Statistics 203
Introduction to Probability Theory
Fall 2021

We have a collect of events S, representing all possible outcomes of an experiment.

An event A is a subset of S.

$P(A)$ denotes the probability of the event A.

Axioms of Probability

1) $P(S) = 1$
2) $P(A) \geq 0$
3) If $A_1, A_2, \ldots$ are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \qquad \text{countable additivity}$$

$A_1, A_2, \ldots$ disjoint means $A_i \cap A_j = \phi$ when $i \neq j$

$x \in \bigcup_{i=1}^{\infty} A_i$ means $x \in A_i$ for at least one i.

$x \in \bigcap_{i=1}^{\infty} A_i$ means $x \in A_i$ for every $i \geq 1$.

A^c = the complement of A = not A.

$A \cup B$ = A or B

$A \cap B$ = A and B

3) $\Rightarrow$ Finite Additivity

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \quad \text{if the } A_i\text{s are disjoint.}$$

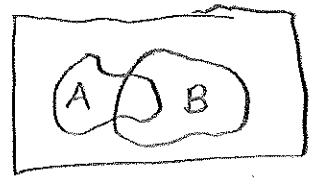
$$\left(\text{take } A_{n+1} = A_{n+2} = \cdots = \phi\right)$$

Fact: $P(\phi) = 0.$

Now $P(A \cup A^c) = P(S) = 1 = P(A) + P(A^c)$

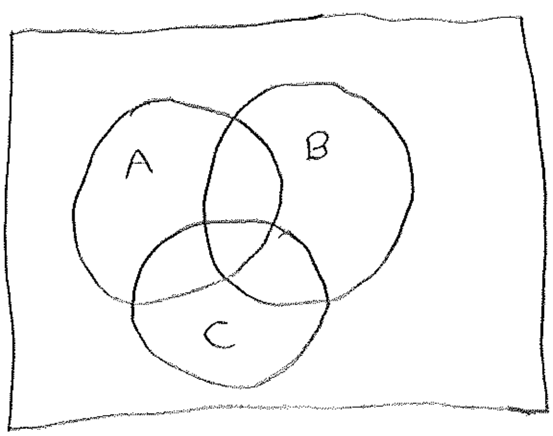
$$\Rightarrow \quad P(A^c) = 1 - P(A)$$

S



$$\begin{aligned} P(A \cup B) &= P(A \cap B^c) + P(A \cap B) + P(B \cap A^c) \\ &= \underbrace{P(A \cap B^c) + P(A \cap B)}_{P(A)} + \underbrace{P(B \cap A^c) + P(B \cap A)}_{P(B)} - P(B \cap A) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

S



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

It gets more complicated with more events.

For any events $A_1, A_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

<u>Why</u> Set $B_1 = A_1$, $B_2 = A_2 \cap A_1^c$, $B_3 = A_3 \cap A_1^c \cap A_2^c$, $\ldots$

Then 1) $B_1 \cup B_2 \cup \dots = A_1 \cup A_2 \cup \dots$
2) The B_is are disjoint.

$$\text{So } P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$
$$= \sum_{i=1}^{\infty} P(A_i \cap A_1^c \cap A_2^c \cap \dots \cap A_{i-1}^c)$$
$$\leq \sum_{i=1}^{\infty} P(A_i)$$

For You: Show that $P(A) \leq P(B)$ when $A \subset B$.

Example

Toss two fair six sided dice.

S

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

C A B

(Roll 1 , Roll 2)

36 equally likely outcomes

$A = \{$ Sum of two rolls is 7 $\}$
$B = \{$ Maximum of the two rolls is 6 $\}$
$C = \{$ Minimum of the two rolls is 1 $\}$

$P(A) = \frac{6}{36}$, $P(B) = \frac{11}{36}$, $P(C) = \frac{11}{36}$,

$P(A \cap B) = \frac{2}{36}$ $P(A \cap C) = \frac{2}{36}$, $P(B \cap C) = \frac{2}{36}$

