BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

Student number: ____________________

First Year Exam: June 11th 2010

INSTRUCTIONS

You must answer both questions in Part A pertaining to courses AMS 205 and AMS 211.

You must also answer all 4 out of the 8 questions in Part B.

PART A

Problem 1 (AMS 205) :

For each one of the following statements, decide if it is true or false. You must briefly justify your answer (short proof, counterexample and/or argument).

1. Let $\Phi(x)$ be the cumulative distribution function of the standard normal distribution and let $Z \sim \mathsf{N}(0,1)$. Then $\mathsf{E}(\Phi(Z)) = 1/2$.

   Answer: True. Note that

   $$E(\Phi(Z)) = \int_{-\infty}^{\infty} \Phi(z)\phi(z)dz = \int_0^1 t\,dt = 1/2$$

2. If $\tilde{\theta}$ is is an unbiased estimator of θ, then $\tilde{\phi} = 1/\tilde{\theta}$ is an unbiased estimator of $\phi = 1/\theta$.

   Answer: False. For a counterexample, take $X_i \sim \mathsf{Exp}(\lambda)$ and $\hat{\lambda} = \bar{X}$. $\mathsf{E}(\hat{\lambda}) = \lambda$ but $E(1/\bar{X}) = n/\{(n-1)\lambda\} \neq \lambda$

3. Consider a statistical model where $Y_{ijk} \sim \mathsf{N}(\alpha_i\beta_j, \sigma^2)$ for $i = 1, \ldots, 5$, $j = 1, \ldots, 8$, $k = 1, \ldots, 10$ and where $\{\alpha_i\}_{i=1}^5$, $\{\beta_j\}_{j=1}^8$ and σ^2 are all unknown. The resulting model is not identifiable.

   Answer: True. For example, $\alpha_i = 2$ and $\beta_j = 3$ lead to the same likelihood as $\alpha_i = 1$ and $\beta_j = 6$.

4. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density $f(x|\theta)$ that depends on a unidimensional parameter θ. If $n > 2$, every minimal sufficient statistic for the problem has dimension $k < n$.

   Answer: False. For example, the double exponential and the t distribution with unknown location parameters have as minimal sufficient statistics the order statistics.

5. Let $X_1, \ldots, X_n$ be a random sample where $X_i \sim p(\cdot|\theta)$. The uniformly most powerful test of level α to contrast $H_0 : \theta = \theta_0$ vs. $H_a : \theta = \theta_1$ that is based on the likelihood ratio test is obtained by rejecting H_0 if

   $$\Lambda = \frac{\prod_{i=1}^n p(x_i|\theta_1)}{\prod_{i=1}^n p(x_i|\theta_0)} > k$$

   for some constant k such that $\mathsf{Pr}(\Lambda > k|H_0) = \alpha$.

   Answer: True. This is, literally Neyman-Pearson Lemma

6. The power and the level of a statistical test must sum up to one.

   Answer: False. Does it need explanation?

Problem 2 (AMS 211) :

1. (50%) Classify the following ordinary differential equation and solve it:

$$x\frac{dy}{dx} + 2(1-x^2)y = 1$$

2. (50%) A function $f(x)$ equals e^{-x} over $0 < x < 1$. Expand $f(x)$ as a Fourier sine series and calculate the Fourier coefficients. Sketch the periodic Fourier solution. Why are the end points of the function of the interval $[0,1]$ excluded?

Solution :

(1) Classification: First order, linear
Solve using integrating factor

$$\tfrac{dy}{dx} + (\tfrac{2}{x} - 2x)y = \tfrac{1}{x}$$

The integrating factor is

$$= e^{\int(\frac{2}{x}-2x)dx} = x^2 e^{x^2}$$

and so

$$\begin{aligned} \tfrac{d(x^2e^{x^2}y)}{dx} &= x^2e^{x^2}\tfrac{1}{x} \\ x^2e^{x^2}y &= -\tfrac{1}{2}e^{-x^2} + C \\ y &= -\tfrac{1}{2x^2} + \tfrac{Ce^{x^2}}{x^2} \end{aligned}$$

