STAT206B Homework #2

Due: 5pm, 01/28 (Friday) via email.

1. Let $X_1, \ldots, X_n$ be an i.i.d. sample such that $X_i \mid \theta, \sigma^2 \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known and θ is unknown. Also, let your prior for θ be a mixture of conjugate priors, i.e.,

$$\pi(\theta) = \sum_{\ell=1}^{K} w_\ell \phi(\theta \mid \mu_\ell, \tau^2)$$

where $\phi(\theta \mid \mu_\ell, \tau^2)$ denotes the Gaussian density with mean μ_ℓ and variance τ^2 and mixture weights $0 < w_\ell < 1$ for all $\ell = 1, \ldots, K$ with $\sum_{\ell=1}^{K} w_\ell = 1$.

Note: This questions is challenging. Use the results from the class example with $X_i \mid \theta \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$ and $\theta \sim \mathrm{N}(\mu, \tau^2)$.

   (a) Find the posterior distribution for θ based on this prior.
   (b) Find the posterior mean.
   (c) Find the prior predictive distribution associated with this model (i.e., the marginal distribution of data).
   (d) Find the posterior predictive distribution associated with this model.

2. Let $X_1, \ldots, X_n$ be an i.i.dsample such that $X_i \mid \theta \overset{iid}{\sim} \mathrm{N}(\theta, 1)$. Suppose that you know that $\theta > 0$, and you want your prior to reflect that fact. Hence, you decide to set $\pi(\theta)$ to be a normal distribution with mean μ and variance τ^2, truncated to be positive, i.e.,

$$\pi(\theta) = \frac{1}{\sqrt{2\pi\tau^2}\Phi(\mu/\tau)} \exp\left\{-\frac{(\theta-\mu)^2}{2\tau^2}\right\} \mathrm{I}_{[0,\infty)}(\theta),$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution and $\mathrm{I}_{[0,\infty)}(\cdot)$ the indicator function.

   (a) Find the posterior distribution for θ based on this prior.
   (b) Find the prior predictive distribution.

3. Let $X_1, \ldots, X_n$ be an i.i.d. sample such that each X_i comes from a truncated normal with unknown mean θ and variance 1,

$$f(X_i \mid \theta) = \frac{1}{\sqrt{2\pi}\Phi(\theta)} \exp\left\{-\frac{(X_i-\theta)^2}{2}\right\} \mathrm{I}_{[0,\infty)}(X_i).$$

If $\theta \sim \mathrm{N}(\mu, \tau^2)$, find the posterior for θ.

4. (Robert 2.28 - slightly reworded) Consider $x \mid \theta \sim \mathrm{Binomial}(n, \theta)$ with n known.

   (a) If the prior is $\theta \sim \mathrm{Be}(\sqrt{n}/2, \sqrt{n}/2)$, give the associate posterior.

(b) What is the estimator that minimizes the posterior expected loss if the loss function is $L(\delta, \theta) = (\theta - \delta)^2$? Call such estimator $\delta^\pi(x)$ and show that its associated risk $R(\theta, \delta^\pi(x))$ is constant.

(c) Let $\delta_0(x) = x/n$. Find the risk for this estimator, i.e., find $R(\theta, \delta_0(x))$. Compare the risks for $\delta^\pi(x)$ and $\delta_0(x)$ for $n = 10, 50$, and 100. Conclude about the appeal of $\delta^\pi(x)$.

5. (Robert 2.30 - slightly reworded). Consider $x \sim \mathrm{N}(\theta, 1)$ and $\theta \sim \mathrm{N}(0, n)$. Let $\delta^\pi(x)$ be the estimator that minimizes the posterior expected loss under the square error loss. Show that the Bayes risk $r(\pi, \delta^\pi)$ is equal to $n/(n+1)$.

6. (Adapted from Robert 2.42) Consider the LINEX loss function defined by

$$L(\theta, d) = e^{c(\theta - d)} - c(\theta - d) - 1.$$

(a) Show that $L(\theta, d) \geq 0$ and plot this loss as a function of $(\theta - d)$ when $c = 0.1, 0.5, 1, 2$.

