BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

Student number: ____________________

First Year Exam: June 15th 2009

INSTRUCTIONS

You must answer both questions in Part A pertaining to courses AMS 205 and AMS 211.

You must also answer 4 out of the 8 questions in Part B. Please indicate clearly which problems you are selecting to be graded.

PART A

Problem 1 (AMS 205) :

For each one of the following statements, decide if it is true or false. You must justify your answer with a short proof, counterexample, or reference to a standard theorem.
(**Note:** All parts of the problem have equal weight.)

1. If $\tilde{\theta}(X)$ is an unbiased estimator for θ, it is also consistent.

   Answer: False. Suppose $\mathsf{E}X = \theta$ and take $\tilde{\theta}(X) = X_1$.

2. If the maximum likelihood estimator exists and is unique, then it is a function of a minimal sufficient statistic for the problem.

   Answer: True. Use the factorization theorem.

3. Let $X_1, \ldots, X_n$ be a random sample where $X_i \sim \mathsf{Uni}(0, \theta)$. Suppose you want to test $H_0 : \theta = \theta$ vs. $H_a : \theta \neq \theta_0$ using the likelihood ratio test. Then, as $n \to \infty$, $-2 \log \Lambda \xrightarrow{D} \chi 2_1$, where Λ is the likelihood ratio.

   Answer: False. The support of the density depends on θ, so the conditions of the theorem do not apply.

4. In hypothesis testing, the p-value is the probability that H_0 is true.

   Answer: False.

5. Let $X_1, \ldots, X_m$ be a random sample where $X_i \sim \mathsf{Bin}(n, \theta)$. The maximum likelihood estimator for $\phi = \log\left(\frac{\theta}{1-\theta}\right)$ is $\hat{\phi} = \log\left(\frac{\bar{x}}{n-\bar{x}}\right)$

   Answer: True. Use the invariance of the MLE.

6. Let $X_1, \ldots, X_n$ be a random sample where $X_i \sim p(\cdot|\theta)$, where p satisfies the usual regularity conditions. If an unbiased estimator $\tilde{\theta}$ for θ has variance equal to $1/\mathsf{E}_\theta\left(\left[\frac{\partial}{\partial\theta} \log p(x_1, \ldots, x_n|\theta)\right] 2\right)$ then it is the minimum variance unbiased estimator.

   Answer: True. This is the Cramer-Rao bound.

Problem 2 (AMS 211) :

(a) [33%] Find and classify the stationary points of the function

$$f(x, y) = 3xy - x^3 - y^3$$

(b) [34%] A discrete dynamical system (or Markov chain) is defined by a transition matrix $\mathbf{A}$ between one state $\mathbf{x}_k$ and the next $\mathbf{x}_{k+1}$ such that $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$. Show that any steady state of the system is described by a particular eigenvalue/eigenvector problem. If

$$A = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix},$$

find the general form of the steady-state solutions $\mathbf{x_s}$.

(c) [33%] Solve

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$$

subject to

$$y(0) = 0,\ y(\pi/3) = 1$$

Solution :

1.

$$\frac{\partial f}{\partial x} = 3y - 3x^2 = 0$$
$$\frac{\partial f}{\partial y} = 3x - 3y^2 = 0$$
$$\rightarrow (x, y) = (0, 0), (1, 1)$$
$$\frac{\partial^2 f}{\partial x^2} = -6x$$
$$\frac{\partial^2 f}{\partial y^2} = -6y$$
$$\frac{\partial^2 f}{\partial x \partial y} = 3$$
$$(0, 0) : (\frac{\partial^2 f}{\partial x \partial y}) - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 9 > 0 \rightarrow \text{saddle}$$
$$(1, 1) : (\frac{\partial^2 f}{\partial x \partial y}) - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = -27 < 0 \text{ and } \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} < 0 \rightarrow \text{maximum}$$

