Chapter 3 Random Variables

A random variable (RV) is a real # associated with an experiment

Ex Toss a fair coin three times, Let X = # heads obtained

Then X is a RV

HHH	$X=3$
HHT	$X=2$
HTH	$X=2$
HTT	$X=1$
THH	$X=2$
THT	$X=1$
TTH	$X=1$
TTT	$X=0$

$P(X=0) = 1/8$

$P(X=1) = 3/8$

$P(X=2) = 3/8$

$P(X=3) = 1/8$

Total $= 1$

A RV is called discrete if it can only take on a countable # of different values.

A RV is called continuous if it can take on any value in the interval (a,b) for some $a<b$.

For a discrete RV X, the probability mass function $f_X(\cdot)$ is defined by

$$f_X(x) = P[X=x]$$

Notes: $0 \le f_X(x) \le 1$ & $\sum_{\text{All } x} f_X(x) = 1$

Ex Binomial Distribution

Toss a coin with heads probability $p \in (0,1)$ independently n times

Let $X = \#$ heads obtained

$X \in \{0, 1, 2, \dots, n\}$

$$\begin{array}{cccccc} \underline{H} & \underline{H} & \underline{T} & \underline{H} & \cdots & \underline{H} \\ 1 & 2 & 3 & 4 & & n \end{array}$$

What is $P(X = k)$ for $k \in \{0, 1, 2, \dots, n\}$

The probability of any string of k heads and $n-k$ tails has chance $p^k (1-p)^{n-k}$ by independence

$$\begin{array}{ccc} \underline{H_1}\ \underline{T_2}\ \underline{H_3} & \underline{H_1}\ \underline{H_2}\ \underline{T_3} & \underline{T_1}\ \underline{H_2}\ \underline{H_3} \\ p \cdot (1-p) \cdot p & p \cdot p \cdot (1-p) & (1-p) \cdot p \cdot p \end{array}$$

There are $\binom{n}{k}$ distinct arrangements of the k heads in n slots.

So $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

Note that $\displaystyle\sum_{\text{All } x} P(X=x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

Bin Thm $= (p + (1-p))^n = \underline{1}$

Ex $Exp(\beta)$, $\beta > 0$ Exponential Distribution

$$f_X(x) = \begin{cases} \beta e^{-\beta x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

For $t > 0$,

$$P[X > t] = \int_t^{\infty} \beta e^{-\beta x} dx$$

$$= -e^{-\beta x} \Big|_{x=t}^{x=\infty} = e^{-\beta t}$$

Note that $\int_{-\infty}^{+\infty} f_X(x)dx = \int_0^{\infty} \beta e^{-\beta x} dx = 1$

Note: For a continuous RV X, $P[X = x] = 0$ for all x.

Cumulative Distribution Functions (CDFs)

$$F_X(x) = P[X \leq x]$$

Properties of CDFs ...

1) $0 \leq F_X(x) \leq 1$ $\forall x$
2) $F_X(x)$ is non-decreasing in x.
3) $\lim_{x \to \infty} F_X(x) = 1$; $\lim_{x \to -\infty} F_X(x) = 0$
4) $F_X(x)$ is right continuous
5) $F_X(\cdot)$ can only have a countable # of discontinuity points
6) If X is cts, $f_X(x) = \frac{d}{dx}\left(F_X(x) \right)$

Many of these results rely on the following:

If $\{A_n\}$ is a sequence of events with $A_n \to A$, then

$$\lim_{n \to \infty} P(A_n) = P(A)$$

For example, $(-\infty, x+h] \to (-\infty, x]$ as $h \downarrow 0$

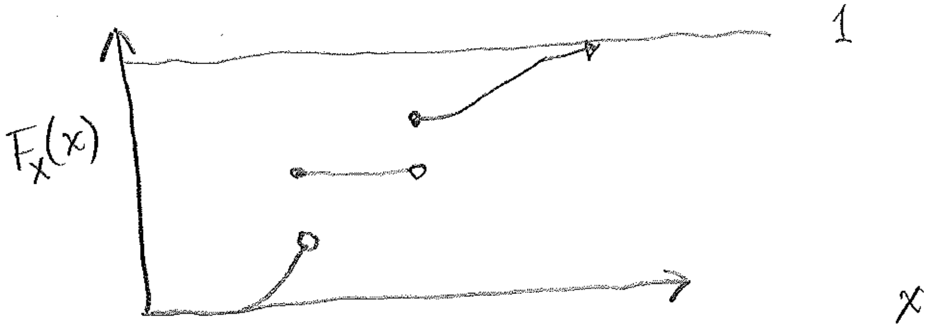
So $\lim_{h \downarrow 0} P[X \leq x+h] = P(X \in (-\infty, x+h])$

$$\to P(X \in (-\infty, x]) \Rightarrow$$

$\lim_{h \downarrow 0} F_X(x+h) = F_X(x)$, which proves 4)

Note that $(-\infty, x-h] \not\to (-\infty, x]$ as $h \downarrow 0$.

For 5), note that $F_X(x)$ looks like



So $F_X(\cdot)$ can only have at most 2 points with a jump $\geq 1/2$

3 points with a jump $\geq 1/3$

$\vdots$

n points with a jump $\geq 1/n$

Let $D_n = \{x : F_X(x) - \lim_{h \downarrow 0} F_X(x-h) > 1/n\}$

Then $\#\{D_n\} \leq n$.