(b) Give the expression of a Bayes estimator $\delta^\pi(x)$ under this loss, i.e., find the estimator that minimizes the posterior posterior loss.

(c) Find $\delta^\pi(x)$ when $x_i \mid \theta \overset{iid}{\sim} \mathrm{N}(\theta, 1)$, $i = 1, \ldots, n$ and $\theta \sim \mathrm{N}(\mu, \tau^2)$.

7. Let $L(\theta, d) = w(\theta)(\theta - d)^2$, with $w(\theta)$ a non-negative function, be the weighted quadratic loss (See CR Corollary 2.5.2). Show that $\delta^\pi(x)$, the estimator that minimizes the posterior expected loss $\rho(\pi, d \mid x)$ has the form $\delta^\pi(x) = \mathrm{E}(w(\theta)\theta \mid x)/\mathrm{E}(w(\theta) \mid x)$.

Hint: Show that any other estimator has a larger posterior expected loss.

8. Let $X \mid \theta \sim \mathrm{Binomial}(n, \theta)$ with $\theta \sim \mathrm{Be}(\alpha, \beta)$. Let $L(\theta, d) = (\theta - d)^2/\{\theta(1-\theta)\}$. Find the estimator that minimizes the posterior expected loss $\rho(\pi, \delta \mid x)$ under this loss function.

9. (Adapted from Robert 2.43). Consider $x \mid \theta \sim \mathrm{N}(\theta, 1)$, $\theta \sim \mathrm{N}(0, 1)$ and the loss $L(\theta, d) = e^{3\theta^2/4}(\theta - d)^2$.

(a) Show that the estimator that minimizes the Bayesian expected posterior loss in this case is $\delta^\pi(x) = 2x$. *Hint:* use results from #7.

(b) Show that $\delta_0(x) = x$ dominates $\delta^\pi(x)$.

10. Assume you have to guess a secret number θ. You know that θ is an integer. You can perform an experiment that would yield either the number before it or the number after it, with equal probability. You perform the experiment twice. More formally, let x_1 and x_2 be independent observations from $f(x = \theta - 1 \mid \theta) = f(x = \theta + 1 \mid \theta) = 1/2$. Consider the 0-1 loss function, i.e.,

$$L(\theta, d) = \begin{cases} 0 & \text{if } \theta = d, \\ 1 & \text{if } \theta \neq d. \end{cases}$$

(a) Find the risks $R(\theta, \delta)$ for the estimators $\delta_0(x_1, x_2) = (x_1 + x_2)/2$ and $\delta_1(x_1, x_2) = x_1 + 1$.

(b) Find the estimator $\delta^\pi(x_1, x_2)$ that minimizes the posterior expected loss.

11. Consider a point estimation problem in which you observe $x_1, \ldots, x_n$ as i.i.d. random variables of the Poisson distribution with parameter θ. Assume a squared error loss and a prior of the form $\theta \sim \mathrm{Gamma}(\alpha, \beta)$.

(a) Show that the Bayes estimator is $\delta^\pi(x) = a + b\bar{x}$ where $a > 0$, $b \in (0,1)$ and $\bar{x} = \sum_{i=1}^n x_i/n$. You may use the fact that the distribution of $\sum_{i=1}^n x_i$ is Poisson with parameter $n\theta$ without proof.

(b) Find the MLE for θ (*Note:* to remind how to find MLEs, read Casella and Berger, Section 7.2.2 – see Def 7.2.4).

(c) Compute and graph the frequentist risks $R(\theta, \delta)$ for $\delta^\pi(x)$ and the MLE of θ.

(d) Compute the Bayes risk of $\delta^\pi(x)$.

(e) Suppose that an investigator wants to collect a sample that is large enough that the Bayes risk after the experiment is half of the Bayes risk before the experiment. Find that sample size.