Winter 22 – STAT206B Homework 4

Due: 5pm on Mar. 8th Tuesday

1. (Wasserman, 2003) A random variable Z has an *inverse Gaussian distribution* if it has density

$$f(z \mid \theta_1, \theta_2) \propto z^{-3/2} \exp\left\{-\theta_1 z - \frac{\theta_2}{z} + 2\sqrt{\theta_1\theta_2} + \log(\sqrt{2\theta_2})\right\}, \quad z > 0,$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are parameters. It can be shown that $E(Z) = \sqrt{\theta_2/\theta_1}$ and $E(1/Z) = \sqrt{\theta_1/\theta_2} + 1/(2\theta_2)$.

   (a) Let $\theta_1 = 1.5$ and $\theta_2 = 2$. Draw a sample of size 1,000 using the independence-Metropolis-Hastings method with a Gamma distribution as the proposal density (note that in an independence-Metropolis-Hastings $q(z^* \mid z) = q(z^*)$). To assess the accuracy of the method, compare the mean of Z and $1/Z$ from the sample to the theoretical means. Try different Gamma distributions to see if you can get an accurate sample.

   (b) Draw a sample of size 1,000 using the random-walk Metropolis method. Since $z > 0$ we cannot just use a Normal density. Let $W = \log(Z)$. Find the density of W. Use the random-walk Metropolis method to get a sample $W_1, \ldots, W_M$ and let $Z_i = e^{W_i}$. Assess the accuracy of the simulation as in the previous part.

2. Consider i.i.d. data $x_1, \ldots, x_n$ such that $x_i|\nu, \theta \sim \text{Gamma}(\nu, \theta)$ where $E(x_i) = \nu/\theta$, and assign priors $\nu \sim \text{Gamma}(3, 1)$ and $\theta \sim \text{Gamma}(2, 2)$.

   (a) Develop a Metropolis-within-Gibbs algorithm to sample from $p(\nu, \theta|x_1, \ldots, x_n)$ using the full conditional distributions $p(\theta|\nu, x_1, \ldots, x_n)$ and $p(\nu|\theta, x_1, \ldots, x_n)$. For the second full conditional, use a random walk proposal on $\log(\nu)$.

   (b) Develop a Metropolis-Hastings algorithm that jointly proposes $\log(\nu)$ and $\log(\theta)$ using a Gaussian random walk centered on the current value of the parameters. Tune the variance-covariance matrix of the proposal using a test run that proposes the parameters independently (but evaluates acceptance jointly).

   (c) Develop a Metropolis algorithm that jointly proposes $\log(\nu)$ and $\log(\theta)$ using independent proposals based on the Laplace approximation of the posterior distribution of $\log(\nu)$ and $\log(\theta)$.

   (d) Run each of the algorithms for the dataset in `my-data.txt` and compute the effective sample sizes associated with each parameter under each of the samplers. Also, construct trace and autocorrelation plots. Report posterior means for each of the parameters of interest, along with 95% symmetric credible intervals. Discuss.

3. (Robert and Casella) Consider a random effects model,

$$y_{i,j} = \beta + u_i + \epsilon_{i,j}, \quad i = 1 : I, j = 1 : J,$$

where $u_i \sim N(0, \sigma^2)$ and $\epsilon_{i,j} \sim N(0, \tau^2)$. Assume a prior of the form

$$\pi(\beta, \sigma^2, \tau^2) \propto \frac{1}{\sigma^2 \tau^2}.$$

(a) Find the full conditional distributions:

i. $\pi(u_i \mid \boldsymbol{y}, \beta, \tau^2, \sigma^2)$;

ii. $\pi(\beta \mid \boldsymbol{y}, \boldsymbol{u}, \tau^2, \sigma^2)$;

iii. $\pi(\sigma^2 \mid \boldsymbol{y}, \boldsymbol{u}, \beta, \tau^2)$;

iv. $\pi(\tau^2 \mid \boldsymbol{y}, \boldsymbol{u}, \beta, \sigma^2)$.

