2015 First Year Exam: June 5, 2015

Solution of Problem 1 (AMS 203):

By definition $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$.

(a) We have

$$1 = \int_0^{40} a x dx = 800a.$$

So $f_X(x) = \frac{x}{800}, \quad 0 \leq x \leq 40$.

(b) From the problem statement $f_{Y|X}(y|x) = \frac{1}{2x}, \quad \text{for} \quad y \in [0, 2x]$. Therefore,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{1600}, & 0 \leq x \leq 40, \quad 0 \leq y \leq 2x \\ 0, & \text{otherwise.} \end{cases}$$

(c) Nic makes a positive profit if $Y > X$. This occurs with probability

$$P(Y > X) = \int\int_{y>x} f_{X,Y}(x,y) dy dx = \int_0^{40} \int_x^{2x} \frac{1}{1600} dy dx = \frac{1}{2}.$$

We could have also arrived at this answer by realizing that for each possible value of X, there is $1/2$ probability that $Y > X$.

(d) The joint density satisfies $f_{X,Z}(x,z) = f_X(x) f_{Z|X}(z|x)$. Since Z is conditionally uniformly distributed given X, $f_{Z|X}(z|x) = \frac{1}{2x}$ for $-x \leq z \leq x$. Therefore $f_{X,Z}(x,z) = 1/1600$ for $0 \leq x \leq 40$ and $-x \leq z \leq x$. The marginal density $f_Z(z)$ is calculated as

$$f_Z(z) = \int_x f_{Z|X}(z|x) dx = \int_{x=|z|}^{40} \frac{1}{1600} dx = \begin{cases} \frac{40-|z|}{1600}, & \text{if } |z| \leq 40, \\ 0, & \text{otherwise.} \end{cases}$$

One can conclude to the same answer, by setting $Z = Y - X$ and $W = X$, finding the jacobian matrix for $X = W$ and $Y = Z + W$,

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

so

$$f_Z(z) = \int f_{X,Y}(w, z+w)dw = \int_{|z|}^{40} \frac{1}{1600}dw = \begin{cases} \frac{40-|z|}{1600}, & \text{if } |z| \leq 40, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$E[Z] = \int_Z f_Z(z)dz = \int_{-40}^{0} \frac{40+z}{1600}dz + \int_0^{40} \frac{40-z}{1600}dz = \frac{1}{1600}\left\{(40z + \frac{z^2}{2})|_{-40}^{0} + (40z - \frac{z^2}{2})|_0^{40}\right\} = 2$$