— 2012

BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

First Year Exam (retake): September 19th, 2012

INSTRUCTIONS

If you are on the Applied Mathematics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 3(AMS 212A), 4(AMS 212B), 5(AMS 213), and 6(AMS 214).

If you are on the Statistics track, you are required to complete problems 1(AMS 203), 2(AMS 211), 7(AMS 205B), 8(AMS 206B), 9(AMS 207), and 10(AMS256).

Please complete all required problems on the supplied exam papers. Write your exam ID number and problem number on each page. Use only the front side of each page.

Problem 1 (AMS 203):

Suppose that X_1 and X_2 are continuous i.i.d. random variables and that each of them has the uniform distribution on the $(0,1)$ interval. Let $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $Z = X_1 - X_2$ and $U = X_1^2$.

1. (35 points) Find the p.d.f. of Y_2. ✓
2. (30 points) Find the p.d.f. of Z.
3. (20 points) Find the p.d.f. of U. ✓
4. (15 points) Find $E(Y_1 Y_2)$.

Problem 2 (AMS 211):

1. [60%] By converting both integrand and surface to standard spherical polar coordinates (r, θ, ϕ), calculate the scalar surface integral

$$\int_S y^2 dS$$

over the surface S defined by $x^2 + y^2 = z^2, 0 \leq z \leq 1$. (If you can remember the scale factors for spherical polars, you can just write them down and use them; otherwise you will have to calculate them).

2. [40%] Solve

$$16\frac{d^2y}{dt^2} - 40\frac{dy}{dt} + 25y = 0$$

subject to boundary conditions

$$y(0) = 3, \quad \frac{dy}{dt}(0) = -\frac{9}{4}$$

Problem 3 (AMS 212A):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

$$T(0, y, z) = T(L, y, z) = 0$$

$$T(x, 0, z) = T(x, L, z) = 0$$

$$T(x, y, 0) = T_0 \quad \text{and} \quad T(x, y, L) = 0 \qquad (1)$$

1. Describe a physical situation that would plausibly be represented by this problem.
2. Give a complete explicit solution of the problem. Explain your calculation step by step.

Problem 4 (AMS 212B):

Use the matched asymptotic expansion to solve the boundary layer problem below.

$$\begin{cases} \varepsilon y'' + y' + y^2 = 0 \\ y(0) = 0\,, \quad y(1) = \frac{1}{2} \end{cases}, \qquad \varepsilon \to 0_+$$

Find the leading term of the composite expansion.

Problem 5 (AMS 213):

Part 1 [40%] Let A be a nonsingular $n \times n$ matrix. Suppose there are two $n \times n$ matrices P and Q, such that P is nonsingular, spectral radius $\rho(P^{-1}Q) < 1$, and $A = P - Q$.

1. Show that the iterates x_k defined by

$$Px_{k+1} = Qx_k + b, \qquad k = 0, 1, 2, \cdots$$

converge to the solution of $Ax = b$ for any starting vector $x_0 \in R^n$. [30%]

2. Find a splitting $A = P - Q$ for the matrix $A = \begin{bmatrix} -100 & 1 \\ 4 & 100 \end{bmatrix}$, so that the iteration is convergent. Justify your answer. [10%]

Part 2 [60%] Consider the following scheme

$$u_k^{n+1} = \frac{1}{2}\mu(1+\mu)u_{k-1}^n + (1-\mu^2)u_k^n - \frac{1}{2}\mu(1-\mu)u_{k+1}^n$$

for the Cauchy problem

$$\begin{aligned} u_t + au_x &= 0, \; -\infty < x < \infty \\ u(x,0) &= g(x), \\ \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx &< \infty, \; \forall t \geq 0 \end{aligned}$$

where a is a constant, $\mu = a\frac{\Delta t}{\Delta x}$, and $(\Delta t, \Delta x)$ are the discretization steps in time and space (assuming uniform grid). Show that the scheme is stable for all $|\mu| \leq 1$.

