Chapter 2 Conditional Probability

Def $P(A|B) = P(A \cap B) / P(B)$ if $P(B) > 0$.

Many times, it is easier to compute a conditional probability than an ordinary probability.

Ex Poker (5-card) Hands

$P(\text{Flush in hearts}) = \binom{13}{5} / \binom{52}{5}$ as before

Let $A_i = \{ i^{\text{th}} \text{ card is a heart} \}$

We want $P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) =$

$$P(A_5 | A_1 \cap A_2 \cap A_3 \cap A_4)\, P(A_1 \cap A_2 \cap A_3 \cap A_4) = \cdots =$$

$$P(A_5 | A_1 \cap A_2 \cap A_3 \cap A_4)\, P(A_4 | A_1 \cap A_2 \cap A_3)\, P(A_3 | A_1 \cap A_2)\, P(A_2 | A_1)\, P(A_1)$$

$$= \frac{13}{52} \frac{12}{51} \frac{11}{50} \frac{10}{49} \frac{9}{48}$$

Def A & B are called independent if $P(A \cap B) = P(A)P(B)$

Note that if $P(B) > 0$ and A & B are independent

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

Likewise $P(B|A) = P(B)$ if $P(A) > 0$

Conditional Probabilites obey the axioms :

1) $P(S \mid B) = P(S \cap B) / P(B) = P(B)/P(B) = 1$

2) $P(A \mid B) = P(A \cap B)/P(B) > 0$

3) If $A_1, A_2, \ldots$ are disjoint

$$P\left(\bigcup_{n=1}^{\infty} A_n \,\middle|\, B \right) = P\left(\left(\bigcup_{n=1}^{\infty} A_n \right) \cap B \right) / P(B)$$

$$= P\left(\bigcup_{n=1}^{\infty} (A_n \cap B) \right) / P(B)$$

since $A_i \cap B$ are disjoint in i

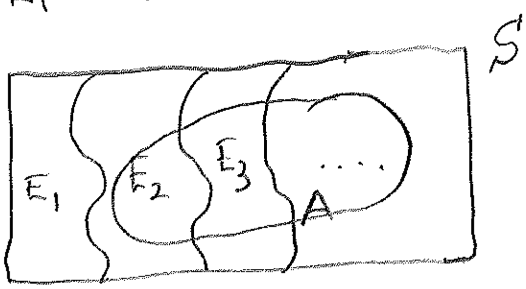
$$= \sum_{n=1}^{\infty} P(A_n \cap B) / P(B)$$

$$= \sum_{n=1}^{\infty} P(A_n \mid B)$$

The Law of Total Probability

Suppose $E_1, E_2, \ldots$ are disjoint and

$E_1 \cup E_2 \cup \ldots = S$



Then $P(A) = P\left(A \cap \left(\bigcup_{n=1}^{\infty} E_n \right)\right)$

$$= P\left(\bigcup_{n=1}^{\infty} (A \cap E_n) \right)$$

$$= \sum_{n=1}^{\infty} P(A \cap E_n) = \sum_{n=1}^{\infty} P(A \mid E_n) P(E_n)$$

Ex

Suppose a fair four-sided die is rolled and then a fair coin is tossed the number of times shown on the die.

$P(\text{Three Heads}) = ?$

$A_i = \{\text{Die toss is } i\}$

$$
\begin{aligned}
P(\text{Three Heads}) = \; & P(\text{Three Heads} \mid A_1)\, P(A_1) \\
& + P(\text{Three Heads} \mid A_2)\, P(A_2) \\
& + P(\text{Three Heads} \mid A_3)\, P(A_3) \\
& + P(\text{Three Heads} \mid A_4)\, P(A_4)
\end{aligned}
$$

$$= 0 \cdot \tfrac{1}{4} + 0 \cdot \tfrac{1}{4} + \tfrac{1}{8} \cdot \tfrac{1}{4} + \tfrac{4}{16} \cdot \tfrac{1}{4} = \tfrac{1}{32} + \tfrac{2}{32} = 3/32$$