(b) Find $\pi(\beta, \tau^2, \sigma^2 \mid \boldsymbol{y})$ up to a proportionality constant.

(c) Find $\pi(\sigma^2, \tau^2 \mid \boldsymbol{y})$ up to a proportionality constant and show that this posterior is not integrable since, for $\tau \neq 0$, it behaves like σ^{-2} in a neighborhood of 0.

Note: This problem shows that even though the full conditional posteriors exist and the Gibbs sampling could be easily implemented, the joint posterior distribution does not exist. Users should be aware of the risks of using the Gibbs sampler in situations like this!

4. (Carlin, Gelfand and Smith, 1992) Let $y_1, \ldots, y_n$ be a sample from a Poisson distribution for which there is a suspicion of a change point m along the observation process where the means change, $m = 1, \ldots, n$. Given m, $y_i \sim \text{Poi}(\theta)$, for $i = 1, \ldots, m$ and $y_i \sim \text{Poi}(\phi)$, for $i = m+1, \ldots, n$. The model is completed with independent prior distributions $\theta \sim \text{Gamma}(\alpha, \beta)$, $\phi \sim \text{Gamma}(\gamma, \delta)$ and m uniformly distributed over $\{1, \ldots, n\}$ where α, β, γ and δ are known constants. The data in file `mining-data.r` consists of counts of coal mining disasters in Great Britain by year from 1851 to 1962.

   (a) Describe a Gibbs sampling algorithm to obtain samples from the joint posterior distribution.

   (b) Specify values of the fixed hyperparameters, α, β, γ and δ and justify your choices.

   (c) Implement your sampling algorithm in part (a) and summarize the posterior distribution using point estimates and interval estimates of the parameters. Provide interpretations of the inference in layman's word.

   (d) Perform prior sensitivity examination. Refit the model with different sets of the fixed hyperparameter values and examine if the prior distributions have an undesired influence on the posterior inference.

5. Souza (1999) considers a number of hierarchical models to describe the nutritional pattern of pregnant women. One of the models adopted was a hierarchical regression model where

$$
\begin{aligned}
y_{i,j} &\sim \text{N}(\alpha_i + \beta_i t_{i,j}, \sigma^2), \\
(\alpha_i, \beta_i)' \mid \alpha, \beta &\sim \text{MVN}_2((\alpha, \beta)', diag(\tau_\alpha^2, \tau_\beta^2)), \\
(\alpha, \beta)' &\sim \text{MVN}_2((0, 0)', diag(P_\alpha^2, P_\beta^2)).
\end{aligned}
$$

Here $y_{i,j}$ and $t_{i,j}$ are the jth weight measurement and visit time of the ith woman with $j = 1 : n_i$ and $i = 1 : I$ for $I = 68$ pregnant women. Here $n = \sum_{i=1}^{I} n_i = 427$. For unknown scale parameters, we assume a priori independence and place a Gamma prior,

$$\sigma^2 \sim \text{IG}(a_\sigma, b_\sigma), \quad \tau_\alpha^2 \sim \text{IG}(a_\alpha, b_\alpha), \text{ and } \tau_\beta^2 \sim \text{IG}(a_\beta, b_\beta).$$

Hyperparameters, $a_\sigma, b_\sigma, a_\alpha, b_\alpha, a_\beta, b_\beta$ P_α^2 P_β^2 are fixed.

(a) Find the joint posterior distribution of all random parameters (up to proportionality)

(b) Find the full conditional distributions of $\alpha, \beta, \tau_\alpha, \tau_\beta$, $\sigma^{-2}, \alpha_i, \beta_i$, and (α_i, β_i).

(c) Describe a Gibbs sampling algorithm to obtain samples from the joint posterior distribution.

(d) Specify values of the fixed hyperparameters, and justify your choices.

(e) Implement your sampling algorithm in part (c) and summarize the posterior distribution using point estimates and interval estimates of the parameters. Provide interpretations of the inference in layman's word.