Math Science 8000, Fall 2018
In Class Test I

NAME:

Directions: Show all work on the test to receive possible partial credit. Unsupported guesses will meet a red pen. You are allowed to use a calculator and an 8×11.5 formula sheet of your own construction.

1. Short Answer (20 points total, 2 points for each part)

a) What is $\int_0^\infty x^{10} e^{-x} dx$?

$\Gamma(11) = 10!$

b) What is the variance of a binomial random variable with 100 trials and success probability 3/4?

$np(1-p) = 100 \times \frac{1}{4} \times \frac{3}{4} = \frac{75}{4}$

c) Yes or No: Do all probability density functions need to be continuous?

p.d.f

X No.

d) What is $P(A|B)$ if A and B are independent?

$P(A)$

e) Give bounds for which the correlation coefficient ρ must lie.

$\rho \in [-1, 1]$

f) True or False: The joint distribution is preferable to know over the marginals.

Since joint $\Rightarrow$ marginal

g) Yes or No: If $X \geq 0$ is a random variable and $E[X^5] = \infty$, does it necessarily follow that $E[X^3] = \infty$?

$$\int_0^\infty x^5 p(x)\, dx = \infty$$

N.

$$\int_0^\infty x^3 p(x) \cdot x^2\, dx$$

h) How does one obtain the cumulative distribution function of X from the joint cumulative distribution function $F_{X,Y}(x, y)$ of the random pair (X, Y)?

$\lim y \to \infty$

i) Yes or No: If $\mathrm{Cov}(X, Y) = 0$, are X and Y necessarily independent?

quadratic.

j) If X and Y are independent, what is $\mathrm{Cov}(X, Y)$?

0

2. (15 points) In a standard 52 card well shuffled deck, what is the probability of being dealt, in a 5 card hand, the following "nice poker hands". (You may leave your answers in terms of combinatorial coefficients).

a) Four of a kind?

$$\frac{\binom{13}{1}\binom{48}{1}}{\binom{52}{5}}$$

b) A full house? $3+2$

$$\frac{\binom{13}{1}\binom{4}{3}\cdot\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

c) Two pairs?

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{52-8}{1}}{\binom{52}{5}}$$

3. (10 points) Which of the following are legitimate probability density functions of univariate or bivariate random variables. Justify your answers.

a) $f_X(x) = 1 - e^{-x}, x > 0$.

Monotone on 1-1 $f^{-1}(y) = -\ln(1-y)$

✓ . 1-1 ✓ .

b) $f_X(x) = x^7, 0 \le x \le 1$.

1-1, Monotone, $f^{-1}(y) = y^{-7}$

c) $f_X(x) = \frac{2-\sin(x)}{4\pi}, 0 \le x \le 2\pi$.

arcsin

Not 1-1. No, (2-4πy)

d) $f_{X,Y}(x, y) = \frac{e^{-\frac{1}{2}\left(x^2+y^2\right)}}{2\pi}, -\infty < x, y < \infty$.

No, not 1-1

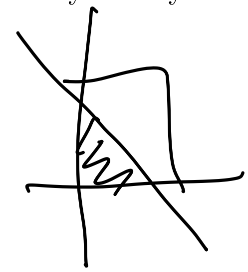
e) $f_{X,Y}(x, y) = \frac{e^{-x^2/2}}{\sqrt{2\pi y^3}}, -\infty < x < \infty, -1 \le y \le 1$.

x is not 1-1 .