2. Steady-state $\rightarrow x_{k+1} = x_k$

   Therefore $x_k = Ax_k$ or $(A - I)x_k = 0$. This is an eigenvalue problem $Ax = \lambda x$ with eigenvalue $\lambda = 1$.

$$A - I = \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix},$$

Therefore, e-vector and steady state of the system is any scalar multiple of

$$x_s = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

3. Auxiliary equation: $\lambda^2 - 6\lambda + 13 = 0$

   Solution $\lambda = 3 \pm i2$

   Therefore general solution $y(t) = Ae^{3x}\cos(2x) + Be^{3x}\sin(2x)$

   Initial conditions: $y(0) = 0 \rightarrow A = 0$ and $y(\pi/3) = 1 \rightarrow B = \frac{2}{\sqrt{3}}e^{-\pi}$

   Particular solution: $y(t) = \frac{2}{\sqrt{3}}e^{-\pi}e^{3x}\sin(2x)$

PART B

Problem 3 (AMS 206) :

Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with both μ and σ^2 unknown, and suppose we use noninformative priors, $f(\mu) \propto 1$ and $f(\sigma^2) \propto \frac{1}{\sigma^2}$. Show that the marginal posterior distribution for μ is a location-scale t with $\nu = n-1$ degrees of freedom. How does this compare to the frequentist result? (You don't need to re-derive the frequentist result.)

Information on distributions that you may find helpful:

Distribution	Density Function
Inverse Gamma	$f(x\|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}$
Normal	$f(x\|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$
Location-Scale t	$f(x\|\nu,\mu,\sigma^2) = \frac{\Gamma\left[\frac{1}{2}(\nu+1)\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi\sigma^2}} \left[1 + \frac{1}{\nu\sigma^2}(x-\mu)^2\right]^{-(\nu+1)/2}$

Solution :

Recall that $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$.

$$
\begin{aligned}
f(\mu, \sigma^2|\mathbf{x}) &\propto \left(2\pi\sigma^2\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\}\left(\sigma^2\right)^{-1} \\
&\propto \left(\sigma^2\right)^{-n/2-1} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\} \\
f(\mu|\mathbf{x}) &\propto \int f(\mu,\sigma^2|\mathbf{x})d\sigma^2 \\
&\propto \frac{\left[\frac{1}{2}\sum(x_i-\mu)^2\right]^{n/2}}{\Gamma(n/2)} \\
&\propto \left[\sum(x_i-\mu)^2\right]^{n/2} \\
&\propto \left[\sum x_i^2 - 2\mu\sum x_i + n\mu^2\right]^{n/2} \\
&\propto \left[n(\mu-\bar{x})^2 + \sum x_i^2 - n\bar{x}^2\right]^{n/2} \\
&\propto \left[n(\mu-\bar{x})^2 + (n-1)s^2\right]^{n/2} \\
&\propto \left[1 + \frac{1}{\frac{n-1}{n}s^2}(\mu-\bar{x})^2\right]^{n/2}
\end{aligned}
$$

Thus the marginal posterior for μ is a location-scale t with $n-1$ degrees of freedom, location parameter $\bar{x}$, and scale parameter s^2/n. This is the same distribution a frequentist would use as their sampling distribution.

Problem 4 (AMS 207) :

Consider the following model

$$\begin{aligned} y_{i,j} &\sim N(\theta_i, v) \;\; i = 1 : I, j = 1 : n_i, && (1) \\ \theta_i &\sim N(\mu, \sigma^2), \;\; i = 1 : I, && (2) \end{aligned}$$

where v, μ and σ^2 are known.

1. (35%) Write the model as a linear regression model and find

$$p(\theta_1, \ldots, \theta_I | \mu, v, \sigma^2, \mathbf{y}).$$

   What is the mode of this distribution?

