AMS 207

Consider the linear model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N_n(\boldsymbol{0}, (1/\phi)\boldsymbol{I})$$

where $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{X}$ is a full rank matrix of dimensions $n \times p$, $\boldsymbol{\beta} \in \mathbb{R}^p$, $\boldsymbol{I}$ is a $n \times n$ identity matrix and $\phi > 0$. Consider the prior $p(\boldsymbol{\beta}|\phi) = \prod_i p(\beta_i|\phi)$ where

$$p(\beta_i|\phi) = \left(1 + \frac{\phi\beta_i^2}{\nu\lambda}\right)^{-(\nu+1)/2} \frac{\phi}{\lambda} \ ,$$

where ν and λ are known, and assume that $p(\phi) \propto 1/\phi$.

1. Write $p(\beta_i|\phi)$ as a scale mixture of normals.
2. Use the above representation to introduce latent variables that facilitate sampling the posterior distribution of all parameters using Gibbs sampling. Write the resulting model.
3. Obtain the full conditionals for all model parameters.

Solution:

1.
$$p(\beta_i|\phi) = \left(1 + \frac{\phi\beta_i^2}{\nu\lambda}\right)^{-(\nu+1)/2} \frac{\phi}{\lambda} \propto \int_0^\infty \frac{\exp\left\{-\frac{1}{\eta_i}\left(\frac{\phi\beta_i^2}{2\lambda} + \frac{\nu}{2}\right)\right\}}{\eta_i^{(\nu+1)/2+1}} d\eta_i$$

Thus,

$$p(\beta_i|\phi) = \int_0^\infty N(\beta_i|0, \lambda/(\phi\eta_i)) IG(\eta_i|\nu/2, \nu/2) d\eta_i$$

2.
$$\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N_n(\boldsymbol{0}, (1/\phi)\boldsymbol{I}) \\
p(\beta_i|\phi) &= N(\beta_i|0, \lambda/(\phi\eta_i)), \quad i = 1, \dots, p \\
p(\eta_i) &= IG(\eta_i|\nu/2, \nu/2) d\eta_i \quad i = 1, \dots, p
\end{aligned}$$

and $p(\phi) \propto 1/\phi$.

3.
$$p(\boldsymbol{\beta}| \dots) = N(\boldsymbol{A}^{-1}\boldsymbol{X}'\boldsymbol{y}, \lambda/\phi(\boldsymbol{X}'\boldsymbol{X} + \boldsymbol{D}_\eta^{-1})^{-1}), \quad \boldsymbol{D} = \text{diag}(\eta_i)$$

where $\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{X} + \boldsymbol{D}_\eta^{-1})$.

$$p(\eta_i| \dots) = IG(\eta_i|(\nu + 1)/2, (\nu + \phi\beta_i^2/\lambda)/2), \quad i = 1, \dots, p$$

$$p(\phi| \dots) = Ga((n + p)/2, ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}||^2 + \boldsymbol{\beta}'\boldsymbol{D}^{-1}\boldsymbol{\beta}/\lambda)$$

Hint:

1.

$$(x-a)'A(x-a)+(x-b)'B(x-b) = (x-c)'(A+B)(x-c)+(a-b)'A(A+B)^{-1}B(a-b)$$

where

$$c = (A+B)^{-1}(Aa+Bb)$$

2.

$$\int_0^\infty \frac{e^{-\beta/x}}{x^{\alpha+1}}\frac{\beta^\alpha}{\Gamma(\alpha)}dx = 1$$