$D = \bigcup_{n=1}^{\infty} D_n$ is all discontinuity points of $F_X(\cdot)$

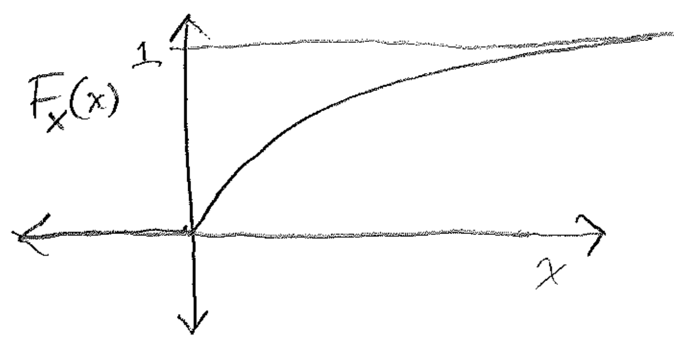
D is countable because it is the countable union of finite sets.

__Ex__ $\quad Exp(\beta) \qquad f_X(x) = \beta e^{-\beta x}, \quad x > 0$

$$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(t)\,dt \qquad x > 0$$

$$= \int_0^x \beta e^{-\beta t}\,dt = -e^{-\beta t}\Big|_{t=0}^{t=x}$$

$$= 1 - e^{-\beta x}$$



$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\beta x}, & x > 0 \end{cases}$$

For $x > 0$, $\quad \frac{d}{dx}\left(F_X(x)\right) = \frac{d}{dx}\left(1 - e^{-\beta x}\right)$

$$= \beta e^{-\beta x}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \qquad f_X(x) = F_X'(x)$$

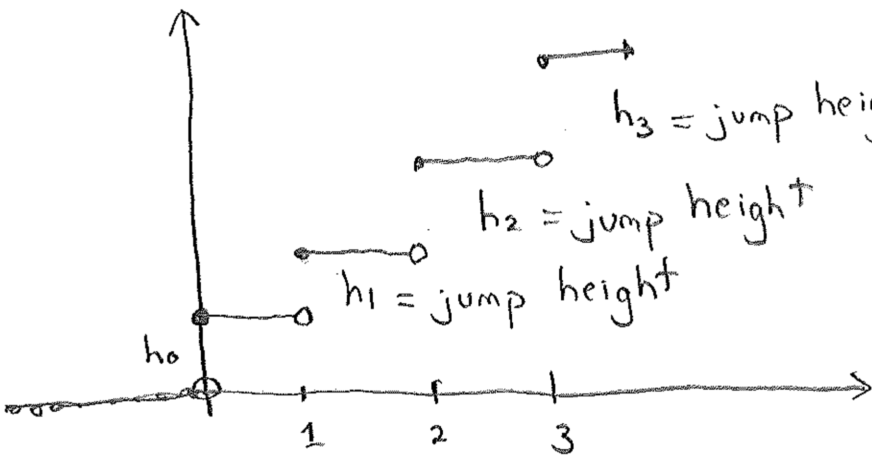
Would you rather have $f_X(\cdot)$ or $F_X(\cdot)$ for a cts RV?

The CDF of a discrete RV is just a step function

Ex X is Poisson(λ)

$$P[X=K] = \frac{e^{-\lambda}\lambda^K}{K!}, \quad K=0,1,2,\ldots$$

Let $h_K = \frac{e^{-\lambda}\lambda^K}{K!}$ $h_0 = e^{-\lambda}$, $h_1 = \lambda e^{-\lambda}$, $h_2 = \frac{\lambda^2 e^{-\lambda}}{2}$,

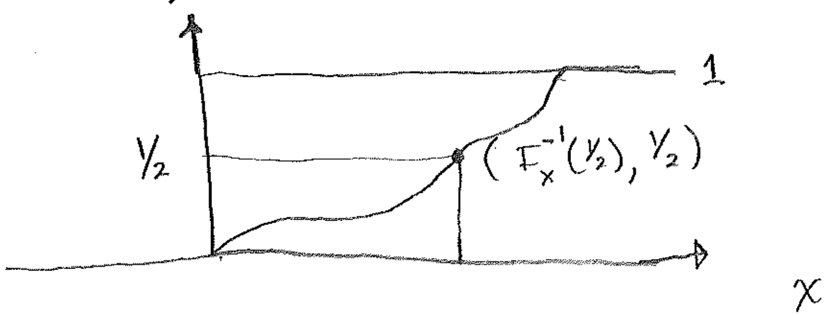


Not very pleasant

Quantiles

Take a cts RV X. For $p \in (0,1)$, define

$$F_X^{-1}(p) = \inf\{x : F_X(x) \geq p\}$$



The median is $F_X^{-1}(1/2)$

Bivariate Distributions

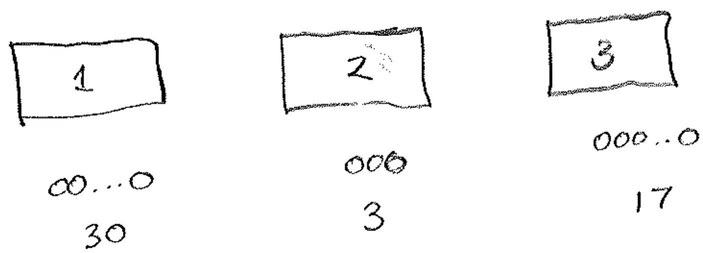
Suppose we have two RVs X & Y.

