Problem 9 (AMS 207):

Consider the following model for $i = 1, \ldots, n$:

$$\begin{aligned} Y_i &= \begin{cases} Z_i & Z_i < 200 \\ 200 & Z_i \geq 200 \end{cases} \\ Z_i|X_i &= \alpha + \beta X_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2), \end{aligned}$$

with n and X_i known for all i, and such that the ϵ_is are independent for all i. In addition, assume that $p(\alpha, \beta, \sigma^2) \propto 1/\sigma^2$.

1. (35 points) Write down $p(\Theta|X, Y)$ up to a proportionality constant, with $\Theta = [\alpha, \beta, \sigma^2]$, $X = [X_1, \ldots, X_n]$, and $Y = [Y_1, \ldots, Y_n]$.

2. (65 points) Now augment your parameter space by considering $\Theta^* = [\Theta, Z_1, \ldots, Z_n]$. Develop a Gibbs sampling scheme to obtain draws from the joint posterior distribution of $(\Theta^*|X, Y)$.

Useful results:

- The p.d.f. of a random variable θ with an inverse-Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ is given by

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}, \;\; \theta > 0.$$

Solution:

1. The posterior density is given by

$$\begin{aligned} p(\Theta|X, Y) &\propto \frac{1}{\sigma^2} \times \prod_{i, Y_i = 200} \Phi\left(\frac{\alpha + \beta X_i - 200}{\sigma}\right) \times \prod_{i, Y_i \neq 200} \frac{1}{\sigma} \exp\left(-\frac{(Y_i - \alpha + \beta X_i)^2}{2\sigma^2}\right), \\ &\propto (\sigma^2)^{-(n_1/2+1)} \times \exp\left(\frac{-\sum_{i, Y_i \neq 200}(Y_i - \alpha + \beta X_i)^2}{2\sigma^2}\right) \times \prod_{i, Y_i = 200} \Phi\left(\frac{\alpha + \beta X_i - 200}{\sigma}\right), \end{aligned}$$

   with $n_1 = \#\{i, Y_i \neq 200\}$, and $\Phi(\cdot)$ the c.d.f. of a standard Normal distribution.

2. Gibbs sampling scheme:

- (25 points) Sampling (α, β) :

$$\left(\begin{array}{c} \alpha \\ \beta \end{array} \middle| X, Y, Z, \sigma^2 \right) \sim N\left((\mathbf{X}\mathbf{X}')^{-1}\mathbf{X}'\mathbf{Z}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1} \right),$$

with

$$\mathbf{X} = \begin{pmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}.$$

- (20 points) Sampling σ^2 :

$$(\sigma^2 | X, Y, Z, \alpha, \beta) \sim IG(a, b),$$

with $a = n/2$ and $b = \sum_{i=1}^{n} (Z_i - \alpha - \beta X_i)^2/2$.

- (20 points) Sampling Z_i, for $i = 1 : n$:

$$(Z_i | \alpha, \beta, X, Y, \sigma^2) = \begin{cases} Y_i, & Y_i \neq 200, \\ TN(\alpha + \beta X_i, \sigma^2, [200, \infty)) & Y_i = 200. \end{cases}$$