BASKIN SCHOOL OF ENGINEERING

Department of Applied Mathematics and Statistics

First Year Exam: June, 2014

INSTRUCTIONS

If you are on the Applied Mathematics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 3(AMS 212A), 4(AMS 212B), 5(AMS 213), and 6(AMS 214).

If you are on the Statistics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 7(AMS 205B), 8(AMS 206B), 9(AMS 207), and 10(AMS256).

Please complete all required problems on the **supplied exam papers**. Write your exam ID number and problem number on each page. Use only the **front side** of each page.

Problem 1 (AMS 203):

Let X_1, X_2, X_3 be independent and identically distributed exponential random variables with density

$$f_X(x) = \begin{cases} \beta e^{-\beta x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_i = X_1 + \ldots + X_i$, for $i = 1 : 3$.

1. (30 points) Find the joint p.d.f. of Y_1 and Y_2.
2. (20 points) Find the p.d.f. of Y_2.
3. (30 points) Find the joint p.d.f. of Y_1, Y_2 and Y_3.
4. (20 points) Show that the p.d.f. of Y_3 is

$$f_{Y_3}(y_3) = \begin{cases} \beta^3 \frac{y_3^2}{2} e^{-\beta y_3} & \text{for } \ y_3 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   In other words, show that Y_3 is a gamma random variable with parameters 3 and β.

Some useful results:

- The p.d.f. of random variable X with a gamma distribution with parameters α and β is

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } \ x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

SOLUTION

1. The joint p.d.f. of Y_1 and Y_2 is given by $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(s_1, s_2)|J|$ with $s_1(Y_1, Y_2) = X_1$, $s_2(Y_1, Y_2) = X_2$ and J the Jacobian of the transformation.

   In this case $X_1 = s_1(Y_1, Y_2) = Y_1$, $X_2 = s_2(Y_1, Y_2) = Y_2 - Y_1$ and

$$J = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1.$$

   Then,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \beta^2 e^{-\beta y_2} & \text{for } \ y_2 > y_1 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. The marginal p.d.f. of Y_2 is

$$f_{Y_2}(y_2) = \int_0^{y_2} f_{Y_1,Y_2}(y_1, y_2) dy_1 = \int_0^{y_2} \beta^2 e^{-\beta y_2} dy_1 = \beta^2 e^{-\beta y_2} y_2$$

for $y_2 > 0$ and $f_{Y_2}(y_2) = 0$ otherwise.

3. Similarly, the joint p.d.f. of Y_1, Y_2, Y_3 is given by

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = f_{X_1,X_2,X_3}(s_1, s_2, s_3)|J|$$

with $X_1 = s_1(Y_1, Y_2, Y_3) = Y_1$, $X_2 = s_2(Y_1, Y_2, Y_3) = Y_2 - Y_1$, and $s_3(Y_1, Y_2, Y_3) = Y_3 - Y_2$ and

$$J = \det \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then,

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = f_{X_1,X_2,X_3}(s_1, s_2, s_3) \times 1 = \beta^3 e^{-\beta y_3}$$

for $y_3 > y_2 > y_1 > 0$ and $f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = 0$ otherwise.

4. We begin by integrating out the first variable:

$$f_{Y_2,Y_3}(y_2, y_3) = \int_0^{y_2} \beta^3 e^{-\beta y_3} dy_1 = \beta^3 y_2 e^{-\beta y_3}$$

for $y_3 > y_2 > 0$ and 0 otherwise.

The next integral gives

$$f_{Y_3}(y_3) = \int_0^{y_3} \beta^3 y_2 e^{-\beta y_3} dy_2 = \beta^3 \frac{y_3^2}{2} e^{-\beta y_3}$$

for $y_3 > 0$ and 0 otherwise.