Final Exam for 206B

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2022/3/17

Question a: Descriptive Statistics

First of all, this data is a longitudinal data, it describes the weight gained by pregnant women during a survey. So I reshaped the data into a wide form instead of a long form for further convenience. And also, I made a plot about the personal trend and overall trend for the women's weight gain in our survey.

For each woman, the weight gain can be plotted as in the figure 1. We can see that as time goes by, the weight gained by women are having a increasing trend for almost all the people in the survey. Also, I provided one overall trend between the average weight and time trend.. Since this data includes longitudinal properties, the mean and the summary of the data will be less informative, but the summary are provided as follows, the average or properties of time is not useful since they are similar to each other. Therefore, only information about weights are provided.

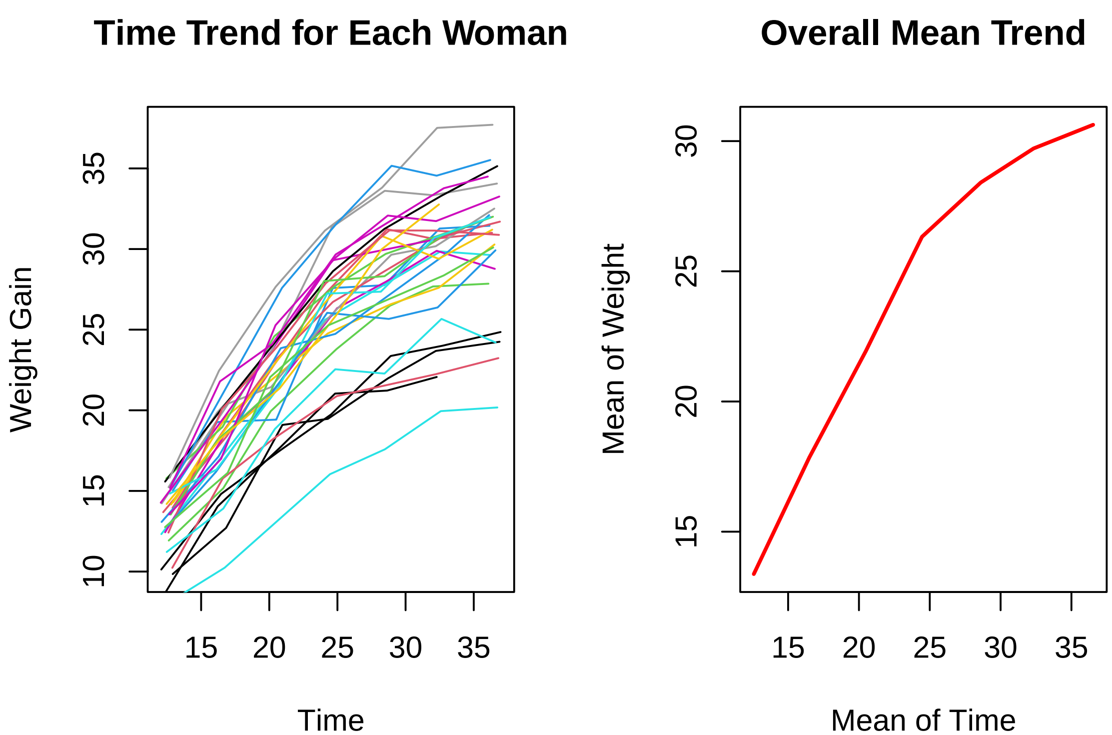

Figure 1: Time Trend of Weight Gain for Each Woman

```
##      weight.1          weight.2         weight.3         weight.4
##  Min.   : 9.846   Min.   :12.71   Min.   :17.49   Min.   :19.47
##  1st Qu.:12.420   1st Qu.:16.22   1st Qu.:20.24   1st Qu.:25.05
##  Median :13.613   Median :18.45   Median :21.67   Median :26.98
##  Mean   :13.374   Mean   :17.85   Mean   :21.94   Mean   :26.33
##  3rd Qu.:14.700   3rd Qu.:19.80   3rd Qu.:23.36   3rd Qu.:27.98
##  Max.   :15.820   Max.   :22.45   Max.   :27.66   Max.   :31.35
##      weight.5         weight.6         weight.7
##  Min.   :21.98   Min.   :23.68   Min.   :24.21
##  1st Qu.:26.51   1st Qu.:27.62   1st Qu.:29.06
##  Median :28.15   Median :30.40   Median :31.10
##  Mean   :28.41   Mean   :29.72   Mean   :30.63
##  3rd Qu.:31.21   3rd Qu.:31.24   3rd Qu.:32.38
##  Max.   :33.81   Max.   :37.51   Max.   :37.70
```

Question b: Joint Posterior Distribution

I used the handwritten form for this question, and the notes has been attached at the end of the file.

For the algorithm part, it is a Gibbs sampler algorithm, but we need MCMC when sampling several parameters. I find the full conditional posterior distribution of β_{0i} and $\bar{\beta_0}$ are a normal distribution, and the full conditional posterior distribution of σ^2 and τ^2 are inverse gamma distribution, which are easy to generate directly from the program. However, the distribution of β_1 and β_2 are not in closed form, indicating that we have to use the MCMC method when we get samples from this distribution.

Algorithm Discription:

Step 1: Specify an initializing value for all parameters

