Math Science 800, Fall 2010
Final Exam

NAME:

Directions: Show all work on the test to receive possible partial credit. Unsupported guesses will meet a red pen. You are allowed to use a calculator and an 8×11.5 formula sheet of your own construction.

1. (10 points) Suppose that X has the geometric distribution $P[X = k] = p(1-p)^{k-1}$ for $k = 1, 2, \ldots$. Show that X is memoryless in that

$$P[X > j + k | X > j] = P[X > k]$$

for every integer $j, k \geq 1$.

2. (10 points) The probability density function of a Beta random variable X has the form

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}, \qquad 0 < x < 1,$$

where α and β are nonnegative parameters. Derive an expression for the mean of this distribution.

3. (10 points) Prove the Weak Law of Large Numbers: If $X_1, X_2, \ldots$, is IID with mean μ and a finite variance $\mathrm{Var}(X_i) \equiv \sigma^2$, then $\bar{X} \xrightarrow{\mathcal{P}} \mu$ as $n \to \infty$.

4. (10 points) The joint density of X and Y is

$$f_{X,Y}(x,y) = c(x^2 - y^2)e^{-x}, \qquad x > 0, \quad -x \leq y \leq x$$

Find the constant c. What is the conditional distribution of Y given that $X = x$?

5. (10 points) Suppose that X and Y are nonnegative random variables with the joint density

$$f_{X,Y}(x, y) = \frac{x}{(1 + x)^2(1 + xy)^2}, \qquad x, y > 0.$$

What is the joint density of X and XY? What is the marginal density of X?

6. (10 points) Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Hint: Think Central Limit Theorem with Poisson random variables.

7. (10 points) Suppose that $X_i, i = 1, \ldots, n$, are independent Poisson random variables and that X_i has mean i. What is the distribution of $\sum_{i=1}^{n} X_i$?

8. (10 points) Suppose that X has a binomial distribution with n trials and success probability $p \in (0,1)$. What is $E[X^4]$?

9. (10 points) Suppose that the random triple $(X_1, X_2, X_3)'$ has a joint normal distribution with mean vector and covariance matrix

$$\vec{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 5 \end{pmatrix}.$$

Determine the joint distribution of $X_1 + X_2 + X_3$ and $X_1 + X_3$.

10. (10 points) The number of meteorites that land in the Atamaca desert each year has a Poisson distribution with mean $\lambda > 0$. The size of each falling meteorite is exponentially distributed with a mean of 4 ounces. What is the probability that no meteorites weighing more than a pound hit the desert in a year's time? You may assume any needed independence to make the problem well posed.

Bonus (10 points) Each cereal box contains one sticker of one of the 32 National Football League teams (assumed these are packed at random). A hungry child wants to collect at least one sticker from each team. How many boxes of cereal does the child expect to eat before obtaining this goal?

Bonus. (1 point) Yes or No: Did you fill out the course evaluation?

Bonus. (1 point) Tell your best joke within the King's English.