(2)

$$\begin{aligned} f(x) &= \textstyle\sum_{n=0}^{\infty} B_n \sin(n\pi x) \\ B_n &= 2\textstyle\int_0^1 e^{-x}\sin(n\pi x)dx \end{aligned}$$

Integrating by parts twice gives

$$\begin{aligned} B_n &= 2n\pi\tfrac{(1-\frac{\cos(n\pi)}{e})}{(1+(n\pi)^2)} \\ &= \tfrac{2n\pi}{e}\tfrac{(e-(-1)^n)}{(1+(n\pi)^2)} \end{aligned}$$

Since a Fourier sine series is used, the extension to a periodic function is an odd extension. An odd extension of the given function will be discontinuous at the end points and therefore give an incorrect value at the end points of the [0,1] range if included.

PART B

Problem 3 (AMS 212A) :

(a) (20%) The following equation

$$x^2 f'' + 2xf' + (x^2 - n(n+1))f = 0 \tag{1}$$

for integer values of n is called the Spherical Bessel Equation.

Show that the Spherical Bessel Equation is a Sturm-Liouville problem, by casting it in the form $(p(x)f')' + q(x)f = -\lambda r(x)f$. You will need to specify what the functions p, q and r are.

(b) (80%) A conducting sphere of radius R has been left at ambient temperature ($t = 20°$) for a long time, and now has a steady, uniform temperature throughough with this value. At time $t = 0$, it is immersed in water at $0°$, and slowly cools down. The aim of this problem is to calculate the temperature profile T in the sphere as a function of position within the sphere and of time. You may assume that the temperature of the water remains $0°$ at all times, and that the conductivity of the sphere is $\kappa = 1$.

(i) Give a complete mathematical description of the problem (governing equation, boundary conditions, initial conditions).

(ii) Is this a hyperbolic, parabolic or elliptic problem? (Pass/Fail question)?

(iii) Using separation of variables, show that the temperature as a function of position and time (for $t > 0$ and interior to the sphere) is

$$T(r, \theta, \phi, t) = \sum_{n=1}^{\infty} a_n j_0(\sqrt{\lambda_n} r) e^{-\lambda_n t} \tag{2}$$

where r is radius, θ is co-latitude and ϕ is azimuth. You must find expressions for a_n and λ_n in terms of known quantities in the problem.

Useful mathematical formula for this problem:

1. The spherical geometry Laplacian is:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{3}$$

in the coordinate system (r, θ, ϕ).

2. Solutions to the Spherical Bessel Equation are the Spherical Bessel Functions:

$$\begin{aligned} &j_n(x) \text{ are regular at } x = 0 \\ &y_n(x) \text{ are singular at } x = 0 \end{aligned} \tag{4}$$