Give values for $\beta_{0i}^{(1)}$, $\beta_1^{(1)}$, $\beta_2^{(1)}$, $\bar{\beta_0}^{(1)}$, ${\sigma^2}^{(1)}$, ${\tau^2}^{(1)}$

Step 2: Use Metroplis Hasting with Gibbs and loop for i from 1 to 12000

Generate $\beta_{0i}^{(i+1)}$, $\bar{\beta_0}^{(i+1)}$, ${\sigma^2}^{(i+1)}$, ${\tau^2}^{(i+1)}$ directly from the full conditional posterior distribution. But for $\beta_1^{(i+1)}$, $\beta_2^{(i+1)}$, I will use Metropolis Hasting algorithm here. My proposal distribution is a **Random Walk**:

$$q(\theta^{(i+1)}|\theta^{(i)}) \propto exp(-\frac{(\theta^{(i+1)} - \theta^{(i)})^2}{2d^2})$$

First, for $\beta_1^{(i+1)}$ generation, use the proposal distribution to propose a new value and denote to ξ, then calculate the transit probability:

$$p = min\{\frac{\pi(\xi)q(\theta^{(i)}|\xi)}{\pi(\theta^{(i)})q(\xi|\theta^{(i)})}, 1\}$$

Since this is a random walk, $q(\theta^{(i)}|\xi) = q(\xi|\theta^{(i)})$, therefore:

$$p = min\{\frac{\pi(\xi)}{\pi(\theta^{(i)})}, 1\}$$

π here is just the full conditional probability function of θ

Generate an index number, for example, k which follows a uniform distribution between 0 and 1. Compare the value of p between k:

$$p > k : \theta^{(i+1)} = \xi$$

$$p \leq k : \theta^{(i+1)} = \theta^{(i)}$$

Then, same method for generate $\beta_2^{(i+1)}$. Until now, we have finished one update for all the parameters in a Gibbs sampler.

Step 3: Burn-in and thinning

Since we have strong auto correlation after generating the raw samples from the posterior distribution, we have to do the burn-in and thinning.

Burn-in: Discard the front 12000 samples

Thinning: For the rest 88000 samples, we get samples every 88 sample, then we get 1000 better samples in total.

Step 4: Make inference

Since we have the 1000 samples after the burn-in and thinning, we can make inference based on these samples, for example, the credible interval, posterior mode, posterior mean, median and so on.

Question c: Hyperparameter Specification

I used this set of hyper parameters: $\bar{\beta}_1 = 10$, $\bar{\beta}_2 = -0.15$, $u_1^2 = 100$, $u_2^2 = 100$, $\mu_0 = 30$, $v^2 = 1000$, $a_\sigma = 1$, $a_\tau = 1$, $b_\sigma = 1$, $b_\tau = 1$. We have many ways to specify the hyper parameter, in my method, I just used the empirical Bayesian. I used a simple random parameter for the mean, but I must set the variance to be large to make sure that the prior information is very weak. Then, I run my algorithm, then it did not work well, but I can approximately get a "proper" range of the hyper parameters by definition of the parameter. Then, I used the ones seems perform well to be my final parameter.