If X & Y are both discrete, set

$$f_{X,Y}(x,y) = P[X=x \cap Y=y]$$

Note: $0 \le f_{X,Y}(x,y) \le 1$ & $\displaystyle\sum\sum_{\text{All } x,y} f_{X,Y}(x,y) = 1$

<u>Ex</u> Throw 50 balls into three boxes at random



Let X = # balls in box 1
Y = # balls in box 2

$$P[X=x \cap Y=y] = \binom{50}{x}\binom{50-x}{y}\left(\frac{1}{3}\right)^{50} \qquad x,y \in \{0,1,2,\dots,50\},\ x+y \le 50.$$

$\underset{1}{\underline{3}}$ $\underset{2}{\underline{1}}$ $\underset{3}{\underline{1}}$ $\underset{4}{\underline{2}}$ .... $\underset{50}{\underline{2}}$ Each string has probability $\left(\frac{1}{3}\right)^{50}$

Note:
$$\sum_{x=0}^{50}\sum_{y=0}^{50-x}\binom{50}{x}\binom{50-x}{y}\left(\frac{1}{3}\right)^{50} =$$

$$\sum_{x=0}^{50}\binom{50}{x}\left(\frac{1}{3}\right)^{x}\sum_{y=0}^{50-x}\binom{50-x}{y}\left(\frac{1}{3}\right)^{50-x} =$$

$$\sum_{x=0}^{50}\binom{50}{x}\left(\frac{1}{3}\right)^{x}\sum_{y=0}^{50-x}\binom{50-x}{y}\left(\frac{1}{3}\right)^{y}\left(\frac{1}{3}\right)^{50-x-y}$$

$$= \sum_{x=0}^{50} \binom{50}{x} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}+\frac{1}{3}\right)^{50-x} \qquad \text{Binomial thm}$$

$$= \sum_{x=0}^{50} \binom{50}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{50-x}$$

$$= \left(\frac{1}{3}+\frac{2}{3}\right)^{50} \qquad \text{Binomial thm}$$

$$= 1$$

Continuous Joint Distributions

We have a joint density function $f_{X,Y}(x,y)$ with

1) $f_{X,Y}(x,y) \geq 0$
2) $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy\, dx = 1$

Probabilities are determined by integration

$$P\big((X,Y) \in (a,b) \times (c,d) \big) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

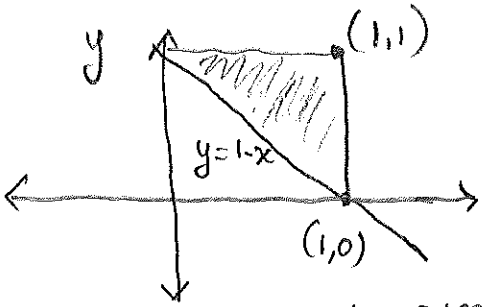
In general if C is a set in $\mathbb{R}^2$,

$$P\big((X,Y) \in C \big) = \iint_C f_{X,Y}(x,y)\, dy\, dx$$

Ex Suppose $f_{X,Y}(x,y) = \begin{cases} K(x+y), & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ & y > 1-x \\ 0, & \text{otherwise} \end{cases}$

Where K is some constant.

a) What is K?



(X, Y) live in shaded region, termed the support set

We must have $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy\,dx = 1 \Rightarrow$

$$\int_0^1\int_{1-x}^1 K(x+y)\,dy\,dx = 1 \Rightarrow$$

$$K\int_0^1 \left[xy + y^2/2 \Big|_{y=1-x}^{y=1} \right] dx = 1 \Rightarrow$$

$$K\int_0^1 \left[\left(x+\tfrac{1}{2}\right) - \left(x(1-x) + \frac{(1-x)^2}{2}\right)\right] dx = 1 \Rightarrow$$

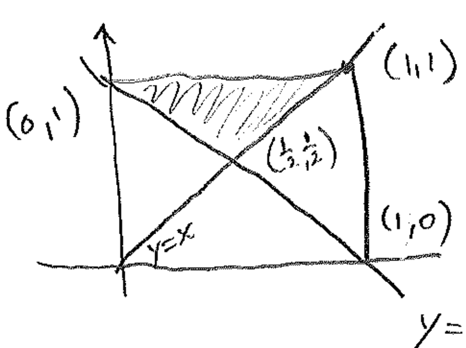
$$K\int_0^1 \left[\tfrac{1}{2} + x^2 - \left(\frac{1-2x+x^2}{2}\right)\right] dx = 1 \Rightarrow$$

$$K\int_0^1 \left(\frac{x^2}{2} + x\right) dx = 1 \Rightarrow K\left[\frac{x^3}{6} + \frac{x^2}{2}\Big|_{x=0}^{x=1}\right] = 1 \Rightarrow$$

$$K\left(\tfrac{2}{3}\right) = 1 \Rightarrow K = 3/2$$

b) What is $P(Y > X)$?

