Question 1 For every collection of events A_i $(i \in I)$, show that

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Proof: Main idea: if $\forall\, a \in A$, we have $a \in B$,

then $A \subseteq B$.

Denote $\left(\bigcup_{i\in I} A_i\right)^c = \Omega_1$, $\bigcap_{i\in I} A_i^c = \Omega_2$

For $\forall\, x \in \Omega_1$, $x \notin \bigcup_{i\in I} A_i$

$\Rightarrow$ for $\forall\, i \in I$, $x \notin A_i$

$\Rightarrow$ for $\forall\, i \in I$. $x \in A_i^c$

$\Rightarrow$ $x \in \bigcap_{i\in I} A_i^c$ Therefore, $\Omega_1 \subseteq \Omega_2$ *

For $\forall\, y \in \Omega_2$, $y \in \bigcap_{i\in I} A_i^c$

$\Rightarrow$ for $\forall\, i \in I$, $y \in A_i^c$

$\Rightarrow$ for $\forall\, i \in I$, $y \notin A_i$

$\Rightarrow$ $y \in \left(\bigcup_{i\in I} A_i\right)^c$ Therefore, $\Omega_2 \subseteq \Omega_1$ **

By * and **, we have $\Omega_1 = \Omega_2$

Similarly:

Denote $\left(\bigcap_{i\in I} A_i\right)^c = S_1$ $\left(\bigcup_{i\in I} A_i^c\right) = S_2$

For any $m \in S_1$, $m \notin \bigcap_{i\in I} A_i$

$\Rightarrow \exists\ j \in I$ such that $m \notin A_j$

$\Rightarrow \exists\ j \in I$ such that $m \in A_j^c$

Also, $A_j^c \subseteq \bigcup_{i\in I} A_i^c$

$\Rightarrow m \in \bigcup_{i\in I} A_i^c = S_2$ $\Rightarrow S_1 \subseteq S_2$ $^{*}_{**}$

For any $n \in S_2$, $\exists\ k \in I$, such that:

$$n \in A_k^c$$

$\Rightarrow \exists\ k \in I$ such that $n \notin A_k$

$\Rightarrow n \notin \bigcap_{i\in I} A_i$ (Because k must exist)

$\Rightarrow n \in \left(\bigcap_{i\in I} A_i\right)^c$ $\Rightarrow S_2 \subseteq S_1$ $^{**}_{**}$

By $^{*}_{**}$ and $^{**}_{**}$, we have $S_1 = S_2$

Question 2

7. Consider two events A and B with $\Pr(A) = 0.4$ and $\Pr(B) = 0.7$. Determine the maximum and minimum possible values of $\Pr(A \cap B)$ and the conditions under which each of these values is attained.

Maximum: $P(A \cap B) = P(A) = 0.4$

Reason: $\begin{cases} P(A \cap B) \subseteq P(A) \\ P(A \cap B) \subseteq P(B) \end{cases}$

$\Rightarrow P(A \cap B) \le \min(P(A), P(B)) = 0.4$

$\Rightarrow P(A \cap B) = 0.4$ iff $A \cap B = A$

When: $A \subset B$

Minimum: $P(A \cap B) = 0.1$

Reason: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = \underset{\text{Fixed}}{P(A)} + \underset{\text{Fixed}}{P(B)} - \underset{\text{Flexible}}{P(A \cup B)}$

So we want $P(A \cup B)$ as much as possible.

However, $\begin{cases} P(A \cup B) \le 1 & \text{[By axiom of probability]} \\ P(A \cup B) \le P(A) + P(B) & \text{[By Subadditivity]} \end{cases}$

$\Rightarrow P(A \cup B) \le 1$

Therefore. $P(A \cap B)$ achieve its minimum when

$P(A \cup B) = 1$, then $P(A \cap B) = 0.4 + 0.7 - 1 = 0.1$.

When: $A \cup B = \Omega$, i.e, $P(A \cup B) = 1$

Question 3 **5.** If four dice are rolled, what is the probability that each of the four numbers that appear will be different?

Sample space #: 6^4 [Since we have 4 dices]

Denote $A = \{$4 numbers of 4 dices appear differently$\}$

$$P(A) = P_{6,4}/6^4 = \frac{6 \times 5 \times 4 \times 3}{6^4} = \frac{60}{6^3} = \frac{10}{36} = 5/18$$

Question 4 **4.** A box contains 24 light bulbs, of which four are defective. If a person selects four bulbs from the box at random, without replacement, what is the probability that all four bulbs will be defective?

Sample space #: $\binom{24}{4}$ [Choose 4 from 24]

Denote $A = \{$All 4 bulbs are defective$\}$.

$$P(A) = \binom{4}{4} / \binom{24}{4} = 1/23 \times 22 \times 21 = 1/10626$$

Question 5 **12.** Suppose that 35 people are divided in a random manner into two teams in such a way that one team contains 10 people and the other team contains 25 people. What is the probability that two particular people *A* and *B* will be on the same team?