$P(A \cap B \cap C) = 2/36$

So $P(A \cup B \cup C) = \frac{6}{36} + \frac{11}{36} + \frac{11}{36} - \frac{2}{36} - \frac{2}{36} - \frac{2}{36} + \frac{2}{36}$

$= \frac{24}{36}$

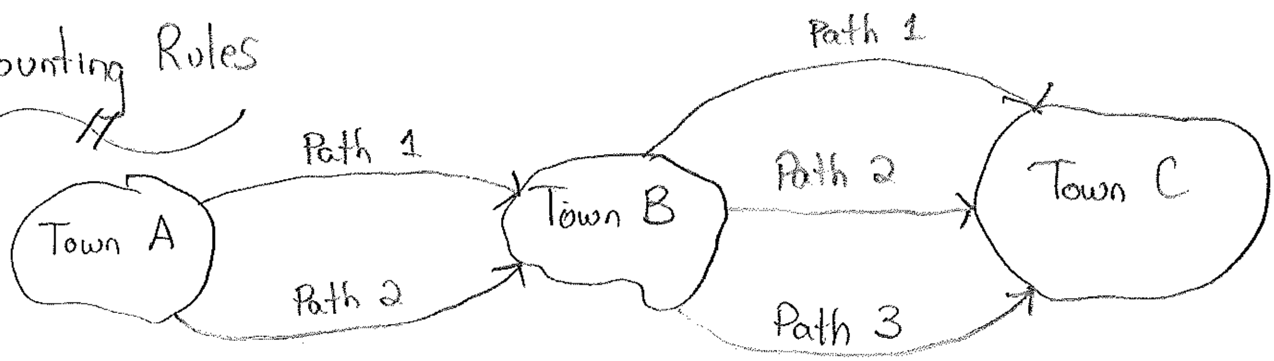
Example

Toss a fair coin three times

H H H	Each equally likely outcome has probability of $\frac{1}{8}$
H H T	
H T H	$P[\text{No Heads}] = \frac{1}{8}$
H T T	
T H H	$P[\text{One Head}] = \frac{3}{8}$
T H T	
T T H	$P[\text{Two Heads}] = \frac{3}{8}$
T T T	$P[\text{Three Heads}] = \frac{1}{8}$

Sums to unity: 1

Counting Rules



Travel from Town A to Town C going through Town B.

If there are m ways to go from A to B and
n ways to go from B to C,

there are $m \times n$ ways to go from A to C.

Example

How many different phone #'s can be made from seven digit phone #'s if the first number cannot be zero?

$\underline{9}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$ $\underline{10}$

Digit 1 . . . . Digit 7

Answer $= 9 \times 10^6 =$ 9 million

Permutations

Suppose we have n distinct letters and want to make words of length $K \leq n$. The words need not make grammatical sense. You cannot reuse a letter. How many different words can we make?

$\underline{n}$ $\underline{n-1}$ $\underline{n-2}$ . . . . $\underline{n-K+1}$

Slot 1 Slot 2 Slot 3 Slot K

Ans $= n \times (n-1) \times \ldots \times (n-K+1) = n!/K!$

Def $P_{n,K} = n!/(n-K)!$

Order matters in permutations . . .

Ex How many words of length 3 can be spelled from the letters J, A, I, L ?

$\underline{4}$ $\underline{3}$ $\underline{2}$

Here, AIL and LAI are different words

Ans $= 24$

What if the letters are repeated?

Example How many distinct 11-letter words can be made from

C O N N E C T I C U T ?

Ans $= 11 \times 10 \times \dots \times 1 = 11!$ if all letters are distinct.

But C is repeated three times, T twice, & N twice.

Let's look at a simpler setting: 3 letters C, A, C

View this as C_1, A, C_2

Answer would be $3! = 6$ if letters were different

1) C_1, A, C_2
2) C_1, C_2, A
3) A, C_1, C_2
4) A, C_2, C_1
5) C_2, C_1, A
6) C_2, A, C_1

But under restriction, 1) = 6), 2) = 5), & 3) = 4)

So there are $3 = \dfrac{3!}{2!} = 3$

Answer to the example $= \dfrac{11!}{3! \times 2! \times 2!}$

Why multiply in denominator?