$$P(Y>X) = \iint_{\{(x,y):\, y>x\}} f_{X,Y}(x,y)\,dy\,dx = \iint_{\text{Shaded Region}} f_{X,Y}(x,y)\,dy\,dx$$



$$= \int_0^{\frac{1}{2}}\int_{1-x}^{1} \tfrac{3}{2}(x+y)\,dy\,dx + \int_{\frac{1}{2}}^{1}\int_{x}^{1} \tfrac{3}{2}(x+y)\,dy\,dx$$

= For You

Joint CDF

$$F_{X,Y}(x,y) = P[X \le x \cap Y \le y]$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds\, dt$$ if a density exists

So $$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} = f_{X,Y}(x,y)$$

Marginal Distributions

Suppose X & Y are discrete and we have

$$f_{X,Y}(x,y) = P[X = x \cap Y = y]$$

Then $$f_X(x) = \sum_{\text{All } y} P[X = x \cap Y = y] = \sum_{\text{All } y} f_{X,Y}(x,y)$$

Likewise $$f_Y(y) = \sum_{\text{All } x} f_{X,Y}(x,y)$$

In the continuous case

$$P[X \le x] = P[X \le x \cap Y \in (-\infty, +\infty)]$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{X,Y}(t,s)\, ds\, dt$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{+\infty} f_{X,Y}(t,s)\, ds \right] dt$$

$\Rightarrow \int_{-\infty}^{+\infty} f_{X,Y}(x,s)\, ds$ is the density of X

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

Likewise, $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$

Note also that $F_X(x) = P[X \le x]$

$$= \lim_{y\to\infty} P[X \le x \cap Y \le y]$$

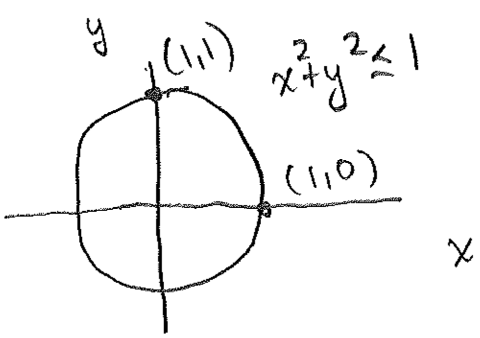
$$= \lim_{y\to\infty} F_{X,Y}(x,y)$$

Also, $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y)$

<u>Ex</u> X is uniform [a,b] means

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Unit dart board



$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & x^2+y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

What is the marginal density of Y?

For $-1 \le y \le +1$, $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx$$

$$= \frac{2}{\pi}\sqrt{1-y^2}\,, \qquad -1 \le y \le 1$$

Independent RVs

X & Y are called indep if $P[\,X \in A \cap Y \in B\,] = P[X \in A]\,P[Y \in B]$ for all sets A & B.

If X & Y are indep., take $A = -(\infty, x]$ & $B = (-\infty, y]$ to get

$$F_{X,Y}(x,y) = P[\,X \in (-\infty, x] \cap Y \in (-\infty, y]\,] =$$
$$P[\,X \in (-\infty, x]\,]\; P[\,Y \in (-\infty, y]\,] =$$
$$F_X(x)\, F_Y(y)$$

So if X & Y have a density

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\, \partial y}\left(F_{X,Y}(x,y) \right) = \frac{\partial^2}{\partial x\, \partial y}\left(F_X(x)\, F_Y(y) \right)$$
$$= \frac{\partial}{\partial x}\left(F_X(x)\, f_Y(y) \right)$$
$$= f_X(x) f_Y(y)$$

In the discrete case,

$$f_{X,Y}(x,y) = P[\,X = x \cap Y = y\,] = P[X = x]\, P[Y = y]$$
$$= f_X(x)\, f_Y(y)$$

<u>Thm</u> X & Y are independent iff $f_{X,Y}(x,y) = h_1(x)\, h_2(y)$ for some functions h_1 & h_2

<u>Proof</u>

Cts case

Suppose $f_{X,Y}(x,y) = h_1(x)\, h_2(y)$.

Then $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \int_{-\infty}^{+\infty} h_1(x)\, h_2(y)\, dy$

$= h_1(x) \cdot C_X$, where $C_X = \int_{-\infty}^{+\infty} h_2(y)\, dy$

So $h_1(x) = \dfrac{f_X(x)}{C_X}$

Likewise $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx = \int_{-\infty}^{+\infty} h_1(x)\, h_2(y)\, dx$

$= C_Y\, h_2(y)$,

where $C_Y = \int_{-\infty}^{+\infty} h_1(x)\, dx$

So $f_{X,Y}(x,y) = h_1(x)\, h_2(y) = \dfrac{f_X(x)}{C_X} \dfrac{f_Y(y)}{C_Y}$

But $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy\, dx = 1 \implies C_X C_Y = 1$

So $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

Now $P[X \in A \cap Y \in B] = \int_A \int_B f_{X,Y}(x,y)\, dy\, dx$

$= \int_A \int_B f_X(x)\, f_Y(y)\, dy\, dx$

$$= \int_A f_X(x)dx \int_B f_Y(y)dy$$

$$= P[X \in A]\, P[Y \in B]$$

So X & Y are independent.

