BASKIN SCHOOL OF ENGINEERING

Department of Applied Mathematics and Statistics

2015 First Year Exam: June 5, 2015

INSTRUCTIONS

If you are on the Applied Mathematics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 3(AMS 212A), 4(AMS 212B), 5(AMS 213), and 6(AMS 214).

If you are on the Statistics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 7(AMS 205B), 8(AMS 206B), 9(AMS 207), and 10(AMS256).

Please complete all required problems on the **supplied exam papers**. Write your exam ID number and problem number on each page. Use only the **front side** of each page.

Problem 1 (AMS 203):

Problem 2 (AMS 211):

Problem 3 (AMS 212A):

Problem 4 (AMS 212B):

Problem 5 (AMS 213):

Problem 6 (AMS 214):

Problem 7 (AMS 205B):

Problem 8 (AMS 206B):

Problem 9 (AMS 207):

Count data with an overabundance of zeros can be modeled through the zero-inflated Poisson model (ZIP). In such model, the zero-inflated Poisson distribution, ZIP(λ, π), is given by the probability mass function

$$f(y|\lambda, \pi) = \pi I(y = 0) + (1 - \pi) f_0(y|\lambda), \quad y = 0, 1, 2, \ldots,$$

where $0 \leq \pi \leq 1$, $\lambda > 0$, $I(\cdot)$ is the indicator function, and

$$f_0(y|\lambda) = \frac{e^{\lambda} \lambda^y}{y!}.$$

1. (10 points) Let $Y \sim ZIP(\lambda, \pi)$, find $Pr(Y = y|\lambda, \pi)$ for $y = 0$ and $Pr(Y = y|\lambda, \pi)$ for $y \neq 0$.

2. Let $Y_1, \ldots, Y_n$ be i.i.d. with $Y_i \sim ZIP(\lambda, \pi)$, $\lambda \sim Gamma(a, b)$, and $\pi \sim Beta(c, d)$. Assume this model is used to describe the number of cigarettes smoked by the respondents of a health survey. More specifically, suppose that out of 10 respondents, the number of cigarettes smoked by each respondent are:

$$y_1 = 0,\ y_2 = 5,\ y_3 = 0,\ y_4 = 0,\ y_5 = 10,\ y_6 = 0,\ y_7 = 0,\ y_8 = 0,\ y_9 = 3,\ y_{10} = 0.$$

   (a) (15 points) Write down $p(\lambda, \pi|y_1, \ldots, y_{10})$ up to a proportionality constant.

   (b) (15 points) Find $p(\lambda|\pi, y_1, \ldots, y_{10})$, and $p(\pi|\lambda, y_1, \ldots, y_{10})$ (up to a proportionality constant in each case). Are these distributions available in closed form?

   (c) (60 points) Introduce the auxiliary variables Z_i such that $Y_i = 0$ if $Z_i = 1$ and $Y_i \sim Poisson(\lambda)$ if $Z_i = 0$ with $Z_i \sim Bernoulli(\pi)$. Derive a MCMC algorithm to obtain samples from $p(\lambda, \pi, Z_1, \ldots, Z_{10}|y_1, \ldots, y_{10})$. Provide the details of the steps in your algorithm, i.e., for Gibbs steps, provide the full conditional distributions, for Metropolis-Hastings steps, provide the proposal distributions and the acceptance probabilities.

Some useful results:

- The p.m.f. of a random variable X with a Poisson distribution with parameter λ is

$$f(x) = \frac{1}{x!} \lambda^x e^{-\lambda}, \quad x = 0, 1, 2, \ldots, \quad \lambda > 0.$$

- The p.d.f. of random variable X with a Gamma distribution with parameters a and b is

$$f(x) = \begin{cases} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- The p.d.f. of a random variable X with Beta distribution with parameters c and d is

$$f(x) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} x^{c-1}(1-x)^{d-1}, \quad x \in [0,1], c > 0, d > 0.$$

SOLUTION

1.

