BASKIN SCHOOL OF ENGINEERING

Department of Applied Mathematics and Statistics

First Year Exam: June 2014

Problem AMS 206B:

Assume a Bernoulli experiment in which you perform n independent trials with success probability θ and X counts the number of successes. Then, $X|\theta \sim Binomial(n, \theta)$ with n known.

1. (20 points) Assume you observe $X = x$. Find the posterior density of θ, $\pi(\theta|x)$, under a uniform prior $\theta \sim U(0, 1)$.

2. (40 points) Consider the loss function defined by

$$L(\theta, \hat{\theta}) = \frac{(\hat{\theta} - \theta)^2}{\theta(1 - \theta)}.$$

   Find $\hat{\theta}(x)$ the estimator that minimizes the Bayesian expected posterior loss under the scenario described in part 1. Note: Assume $x \neq 0$.

3. (40 points) Find Jeffreys prior on θ and the corresponding posterior distribution under such prior.

Some useful information:

- If $X \sim Binomial(n, \theta)$, X has probability mass function given by

$$Pr(X = x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}, \;\; x = 0, 1, \ldots, n.$$

- Beta distribution. If $X \sim Beta(\alpha, \beta)$ its pdf is given by

$$p(x|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}, \;\; 0 \leq x \leq 1.$$

  In addition, $E(X) = \alpha/(\alpha + \beta)$, and $V(X) = \alpha\beta/((\alpha + \beta)^2(\alpha + \beta + 1))$.

- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

SOLUTION

- *Part (a):*

$$\pi(\theta|x) \propto \theta^x(1-\theta)^{n-x},$$

therefore, the posterior is a $Beta(x+1, n-x+1)$.

- *Part (b):* The expected posterior loss is

$$E(L(\theta,\hat{\theta})|X=x) = \int_0^1 c\frac{(\hat{\theta}-\theta)^2}{\theta(1-\theta)}\theta^{x+1-1}(1-\theta)^{n-x+1-1}d\theta,$$

with $c = \Gamma(n+2)/(\Gamma(x+1)\Gamma(n-x+1))$. To minimize this function we take derivatives w.r.t. $\hat{\theta}$ and so,

$$\frac{dE(L(\theta,\hat{\theta})|X=x)}{d\hat{\theta}} = 2c\int_0^1 (\hat{\theta}-\theta)\theta^{x-1}(1-\theta)^{n-x-1}d\theta = 0,$$

implies that the optimal Bayes estimator is $\hat{\theta} = x/n$.

Note that the second derivative is

$$2c\int_0^1 \theta^{x-1}(1-\theta)^{n-x-1}d\theta > 0$$

- *Part (c):* Jeffreys prior is given by $\pi(\theta) = |I(\theta)|^{1/2}$ with

$$I(\theta) = -E_{X|\theta}\frac{d^2\log f(x|\theta)}{d\theta^2}.$$

In this case we obtain that $\pi(\theta) \propto \theta^{-1/2}(1-\theta)^{1/2}$, which corresponds to a $Beta(1/2, 1/2)$. The corresponding posterior is a $Beta(x+1/2, n-x+1/2)$.