm{Var}(\mathbf{X}) = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 5 \end{pmatrix}.$$

Determine the joint moment generating function of $X_1 + X_2 + X_3$ and $X_1 + X_3$.

9. (10 points) Prove the Weak Law of Large Numbers: If $X_1, X_2, \ldots,$ is IID with mean μ and variance $\mathrm{Var}(X_i) \equiv \sigma^2 \in (0, \infty)$, then $\bar{X} \xrightarrow{\mathcal{P}} \mu$ as $n \to \infty$, where $\bar{X} = n^{-1}(X_1 + \cdots + X_n)$.

10. (10 points) Suppose that $X_1, X_2, \ldots$ are IID Uniform[0,1] random variables. Show that

$$\min_{1 \le k \le n} X_k \xrightarrow{P} 0$$

as $n \to \infty$.

Bonus (10 points) Each cereal box contains one sticker of one of the 32 National Football League teams (assumed these are packed at random). A hungry student wants to collect at least one sticker from each team. How many boxes of cereal does the student expect to eat before obtaining this goal?

Bonus (1 point) Yes or No: did you fill out the course evaluation?

Bonus (1 point) Tell your best joke within the King's English.