BASKIN SCHOOL OF ENGINEERING

Department of Applied Mathematics and Statistics

2015 First Year Exam Retake: September 16, 2015

INSTRUCTIONS

If you are on the Applied Mathematics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 3(AMS 212A), 4(AMS 212B), 5(AMS 213), and 6(AMS 214).

If you are on the Statistics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 7(AMS 205B), 8(AMS 206B), 9(AMS 207), and 10(AMS256).

Please complete all required problems on the **supplied exam papers**. Write your exam ID number and problem number on each page. Use only the **front side** of each page.

Problem 1 (AMS 203):

Andrew shows up at BE 354 at time zero and spends his time exclusively typing e-mails. The times that his e-mails are sent follow a Poisson process with rate λ_A per hour. Let Y_1 and Y_2 be the times that Andrew's first and second e-mails were sent.

1. (30%) Find the probability density function (p.d.f.) of Y_1^2.
2. (20%) Find the conditional expectation of Y_2 given Y_1.
3. (30%) Find the joint p.d.f. of Y_1 and Y_2.
4. (20%) Let N be the number of e-mails sent by Andrew during the interval [0,1]. You are now told that λ_A is actually the realized value of an exponential random variable Λ with p.d.f. given by $f_\Lambda(\lambda) = 2e^{-2\lambda}$, $\lambda \geq 0$. Find $\mathrm{E}(N^2)$.

Hint: The inter-arrival times of a Poisson process follow an Exponential distribution with the same rate.

Problem 2 (AMS 211):

Question 1: Consider the function $f(x) = x$ on the interval $[0, 1]$.

(a) (15%) Describe the *even* and *odd* periodic extensions of $f(x)$ to a periodic function of period $L = 2$ on the entire real line.

(b) (10%) Which of the the two periodic functions you described in (a) will have a more rapidly converging Fourier series? Why?

(c) (25%) Compute the Fourier series of the two periodic functions from (a) to confirm your answer to (b).

Question 2: (25%) Find the solution $y(x)$ of the initial value problem

$$\frac{dy}{dx} + \frac{y}{x} = 2\sin x; \qquad y(\pi) = 1.$$

Question 3: (25%) Find the general solution of the differential equation

$$y'' + 2y' + 5y = 7$$

and show that $\lim_{t\to\infty} y(t)$ is independent of the initial values $y(0)$ and $y'(0)$.

Problem 3 (AMS 212A):

Solve the eigenvalue problem

$$\begin{cases} x^2 A_{xx} + x\, A_x + \lambda x^3 A = 0 \\ A(0) \text{ is finite}\ , \quad A(L) = 0 \end{cases}$$

(a) (50%) Express the eigenvalues $\{\lambda_n\}$ in terms of the zeros of Bessel function(s).

(b) (30%) Express the eigenfunctions $\{A_n(x)\}$ in terms of Bessel function(s).

(c) (20%) Write out the inner product with respect to which the eigenfunctions are orthogonal to each other.

Hint: You may need the information about Bessel functions given below.

The Bessel equation of order n,

$$x^2 f_{xx} + x\, f_x + (x^2 - n^2) f = 0,$$

has two independent solutions:

- The Bessel function of the first kind, $J_n(x)$, is finite at $x = 0$, and has an infinite sequence of zeros $\left\{ z_m^{(n)}, \quad m = 0, 1, 2, \ldots \right\}$;
- The Bessel function of the second kind, $Y_n(x)$, diverges to infinity at $x = 0$.

Problem 4 (AMS 212B):

Question 1: Consider the equation

$$x^2 - 4x = -2\epsilon$$

(a) (20%) Find the two-term expansion of each solution in the limit of small ϵ.

(b) (20%) Find the true solutions, and show that they recover these approximate solutions in the limit of small ϵ.