2. (65%) Now assume that μ is unknown, with

$$\begin{aligned} \mu &\sim N(m, \phi), \\ \phi &\sim Inv - \chi^2(\nu), \end{aligned}$$

   and m and ν known. Write down the joint posterior $p(\theta_1, \ldots, \theta_I, \mu, \phi | \mathbf{y})$, (up to a proportionality constant) and show that samples from this posterior distribution can be obtained via Gibbs sampling. Provide the details of such algorithm (i.e., find the all the full conditional distributions).

Some useful distributions.

- inverse-χ^2 : $x \sim Inv - \chi^2(n)$,

$$p(x) \propto x^{-(n/2+1)} e^{-1/2x};$$

- scaled-inverse-χ^2 : $x \sim Inv - \chi^2(n, s^2)$,

$$p(x) \propto x^{-(n/2+1)} e^{-ns^2/(2x)}.$$

Solution :

1. Linear model representation:

$$\begin{aligned} \mathbf{y} &= \mathbf{X}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \;\; \boldsymbol{\epsilon} \sim N(\mathbf{0}, \Sigma_y) \\ \boldsymbol{\theta} &= \mathbf{1}\mu + \boldsymbol{\xi}, \;\; \boldsymbol{\xi} \sim N(\mathbf{0}, \Sigma_\theta), \end{aligned}$$

   with $\Sigma = v\mathbf{I}_N$, $\Sigma_\theta = \sigma^2 \mathbf{I}_I$, and $N = \sum_i n_i$.

   The posterior distribution is multivariate normal. The mode could be found by maximizing the log-posterior, by writing the model above as a linear regression model with a non-informative prior and using some results from class, or by using properties of the multivariate normal distribution. The first and second options appear below.

(a) The log-posterior is given by

$$\log(p(\theta_1,\ldots,\theta_I|\mathbf{y})) = K - \frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i}(y_{i,j}-\theta_i)^2}{2v} - \frac{\sum_{i=1}^{I}(\theta_i-\mu)^2}{2\sigma^2}.$$

Now,

$$\frac{\partial \log(p(\boldsymbol{\theta}|\mathbf{y}))}{\partial\theta_i} = \frac{\sum_{j=1}^{n_i}(y_{i,j}-\theta_i)}{v} - \frac{(\theta_i-\mu)}{\sigma^2},$$

and so,

$$\hat{\theta}_i = \frac{\mu/\sigma^2 + \sum_{j=1}^{n_i} y_{i,j}/v}{1/\sigma^2 + n_i/v}.$$

(b) Write the model as $\mathbf{y}_* = \mathbf{X}_*\boldsymbol{\theta} + \boldsymbol{\epsilon}_*$, with $\boldsymbol{\epsilon}_* \sim N(\mathbf{0}, \Sigma_*)$, $p(\boldsymbol{\theta}) \propto 1$ and

$$\mathbf{y}_* = \begin{pmatrix} \mathbf{y} \\ \mu \\ \vdots \\ \mu \end{pmatrix}, \quad \mathbf{X}_* = \begin{pmatrix} \mathbf{X} \\ \mathbf{I}_I \end{pmatrix}, \quad \Sigma_* = \text{blockdiag}(\Sigma_y, \Sigma_\theta).$$

Then, the posterior distribution is $p(\boldsymbol{\theta}|\mathbf{y}) = N(\hat{\boldsymbol{\theta}}, \mathbf{V}_\theta)$, with

$$\hat{\boldsymbol{\theta}} = (\mathbf{X}_*'\Sigma_*^{-1}\mathbf{X}_*)^{-1}\mathbf{X}_*'\Sigma_*^{-1}\mathbf{y}_*,$$

and $\mathbf{V}_\theta = (\mathbf{X}_*'\Sigma_*^{-1}\mathbf{X}_*)^{-1}$, and so, the posterior mode is $\hat{\boldsymbol{\theta}}$.