You might need to use the following trigonometric identities

$$\begin{aligned} 1 - \cos\alpha &= 2\sin^2\frac{\alpha}{2} \\ \sin\alpha &= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}. \end{aligned}$$

Problem 6 (AMS 214):

Suppose that rabbits and sheep are competing for the same food supply (grass) and the amount available is limited. Each species would grow to its carrying capacity in the absence of the other. However, in the presence of the both, the population sizes of rabbits x and sheep y obey

$$\dot{x} = x(3 - 2x - 2y) \tag{2}$$
$$\dot{y} = y(2 - x - y) \tag{3}$$

where $x \geq 0$ and $y \geq 0$.

a) (10%) Sketch the nullclines, i.e., the lines in the phase plane where $\dot{x} = 0$, or $\dot{y} = 0$.

b) (10%) Find all fixed points of the dynamical system. (Hint: you may find the sketch of the nullclines useful to answer this question).

c) (35%) Find the eigenvalues and eigenvectors of the linearized dynamical system at the fixed points (Hint: Note that for 2×2 diagonal matrices, the eigenvectors are $[1\ 0]^T$ and $[0\ 1]^T$).

d) (30%) Sketch the phase diagram.

e) (15%) What is the carrying capacity of rabbits in the absence of sheep? What is the carrying capacity of sheep in the absence of rabbits? Discuss if the long term coexistence of rabbits and sheep is possible.

Problem 7 (AMS 205B):

Consider a model in which the elements of your data vector $y = (y_1, \ldots, y_n)$ are conditionally IID (given an unknown $-1 < \theta < 1$) with marginal sampling distribution

$$p(y_i|\theta) = \frac{1}{2}(1 + \theta\, y_i) \quad \text{for} \quad -1 < y_i < 1 \tag{4}$$

and 0 otherwise.

1. (15 points) Sketch the marginal sampling distribution $p(y_i|\theta)$ in (4) as a function of y_i for fixed θ, taking care to distinguish its various shapes as θ varies from -1 to 1.

2. (20 points) Work out the repeated-sampling mean $E(y_i|\theta)$ of this distribution, and use this to show that the method-of-moments estimator of θ in this model is $\hat{\theta}_{MoM} = 3\,\bar{y}_n$, where $\bar{y}_n = \frac{1}{n}\sum_{i=1}^n y_i$. Briefly explain what may go wrong with this estimator, and identify when this unpleasant behavior is most likely to occur.

3. (10 points) Is $\hat{\theta}_{MoM}$ consistent for θ in this model? Briefly justify your answer.

4. (25 points) Work out the log-likelihood function based on (4), and briefly explain how you would use it to find numerically the maximum likelihood estimator $\hat{\theta}_{MLE}$ (don't try to solve for the MLE in closed form). Is there a one-dimensional sufficient statistic here? Explain briefly.

5. (15 points) It can be shown (you don't have to show this) that the repeated-sampling variance $V(y_i|\theta)$ of y_i in this model is

$$V(y_i|\theta) = \frac{1}{3} - \frac{\theta^2}{9}. \tag{5}$$

   Use this fact to work out an estimated standard error $\widehat{SE}\left(\hat{\theta}_{MoM}\right)$ based on $\hat{\theta}_{MoM}$. Briefly explain what may go wrong with this estimated standard error, and identify when this unpleasant behavior is most likely to occur.

6. (15 points) Write down a closed-form expression for a large-sample estimated standard error $\widehat{SE}\left(\hat{\theta}_{MLE}\right)$ for $\hat{\theta}_{MLE}$. Appealing to basic properties of the MLE, how do you expect $\widehat{SE}\left(\hat{\theta}_{MLE}\right)$ to compare with $\widehat{SE}\left(\hat{\theta}_{MoM}\right)$ for large n? Explain briefly.

Problem 8 (AMS 206B):

Assume that, given unknown parameter $\alpha > 0$, your observations $y = (y_1, \ldots, y_n)$ arise i.i.d. from a Pareto distribution with density function

$$f(y \mid \alpha) = \alpha y^{-(\alpha+1)}, \quad \text{for } y > 1.$$

1. (25 points) Derive the Jeffreys prior for α. (The Jeffreys prior is assumed for the remaining parts of the problem.)

2. (25 points) Show that the posterior for α is given by a gamma distribution and obtain its parameters.

3. (25 points) Derive the closed-form expression for the posterior predictive density, $p(y_{new} \mid y)$, corresponding to a new observation y_{new}.