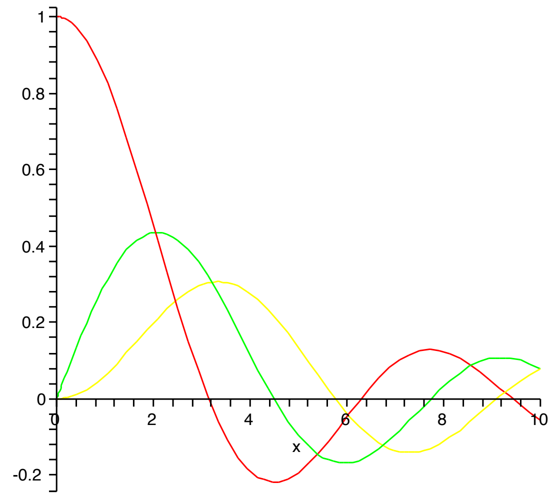

Figure 1: The functions $j_0(x)$, $j_1(x)$ and $j_2(x)$

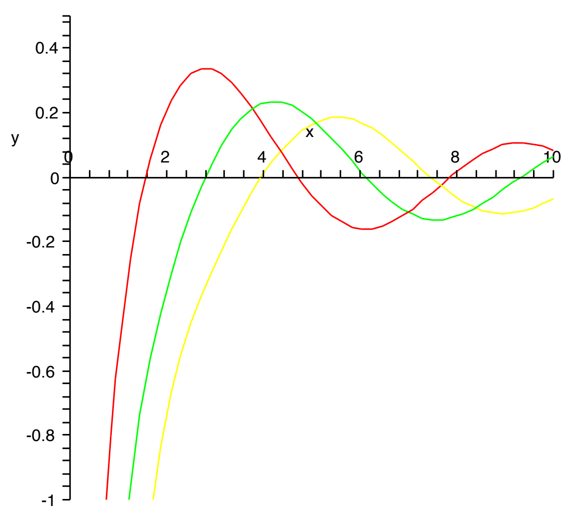

Figure 2: The functions $y_0(x)$, $y_1(x)$ and $y_2(x)$

Spherical Bessel Functions oscillate about 0 (see Figure). The $m-$th zero of the $j_n(x)$ function is denoted as $z_{n,m}$, and the m-th of the $y_n(x)$ function is denoted as $\zeta_{n,m}$.

Solution :

See next page

AMS 212A

(a) $x^2 f'' + 2x f' + (x^2 - (n(n+1)))f = 0$

$\Leftrightarrow \quad [p(x)f']' + q(x)f = -\lambda r(x) f$

provided
$$p(x) = x^2$$
$$q(x) = x^2$$
$$r(x) = 1$$

(b) (i) Equation: $\dfrac{\partial T}{\partial t} = \nabla^2 T$ for $0 < r < R$, $\theta \in [0,\pi]$, $\phi \in [0, 2\pi]$

BC: $\begin{cases} T(R,\theta,\phi,t) = 0 \\ T \text{ regular} (@\ r=0 \text{ and at } \theta = 0,\pi) \end{cases}$

IC: $T(r,\theta,\phi,0) = 20$ for $0 \leq r < R$

(ii) A parabolic problem.

(iii) BCs & ICs are spherically symmetric (no dependence on θ, ϕ) $\Rightarrow$ $\partial/\partial\theta$, $\partial/\partial\phi = 0$ in the problem. Then, the governing eq. reduces to

$$\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

By separation of variables seek solutions
$$T \sim A(r) B(t)$$

$$\Rightarrow \begin{cases} \dfrac{1}{A}\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2 \dfrac{dA}{dr}\right) = K \\ \dfrac{1}{B}\dfrac{dB}{dt} = K \end{cases}$$

Possible values of K

- K cannot be > 0 otherwise we have

- $K=0$ case $\Rightarrow$ Steady state solution towards which the system relaxes. Here clearly uniform solution with $T=0$ everywhere

- $K<0$ case : let $\lambda = -K$ then

$$B(t) \propto e^{-\lambda t}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) = -\lambda A$$

$$\Leftrightarrow \quad r^2 A'' + 2rA' = -\lambda r^2 A$$

$$\Leftrightarrow \quad r^2 A'' + 2rA' + \lambda r^2 A = 0$$

If $x^2 = \lambda r^2$ then by change of variable

$$x^2 A_{xx} + 2x A_x + x^2 A = 0$$

$\Rightarrow$ A is solution to a spherical Bessel equation with $n=0$.

$A =$ a linear combination of $\begin{cases} j_0(x) = j_0(\sqrt{\lambda}\, r) \\ y_0(x) = y_0(\sqrt{\lambda}\, r) \end{cases}$

- regularity condition @ $r=0$ $\Rightarrow$ only keep j_0
- $A(R) = 0 \Rightarrow j_0(\sqrt{\lambda}\, R) = 0$

$\Rightarrow$ many solutions for evalue λ_n with $\Rightarrow$ $\sqrt{\lambda_n}\, R = z_{0n}$

$$\Rightarrow \quad \lambda_n = \left(\frac{z_{0n}}{R}\right)^2$$

So finally we can write T as a linear combination of all solutions

$$T(r,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} j_0(\sqrt{\lambda_n}\, r)$$

with $\lambda_n = \left(\frac{z_{0n}}{R}\right)^2$.