2. Gibbs sampling.

(a) $p(\theta_i|\boldsymbol{\theta}_{(-i)}, \mu, \phi, \mathbf{y}) = N(\theta_i|\hat{\theta}_i, V_{\theta_i})$, with

$$\begin{aligned} V_{\theta_i} &= \left(\frac{1}{\sigma^2} + \frac{n_i}{v}\right)^{-1} \\ \hat{\theta}_i &= V_{\theta_i}\left(\frac{\mu}{\sigma^2} + \frac{\sum_{j=1}^{n_i} y_{i,j}}{v}\right). \end{aligned}$$

(b) $p(\mu|\boldsymbol{\theta}, \phi, \mathbf{y}) = N(\mu|m_\mu, V_\mu)$, with

$$\begin{aligned} V_\mu &= \left(\frac{I}{\sigma^2} + \frac{1}{\phi}\right)^{-1}, \\ m_\mu &= V_\mu\left(\frac{\sum_{i=1}^{I}\theta_i}{\sigma^2} + \frac{m}{\phi}\right) \end{aligned}$$

(c) $p(\phi|\mu, \boldsymbol{\theta}, \mathbf{y}) = Inv-\chi^2(\phi|\nu+1, (\mu-m)^2+1)$.

Problem 5 (AMS 212A) :

The problem contains 2 questions: (A) and (B). Wrong answers to a pass/fail (P/F) question will give you 0 on the whole AMS212A Problem. (B = 60% - 80%; A = 80% - 100%).

(A) (20%) Consider the linear PDE

$$a u_{xx} + b u_{xy} + c u_{yy} = 0 \tag{3}$$

- Describe how to use coefficients a, b and c to determine the type of the PDE. What condition needs to be satisfied for this PDE to be (a) elliptic (b) parabolic and (c) hyperbolic.
- (P/F question) For each of the 3 types of equations named above, give one well-known example of real physical equation which is of the same type .

(B) (80%) Solve the following problem using whichever method you prefer, for $\omega \neq 2c$.

$$\begin{aligned}
&u_{tt} = c^2 u_{xx} + \sin(\omega t)\sin(2x) \\
&u(0,t) = u(\pi,t) = 0 \\
&u(x,0) = 0 \\
&u_t(x,0) = 0
\end{aligned}$$

(c, ω are known constants). What happens for $\omega = 2c$? You may answer the question by solving the problem mathematically or by explaining what happens physically.

Solution :

Problem 6 (AMS 212B) :

This problem contains 2 questions: (A) and (B).

(A) (50%) Find **the leading term** expansion of

$$I = \int_0^2 t \cdot \exp\left[\lambda(3t - t^3)\right] dt \qquad \text{as } \lambda \to +\infty$$

Hint: You may need the expansion formulas for the Laplace method. If you do not remember these expansion formulas, you can derive them from $\int_{-\infty}^{+\infty} \exp(x^2)dx = \sqrt{\pi}$.

Solution :

Integral I is of the form $\int_0^1 f(t) \exp\left[\lambda h(t)\right] dt$ with $h(t) = 3t - t^3$. $h(t)$ satisfies

$$h'(t) = 3(1-t^2) = \begin{cases} > 0, & 0 \le t < 1 \\ = 0, & t = 1 \\ < 0, & t > 1 \end{cases}$$

$$h''(1) = -6$$

Thus, function $h(t)$ attains a turning point maximum at $t = 1 \in (0, 2)$ with $h''(1) < 0$. Applying the Laplace method, we obtain

$$\begin{aligned} I &\sim \int_0^2 f(1) \exp\left[\lambda\left(h(1) + \frac{h''(1)}{2}(t-1)^2\right)\right] dt \\ &\sim \exp(\lambda h(1)) f(1) \sqrt{\frac{2\pi}{(-h''(1))}} \frac{1}{\sqrt{\lambda}} \\ &= \exp(2\lambda) \sqrt{\frac{\pi}{3}} \frac{1}{\sqrt{\lambda}} \end{aligned} \tag{4}$$