Question 2: Consider the equation and associated initial conditions:

$$f'' + f + \left(e^{\epsilon f} - 1\right) = 0 \ , \ f(0) = 1 \text{ and } f'(0) = 0$$

(a) (30%) Find the non-uniform two-term expansion of the solution in the limit of small ϵ.

(b) (30%) Using renormalization, or any other method of your choice, find the uniform one-term expansion of the solution in the limit of small ϵ.

Problem 5 (AMS 213):

Question 1: (50%) Consider the QR algorithm in order to find eigenvalues of a matrix. Perform one step of the QR algorithm using Householder transformation on

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 12 & 7 \end{bmatrix}.$$

Use the characteristic polynomial to obtain the two analytic eigenvalue of $\mathbf{A}$ and use them to compare with the result of the one step QR.

Question 2: (50%) Consider a general form of three-step ODE method for an IVP

$$u'(t) = f(t, u), u(0) = u_0,$$

given by

$$U^{n+2} = \alpha_1 U^n + \Delta t \Big\{ \alpha_2 f(t^n, U^n) + \alpha_3 f(t^{n+1}, U^{n+1}) + \alpha_4 f(t^{n+2}, U^{n+2}) \Big\}.$$

(a) Determine the coefficients $\alpha_i, i = 1, 2, 3, 4$ that ensure the method is third-order.

(b) Verify the resulting method is consistent.

Problem 6 (AMS 214):

Question 1: Consider the following system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + y(2 - 3x^2 - 2y^2).\end{aligned}$$

(a) (20%) Find the fixed point and discuss its stability.

(b) (30%) Use Poincaré-Bendixson's Theorem to show that the system has a closed orbit.

Question 2: Consider the system

$$\begin{aligned}\dot{x} &= x^2 + x - y \\ \dot{y} &= y(x + 2).\end{aligned}$$

(a) (20%) Find all fixed points and study their stability.

(b) (20%) Discuss the behavior of the trajectory starting from an arbitrary initial condition on the line $x + y = 0$. Hint: let $z(t) = x(t) + y(t)$ and show that $\dot{z} = z(x + 1)$.

(c) (10%) Can this system have a closed orbit? Explain your answer.

Problem 7 (AMS 205B):

Suppose that $X_1, \ldots, X_n \stackrel{IID}{\sim}$ Bernoulli(p_1) and $Y_1, \ldots, Y_m \stackrel{IID}{\sim}$ Bernoulli(p_2), where $n \geq 3$, $m \geq 4$, and $p_1, p_2 \in (0,1)$. Moreover, assume that the X_i and Y_j are mutually independent.

(a) (30%) Derive a set of complete sufficient statistics for (p_1, p_2). Is this collection of statistics complete under the additional assumption that $p_1 = p_2$? Justify your answer.

(b) (40%) Find the UMVUE for $h_1(p_1)h_2(p_2)$, where $h_1(p_1) = P_{p_1}(\sum_{i=1}^{n-2} X_i > X_{n-1} + X_n)$ and $h_2(p_2) = P_{p_2}(\sum_{j=1}^{m-2} Y_j > Y_{m-1} + Y_m)$.
(Hint: Start with an unbiased estimator $I(\sum_{i=1}^{n-2} X_i > X_{n-1} + X_n)$ of $h_1(p_1)$, where $I(\cdot)$ is the indicator function. You may find Rao-Blackwell and Lehman-Scheffe theorems useful.)

(c) (30%) Consider two different hypotheses tests: (i) $H_{0A} : p_1 = \frac{1}{2}$ vs. $H_{1A} : p_1 \neq \frac{1}{2}$; and (ii) $H_{0B} : p_2 = \frac{1}{3}$ vs. $H_{1B} : p_2 \neq \frac{1}{3}$. Let

$$
\begin{aligned}
\phi_1(X_1, ..., X_n) &= 1 \text{ if } X_1 + X_2 + X_3 > 1; \text{ and } \phi_1(X_1, ..., X_n) = 0 \text{ o.w.} \\
\phi_2(Y_1, ..., Y_m) &= 1 \text{ if } Y_1 + Y_2 + Y_3 + Y_4 > C; \text{ and } \phi_2(Y_1, ..., Y_m) = 0 \text{ o.w.}
\end{aligned}
$$

where ϕ_1 and ϕ_2 are the test functions for hypothesis (i) and (ii), respectively. Let α be the size of ϕ_1. Compute α and determine C so that ϕ_2 has level α. You do not need to simplify, even an equation with C as the only variable should be enough.

