Two-way ANOVA Model

- Full model with interactions

$$y_{i,j,k} = \mu + \alpha_i + \beta_j + \underset{\sim}{\gamma}_{i,j} + \varepsilon_{i,j,k}$$

$$\varepsilon_{i,j,k} \overset{iid}{\sim} N(0,\sigma^2)$$

$i = 1:a$

$j = 1:b$

$k = 1:n_{i,j}$

- Model with no interactions

$$y_{i,j,k} = \mu + \alpha_i + \beta_j + \varepsilon_{i,j,k}$$

$$\underset{N}{y} = \underbrace{X}_{(N \times p)} \underbrace{\beta}_{p} + \varepsilon \qquad \hat{\beta} = \underbrace{(X'X)}^{-1} X' y$$

$$\underbrace{\begin{pmatrix} y_{1,1,1} \\ \vdots \\ y_{1,1,n_{11}} \\ \vdots \\ y_{a,b,1} \\ \vdots \\ y_{a,b,n_{ab}} \end{pmatrix}}_{y} \qquad \underset{\sim}{\beta} = \begin{pmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \\ \beta_1 \\ \vdots \\ \beta_b \end{pmatrix}$$

$$N = n_{1,1} + n_{1,2} + \cdots + n_{a,b}$$

These models are not full rank unless we add some restrictions.
Usually:

- $\alpha_{i_0} = 0$ for one i_0 } **or**

  $\sum \alpha_i = 0$ }

- β_{j_0} for one j_0 } **or**

  $\sum \beta_j = 0$ }

- $\gamma_{i_0,j} = 0$ for one i_0 **and** all j **and** γ_{i,j_0} for one j_0 **and** all i }

  or

  $\sum_{i=1}^{a} \gamma_{i,j} = 0$ for all j **and**

  $\sum_{j=1}^{b} \gamma_{i,j} = 0$ for all i }

Estimates under restrictions are unique. For example in the balanced case with $n_{i,j} = n$ with zero sum constraints are:

$\hat{\mu} = \bar{y}_{\cdot\cdot\cdot}$ $\hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot}$

$\hat{\beta}_j = \bar{y}_{\cdot j \cdot} - \bar{y}_{\cdot\cdot\cdot}$ and

$\hat{\gamma}_{i,j} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j \cdot} + \bar{y}_{\cdot\cdot\cdot}$

You can also consider

$y_{i,j,k} = \mu_{i,j} + \varepsilon_{i,j,k}$

$\hat{\mu}_{i,j} = \bar{y}_{ij\cdot}$