If we do not use the formula for the Laplace method directly, we have

$$\begin{aligned} I &\sim \int_0^2 f(1) \exp\left[\lambda\left(h(1) + \frac{h''(1)}{2}(t-1)^2\right)\right] dt \\ &= \exp(\lambda h(1)) f(1) \int_{-1}^1 \exp\left[\lambda \frac{h''(1)}{2} s^2\right] ds \\ &\sim \exp(\lambda h(1)) f(1) \sqrt{\frac{2}{(-h''(1))\lambda}} \int_{-\infty}^{+\infty} \exp\left[-x^2\right] dx \\ &= \exp(\lambda h(1)) f(1) \sqrt{\frac{2\pi}{(-h''(1))}} \frac{1}{\sqrt{\lambda}} \\ &= \exp(2\lambda) \sqrt{\frac{\pi}{3}} \frac{1}{\sqrt{\lambda}} \end{aligned} \tag{5}$$

(B) (50%) Find <u>**the leading term**</u> expansion of

$$I = \int_0^1 \sqrt{1+t} \cdot \sin\left[\lambda(e^t - t)\right] dt \qquad \text{as } \lambda \to +\infty$$

<u>Hint</u>: You may need the expansions of $\int_0^a \cos(\lambda s^2)\, ds$ and $\int_0^a \sin(\lambda s^2)\, ds$. If you do not remember these expansions, you can derive them from $\int_0^{+\infty} \cos(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ and $\int_0^{+\infty} \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

<u>**Solution :**</u>
Integral I is of the form $\int_0^1 f(t) \sin[\lambda h(t)]\, dt$ with $h(t) = e^t - t$. $h(t)$ satisfies

$$h'(t) = e^t - 1 = \begin{cases} = 0, & t = 0 \\ > 0, & t > 0 \end{cases}$$

Thus, $t = 0$ is the only stationary point, and it satisfies $h''(0) = 1 \neq 0$. Using the method of stationary phase, we have

$$\begin{aligned} I &\sim \int_0^1 \sin\left[\lambda\left(1 + \frac{1}{2}t^2\right)\right] dt \\ &= \sin(\lambda) \int_0^1 \cos\left(\frac{\lambda}{2}t^2\right) dt + \cos(\lambda) \int_0^1 \sin\left(\frac{\lambda}{2}t^2\right) dt \\ &\sim [\sin(\lambda) + \cos(\lambda)]\, \frac{1}{2}\sqrt{\pi}\frac{1}{\sqrt{\lambda}} \end{aligned}$$

Problem 5 (AMS 213) :

Problem 5.1 [20%] Consider the following Adams Bashforth method

$$x_{k+1} = x_k + \frac{h}{2}\left(3f(x_k, t_k) - f(x_{k-1}, t_{k-1})\right), \qquad k = 1, 2, \cdots$$

for the initial value problem

$$\frac{dx}{dt} = f(x,t), \qquad x(0) = x_0$$

Is this scheme explicit or implicit, single step or multistep? What is the order of the local truncation error? (only a brief explanation is needed.)

Problem 5.2 [40%] Let A be a nonsingular matrix. Consider the following iterative process for solving $Ax = b$.

$$Px^{k+1} = b - Qx^k, \quad k = 0, 1, 2, \cdots$$

where P and Q are $n \times n$ matrices satisfying

$$\begin{aligned} P + Q &= A \\ \|P^{-1}\| \cdot \|Q\| &= \delta \; < \; 1 \end{aligned}$$

for some induced norm $\|\cdot\|$. Show that, from any x^0, x^k converges to the solution of $Ax = b$.

Problem 5.3 [40%] Let $u(x,t)$ be a sufficiently smooth function of x and t. Denote $u_i^n = u(i\Delta x, n\Delta t)$, where Δx and Δt are the mesh sizes of a rectangular grid. Show that the finite difference scheme

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} = 0$$

1. is a consistent approximation of $u_t - u_{xx} = 0$, if $\Delta t = \Delta x^2$;
2. is a consistent approximation of $u_t + u_{tt} - u_{xx} = 0$, if $\Delta t = \Delta x$.