4. (25 points) Consider the special case of a sample consisting of a single observation, i.e., $n = 1$ and $y = y_1$. Obtain the 90% highest posterior density interval for α.

Problem 9 (AMS 207):

Ten water samples are taken at a given site in the Monterey Bay and temperature, y_1, and salinity, y_2, are recorded. As a result we have the sample $y_i = (y_{1,i}, y_{2,i})$, for $i = 1, \ldots, 10$. Assume that all y_i are independent and that their distribution corresponds to a bivariate normal with mean $\mu \in \mathbb{R}^2$ and covariance matrix $V \in \mathbb{R}^{2\times 2}$.

1. (35 points) Write the likelihood function for μ and V and find the sufficient statistics.

2. (65 points) Consider the prior $p(\mu, V) \propto p(V)$, where $p(V)$ is an inverse Wishart distribution with parameters ν and S_0. Thus, the prior is proportional to

$$|V|^{-(\nu+k+1)/2} \exp\left\{-\frac{1}{2} S_0 V^{-1}\right\},$$

   where k is the dimension of V, in this case $k = 2$.

   Assume that observation y_{22} is missing. We resort to a Markov chain Monte Carlo approach to obtain samples of μ, V, and the missing observation. For that purpose, write the full conditionals that correspond to the Gibbs sampler.

Hints:

- Recall that the trace of a product of matrices is invariant under cyclic permutations; for instance, for three matrices $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times k}$, and $C \in \mathbb{R}^{k\times n}$, we have $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$.

- Consider a bivariate normal vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} v_x & v_{xy} \\ v_{yx} & v_y \end{pmatrix}\right),$$

then

$$p(x|y) = N_1\left(\mu_x + \frac{v_{xy}}{v_y}(y-\mu_y), v_x - \frac{v_{xy}^2}{v_y}\right)$$

Problem 10 (AMS 256):

Suppose that $\mathbf{y} = (y_1, \dots, y_n)'$ is generated from a model (true model) given by

$$\mathbf{y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \epsilon, \tag{6}$$

where $\epsilon \sim N_n(0, \sigma^2\mathbf{I})$ with σ^2 known and $\mathbf{X}_1$, $\mathbf{X}_2$ full rank matrices of dimensions $n \times p_1$ and $n \times p_2$, respectively.

1. Part I. Assume that the model

$$\mathbf{y} = \mathbf{X}_1\beta_1^* + \epsilon^*,$$

is fitted to the data, with $\epsilon^* \sim N_n(0, \sigma^2\mathbf{I})$.

(a) (15 points) Let $\hat{\beta}_1^*$ be the LSE of β_1^*. Is $\hat{\beta}_1^*$ an unbiased estimator of β_1 in general? Please justify your answer providing a detailed calculation.

(b) (10 points) Now assume that the columns of $\mathbf{X}_1$ are orthogonal to the columns of $\mathbf{X}_2$. Is $\hat{\beta}_1^*$ an unbiased estimator of β_1 in this case? Please justify your answer.

(c) (10 points) Find $Cov(\hat{\beta}_1^*)$.

(d) (25 points) Let $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$ be the LSE of $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ for the model in (6) written as $\mathbf{y} = \mathbf{X}\beta + \epsilon$ with $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$. Once again, assume that the columns of $\mathbf{X}_1$ are orthogonal to the columns of $\mathbf{X}_2$. Is $Cov(\hat{\beta}_1) = Cov(\hat{\beta}_1^*)$ in this case? Are $\hat{\beta}_1$ and $\hat{\beta}_2$ independent? Please justify your answers.

2. Part II. Suppose that data $y_1, \dots, y_n$ are generated from the model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, but the model $y_i = \beta_1^* x_i + \epsilon_i^*$ is fitted to the data instead. Once again assume that $\epsilon \sim N_n(0, \sigma^2\mathbf{I})$ and $\epsilon^* \sim N_n(0, \sigma^2\mathbf{I})$ with σ^2 known.

(a) (15 points) Find $\hat{\beta}_1^*$, the LSE of β_1^*.

(b) (25 points) Derive the statistic to perform the following test:

$$H_0: \beta_1^* = 0 \quad H_1: \beta_1^* \neq 0.$$