For the other direction, if X&Y are indep. and continuous, we have shown that $f_{X,Y}(x,y) = f_X(x) f_Y(y)$. ∎

<u>Result</u> If X & Y are indep. and g_1 & g_2 are any functions, then $g_1(X)$ & $g_2(Y)$ are indep.

<u>Why</u>

$$P[g_1(X) \in A \cap g_2(Y) \in B] =$$

$$P[X \in g_1^{-1}(A) \cap Y \in g_2^{-1}(B)] =$$

$$P[X \in g_1^{-1}(A)]\, P[Y \in g_2^{-1}(B)] =$$

$$P[g_1(X) \in A]\, P[g_2(Y) \in B]$$ ∎

Here, $g^{-1}(A) = \{x : g(x) \in A\}$ is the inverse set.

Conditional Distributions

Take X&Y jointly discrete

$$f_{X|Y=y}(x) \overset{\text{def}}{=} P[X = x \mid Y = y] \qquad f_{Y|X=x}(y) = P[Y = y \mid X = x]$$

Note $$f_{X|Y=y}(x) = P[X=x \mid Y=y]$$
$$= P[X=x \cap Y=y] \,/\, P[Y=y]$$
$$= \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Likewise, $$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

<u>Continuous Case</u>

We define $$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

& $$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Notes $$f_{X,Y}(x,y) = f_{X|Y=y}(x)\, f_Y(y)$$
$$= f_{Y|X=x}(y)\, f_X(x)$$

<u>Ex</u> Toss a fair 6-sided die and then flip a fair coin the # of times shown on the die.

Let D = die toss
X = # heads

$X|d=d$ has a binomial distribution with d trials and success probability p.

$D \in \{1,2,\dots,6\}$ $X \in \{0,1,2,\dots,6\}$

$$P(X=K \mid D=d) = \binom{d}{K}\left(\frac{1}{2}\right)^{K}\left(1-\frac{1}{2}\right)^{d-K}$$

$$K = 0, 1, 2, \ldots, d$$

$$P(D=d) = \frac{1}{6}, \quad d = 1, 2, \ldots, 6$$

$$P(X=K \cap D=d) = P(X=K \mid D=d)\,P(D=d) \qquad d \in \{1, 2, \ldots, 6\}$$

$$= \binom{d}{K}\left(\frac{1}{2}\right)^{K}\left(1-\frac{1}{2}\right)^{d-K}\frac{1}{6}, \qquad K \in \{0, 1, \ldots, d\}$$

$$P(X=K) = \sum_{d=1}^{6} P(X=K \mid D=d)\,P(D=d)$$

$$= \sum_{d=K}^{6} P(X=K \mid D=d)\,\tfrac{1}{6} = \sum_{d=K}^{6} \binom{d}{K}\left(\frac{1}{2}\right)^{d} \cdot \frac{1}{6}$$

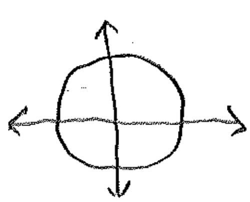
$$= \left(\frac{1}{2}\right)^{d}\frac{1}{6}\sum_{d=K}^{6}\binom{d}{K}$$

$$P(D=d \mid X=K) = \frac{P(D=d \cap X=K)}{P(X=K)} = \frac{\frac{1}{6}\binom{d}{K}\left(\frac{1}{2}\right)^{d}}{\frac{1}{6}\left(\frac{1}{2}\right)^{d}\sum_{d=K}^{6}\binom{d}{K}}$$

$$= \frac{\binom{d}{K}}{\binom{d}{K} + \binom{d+1}{K} + \cdots \binom{6}{K}}$$

Ex Back to the dartboard

$x^2+y^2 \le 1$



$$f_{X,Y}(x,y) = \begin{cases} 1/\pi, & x^2+y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{X,Y}(x,y)\,dx = \frac{2}{\pi}\sqrt{1-y^2}, \quad -1 \le y \le +1$$

$$\text{So } f_{X|Y=y}(x) = \frac{\frac{1}{\pi}}{\frac{2}{\pi}\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}}$$

Limits $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ Uniform$\left(\frac{-1}{2\sqrt{1-y^2}}, \frac{+1}{2\sqrt{1-y^2}}\right)$

Mutivoriate Generalities

$X_1, X_2, \ldots, X_n$ are n RVs.