Furthermore, the length of a step for a proposal distribution should also be selected, i.e., the standard deviation of the random walk kernel. I get the standard deviation by the approximate posterior sampling by a random selected hyper parameter and then fixed it little by little to fix it with the greatest effective sample size. This is really hard in my project, I used almost 60% of time to get the standard deviation of the random walk kernel. I think I can use adaptive MH but I am not sure whether it is cheating since selecting the variance is also a core part of the MH algorithm. Finally, I chose the standard deviation of beta_1 proposal to be 0.4 and 0.003 for beta_2.

Question d: MCMC within Gibbs Coding

First, I will check for convergence, from the trace plot in figure 2, we can see that after the burn-in and thinning process, the chains mixed well.q

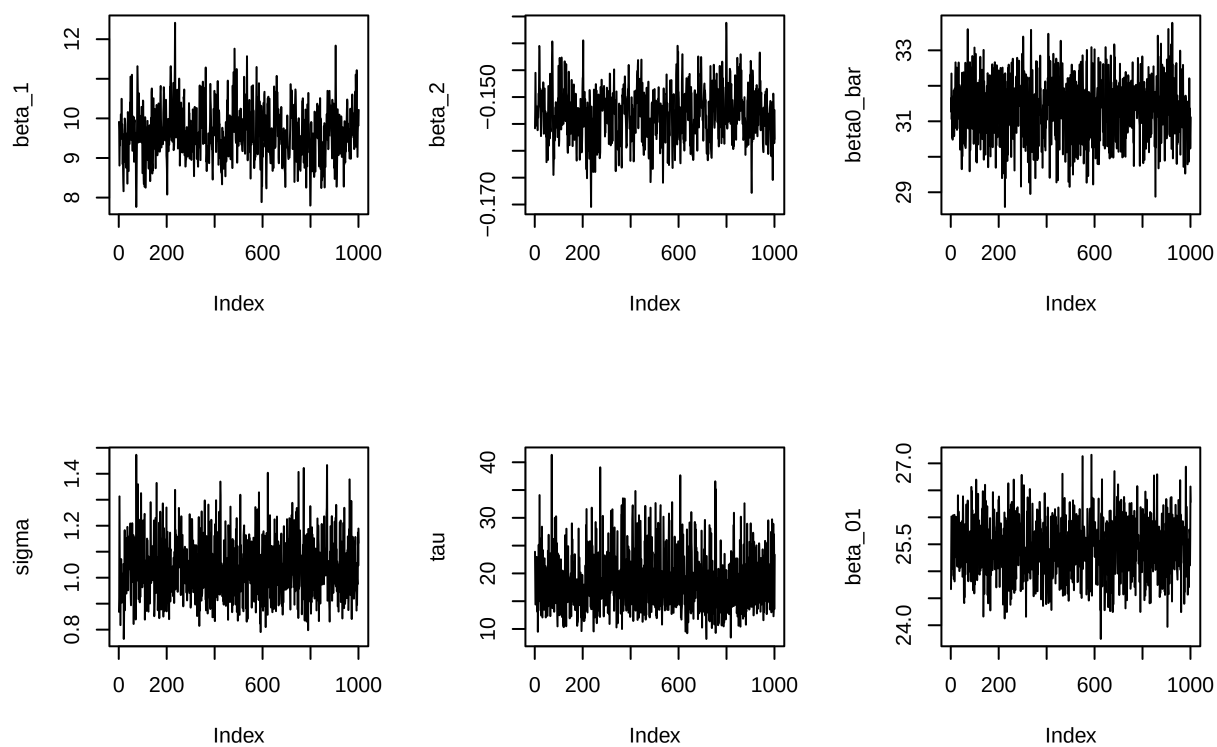

Figure 2: Trace Plot

Question e: Interpretation

To interpret the parameters of the other parameters except β_{0i}, I simply focused on three concepts, one is posterior mean, the other two is 2.5% quantile and 97.5% quantile, for $\beta_1, \beta_2, \bar{\beta_0}, \sigma, \tau$, I reported a posterior mean and a lower bound and an upper bound. They are got from the Monte Carlo simulation, since we got 1000 samples after burn-in and thinning the chain, the mean of them is just the mean of these samples. By LLN, it converges to the true posterior mean as sample size goes to infinity. And the 2.5% quantile and the 97.5% quantile can construct an interval with a lower bound and an upper bound, this is a 95% credible interval since they are uni-modal distribution (I checked the histogram for β_1 and β_2), so the center is the HPD. The credible interval means, the probability of the true parameter being in this interval is 0.95.

