AMS206B

1. $$p(x \mid \theta) \propto \theta^{x}(1-\theta)^{(m-x)}.$$

   So,

$$\begin{aligned}
\ell(\theta) &\propto x\log(\theta) + (m-x)\log(1-\theta), \\
\frac{\partial \ell(\theta)}{\partial \theta} &= \frac{x}{\theta} - \frac{(m-x)}{1-\theta}, \\
\frac{\partial^2 \ell(\theta)}{\partial \theta^2} &= \frac{x}{\theta^2} - \frac{(m-x)}{(1-\theta)^2}, \\
I(\theta) &= -\mathrm{E}(-\frac{x}{\theta^2} - \frac{m-x}{(1-\theta)^2} = m\theta^{-1}(1-\theta)^{-1}.
\end{aligned}$$

   Therefore, the Jeffery's prior is

$$\pi(\theta) \propto \sqrt{I(\theta)} = \theta^{-1/2}(1-\theta)^{-1/2},$$

   that is, $\mathrm{Be}(0.5, 0.5)$.

   Due to the conjugacy, $\pi(\theta \mid x)$ is $\mathrm{Be}(x+1/2, m-x+1/2)$.

2. (a) Consider a 1-to-1 transformation of θ, $\phi = g(\theta)$. The corresponding prior on ϕ, $\pi^{\star}(\phi) = \pi(g^{-1}(\phi))|\frac{dg^{-1}(\phi)}{d\phi}|$ is also noninformative.

   (b) $\phi = \log(\frac{\theta}{1-\theta}) \Leftrightarrow \theta = \frac{e^{\phi}}{1+e^{\phi}}$. Thus,

$$\begin{aligned}
\pi^{\star}(\phi) &\propto \left(\frac{e^{\phi}}{1+e^{\phi}}\right)^{-1/2}\left(\frac{1}{1+e^{\phi}}\right)^{-1/2}\frac{e^{\phi}}{(1+e^{\phi})^2} \\
&= \frac{e^{\phi/2}}{1+e^{\phi}}.
\end{aligned}$$

3. Due to the conjugacy, $\pi(\theta \mid x)$ is $\mathrm{Be}(a+x, b+m-x)$.

$$\begin{aligned}
p(y \mid x) &= \int_0^1 \frac{\Gamma(a+b+m)}{\Gamma(a+x)\Gamma(b+m-x)}\theta^{a+x-1}(1-\theta)^{b+m-x-1}\binom{n}{x}\theta^{y}(1-\theta)^{n-y}d\theta \\
&= \frac{\Gamma(a+b+m)}{\Gamma(a+x)\Gamma(b+m-x)}\frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} \\
&\quad \times \frac{\Gamma(a+x+y)}{\Gamma(b+m-x+n-y)\Gamma(a+b+m+n)}.
\end{aligned}$$

AMS256

1.
$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1n} \\ y_{21} \\ \vdots \\ y_{2n} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \theta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1n} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2n} \end{bmatrix}.$$

The rannk of $\boldsymbol{X}$ is 2 (full rank).

2.
$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} = \begin{bmatrix} \bar{y}_2 \\ \bar{y}_1 - \bar{y}_2 \end{bmatrix}.$$

The distribution of $\hat{\boldsymbol{\beta}}$ is
$$\hat{\boldsymbol{\beta}} \sim \mathrm{N}(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}),$$
where
$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \frac{1}{n}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

3. $SSE = (\boldsymbol{y} - \hat{\boldsymbol{y}})^T(\boldsymbol{y} - \hat{\boldsymbol{y}}) = \sum_{i,j}(y_{ij} - \hat{y}_{ij})^2$ where $\hat{y}_{1j} = \bar{y}_1$ and $\hat{y}_{2j} = \bar{y}_2$. So,
$$\hat{\sigma}^2 = \frac{SSE}{2n-2} = \frac{\sum_{i,j}(y_{ij} - \bar{y}_i)^2}{2n-2}.$$

$$\frac{(2n-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(2n-2).$$

4. $\boldsymbol{X}$ has the full rank so any linear function of $\boldsymbol{\beta}$ is estimable.
$$\theta_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}\boldsymbol{\beta}$$

5.
$$\hat{\theta}_1 \sim \mathrm{N}(\theta_1, \frac{2\sigma^2}{n}).$$

Thus, under $H0 : \theta_1 = 0$,
$$F = \frac{(\hat{\theta}_1)^2}{2\hat{\sigma}^2} \sim F(1, 2n-2).$$