Joint CDF: $F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$

If a PDF exists

$$f_{X_1,X_2,\ldots X_n}(x_1, \ldots, x_n) = \frac{\partial^n F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)}{\partial x_1 \, \partial x_2 \cdots \partial x_n}$$

All lower dimensional CDFs can be obtained. For example, if one wants the joint CDF of X_1, X_2, and X_5 from that of X_1, X_2, X_3, X_4, X_5

$$F_{X_1,X_2,X_5}(x_1, x_2, x_5) = \lim_{x_3 \to \infty,\ x_4 \to \infty} F_{X_1,X_2,X_3,X_4,X_5}(x_1, x_2, x_3, x_4, x_5)$$

When a density exists,

$$f_{X_1,X_2,X_5}(x_1,x_2,x_5) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X_1,X_2,X_3,X_4,X_5}(x_1,x_2,x_3,x_4,x_5)\,dx_4\,dx_3$$

In the case of jointly discrete RVs $X_1, X_2, \dots, X_n$

$$f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = P[X_1 = x_1 \cap X_2 = x_2 \cap \dots \cap X_n = x_n]$$
$$= P[\vec{X} = \vec{x}]$$

$$\vec{X} = (X_1,\dots,X_n)^T, \quad \vec{x} = (x_1,\dots,x_n)^T$$

When $X_1, X_2, \dots, X_n$ are mutually independent,

$$f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n)$$
$$F_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = F_{X_1}(x_1)F_{X_2}(x_2)\cdots F_{X_n}(x_n)$$

In general, you cannot recover the joint distribution from lower order marginal distributions unless there is independence.

Conditional distributions behave as before. For example,

$$f_{X_1,X_4 \mid X_2=x_2, X_3=x_3}(x_1,x_4) = \frac{f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)}{f_{X_2,X_3}(x_2,x_3)}$$

<u>Ex</u> Suppose we have n IID light bulbs in a cave.
Each light bulb lasts an $\text{Exp}(\beta)$ amount of time

What is the distribution of $\max(X_1, \dots, X_n)$?

$$F_{X_i}(x_i) = \begin{cases} 1 - e^{-\beta x_i}, & x_i > 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $Y = \max(X_1, \dots, X_n)$.

For $y > 0$,
$$\begin{aligned} P[Y \le y] &= P[\max(X_1, X_2, \dots, X_n) \le y] \\ &= P[X_1 \le y \cap X_2 \le y \cap \dots \cap X_n \le y] \\ &= P[X_1 \le y]\, P[X_2 \le y] \dots P[X_n \le y] \\ &= (1 - e^{-\beta y})(1 - e^{-\beta y}) \dots (1 - e^{-\beta y}) \end{aligned}$$

So
$$F_Y(y) = \begin{cases} (1 - e^{-\beta y})^n, & y > 0 \\ 0, & y \le 0 \end{cases}$$

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left((1 - e^{-\beta y})^n\right) = n(1 - e^{-\beta y})^{n-1} \beta e^{-\beta y}$$
when $y > 0$

What is the distribution of $\min(X_1, \dots, X_n)$?

$Y = \min(X_1, \dots, X_n)$

$$\begin{aligned} F_Y(y) = {} & 1 - P[\min(X_1, \dots, X_n) > y] = \\ & 1 - P[X_1 > y \cap \dots \cap X_n > y] = \\ & 1 - P[X_1 > y]\, P[X_2 > y] \dots P[X_n > y] = \\ & 1 - e^{-\beta y}\, e^{-\beta y} \dots e^{-\beta y} = \\ & 1 - e^{-n\beta y}, \quad y > 0 \end{aligned}$$

So $f_Y(y) = \frac{d}{dy}\left(1 - e^{-n\beta y}\right) = n\beta e^{-n\beta y}$

In stats notation, $X_1, X_2, \ldots, X_n \sim \text{IID Exp}(\beta) \implies$
$\min(X_1, \ldots, X_n) \sim \text{Exp}(n\beta)$

<u>Functions of a RV</u>

If X is a RV and $g(\cdot)$ a function, then $g(X)$ is also a RV.

How do we get its distribution?

<u>Ex</u> Suppose $X \sim \text{Unif}[0,1]$. What is the pdf of X^α, where $\alpha > 0$.

Note that X^α also lives in $[0,1]$. With $Y = X^\alpha$,

$$F_Y(y) = P[Y \le y] = P[X^\alpha \le y] = P[X \le y^{1/\alpha}]$$
$$= \int_0^{y^{1/\alpha}} f_X(x)\,dx$$
$$= \int_0^{y^{1/\alpha}} 1 \cdot dx = y^{1/\alpha}$$

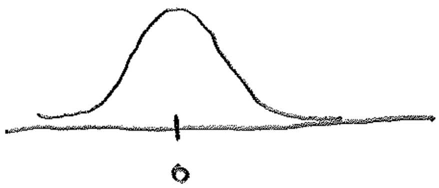
So $f_Y(y) = \frac{d}{dy}\left(y^{\frac{1}{\alpha}}\right)$
$$= \frac{1}{\alpha} y^{\frac{1}{\alpha} - 1}, \quad 0 \le y \le 1$$

If $\alpha = 2$, $f_Y(y) = \frac{1}{2} y^{-1/2}$, which is unbounded at zero.