4. (20 points) Suppose that the random pair (X, Y) has the joint probability density

$$f_{X,Y}(x, y) = cx, \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 1,$$

where c is some constant.



a) What is c?

$$\int_0^1 \int_0^{1-x} cx\,dy\,dx$$

$$= \int_0^1 cx(1-x)\,dx = c \cdot -\frac{x^3}{3} + \frac{1}{2}x^2 \Big|_0^1$$

$$= \frac{c}{6} \qquad c = 6$$

b) What is the conditional density of X given that $Y = y$?

$$f_Y(y) = \int_0^{1-y} 6x\,dx = 3x^2 \Big|_0^{1-y} = 3(1-y)^2$$

$$f_{X|Y}(x|y) = \frac{6x}{3(1-y)^2} = \frac{2x}{(1-y)^2}$$

c) Calculate $E[X|Y = y]$.

$$E(X|Y=y) = \int_0^{1-y} \frac{1}{(1-y)^2} 2x^2\,dx$$

$$= \frac{1}{1-y^2} \frac{2}{3} x^2 \Big|_0^{1-y} = 2/3$$

d) Evaluate $\text{Var}(X|Y = y)$

$$Z(x^2|Y=y) = \int_0^{1-y} \frac{2x^3}{(1-y)^2}\,dx = \frac{1}{(1-y)^2}(1-y)^4 x \frac{1}{2}$$

$$= \frac{(1-y)^2}{2}$$

$$Var(X|Y) = \frac{(1-y)^2}{2} - (2/3)^2$$

5. (15 points) A fair L-sided die is tossed and then a fair coin is flipped independently the number of times shown on the die. Let D be the die toss and X the number of heads obtained in this experiment. Compute $E[D]$, $\mathrm{Var}(D)$, $E[X]$, and $\mathrm{Var}(X)$.

$D \sim \frac{1}{L}$ (Discrete Unif)

$X \sim \mathrm{Bin}(D, \frac{1}{2})$

$E(D) = \dfrac{\frac{1}{2}(1+L)\cdot L}{L} = \dfrac{1+L}{2}$

$\mathrm{Var}(D) = E(D^2) - E(D)^2$

$= \sum_{i=1}^{L} \dfrac{1^2 + \cdots + L^2}{L} - \left(\dfrac{1+L}{2}\right)^2$

$= \dfrac{\cancel{l}(l+1)(2l+1)}{6\cancel{l}} - \left(\dfrac{1+L}{2}\right)^2$

$= \dfrac{(l+1)(2l+1)}{6} - \dfrac{(1+l)^2}{4}$

$= \dfrac{2l^2+3l+1}{6} - \dfrac{l^2+2l+1}{4}$

$= \dfrac{4l^2+6\cancel{l}+2-3l^2-6l-3}{12} = \dfrac{l^2-1}{12}$

$E(X)$

$= E E(X|D)$

$= E\left(\dfrac{D}{2}\right)$

$= \dfrac{1+l}{4}$

$\mathrm{Var}(X) =$

$E(\mathrm{Var}(X|D)) + \mathrm{Var}(E(X|D))$

$= E\left(D\cdot\frac{1}{2}\cdot\frac{1}{2}\right) + \mathrm{Var}\left(\dfrac{D}{2}\right)$

$= E\left(\dfrac{D}{4}\right) + \mathrm{Var}\left(\dfrac{D}{2}\right)$

$= \dfrac{1}{4}(E(D) + \mathrm{Var}(D))$

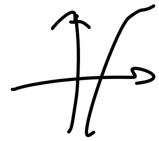
$= \dfrac{1}{4}\left(\dfrac{1+l}{2} + \dfrac{l^2-1}{12}\right)$

$= \dfrac{6+6l+l^2-1}{48}$

$$= \frac{\ell^2 + 6\ell + 5}{48}$$

6. (10 points) Suppose that X_1 and X_2 are two independent Gamma random variables, each having the joint density

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \qquad x > 0,$$



where $\alpha > 0$ and $\lambda > 0$ are parameters.

a) What is the density of $Y_1 = \ln(X_1)$?

$Y = \ln(x)$, $Y \in \mathbb{R}$

$x = e^{y}$ $\quad \left|\frac{dx}{dy}\right| = |e^{y}|$

$$f_Y(y) = f_X(e^{y})\,|e^{y}| = e^{y} \cdot \frac{\lambda e^{-\lambda e^{y}}(\lambda e^{y})^{\alpha-1}}{\Gamma(\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \left(\lambda^{\alpha} \quad e^{\alpha y - \lambda e^{y}}\right)$$

b) What is the joint density of Y_1 and $Y_2 = \ln(X_1) + \ln(X_2)$?