To get the coefficients a_n, we project this solution onto the IC @ $t=0$:

$$20 = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r)$$

By orthogonality property of the eigenfunctions we know that

$$\int_0^R J_0(\sqrt{\lambda_n}\, r)\, J_0(\sqrt{\lambda_m}\, r)\, dr = \delta_{mn} \| J_0(\sqrt{\lambda_n}\, r) \|^2$$

recall weight function is 1 for SB problem.

$$\Rightarrow \quad a_n = \frac{\int_0^R 20\, J_0(\sqrt{\lambda_n}\, r)\, dr}{\int_0^R J_0^2 (\sqrt{\lambda_n}\, r)\, dr}.$$

Problem 4 (AMS 212B) :

Use the method of multiple scale expansion to solve the initial value problem

$$\begin{cases} y'' + \varepsilon\,(1+y^2)\,y' + y = 0 \\ y(0) = 1\ , \quad y'(0) = 0 \end{cases} \qquad , \qquad \varepsilon \to 0_+$$

Find the leading term in the expansion.

Solution :

Let $T_0 = t$ and $T_1 = \varepsilon t$. We write y as a function of (T_0, T_1) and use chain rule to calculate derivatives with respect to t.

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial y}{\partial T_0} + \varepsilon \frac{\partial y}{\partial T_1} \\ \frac{d^2 y}{dt^2} &= \frac{\partial^2 y}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 y}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2 y}{\partial T_1^2} \end{aligned}$$

We seek an expansion of the form

$$y(T_0, T_1) = a_0(T_0, T_1) + \varepsilon a_1(T_0, T_1) + \cdots$$

Substituting into equation and collecting $O(1)$ and $O(\varepsilon)$ terms, we have

$$\left[\frac{\partial^2 a_0}{\partial T_0^2} + a_0\right] + \varepsilon \left[\frac{\partial^2 a_1}{\partial T_0^2} + a_1 + 2\frac{\partial^2 a_0}{\partial T_0 \partial T_1} + (1+a_0^2)\frac{\partial a_0}{\partial T_0}\right] + \cdots = 0$$

Equating the coefficient of $O(1)$ term to zero yields

$$\frac{\partial^2 a_0}{\partial T_0^2} + a_0 = 0$$

With the constraint that solution is real, a general solution is given by

$$a_0(T_0, T_1) = A(T_1)e^{iT_0} + \bar{A}(T_1)e^{-iT_0}$$

Setting the coefficient of $O(\varepsilon)$ term to zero gives us

$$\begin{aligned} \frac{\partial^2 a_1}{\partial T_0^2} + a_1 &= -2\frac{\partial^2 a_0}{\partial T_0 \partial T_1} - (1+a_0^2)\frac{\partial a_0}{\partial T_0} \\ &= -2i\left(A'(T_1)e^{iT_0} - \bar{A}'(T_1)e^{-iT_0}\right) - i\left(A(T_1)e^{iT_0} - \bar{A}(T_1)e^{-iT_0}\right) \\ &\quad \times \left(1 + A^2(T_1)e^{i2T_0} + \bar{A}^2(T_1)e^{-i2T_0} + 2A(T_1)\bar{A}(T_1)\right) \\ &= c_0 + c_1 e^{iT_0} + \bar{c}_1 e^{-iT_0} + c_2 e^{i2T_0} + \bar{c}_2 e^{-i2T_0} + \cdots \end{aligned}$$

where coefficient c_1 has the expression

$$c_1 = -2iA'(T_1) - iA(T_1) - iA^2(T_1)\bar{A}(T_1)$$

To kill secular terms in $a_1(T_0, T_1)$, coefficient c_1 must be zero, which leads to a differential equation for $A(T_1)$:

$$2A'(T_1) = -A(T_1) - A^2(T_1)\bar{A}(T_1)$$

The initial condition for $A(T_1)$ is obtained from initial conditions for $y(t)$.

$$\begin{aligned}
y(0) = 1 \qquad &\Longrightarrow \qquad a_0(0,0) = 0 \\
&\Longrightarrow \qquad A(0) + \bar{A}(0) = 1 \\
y'(0) = 0 \qquad &\Longrightarrow \qquad \frac{\partial a_0(0,0)}{\partial T_0} = 0 \\
&\Longrightarrow \qquad A(0) - \bar{A}(0) = 0
\end{aligned}$$

which leads to $A(0) = \frac{1}{2}$. To solve for $A(T_1)$, we write $A(T_1)$ in the polar form

$$A(T_1) = R(T_1)e^{i\theta(T_1)}$$

$$A(0) = \frac{1}{2} \qquad \Longrightarrow \qquad R(0) = \frac{1}{2}\,, \quad \theta(0) = 0.$$