Standard Normal RV $\quad X \sim N(0,1)$

The bell curve

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad -\infty < x < +\infty$$



$f_X(x)$

Let $Y = X^2$. Then $Y \geq 0$. Let $y > 0$. Then

$$P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq +\sqrt{y}]$$

$$= \int_{-\sqrt{y}}^{+\sqrt{y}} f_X(x)\,dx$$

$$= 2\int_0^{\sqrt{y}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

$$f_Y(y) = \frac{d}{dy}\left(2\int_0^{\sqrt{y}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right) = \frac{2}{\sqrt{2\pi}}\, e^{-x^2/2}\Big|_{x=\sqrt{y}} \frac{d}{dy}(\sqrt{y})$$

$$= \frac{2}{\sqrt{2\pi}}\, e^{-y/2}\, \frac{1}{2} y^{-1/2}$$

$$= \begin{cases} \dfrac{e^{-y/2}}{\sqrt{2\pi}\,\sqrt{y}}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

This will be called a chi-squared(1) $= \chi^2(1)$ density later

Thm If X is a RV with cts CDF $F_X(\cdot)$, then $F_X(X) \sim \text{Unif}[0,1]$
(Probability Integral Transformation)

Why
Let $Y = F_X(X)$. Then $Y \in [0,1]$. For $y \in (0,1)$,

$$P[Y \leq y] = P[F_X(X) \leq y]$$

$$= P[X \le F_X^{-1}(y)] =$$

$$\int_{-\infty}^{F_X^{-1}(y)} f_X(x)\,dx =$$

$$F_X(x) \Big|_{x=-\infty}^{x=F_X^{-1}(y)} = F_X(F_X^{-1}(y)) = y$$

This is the uniform[0,1] CDF!

<u>Generalities</u>

Suppose $g(x)$ is increasing in x and is nice. $Y = g(X)$

Then $g^{-1}(\cdot)$ exists

$$P[Y \le y] = P[g(X) \le y]$$
$$= P[X \le g^{-1}(y)]$$
$$= \int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx = F_X(x) \Big|_{x=-\infty}^{x=g^{-1}(y)}$$
$$= F_X(g^{-1}(y))$$

So
$$f_Y(y) = \frac{d}{dy}\left(F_X(g^{-1}(y)) \right)$$
$$= F_X'(g^{-1}(y)) \frac{d}{dy}\left(g^{-1}(y)\right)$$
$$= f_X(g^{-1}(y)) \frac{d}{dy}\left(g^{-1}(y)\right)$$

≥ 0 since $g^{-1}(y) \uparrow$ in y

If $g(x)$ is decreasing,

$$P[Y \le y] = P[g(X) \le y] = P[X \ge g^{-1}(y)]$$

$$= \int_{g^{-1}(y)}^{\infty} f_X(x)\,dx = F_X(x) \Big|_{x=g^{-1}(y)}^{\infty}$$

$$= 1 - F_X(g^{-1}(y))$$

So $f_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right) = f_X(g^{-1}(y)) \underbrace{-\frac{d}{dy}(g^{-1}(y))}_{\geq 0 \text{ since } g^{-1}(y) \downarrow y}$

DO NOT USE THESE if g is not $\uparrow$ or $\downarrow$.

Ex Suppose $X \sim \text{Unif}[0,1]$ and $Y = -\ln(X)$. Then $g(x) = -\ln(x)$ $\downarrow$ in x

$$f_Y(y) = f_X(g^{-1}(y))\left[-\frac{d}{dy}(g^{-1}(y))\right] \qquad f_X(x) = 1,\ 0<x<1$$

$$= 1 \cdot -\frac{d}{dy}(g^{-1}(y)) \qquad y>0$$

$g(x) = -\ln(x) \qquad g^{-1}(x) = e^{-x}$

So $f_Y(y) = 1 \cdot -\frac{d}{dy}(e^{-y}) = e^{-y},\ y>0$

$Y \sim \text{Exp}(\beta=1)$. $\boxtimes$

The scenario is harder in several dimensions

Suppose $X_1, X_2, \ldots, X_n$ are n RVs and
$Y_1 = g_1(X_1, \ldots, X_n)$, $Y_2 = g_2(X_1, \ldots, X_n)$, $\ldots$, $Y_n = g_n(X_1, \ldots, X_n)$

are n RVs.

Assume that this transformation is 1-to-1 and onto.

Inverse transformation:

$$X_1 = S_1(Y_1, \ldots, Y_n), \quad X_2 = S_2(Y_1, \ldots, Y_n), \quad \ldots \quad X_n = S_n(Y_1, \ldots, Y_n)$$

Let

$$J = \left| \det \begin{pmatrix} \frac{\partial S_1}{\partial y_1} & & \frac{\partial S_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial S_n}{\partial y_1} & & \frac{\partial S_n}{\partial y_n} \end{pmatrix} \right|$$

Thm

$$f_{Y_1, Y_2, \ldots, Y_n}(y_1, y_2, \ldots, y_n) = f_{X_1, \ldots, X_n}(S_1, S_2, \ldots, S_n)\, |J|$$

No proof Essentially is a change of variables formula

Example Suppose X_1 & X_2 are IID $Exp(\beta)$ variates.