The interpretation for β_{0i} are a little different, because it is a longitudinal model, each person has its own β_0, so I presented the table about each β_{0i}. The number of each row corresponds to a specified person with the certain ID (the original order in the data set) at the beginning of this row. It could be interpreted as the posterior mean, 2.5% and 97.5 quantile of the posterior distribution of β_{0i} of patient i, and there is a 0.95 probability that the β_{0i} of patient i is in this interval.

Table 1: Summary of Beta_0i

Patient ID	Posterior Mean	2.5% Quantile	97.5% Quantile
1	25.44486	24.41281	26.43446
2	33.13857	32.17397	34.18924
3	29.00407	27.96091	30.01486
4	32.30057	31.23780	33.33045
5	31.16640	30.01770	32.30795
6	31.14530	30.07335	32.28220
7	30.78991	29.71451	31.86817
8	32.81696	31.78994	33.91261
9	24.39617	23.29745	25.57270
10	32.69151	31.56908	33.84870
11	33.81740	32.64108	35.04703
12	31.94747	30.75654	33.11219
13	20.35037	19.30680	21.37550
14	33.88521	32.63920	35.02479
15	32.80123	31.81788	33.86894
16	36.18852	35.08034	37.29795
17	35.44701	34.39122	36.50632
18	33.50106	32.44930	34.52689
19	32.71862	31.70113	33.72264
20	29.99048	28.99245	31.04918
21	32.20015	31.22083	33.23413
22	34.52761	33.48759	35.57734
23	32.88118	31.67867	34.14022
24	38.98269	37.90681	40.11786
25	24.83323	23.85656	25.90365
26	24.96274	23.87889	26.10766
27	30.72542	29.64242	31.86827
28	37.65172	36.60312	38.78132
29	26.21596	25.22183	27.23138
30	35.92422	34.72867	37.11669

Table 2: Other Parameters Summary

	Posterior Mean	2.5% Lower Bound	97.5% Upper Bound
beta_1	9.6289327	8.4156206	10.896241
beta_2	-0.1531215	-0.1619737	-0.143585
beta_0 bar	31.3489042	29.7064696	32.837960
sigma	1.0403348	0.8477839	1.271401
tau	18.4801297	10.7971255	30.768870