Solution :

Problem 5.1 It is an explicit, multistep scheme (half of the credit). The local truncation error is of order 3 (half of the credit).

Problem 5.2 Let $e^k = x^k - x$. Based on iterative formula and the equation $P + Q = A$,

$$\begin{aligned} e^{k+1} &= P^{-1}b - P^{-1}Qx^k - x \\ &= P^{-1}Ax - P^{-1}(A - P)x^k - x \\ &= -P^{-1}Ae^k + e^k \\ &= -P^{-1}Qe^k \end{aligned}$$

Therefore;

$$\|e^{k+1}\| \;\leq\; \|P^{-1}\|\|Q\| \cdot \|e^k\| \;\leq\; \delta\|e^k\| \;\leq\; \delta^k\|e^0\|$$

(If the student write down the error equation correctly, partial credit can be given; say 50% out of total 40%).

Problem 5.3 From Taylor expansions of u_{i+1}^n, u_i^{n+1}, u_i^{n-1} and u_{i-1}^n at (x_i, t_n) point,

$$\begin{aligned}
\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} &= u_t\Big|_{(x_i,t_n)} + O(\Delta t^2) \\
\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} &= u_{xx}\Big|_{(x_i,t_n)} - \frac{\Delta t^2}{\Delta x^2} u_{tt}\Big|_{(x_i,t_n)} + O(\Delta x) + \frac{1}{\Delta x^2} O(\Delta t^3)
\end{aligned}$$

If $\Delta t = \Delta x^2$,

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} = u_t\Big|_{(x_i,t_n)} - u_{xx}\Big|_{(x_i,t_n)} + O(\Delta x)$$

If $\Delta t = \Delta x$,

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} = u_t\Big|_{(x_i,t_n)} + u_{tt}\Big|_{(x_i,t_n)} - u_{xx}\Big|_{(x_i,t_n)} + O(\Delta x)$$

(This problem is to test the basic concept of consistency and how to analysis the consistency of a finite difference scheme. If the students provide correct Taylor expansion formulas, partial credit, say 20% out of 40%, is deserved. Then, 10% is for $\Delta t = \Delta x^2$ case; and the other 10% is for $\Delta t = \Delta x$ case.)

Problem 8 (AMS 214) :

The problem contains 3 questions: (A), (B) and (C). They are all short questions, you should not be spending more than 10 minutes on each. Wrong answers to a pass/fail (P/F) question will give you 0 on the whole AMS214 Problem. (B = 80% - 90%; A = 90% - 100%).

(A) (20%) Consider the 1D system $\dot{x} = 5 - re^{-x^2}$.

- Find the expression for the fixed points as a function of r, and determine their stability (P/F question).
- Find the bifurcation point.
- Draw a complete annotated bifurcation diagram.
- Name the bifurcation.

(B) (40%) Consider the system

$$\dot{R} = aR - bRF$$
$$\dot{F} = -cF + dRF$$

where a,b,c and d are positive. You can also assume that $R \geq 0$ and $F \geq 0$.

- Find the fixed points of the model and study their stability.
- Show formally that the model predicts many periodic orbits. (Hint: you may want to find a conserved quantity).
- Draw a complete annotated phase portrait for the system.
- For some initial condition (R_0, F_0) that is not one of the fixed points, sketch on the same diagram $R(t)$ and $F(t)$.

(C) (40%) Consider the Tent map

$$x_{n+1} = rx_n \text{ if } x_n \in [0, 1/2]$$
$$x_{n+1} = r - rx_n \text{ if } x_n \in (1/2, 1]$$

where $r > 0$.

- Find the fixed points and study their stability (P/F question).
- At what value of r (call it r_c) do the fixed points bifurcate? Draw two cobweb diagrams: one for r slightly below r_c and one for r slightly above r_c.
- Prove that for $r > r_c$ nearly all trajectories are chaotic.

