AMS 203 Solutions, FYE September 2015

1. Let $Z = Y_1^2$. Then we first find the cdf of Z and differentiate to find the pdf of Z:

$$F_Z(z) = P(Y_1^2 \le z) = P(-\sqrt{z} \le Y_1 \le \sqrt{z}) = F_{Y_1}(\sqrt{z}) - F_{Y_1}(-\sqrt{z})$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = f_{Y_1}(\sqrt{z})\frac{1}{2}z^{-1/2} + f_{Y_1}(-\sqrt{z})\frac{1}{2}z^{-1/2} = \frac{\lambda_A}{2z^{-1/2}}(e^{-\lambda_A\sqrt{z}} + e^{\lambda_A\sqrt{z}})$$

$$f_Z(z) = \begin{cases} \frac{\lambda_A}{2z^{-1/2}}(e^{-\lambda_A\sqrt{z}} + e^{\lambda_A\sqrt{z}}) & x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. We define T_2 as the second inter-arrival time in Andrew's Poisson process. Then

$$Y_2 = Y_1 + T_1$$

$$E[Y_2|Y_1] = E[Y_1 + T_2|Y_1] = Y_1 + E[T_2] = Y_1 + \frac{1}{\lambda_A}.$$

3.

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{Y_1}(y_1) f_{Y_2|Y_1}(y_2|y_1) \\ &= f_{Y_1}(y_1) f_{T_2}(y_2 - y_1) \\ &= \lambda_A e^{-\lambda_A y_1} \lambda_A e^{-\lambda_A(y_2 - y_1)} \\ &= \begin{cases} \lambda_A^2 e^{-\lambda_A y_2} & y_2 \ge y_1 \ge 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

4.

$$\begin{aligned} E[N^2] = E[E[N^2|\Lambda]] &= E[\text{Var}(N|\Lambda) + (E[N|\Lambda])^2] \\ &= E[\Lambda + \Lambda^2] \\ &= E[\Lambda] + \text{Var}(\Lambda) + (E[\Lambda])^2 \\ &= \frac{1}{2} + \frac{2}{2^2} = 1. \end{aligned}$$