Problem 8 (AMS 206B):

Suppose we observe x positive responses out of m Bernoulli trials, each assumed conditionally independent given an unknown, common success probability θ.

1. (40%) Show that the Jeffreys prior for θ is given by a Beta$(0.5, 0.5)$ distribution. Assume the Jeffreys prior and derive the posterior distribution of θ given x.

2. (30%) Consider $\phi = \log(\theta/(1-\theta))$.

   (a) (10%) State the invariance property of Jeffreys prior distributions.

   (b) (20%) Find the Jeffreys prior for ϕ.

3. (30%) We plan to observe an additional n Bernoulli trials. Assume that our prior distribution for θ is Beta(a, b), where a and b are positive real numbers. Find the predictive distribution for the future number of successes Y.

A useful formula:

- The p.d.f. of a random variable X with a Beta(a, b) distribution is given by

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad x \in (0, 1).$$

Problem 9 (AMS 207):

Consider the following model for $i = 1, \dots, n$:

$$
\begin{aligned}
Y_i &= \begin{cases} Z_i, & \text{if } Z_i < 200 \\ 200, & \text{if } Z_i \geq 200 \end{cases} \\
Z_i \mid X_i &= \alpha + \beta X_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2)
\end{aligned}
$$

with n fixed, X_i known for all i, and the ϵ_i, $i = 1, ..., n$, independent. In addition, assume the prior: $p(\alpha, \beta, \sigma^2) \propto 1/\sigma^2$.

1. (35%) Write down $p(\Theta \mid X, Y)$ up to a proportionality constant, where $\Theta = [\alpha, \beta, \sigma^2]$, $X = [X_1, \dots, X_n]$, and $Y = [Y_1, \dots, Y_n]$.
2. (65%) Now augment your parameter space by considering $\Theta^* = [\Theta, Z_1, \dots, Z_n]$. Develop a Gibbs sampling scheme to obtain draws from the joint posterior distribution of $(\Theta^* \mid X, Y)$.

A useful formula:

- The p.d.f. of a random variable θ with an inverse-gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ is given by

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}, \quad \theta > 0.$$

Problem 10 (AMS 256):

Two groups of n observations are fitted using the following fixed-effects model:

$$y_{ij} = \mu + \theta_i + \epsilon_{ij}, \quad \epsilon_{ij} \overset{iid}{\sim} \mathrm{N}(0, \sigma^2),$$

where $i = 1, 2$ and $j = 1, \ldots, n$. To avoid identifiability issues, we set $\sum_{i=1}^{2} \theta_i = 0$ and remove θ_2 from the above formulation, that is, the model parameters are μ and θ_1.

1. (20%) Express these observations into a general linear model form, $\boldsymbol{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{\beta} = [\mu, \theta_1]^T$. That is, specify your $\boldsymbol{y}$, $\mathbf{X}$ and $\boldsymbol{\epsilon}$. Specify the rank of your design matrix.
2. (20%) Derive the least squares estimates (LSE) of μ and θ_1. Find their joint distribution.
3. (15%) Derive the LSE of σ^2 and find its distribution.
4. (15%) Express θ_1 as a linear function of $\boldsymbol{\beta}$. Is θ_1 estimable? Justify your answer.
5. (30%) Derive an F-test for $H_0 : \theta_1 = 0$ against $H_a : \theta_1 \neq 0$. Show the details. Be concise in your expression for the test statistic and specify its number of degrees of freedom under the null.