$P(\text{Three Heads} \mid A_3) = \frac{1}{8}$

HHH ✓
HHT
HTH
HTT
THH
THT
TTH
TTT

$P(\text{Three Heads} \mid A_4) = 4/16$

H HHH
H H HT ✓
H H TH ✓
H H TT
H T HH ✓
H T HT
H T TH
H T TT

T HHH ✓
T HHT
T HTH
T HTT
T T HH
T T HT
T T TH
T T TT

Bayes Rule

$E_1, E_2, \ldots$ disjoint with

$$\bigcup_{n=1}^{\infty} E_n = S$$

$$P(E_j \mid A) = P(E_j \cap A) / P(A)$$

$$= \frac{P(E_j \cap A)}{\sum_{i=1}^{\infty} P(E_i \cap A)} = \frac{P(E_j \mid A)\,P(A)}{\sum_{i=1}^{\infty} P(E_i \mid A)\,P(A)}$$

Example

$P(\text{Covid}) = \frac{1}{100} = .01$

$P(\text{Test} + \mid \text{You have Covid}) = .95$

$P(\text{Test} + \mid \text{You don't have Covid}) = .05$

$E_1 =$ You have Covid

$E_2 =$ You don't have Covid

You take a test and it is +.

$P(\text{Covid} \mid \text{Test} +) =$

$P(\text{Covid} \cap \text{Test} +) \,/\, P(\text{Test} +) =$

$$\frac{P(\text{Test} + \mid \text{Covid})\,P(\text{Covid})}{P(\text{Test} + \mid \text{Covid})\,P(\text{Covid}) + P(\text{Test} + \mid \text{No Covid})\,P(\text{No Covid})} =$$

$$\frac{(.95)(.01)}{(.95)(.01) + (.05)(.99)} = \frac{.0095}{.0095 + .0495}$$

$$= .16107$$

Paradox of False Positives

Ex Roll two fair dice independently until a sum of 11 or 7 is achieved.
What is the chance the game ends with a sum of 7?

$$P(7 \text{ b-4 } 11) = \sum_{i=2}^{12} P(7 \text{ b-4 } 11 \mid \text{Sum on trial 1 is } i) \times P(\text{Sum on trial 1 is } i)$$

Unless Sum is 7 or 11 on Trial 1,

$$P(7 \text{ b-4 } 11 \mid \text{Sum on trial 1 is } i) = P(7 \text{ b-4 } 11)$$

$$P(7 \text{ b-4 } 11 \mid \text{Sum on trial 1 is } 7) = 1$$

$$P(7 \text{ b-4 } 11 \mid \text{Sum on trial 1 is } 11) = 0$$

So
$$P(7 \text{ b-4 } 11) = 1\, P(\text{Sum on trial 1 is } 7) + \left[\sum_{\substack{i=2 \\ i \neq 7,11}}^{12} P(\text{Sum on trial 1 is } i)\right] P(7 \text{ b-4 } 11)$$

$$\sum_{\substack{i=2 \\ i \neq 7,11}}^{12} P(\text{Sum on trial 1 is } i) = 1 - \frac{6}{36} - \frac{2}{36} = \frac{28}{36}$$

So
$$\left(1 - \frac{28}{36}\right) P(7 \text{ b-4 } 11) = P(\text{Sum on trial 1 is } 7)$$

$$P(7 \text{ b-4 } 11) = \frac{\frac{6}{36}}{\frac{8}{36}} = \frac{\frac{6}{36}}{\frac{6}{36} + \frac{2}{36}} = 3/4$$

Mutually Independent Events

$A_1, A_2, \dots, A_n$ are called mutually independent if

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_K}) = P(A_{i_1}) P(A_{i_2}) \dots P(A_{i_K}) \qquad K \leq n$$

for all choices $i_1, i_2, \dots, i_K \in \{1, 2, \dots, n\}$

Caution: There exist examples of events that are pairwise independent but not mutually independent:

Ex An urn has 4 balls, labeled 110, 101, 011, 000

$A_1 = \{$Draw ball with 1 in digit 1$\}$
$A_2 = \{$Draw ball with 1 in digit 2$\}$
$A_3 = \{$Draw ball with 1 in digit 3$\}$

$P(A_1) = 2/4 \quad P(A_2) = 2/4 \quad P(A_3) = 2/4$

$$P(A_1 \cap A_2) = \frac{1}{4} = P(A_1)P(A_2)$$
$$P(A_1 \cap A_3) = \frac{1}{4} = P(A_1)P(A_3)$$
$$P(A_2 \cap A_3) = \frac{1}{4} = P(A_2)P(A_3)$$

So the A_is are pairwise independent

But $P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3) = 1/8$

So the A_is are not mutually independent.