Substituting the polar form into the differential equation for $A(T_1)$, we obtain

$$\begin{cases} \theta'(T_1) = 0 \\ \theta(0) = 0 \end{cases}$$

and

$$\begin{cases} 2R'(T_1) = -\left[R(T_1) + R^3(T_1)\right] \\ R(0) = \frac{1}{2} \end{cases}$$

The initial value problem for $\theta(T_1)$ yields $\theta(T_1) = 0$. The differential equation for $R(T_1)$ is a separable equation and is solved as

$$\begin{aligned}
&2\frac{dR}{dT_1} = -\left(R + R^3\right) \\
&\Longrightarrow \qquad \frac{-2}{R + R^3}dR = dT_1 \\
&\Longrightarrow \qquad \left(\frac{2R}{R^2+1} - \frac{2}{R}\right) dR = dT_1 \\
&\Longrightarrow \qquad \log(R^2+1) - \log(R^2) = T_1 + c \\
&\Longrightarrow \qquad \frac{R^2+1}{R^2} = ce^{T_1} \\
&R(0) = \frac{1}{2} \qquad \Longrightarrow \qquad c = 5 \\
&\Longrightarrow \qquad R(T_1) = \sqrt{\frac{1}{5e^{T_1} - 1}}
\end{aligned}$$

$a_0(T_0, T_1)$ has the expression

$$a_0(T_0, T_1) = 2\sqrt{\frac{1}{5e^{T_1} - 1}}\cos(T_0)$$

The leading term multi-scale expansion for $y(t)$ is

$$y(t) = 2\sqrt{\frac{1}{5e^{\varepsilon t} - 1}}\cos(t) + \cdots$$

Problem 5 (AMS 213) :

Problem 5.1 [20%] The initial value problem

$$\frac{dy}{dt} = y^{1/3},\ y(0) = 0$$

has a solution

$$y(t) = (2t/3)^{3/2} \tag{5}$$

Explain why Euler method $y_{k+1} = y_k + h(y_k)^{1/3}$, $y_0 = 0$, does not converge to solution (5).

Problem 5.2 [30%] Consider the following steepest descent method for solving $Ax = b$.

$$\begin{aligned} p_k &= b - Ax_k \\ \alpha_k &= \frac{p_k^T p_k}{p_k^T A p_k} \\ x_{k+1} &= x_k + \alpha_k p_k \end{aligned}$$

Show that the error vector, $e_{k+1} = A^{-1}b - x_{k+1}$, always conjugate to the search direction p_k, i.e., $p_k^T A e_{k+1} = 0$.

Problem 5.3 [50%] Consider the following Cauchy problem

$$\begin{aligned} u_t + au_x &= 0,\ -\infty < x < \infty \\ u(x,0) &= g(x), \\ \int_{-\infty}^{\infty} |u(x,t)|^2\,dx &< \infty,\ \forall t \geq 0 \end{aligned}$$

where $a > 0$ is a constant; and the following finite difference scheme

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} + a\frac{u_k^n - u_{k-1}^n}{\Delta x} = 0$$

where u_k^n is the discrete solution at $x = k\Delta x$ and $t = n\Delta t$.

1. Show that the scheme is consistent; [25%]
2. Use Fourier transform to show that for $a\frac{\Delta t}{\Delta x} \leq 1$, the scheme is stable. [25%]

Solution :

Problem 5.1 The function $y^{1/3}$ is not Lipschitz continuous at $y = 0$. Therefore, the convergence of Euler's method cannot be guaranteed. Indeed, the considered initial value problem has infinite number of solutions including $y(t) = 0$, $\forall t$.