<u>Combinations</u>

How many distinct teams can be made of K players from a total pool of n players?

Here, order does not matter! A team of

Angus, Malcom, Blair is the same as the team

Blair, Malcom, Angus

$$\text{Answer} = \frac{n!}{(n-k)!} \Big/ k! = \frac{n!}{k!\,(n-k)!}$$

$$\text{Def: } C_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

This uses the operative definition that Team A & B differ if there is one or more players on one team that are not on the other team.

Comment: Combinations are a statistical workhorse.

<u>Example</u> Poker (5 card hands)

In a 52 card deck, there are $\binom{52}{5} = 2{,}598{,}960$ distinct poker hands.

a) What are the chances of getting a flush in hearts?

$$\text{Ans} = \frac{\binom{13}{5}}{\binom{52}{5}} = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48}$$

b) Any flush?

$$\text{Ans} = 4 \times \frac{\binom{13}{5}}{\binom{52}{5}}$$

Flushes in distinct suits are disjoint!

c) 4 of a Kind

$$\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{13 \times 12 \times 4}{\binom{52}{5}}$$

d) Two Pairs

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

e) Full House

$$\frac{\binom{13}{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \times 2 \longleftarrow \text{Why}$$

Properties of combinatorial coefficients

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1 \qquad 0! = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,[n-(n-k)]!} = \frac{n!}{(n-k)!\;k!} = \binom{n}{k}$$

$$\binom{n+m}{k} = \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \dots + \binom{n}{k}\binom{m}{0}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \qquad \text{Binomial Thm}$$

For You: Prove this by induction

Example (Birthday problem)

Suppose K people are gathered. What are the chances that two or more people share a common birthdate?

Ignore Leap year births.

$$P(\text{2 Share a B-D}) = 1 - P(\text{Don't share a B-D})$$
$$= 1 - \frac{365}{365}\ \frac{364}{365}\ \frac{363}{365} \cdots \frac{365-(K+1)}{365}$$

This is surprisingly lage when K is even moderate, breaking ½ when K = 23.

Ans = .970 when K = 50.

Example (Tennis tournament)

We have n equally skilled tennis players. We play a tournament with single elimination. What are the chances players A & B meet?

First, every match eliminates one player. Since all but 1 player is eliminated in the tourney, n-1 total matches are played in the tournament.

Given that everyone is equally skilled, and that A & B cannot meet in two distinct matches, the chance that A & B meet must be

$$\frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}$$

This uses that A & B are equally likely to meet in any match

Multinomial Coefficients

$$\binom{n}{n_1, n_2, \ldots, n_K} = \frac{n!}{n_1!\, n_2! \cdots n_K!}, \qquad n_1 + n_2 + \cdots + n_K = n$$

$$\binom{n}{K} = \binom{n}{K, n-K}$$

This counts the number of ways to partition n_1 players to team 1, n_2 players to team 2, …, n_K players to team K.

$$\binom{n}{n_1, n_2, \ldots, n_K} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{K-1}}{n_K}$$

<u>Example</u> 4-player bridge hands deal 13 cards to each player

There are $\binom{52}{13,13,13,13}$ distinct deals.

$$\binom{52}{13,13,13,13} = \binom{52}{13}\binom{39}{13}\binom{26}{13}\cancel{\binom{13}{13}}^{\,1} \approx \text{Big!}$$

The chance one player gets all 13 hearts is $\dfrac{4 \times \binom{13}{13}}{\binom{52}{13}} = 4 \Big/ \binom{52}{13}$

$$= 4 \cdot \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\cdots\left(\frac{1}{40}\right)$$

Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_K)^n = \sum_{\substack{\text{All } n_1, n_2, \ldots, n_K \\ \text{with } n_1 + \cdots + n_K = n}} \binom{n}{n_1, n_2, \ldots, n_K} x_1^{n_1} x_2^{n_2} \cdots x_K^{n_K}$$