Solution :

Problem 9 (AMS 256) :

Consider the model defined as follows. For $i = 1 : N$, let

$$\begin{aligned} y_i &= \epsilon_i, \quad i \neq k, i \neq k+1, i \neq k+2 \\ y_k &= \lambda_1 + \epsilon_k, \\ y_{k+1} &= -\lambda_1 + \lambda_2 + \epsilon_{k+1}, \\ y_{k+2} &= -\lambda_2 + \epsilon_{k+2}, \end{aligned}$$

where k is a fixed integer, $1 \leq k \leq N-2$ and the ϵ_is are iid $N(0, \sigma^2)$ variables. Let $\boldsymbol{\beta} = (\lambda_1, \lambda_2)'$ and suppose σ^2 is known.

1. (40%) Derive the LSE of $\boldsymbol{\beta}$, denoted as $\hat{\boldsymbol{\beta}}$.
2. (30%) Find the distribution of $(\hat{\boldsymbol{\beta}}|\sigma^2)$. Are $\hat{\lambda}_1$ and $\hat{\lambda}_2$ independent? (Justify your answer).
3. (30%) Find a 95% confident interval for $\lambda_1 + \lambda_2$.

Solution :

1. Writing the model in matrix form $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, we have that

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} y_k - y_{k+1} \\ y_{k+1} - y_{k+2} \end{pmatrix},$$

and so

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix} = \begin{pmatrix} (2y_k - y_{k+1} - y_{k+2})/3 \\ (y_k - y_{k+1} + 2y_{k+2})/3 \end{pmatrix}.$$

2. $(\hat{\boldsymbol{\beta}}|\sigma^2) \sim N(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$ and so, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are not independent.
3. 95% CI for $\lambda_1 + \lambda_2$ is given by

$$\hat{\lambda}_1 + \hat{\lambda}_2 \pm 1.96\sqrt{2\sigma^2}.$$

Problem 10 (AMS 274) :

1. Consider the Poisson distribution with mean μ (denoted by Poisson$(\cdot;\mu)$).
   • Write the probability mass function of the distribution in the exponential dispersion family form, and provide the expressions for the natural parameter and the variance function.

   (20%)

2. Let y_i be realizations of independent random variables Y_i with Poisson$(\cdot;\mu_i)$ distributions, for $i = 1,\ldots,n$ (where $\mathrm{E}(Y_i) = \mu_i$). Consider the Poisson generalized linear model (glm) based on link function $\log(\mu_i) = \boldsymbol{x}_i^T\boldsymbol{\beta}$, where $\boldsymbol{x}_i$ is the covariate vector for response y_i and $\boldsymbol{\beta}$ is the vector of regression coefficients.
   • Obtain the deviance for the comparison of the Poisson glm above with the *full* model that includes a different μ_i for each y_i.

   (35%)

3. Consider a Bayesian formulation for the special case of the Poisson glm from part 2 based on a single covariate with values x_i. That is,

   $$y_i \mid \beta_1, \beta_2 \overset{ind.}{\sim} \text{Poisson}(y_i; \mu_i = \exp(\beta_1 + \beta_2 x_i)), \quad i = 1, ..., n$$

   with $\mathrm{N}(a_j, b_j^2)$ priors for β_j, $j = 1, 2$.
   • Develop a Metropolis-Hastings algorithm to sample from $p(\beta_1, \beta_2 \mid \text{data})$, the posterior distribution of (β_1, β_2), where data $= \{(y_i, x_i) : i = 1, ..., n\}$. Moreover, provide the expression for $p(y_0 \mid x_0, \text{data})$, the posterior predictive distribution of new (unobserved) response y_0 associated with a specified covariate value x_0.

   (45%)

Solution :

1. For the Poisson p.m.f. with mean μ, we have

   $$f(y;\mu) = \mu^y \exp(-\mu)(y!)^{-1} = \exp\{y\log(\mu) - \mu - \log(y!)\}, \quad y \in \{0, 1, 2, ...\}$$

   which is of the EDF form with natural parameter $\theta = \log(\mu)$, $b(\theta) = \exp(\theta)$, and dispersion parameter $\phi = 1$. The variance function is given by $b''(\theta) = \exp(\theta)$ $(= \mu)$.

