AMS 207

Consider a set of observations $x_1, \ldots, x_n, x^*_{n+1}, \ldots, x^*_{n+k}$. Ignore for now the asterisks, and assume that they follow the distributions $Exp(\lambda_i), i = 1, \ldots, n+k$. Consider the prior

$$p(\lambda_1, \ldots, \lambda_{n+k}|\beta) = \prod_i p(\lambda_i|\beta) = \prod_i Ga(\lambda_i|\alpha, \beta),$$

where α is a known positive constant, and $p(\beta) = Ga(\beta|a, b)$, where a and b are both known.

1. Find the unconditional density $p(\lambda_1, \ldots, \lambda_{n+k})$. Are the λ_i independent?

   (20%)

2. Find the posterior distribution of β given the sample.

   (20%)

Suppose that the observations with an asterisk are right censored. That is, the true value is missing, but we know that it is larger than the value recorded with and asterisk.

3. Use the missing value approach to obtain the likelihood that accounts for the censoring.

   (15%)

4. Find the posterior distribution of β in this case. Is there any difference with respect to the case where the censoring is ignored?

   (15%)

5. Introduce latent variables to obtain a Gibbs sample, in order to explore the posterior distribution of all model parameters. Write, explicitly, all the full conditionals, and identify them.

   (30%)

Hint: The density of a $Ga(\alpha, \beta)$ is

$$e^{-x\beta} x^{\alpha-1} \frac{\beta^\alpha}{\Gamma(\alpha)} \ .$$

An exponential density is a gamma with $\alpha = 1$.

Solution:

1. $$p(\lambda_1, \ldots, \lambda_{n+k}) \propto \int_0^\infty \prod_i \lambda_i^{\alpha-1} e^{-\lambda_i \beta} \beta^{\alpha+a-1} e^{-b\beta} d\beta =$$

$$\prod_i \lambda_i^{\alpha-1} \int_0^\infty \beta^{\alpha+a-1} e^{-\beta((n+k)\overline{\lambda}+b)} d\beta = \prod_i \lambda_i^{\alpha-1} \left((n+k)\overline{\lambda} + b)\right)^{-(\alpha+a)}$$

   There is no independence, due to the $\overline{\lambda}$ term. But there is exchangeability, as the ordering of the λs does not affect the value of the density.

2. The posterior distributions is proportional to

$$\prod_i \lambda_i e^{-\lambda_i x_i} \lambda_i^{\alpha-1} e^{-\lambda_i \beta} \beta^{\alpha+a-1} e^{-b\beta}$$

   Integrating out all λ_i we have

$$\beta^{\alpha+a-1} e^{-b\beta} \prod_i \int_0^\infty e^{-\lambda_i(x_i+\beta)} \lambda_i^{\alpha+1-1} d\lambda_i = \beta^{\alpha+a-1} e^{-b\beta} \prod_{i=1}^{n+k} (x_i + \beta)^{-(\alpha+1)}$$

3. We can assume that the censored observations correspond to missing values $\xi_j, j = 1, \ldots, k$, so that the full likelihood of x_i and ξ_j is given as

$$\prod_i \lambda_i e^{-\lambda_i x_i} \prod_j \lambda_{n+j} e^{-\lambda_{n+j} \xi_j}.$$

   The integrated likelihood is given as

$$\prod_i \lambda_i e^{-\lambda_i x_i} \prod_j \int_{x^*_{n+j}}^\infty \lambda_{n+j} e^{-\lambda_{n+j} \xi_j} d\xi_j = \prod_i \lambda_i e^{-\lambda_i x_i} \prod_j e^{-\lambda_{n+j} x^*_j}.$$

4. Using the integrated likelihood in the previous question, we obtain the joint posterior for all parameters as

$$\beta^{\alpha+a-1} e^{-b\beta} \prod_i \lambda_i e^{-\lambda_i x_i} \lambda_i^{\alpha-1} e^{-\lambda_i \beta} \prod_j \lambda_{n+j}^{\alpha-1} e^{-\lambda_{n+j} \beta} e^{-\lambda_{n+j} x^*_j} =$$

$$\beta^{\alpha+a-1}e^{-b\beta}\prod_i \lambda_i^{\alpha+1-1}e^{-\lambda_i(x_i+\beta)}\prod_j \lambda_{n+j}^{\alpha-1}e^{-\lambda_{n+j}(x_j^*+\beta)}.$$

Integrating λ_i out $\forall i$ and j, we have

$$p(\beta|\boldsymbol{x},\boldsymbol{x}^*) \propto \beta^{\alpha+a-1}e^{-b\beta}\prod_{i=1}^{n}(x_i+\beta)^{-(\alpha+1)}\prod_{j=1}^{k}(x_j^*+\beta)^{-\alpha}.$$

Compared to the posterior that we obtained by ignoring the censoring, we have that the last k terms in the product have exponent $-\alpha$ instead of $-(\alpha+1)$.

5. The latent variables are already introduced in Question 3. The full conditionals are:

   - $$p(\lambda_i|\ldots) \propto \lambda_i^{\alpha+1-1}e^{-\lambda_i(z_i+\beta)},\ \ i=1,\ldots,n+k$$

     where $z_i = x_i$ if $i \leq n$, and $z_n + j = \xi_j, j = 1,\ldots,k$. This corresponds to a $Ga(\lambda_i|\alpha+1, z_i+\beta)$.

   - $$p(\beta|\ldots) \propto \beta^{\alpha+a-1}e^{-\beta(b+(n+k)\overline{\lambda})}.$$

     This corresponds to a $Ga(\beta|\alpha+a, b+(n+k)\overline{\lambda})$.

   - $$p(\xi_j|\ldots) \propto \lambda_{n+j}e^{-\lambda_{n+j}\xi_j},\ \ \text{for}\ \ \xi_j \geq x_{n+j}^*\ \ j=1,\ldots,k.$$

     This corresponds to a truncated exponential with parameter λ_{n+j} and truncation x_j^*.