$$Pr(Y = y|\lambda, \pi) = \begin{cases} \pi + (1-\pi)e^{-\lambda} & y = 0, \\ (1-\pi)\frac{e^{-\lambda}\lambda^y}{y!}, & y > 0. \end{cases}$$

2 (a)

$$\begin{aligned} p(\lambda, \pi|y_1, \ldots, y_{10}) &\propto [\pi + (1-\pi)e^{-\lambda}]^7[(1-\pi)e^{-\lambda}]^3\lambda^{18}\lambda^{a-1}e^{-b\lambda}\pi^{c-1}(1-\pi)^{d-1}, \\ &\propto [\pi + (1-\pi)e^{-\lambda}]^7(1-\pi)^{3+d-1}\pi^{c-1}e^{-\lambda(3+b)}\lambda^{18+a-1} \end{aligned}$$

2 (b)

$$p(\lambda|\pi, y_1, \ldots, y_{10}) \propto [\pi + (1-\pi)e^{-\lambda}]^7 e^{-\lambda(3+b)}\lambda^{18+a-1}$$

and

$$p(\pi|\lambda, y_1, \ldots, y_{10}) \propto [\pi + (1-\pi)e^{-\lambda}]^7(1-\pi)^{3+d-1}\pi^{c-1}.$$

These are not available in close form.

2 (c)

$$\begin{aligned} p(\lambda, \pi, Z_1, \ldots, Z_{10}|y_1, \ldots, y_{10}) &\propto \prod_{i=1}^{10} \pi^{Z_i}[(1-\pi)Pr(Y_i = y_i|Z_i = 0)]^{1-Z_i}\lambda^{a-1}e^{-b\lambda}\pi^{c-1}(1-\pi)^{d-1} \\ &\propto \prod_{i=1}^{10} \pi^{Z_i}[(1-\pi)\frac{\lambda_i^y}{y_i!}e^{-\lambda}]^{1-Z_i}\lambda^{a-1}e^{-b\lambda}\pi^{c-1}(1-\pi)^{d-1} \end{aligned}$$

Then, the full conditionals are as follows:

- For $i = 1, \ldots, 10$, $Pr(Z_i|\lambda, \pi, y_1, \ldots, y_{10})$.

$$Pr(Z_i = 1|\lambda, \pi, y_1, \ldots, y_{10}) = \frac{\pi y_i!}{\pi y_i! + (1-\pi)e^{-\lambda}\lambda^{y_i}},$$

  and

$$Pr(Z_i = 0|\lambda, \pi, y_1, \ldots, y_{10}) = \frac{(1-\pi)e^{-\lambda}\lambda^{y_i}}{\pi y_i! + (1-\pi)e^{-\lambda}\lambda^{y_i}},$$

- $(\lambda|\pi, Z_1, \ldots, Z_{10}, y_1, \ldots, y_{10})$ is drawn from a Gamma distribution, i.e.,

$$\begin{aligned} p(\lambda|\pi, Z_1, \ldots, Z_{10}, y_1, \ldots, y_{10}) &\propto \prod_{i, Z_i = 0} e^{-\lambda}\lambda^{y_i} e^{-b\lambda}\lambda^{a-1} \\ &\propto e^{-\lambda(n_0+b)}\lambda^{\sum_{i,Z_i=0} y_i + a - 1}, \end{aligned}$$

  which corresponds to a $Gamma((\sum_{i,Z_i=0} y_i + a), n_0 + b)$, with n_0 the number of Z_i equal to zero.

- $(\pi|\lambda, Z_1, \ldots, Z_{10}, y_1, \ldots, y_{10})$ is drawn from a Beta distribution, i.e.,

$$p(\pi|\lambda, Z_1, \ldots, Z_{10}, y_1, \ldots, y_{10}) \propto \pi^{\sum_{i=1}^{10} Z_i}(1-\pi)^{10-\sum_{i=1}^{10} Z_i}\pi^{c-1}(1-\pi)^{d-1},$$

  which corresponds to a $Beta((\sum_{i=1}^{10} Z_i + c), ((10 - \sum_{i=1}^{10} Z_i) + d))$.

Problem 10 (AMS 256):