2. Under the full model, the ML estimates are $\tilde{\mu}_i = y_i$ and thus, $\tilde{\theta}_i = \log(y_i)$. Under the Poisson glm with the logarithmic link function, we have $\hat{\theta}_i = \log(\hat{\mu}_i) = \boldsymbol{x}_i^T\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is the vector of ML estimates for $\boldsymbol{\beta}$.
   The deviance is given, in general, by $D = -2\left[\ell(\hat{\boldsymbol{\theta}};\text{data}) - \ell(\tilde{\boldsymbol{\theta}};\text{data})\right]$, where $\hat{\boldsymbol{\theta}} = \{\hat{\theta}_i : i = 1, ..., n\}$, $\tilde{\boldsymbol{\theta}} = \{\tilde{\theta}_i : i = 1, ..., n\}$, and $\ell(\boldsymbol{\theta};\text{data})$ is the log-likelihood function

(expressed as a function of the natural parameters). Using the Poisson likelihood form and the expressions for $\tilde{\theta}_i$ and $\hat{\theta}_i$ above, we obtain

$$D = 2\sum_{i=1}^{n}\left[y_i \log\left(\frac{y_i}{\hat{\mu}_i}\right) - (y_i - \hat{\mu}_i)\right] = 2\sum_{i=1}^{n}\left[(y_i \log(y_i) - y_i) + (\exp(\boldsymbol{x}_i^T\hat{\boldsymbol{\beta}}) - y_i(\boldsymbol{x}_i^T\hat{\boldsymbol{\beta}}))\right]$$

3. We have

$$\begin{aligned} p(\beta_1, \beta_2 \mid \text{data}) &\propto \exp\left[-\tfrac{(\beta_1 - a_1)^2}{2b_1^2} - \tfrac{(\beta_2 - a_2)^2}{2b_2^2} + \textstyle\sum_{i=1}^{n}(y_i(\beta_1 + \beta_2 x_i) - \exp(\beta_1 + \beta_2 x_i))\right] \\ &= h(\beta_1, \beta_2). \end{aligned}$$

A random walk Metropolis-Hastings algorithm, with a normal proposal distribution, would proceed as follows:
• Let $(\beta_1^{(t)}, \beta_2^{(t)})$ be the current state of the chain.
• Draw (β_1^*, β_2^*) from a normal distribution with mean vector $(\beta_1^{(t)}, \beta_2^{(t)})$ and covariance matrix D, which can be specified through the inverse of the expected Fisher information matrix estimated at the MLEs for (β_1, β_2).
• Set $(\beta_1^{(t+1)}, \beta_2^{(t+1)}) = (\beta_1^*, \beta_2^*)$ with probability $\min\{1, h(\beta_1^*, \beta_2^*)/h(\beta_1^{(t)}, \beta_2^{(t)})\}$, and $(\beta_1^{(t+1)}, \beta_2^{(t+1)}) = (\beta_1^{(t)}, \beta_2^{(t)})$ otherwise.

The full Bayesian model that includes the distribution component for (y_0, x_0) is given by

$$p(y_0, \beta_1, \beta_2 \mid x_0, \text{data}) = \text{Poisson}(y_0; \mu_0 = \exp(\beta_1 + \beta_2 x_0))p(\beta_1, \beta_2 \mid \text{data})$$

and, therefore, the posterior predictive distribution of new response y_0, associated with specified covariate value x_0, is given by

$$p(y_0 \mid x_0, \text{data}) = \int\int \text{Poisson}(y_0; \exp(\beta_1 + \beta_2 x_0))p(\beta_1, \beta_2 \mid \text{data})\, \mathrm{d}\beta_1 \mathrm{d}\beta_2.$$