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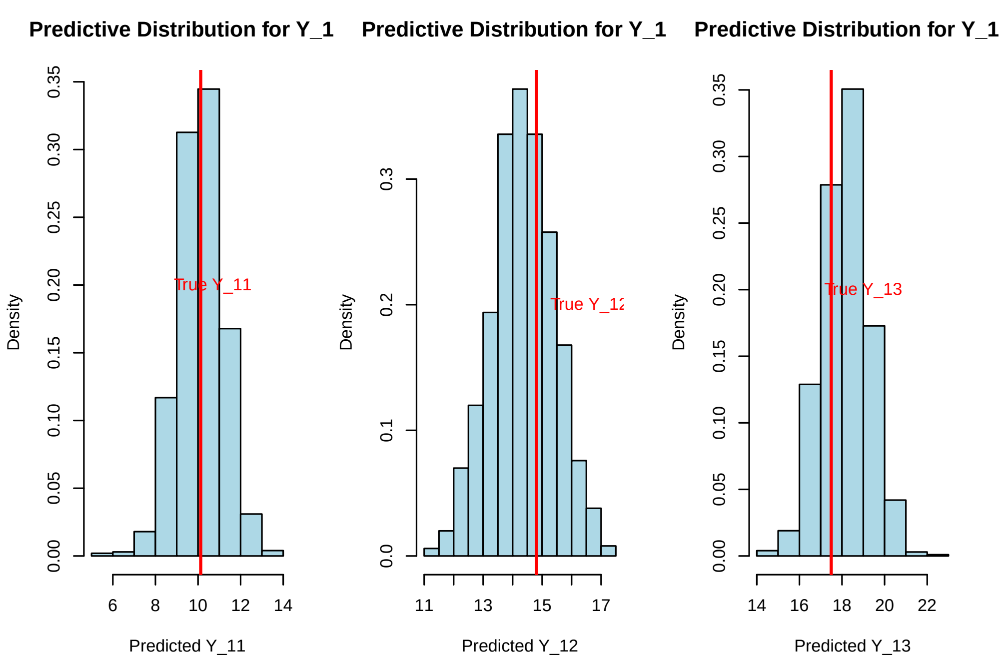
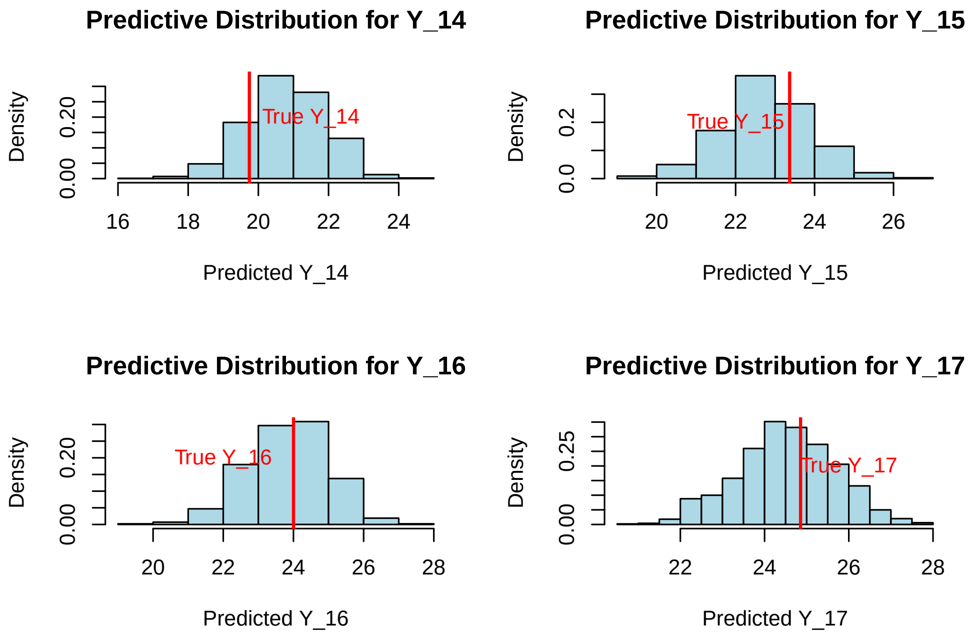
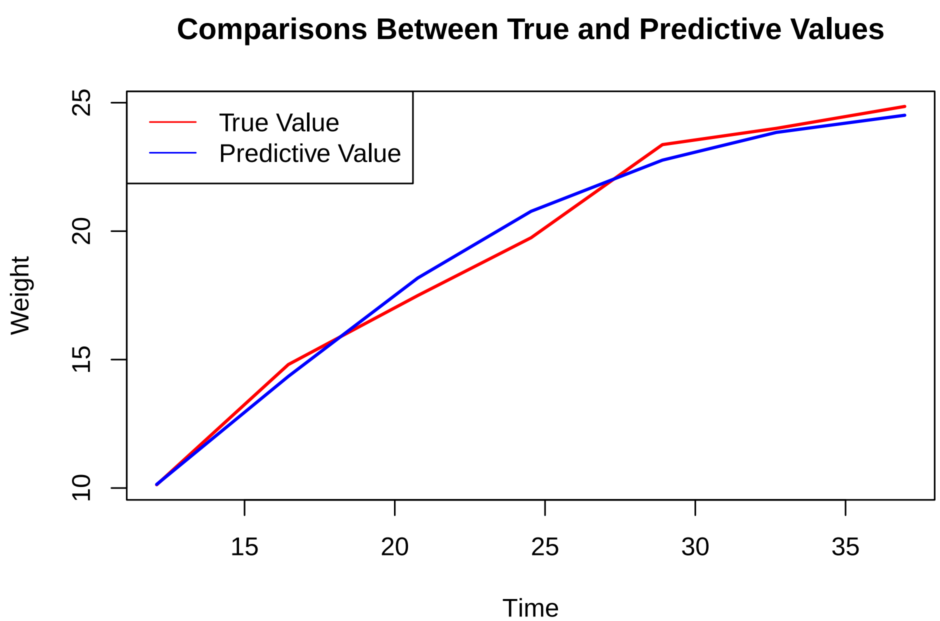
Question f: Pridictive distribution and model checking

