BASKIN SCHOOL OF ENGINEERING

Department of Applied Mathematics and Statistics

First Year Exam: September 2014

Problem AMS 206B:

Consider the estimation of the parameter $\theta \in (0, \infty)$ under the loss function

$$L(\theta, d) \quad = \quad \frac{(\theta - d)^2}{\theta(\theta + 1)}, \tag{1}$$

based on one observation X from the negative binomial distribution parameterized as

$$f(x|\theta) = \binom{n + x - 1}{x} \theta^x (\theta + 1)^{-(n+x)}.$$

Note that under this parameterization $E(X|\theta) = n\theta$ and $Var(X|\theta) = n\theta(\theta + 1)$.

1. (15 points) Determine the risk function of the unbiased estimator $\delta_0(x) = x/n$.

2. (15 points) Determine the risk function of the estimator $\delta_1(x) = x/(n + 1)$.

3. (10 points) Which of the two estimators ($\delta_0(x)$ or $\delta_1(x)$) has lower maximum risk? Justify your answer.

4. (60 points) Find the Bayes rule (estimator) under loss function in (1), and the class of priors

$$\pi(\theta) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (\theta + 1)^{-(a+b)}, \; a > 0, \; b > 0.$$

Some facts that may be useful:

- Under the squared error loss function $L(\theta, d) = (\theta - d)^2$ we have that

$$R(\theta, \delta(x)) = Bias^2(\delta(x)) + Variance(\delta(x)),$$

  with

$$R(\theta, \delta(x)) = \int L(\theta, \delta(x)) f(x|\theta) dx.$$

- For the priors in part (4) we have that

$$E(\theta^c(\theta+1)^{-d}) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+c)\Gamma(b+d-c)}{\Gamma(a+b+d)}.$$

SOLUTION.

1. First note that the risk under the loss function in (1) is the same as the risk under the SEL divided by $\theta(\theta+1)$. Therefore,

$$\begin{aligned} R(\theta, \delta_0(x)) &= \frac{Bias^2(\delta_0(x)) + Variance(\delta_0(x))}{\theta(\theta+1)} \\ &= \frac{0 + \theta(\theta+1)/n}{\theta(\theta+1)} \\ &= \frac{1}{n}. \end{aligned}$$

2. Similarly, for $\delta_1(x) = x/(n+1)$ we have

$$\begin{aligned} Bias(\delta_1(x))^2 &= (E(\delta_1(x)) - \theta)^2 \\ &= \frac{\theta^2}{(n+1)^2}, \\ Variance(\delta_1(x)) &= \frac{n\theta(\theta+1)}{(n+1)^2}. \end{aligned}$$

   Therefore,

$$\begin{aligned} R(\theta, \delta_1(x)) &= \frac{Bias^2(\delta_1(x)) + Variance(\delta_1(x))}{\theta(\theta+1)} \\ &= \frac{1}{(n+1)^2}\left(\frac{\theta}{(\theta+1)} + n\right). \end{aligned}$$

3. $\theta < \theta + 1$ and this implies that

$$R(\theta, \delta_1(x)) < \frac{(n+1)}{(n+1)^2} = \frac{1}{(n+1)} < \frac{1}{n} = R(\theta, \delta_0(x)).$$

4. First we need to find the posterior distribution for θ. It can be shown that the posterior has the form of the prior with parameters $a^* = a + x$ and $b^* = b + n$.

   The Bayes estimator can be found by minimizing the expected posterior loss, i.e., by finding $\hat{\delta}$ that minimizes

$$\begin{aligned} E(L(\theta, \delta|x) &= \int \frac{(\theta-\delta)^2}{\theta(\theta+1)} \pi_{a^*,b^*}(\theta|x) d\theta. \\ &= C - 2\delta E((\theta+1)^{-1}|a^*, b^*) + \delta^2 E(\theta^{-1}(\theta+1)^{-1}|a^*, b^*), \end{aligned}$$

where C is a constant that does not depend on δ. Taking the first derivative with respect to δ and making it equal to zero we obtain that

$$\begin{aligned}\hat{\delta} &= \frac{E((\theta+1)^{-1}|a^*,b^*)}{E(\theta^{-1}(\theta+1)^{-1}|a^*,b^*)} \\ &= \frac{a+x-1}{b+n+1},\end{aligned}$$

and the second derivative is positive, so the Bayes estimator is $\delta_{Bayes}(x) = \frac{a+x-1}{b+n+1}$.