$Y_1 = \ln(x_1)$ $\quad Y_2 = \ln(x_1) + \ln(x_2)$, $\quad -\infty < y_1, y_2 \leq \infty$

Solve for x_1, x_2

$$\begin{cases} x_1 = e^{y_1} \\ x_2 = e^{y_2 - y_1} \end{cases}$$

$$|J| = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix} = \begin{vmatrix} e^{y_1} & 0 \\ -e^{y_2-y_1} & e^{y_2-y_1} \end{vmatrix}$$

$$= e^{y_2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(e^{y_1}, e^{y_2-y_1})\,|e^{y_2}|$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\alpha)} \cdot \left(\lambda e^{-\lambda e^{y_1}}(\lambda e^{y_1})^{\alpha-1}\right) \cdot \left(\lambda e^{-\lambda e^{y_2-y_1}}(\lambda e^{y_2-y_1})^{\alpha-1}\right) e^{y_2}$$

$$= \frac{1}{(\Gamma(\alpha))^2} \cdot \lambda^{\alpha}\, e^{-\lambda e^{y_1}}\, e^{y_1(\alpha-1)} \cdot \lambda^{\alpha}\, e^{-\lambda e^{y_2-y_1}}\, e^{(y_2-y_1)(\alpha-1)}\, e^{y_2 - 2y_1}$$

$\lambda^{\alpha}\lambda^{\alpha} \quad e^{-\lambda(e^{y_1} + e^{y_2-y_1})}$

$$= \frac{\eta^{\alpha}\eta^{\alpha}}{\Gamma(\alpha)} \cdot e^{-\eta(e^{y_1}+e^{y_2-y_1})} e^{\alpha y_2} \qquad Y_1, Y_2 \in R$$

7. (10 points) The probability that a perpetually hungry student brings a bag of trail mix to each and every class is $p_{\text{mix}} \in (0,1)$. If the student brings the trail mix bag to class, the chance that he eats from it during class is $p_{\text{eat}} \in (0,1)$. And if the student eats trail mix, the chances that the student becomes so thirsty as to need to borrow a classmate's water bottle is $p_{\text{water}} \in (0,1)$. Let X denote the number of times the student needs to borrow a water bottle during an n class semester. What is the distribution of X? You may assume that the student does not ask for water unless he eats trail mix and that the individual classes are statistically independent.

P(Ask for water)

= P(Thirsty | eat, bring) · P(eat | bring) · P(bring)

$= P_{water} \cdot P_{eat} \cdot P_{mix}$

for only 1 class. Denot $X_1, X_2 \cdots X_n$ are i.i.d Bernolli (Ask for water $= P_w P_e P_m$).

P(Borrow) $= P_w P_e P_M$

$\sum_{i=1}^{n} X_i \sim Bin(N, P_w P_e P_m)$

$\Rightarrow E(\sum_{i=1}^{n}) = N \cdot (P_w P_e P_m)$.

Bonus (10 points) You have become marooned on a desert island in the middle of the Pacific Ocean. Salt water has ruined your trusty calculator and a parrot is now your only means of companionship. You will either walk south or north along the beach, in search of civilization, but don't know which direction to choose. After some thought, you have decided to choose a north/south direction at random, but want each to have an equal probability. Unfortunately, the only way to generate any type of randomness is by flipping (tossing) the calculator high in the air, which is known to land face up more often than face down. Explain how to choose a direction at random (with probability 1/2 each) by flipping the calculator as you see fit.

↑ N

↓ S

First: Toss 100 times, record the N_{head} and N_{tail}. Then, $\hat{P}_{head} = \frac{N_{head}}{N_{head}+N_{tail}}$

then. toss 10 more times,

If # of head appears in this 10 times is $\geq \hat{P}_{head}$, then go north, o.w. go south.

Bonus. (Trivia, 1 point) Why is the Arctic circle placed where it is?

If I am the first man creating the map, I will put it anywhere as my wish.