Problem 5.2

$$\begin{aligned} p_k^T A e_{k+1} &= p_k^T A(A^{-1}b - x_{k+1}) \\ &= p_k^T b - p_k^T A(x_k + \alpha_k p_k) \\ &= p_k^T (b - Ax_k) - p_k^T p_k \;=\; 0 \end{aligned}$$

Problem 5.3

1. The difference scheme is consistent, since

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} + a\frac{u_k^n - u_{k-1}^n}{\Delta x} \quad = \quad (u_t + au_x)\big|_{(x_k, t_n)} + O(\Delta t, \Delta x)$$

2. Let $\mu = \frac{\Delta t}{\Delta x}$. Then, $u_k^{n+1} = (1 - a\mu)u_k^n + a\mu u_{k-1}^n$. Therefore,

$$\begin{aligned} u_k^{n+1} e^{-ik\theta} &= (1 - a\mu) u_k^n e^{-ik\theta} + a\mu u_{k-1}^n e^{-ik\theta} \quad \Longrightarrow \\ \sum_{k=-\infty}^{\infty} u_k^{n+1} e^{-ik\theta} &= (1 - a\mu) \sum_{k=-\infty}^{\infty} u_k^n e^{-ik\theta} + a\mu e^{-i\theta} \sum_{k=-\infty}^{\infty} u_k^n e^{-ik\theta} \quad \Longrightarrow \\ \hat{u}^{n+1}(\theta) &= \left[(1 - a\mu) + a\mu e^{-i\theta}\right] \hat{u}^n(\theta) \end{aligned}$$

For the scheme to be stable, we need

$$\begin{aligned} \left|(1 - a\mu) + a\mu e^{-i\theta}\right|^2 &= 1 + 2a\mu(a\mu - 1)(1 - \cos\theta) \;\le 1,\;\; \theta \in [0, 2\pi], \\ \Longrightarrow\; a\mu \;\le\; 1 \end{aligned}$$

Problem 6 (AMS 214) :

Question 6.1: [50%] Consider the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

Using a change of variable from polar to Cartesian to show that

$$r - r^3 \leq \dot{r} \leq r - \frac{1}{2}r^3 \tag{6}$$

where $r^2 = x^2 + y^2$. Hint: you may need to use the trigonometric relation

$$\cos^4\theta + \sin^4\theta = \frac{3}{4} + \frac{1}{4}\cos 4\theta \tag{7}$$

By constructing an appropriate trapping region, show the existence of a limit cycle in this system.

Question 6.2: [50%]
(a) Consider the non-dimensional equation:

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$

where $b > 0$. What physical system could this represent? Cast this equation into a system of two first-order ODEs.

(b) Find the fixed points, and study their stability. Show that there is a critical value of the parameter, b_c, where the qualitative behavior of the solutions changes.

(c) Draw an accurate phase portrait for two values of b, one larger than b_c and one smaller than b_c.

(d) Explain in words how these solutions relate to the original physical system studied.

(e) Can this system exhibit chaos? If yes, discuss the kind of solutions expected in the chaotic regime. If no, explain why.

Solution :

(Q6 cont)

Problem 7 (AMS 207) :

Consider the following random-effects hierarchical model, which is useful in meta-analysis and other applications:

$$\begin{aligned}(\mu, \sigma^2) &\sim p(\mu, \sigma^2) \\ (\gamma_i|\mu, \sigma^2) &\overset{\text{IID}}{\sim} N(\mu, \sigma^2) \\ (y_i|\gamma_i) &\overset{\text{indep}}{\sim} N(\gamma_i, \tau_i^2),\end{aligned} \tag{8}$$

for $i = 1, \ldots, I$, in which the τ_i^2 are assumed known.

(a) With $\theta = (\mu, \sigma^2)$ and an appropriate choice for latent data z, specify the two distributions $p(\theta|z, y)$ and $p(z|\theta, y)$ needed to carry out an EM algorithm to find the posterior mode of θ given $y = (y_1, \ldots, y_I)$, making an appropriate conditionally conjugate choice for the prior distribution on θ, and use this to specify the E and M steps of your EM algorithm.

(b) Specify a Gibbs sampler to make random draws from the augmented posterior distribution $p(\mu, \sigma^2, z|y)$, by providing details on the full-conditional distributions $p(\mu|\sigma^2, z, y)$, $p(\sigma^2|\mu, z, y)$ and $p(z|\mu, \sigma^2, y)$.

Solution :