Here are plots supporting my models. My predictive distribution performs well for the time point 1, 2, 5, 6, 7, and a little away from the true value for the time point 4, but it is okay from the figure about the trend between value and time.




Predictive Distribution for Y_14
True Y_14
Density
Predicted Y_14
Predictive Distribution for Y_15
True Y_15
Density
Predicted Y_15
Predictive Distribution for Y_16
True Y_16
Density
Predicted Y_16
Predictive Distribution for Y_17
True Y_17
Density
Predicted Y_17


Comparisons Between True and Predictive Values
True Value
Predictive Value
Weight
Time

(b) $L(\theta|y) = \prod_{i=1}^{I}\prod_{j=1}^{n_i}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\left(y_{ij}-\frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{2\sigma^2}\right)\right)$

$\pi(\beta_{0i}|\bar{\beta}_0) = \frac{1}{\sqrt{2\pi\tau^2}}\exp\left(-\frac{(\beta_{0i}-\bar{\beta}_0)^2}{2\tau^2}\right)$

$\pi(\beta_p|\bar{\beta}_p) = \frac{1}{\sqrt{2\pi u_p^2}}\exp\left(-\frac{(\beta_p-\bar{\beta}_p)^2}{2u_p^2}\right)$

$\pi(\sigma^2) = \frac{b_\sigma^{a_\sigma}}{\Gamma(a_\sigma)}(\sigma^2)^{-a_\sigma-1}\exp\left(-\frac{b_\sigma}{\sigma^2}\right)$

$\pi(\bar{\beta}_0) = \frac{1}{\sqrt{2\pi v^2}}\exp\left(-\frac{(\bar{\beta}_0-\mu_0)^2}{2v^2}\right)$

$\pi(\tau^2) = \frac{b_\tau^{a_\tau}}{\Gamma(a_\tau)}(\tau^2)^{-a_\tau-1}\exp\left(-\frac{b_\tau}{\tau^2}\right)$

$$\pi(\theta|y) \propto (\sigma^2)^{-\frac{1}{2}N}\exp\left(-\frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left(y_{ij}-\frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{2\sigma^2}\right)$$
$$(\tau^2)^{-\frac{1}{2}I}\exp\left(-\frac{\sum_{i=1}^{I}(\beta_{0i}-\bar{\beta}_0)^2}{2\tau^2}\right)$$
$$\exp\left(-\frac{(\beta_1-\bar{\beta}_1)^2}{2u_p^2}\right)\exp\left(-\frac{(\beta_2-\bar{\beta}_2)^2}{2u_p^2}\right)$$
$$(\sigma^2)^{-a_\sigma-1}\exp\left(-\frac{b_\sigma}{\sigma^2}\right)\exp\left(-\frac{(\bar{\beta}_0-\mu_0)^2}{2v^2}\right)$$
$$(\tau^2)^{-a_\tau-1}\exp\left(-\frac{b_\tau}{\tau^2}\right)$$