Let $Y_1 = X_1 + X_2$, $Y_2 = X_2$

Then $X_2 = Y_2$

$X_1 = Y_1 - X_2 = Y_1 - Y_2$

So $|J| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

& $f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 - y_2, y_2)\, |J|$

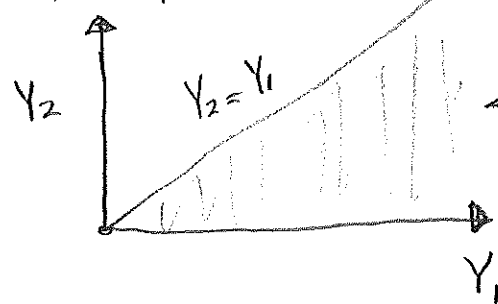
But $f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \beta e^{-\beta x_1} \beta e^{-\beta x_2}$

$$= \beta^2 e^{-\beta(x_1+x_2)}$$

So $$f_{Y_1,Y_2}(y_1,y_2) = \beta^2 e^{-\beta(y_1 - y_2 + y_2)}$$
$$= \beta^2 e^{-\beta y_1}$$

But where do (Y_1, Y_2) live. $Y_2 \in (0,\infty)$ is obvious

But $Y_1 = X_1 + X_2$ is also ≥ 0. However, note that $Y_1 \geq Y_2$

So 

$\longleftarrow$ (Y_1, Y_2) live here

Note that $$\int_0^\infty \int_0^{y_1} f_{Y_1,Y_2}(y_1,y_2)\, dy_2\, dy_1 =$$
$$\int_0^\infty \int_0^{y_1} \beta^2 e^{-\beta y_1} dy_2\, dy_1 =$$
$$\int_0^\infty \beta^2 e^{-\beta y_1} y_1\, dy_1 = 1 \quad \text{(For You)}$$

<u>Example</u> Box Muller Transform.

Let $U_1, U_2 \sim$ IID Uniform$[0,1]$

Set $$Y_1 = \sqrt{-2\ln(U_1)}\ \cos(2\pi U_2)$$
$$Y_2 = \sqrt{-2\ln(U_1)}\ \sin(2\pi U_2)$$

To invert this note that $\dfrac{Y_2}{Y_1} = \tan(2\pi U_2) \Rightarrow U_2 = \dfrac{\tan^{-1}(Y_2/Y_1)}{2\pi}$

(27)

$$Y_1^2 + Y_2^2 = -2\ln(U_1) \Rightarrow U_1 = e^{-\frac{1}{2}(Y_1^2+Y_2^2)}$$

$$\frac{\partial U_1}{Y_1} = e^{-\frac{1}{2}(Y_1^2+Y_2^2)}(-Y_1) \qquad \frac{\partial U_1}{\partial Y_2} = e^{-\frac{1}{2}(Y_1^2+Y_2^2)}(-Y_2)$$

$$\frac{\partial U_2}{\partial Y_1} = \frac{d}{dY_1}\left(\tan^{-1}(Y_2/Y_1)\right)/2\pi =$$

$$\frac{\frac{1}{1+(Y_2/Y_1)^2}\frac{d}{dy_1}\left(\frac{Y_2}{Y_1}\right)}{2\pi} = \frac{-Y_2\, y_1^{-2}}{1+\left(\frac{Y_2}{Y_1}\right)^2}\Big/(2\pi)$$

$$= \frac{-Y_2}{Y_1^2+Y_2^2}\Big/(2\pi)$$

$$\frac{\partial U_2}{\partial y_2} = \frac{d}{dy_2}\left(\tan^{-1}(Y_2/y_1)\right)/2\pi = \frac{1}{1+\left(\frac{Y_2}{y_1}\right)^2}\frac{d}{dy_2}\left(y_2/y_1\right)$$

$$= \frac{\frac{1}{Y_1}}{1+\left(\frac{Y_2}{Y_1}\right)^2}\Big/(2\pi)$$

$$J = \begin{vmatrix} e^{-\frac{1}{2}(Y_1^2+Y_2^2)}(-Y_1) & e^{-\frac{1}{2}(Y_1^2+Y_2^2)}(-Y_2) \\ \frac{-Y_2}{Y_1^2+Y_2^2}\Big/(2\pi) & \frac{\frac{1}{Y_1}}{1+\left(\frac{Y_2}{Y_1}\right)^2}\Big/(2\pi) \end{vmatrix} =$$

$$\left|\frac{-e^{-\frac{1}{2}(Y_1^2+Y_2^2)}}{1+\left(\frac{Y_2}{Y_1}\right)^2}\Big/2\pi - \frac{e^{-\frac{1}{2}(Y_1^2+Y_2^2)}\,Y_2^2}{Y_1^2+Y_2^2}\Big/2\pi\right| =$$

$$\frac{e^{-\frac{1}{2}(Y_1^2+Y_2^2)}}{2\pi}\left[\frac{1}{1+\left(\frac{Y_2}{Y_1}\right)^2} + \frac{Y_2^2}{Y_1^2+Y_2^2}\right] = \frac{e^{-\frac{1}{2}(Y_1^2+Y_2^2)}}{2\pi}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{U_1,U_2}(s_1,s_2)\,|J|$$

But $f_{U_1,U_2}(u_1,u_2) = \underset{[0,1]}{1}(u_1)\ \underset{[0,1]}{1}(u_2)$

Hence, $$f_{Y_1,Y_2}(y_1,y_2) = |J| = \frac{e^{-\frac{1}{2}(y_1^2+y_2^2)}}{\sqrt{2\pi}} = \frac{e^{-y_1^2/2}}{\sqrt{2\pi}}\ \frac{e^{-y_2^2/2}}{\sqrt{2\pi}}$$

Two independent $N(0,1)$ densities.

$$Y_1, Y_2 \sim \text{IID } N(0,1)$$