Full conditional conditions:

$$\pi(\beta_{0i}|y,\text{others}) \propto \exp\left(-\frac{\sum_{j=1}^{n_i}\left(y_{ij}-\frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{2\sigma^2}-\frac{(\beta_{0i}-\bar{\beta}_0)^2}{2\tau^2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta_{0i}^2\left(\frac{\sum_{j=1}^{n_i}\left(\frac{1}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{\sigma^2}+\frac{1}{\tau^2}\right)-2\beta_{0i}\left(\frac{\sum_{j=1}^{n_i}\left(\frac{y_{ij}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)}{\sigma^2}+\frac{\bar{\beta}_0}{\tau^2}\right)\right]\right)$$

$$\beta_{0i}|y,\text{others} \sim N\left(\frac{\frac{\sum_{j=1}^{n_i}\frac{y_{ij}}{1+\beta_1\exp(\beta_2 t_{ij})}}{\sigma^2}+\frac{\bar{\beta}_0}{\tau^2}}{\frac{\sum_{j=1}^{n_i}\left(\frac{1}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{\sigma^2}+1/\tau^2},\ \left[\frac{\sum_{j=1}^{n_i}\left(\frac{1}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{\sigma^2}+1/\tau^2\right]^{-1}\right)$$

$$\pi(\beta_1 \mid y, \text{others}) \propto \exp\left(-\frac{(\beta_1-\bar{\beta}_1)^2}{2u_1^2} - \frac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(y_{ij} - \frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{2\sigma^2}\right)$$

It's not a closed form, we should use MCMC.

$$\pi(\beta_2 \mid y, \text{others}) \propto \exp\left(-\frac{(\beta_2-\bar{\beta}_2)^2}{2u_2^2} - \frac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(y_{ij} - \frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{2\sigma^2}\right)$$

Still, should be a MCMC approach.

$$\pi(\sigma^2 \mid \text{others}) \propto (\sigma^2)^{-\frac{N}{2}-a_\sigma-1}\exp\left(-\frac{1}{\sigma^2}\left[\frac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(y_{ij} - \frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2}{2} + b_\sigma\right]\right)$$

It's an inverse gamma $IG\left(\frac{N}{2}+a_\sigma,\ \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(y_{ij} - \frac{\beta_{0i}}{1+\beta_1\exp(\beta_2 t_{ij})}\right)^2 + b_\sigma\right)$

$$\pi(\bar{\beta}_0 \mid \text{others}) \propto \exp\left(-\frac{\sum_{i=1}^{2}(\beta_{0i}-\beta_0)^2}{2\tau^2} - \frac{(\beta_0-\mu_0)^2}{2v^2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta_0^2\left(\frac{2}{\tau^2}+\frac{1}{v^2}\right) - 2\beta_0\left(\frac{\sum_{i=1}^{2}\beta_{0i}}{\tau^2}+\frac{\mu_0}{v^2}\right)\right]\right)$$

$$\bar{\beta}_0 \mid \text{others} \sim N\left(\frac{\frac{\sum_{i=1}^{2}\beta_{0i}}{\tau^2}+\frac{\mu_0}{v^2}}{\frac{2}{\tau^2}+\frac{1}{v^2}},\ \left(\frac{2}{\tau^2}+\frac{1}{v^2}\right)^{-1}\right)$$

$$\pi(\tau^2 \mid \text{others}, y) \propto (\tau^2)^{-a_\tau-\frac{2}{2}-1}\exp\left(-\frac{1}{\tau^2}\left(\frac{\sum_{i=1}^{2}(\beta_{0i}-\bar{\beta}_0)^2}{2}+b_\tau\right)\right)$$

It's an inverse gamma $IG\left(a_\tau+\frac{2}{2},\ \frac{\sum_{i=1}^{2}(\beta_{0i}-\bar{\beta}_0)^2}{2}+b_\tau\right)$