Sample space #: $\binom{35}{10} = \binom{35}{25}$

[Choose 10 or 25 people from all]

Denote $M = \{$ A and B are in the same team $\}$.

$M_1 = \{$ A and B are in team with 10 people $\}$

$M_2 : \{$ A and B are in team with 25 people $\}$

$M_1 \cup M_2 = M$, $M_1 \cap M_2 = \phi$ (Disjoint)

of M_1 : $\binom{2}{2} \cdot \binom{35-2}{10-2}$

of M_2 : $\binom{2}{2} \cdot \binom{35-2}{25-2}$

$$\Rightarrow P(M) = \frac{\#\text{of } M_1 + \#\text{ of } M_2}{\#\text{ of Sample Space}} = \frac{\binom{33}{8} + \binom{33}{23}}{\binom{35}{10}}$$

Question 6

15.

a. Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

b. Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0.$$

Hint: Use the binomial theorem.

Binomial Theorem: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

Proof: $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{\text{we have } n\ (x+y)}$

in each brackets, we can choose x or y,

we choose n times, # of sample space is 2^n.

But this is the case with order, meaning that

for example we regard $x^i y^{n-i}$ and $x^{i-1} y^{n-i} x$ as

different. There should be 2^n items anyway.

According to binomial theorem: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

In this formula, items with same power of x and y

has been gathered together, however, the # of all items

are still the same $\Rightarrow \sum_{i=0}^{n} \binom{n}{i} = 2^n$

$n \geq k \geq 0$

Also, $(x-y)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i y^i x^{n-i}$

We pick up any item in it: $\binom{n}{k} (-1)^k y^k x^{n-k}$

There must be another item: $\binom{n}{n-k} \cdot (-1)^{n-k} y^{n-k} x^k$

If n is odd: $\binom{n}{k} (-1)^k + \binom{n}{n-k} (-1)^{n-k} = 0$

If n is even: It's a little tricky.

I'm going to use induction here:

From observation:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} = \overset{1^{st}}{1} - \overset{2^{nd}}{n} + \overset{3^{rd}}{\frac{n(n-1)}{2!}} + \cdots\cdots$$

$$= -(n-1) + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \cdots\cdots$$

$$= \sum_{j=2}^{n} (-1)^j \binom{n}{j} - (n-1)$$

Assuming $\sum_{m=2}^{M} (-1)^m \binom{n}{m} - (n-1) = (-1)^M \binom{n-1}{M}$

When $M=2$: $\binom{n}{2} - (n-1) = \frac{n(n-1)}{2!} - (n-1) = \frac{(n-2)(n-1)}{2!}$

$$= \binom{n-1}{2} \cdot (-1)^2$$

Assuming when $M=K$, it is correct:

$$\sum_{m=2}^{K} (-1)^m \binom{n}{m} - (n-1) = (-1)^K \binom{n-1}{K}$$

when $M=k+1$:

$$\sum_{m=2}^{k+1} (-1)^m \binom{n}{m} - (n-1) = (-1)^K \binom{n-1}{K} + (-1)^{k+1} \binom{n}{k+1}$$

$$= (-1)^K \binom{n-1}{K} + (-1)^{k+1} \overbrace{\binom{n}{k+1}}^{\frac{n!}{[n-(k+1)]!\,(k+1)!}}$$

$$= (-1)^k \frac{(n-1)!}{(n-1-k)!\,k!} + (-1)^{k+1} \left(\frac{(n-1)!}{(n-1-k)!\,k!} \cdot \frac{n}{k+1} \right)$$

$$= \frac{(n-1)!}{(n-k-1)!\,k!} \left((-1)^{k+1} \frac{n}{k+1} + (-1)^k \right)$$

If k is odd:

$$\frac{(n-1)!}{(n-k-1)!\,k!}\left((-1)^{k+1}\cdot\frac{n}{k+1}+(-1)^k\right)$$

$$=\frac{(n-1)!}{(n-k-1)!\,k!}\left(\frac{n}{k+1}-1\right)$$

$$=\frac{(n-1)!}{(n-k-1)!\,k!}\cdot\frac{n-k-1}{k+1}=\frac{(n-1)!}{(n-k-2)!\,(k+1)!}=\binom{n-1}{k+1}(-1)^{k+1}$$

If k is even:

$$\frac{(n-1)!}{(n-k-1)!\,k!}\left((-1)^{k+1}\cdot\frac{n}{k+1}+(-1)^k\right)$$

$$=\frac{(n-1)!}{(n-k-1)!\,k!}\left(1-\frac{n}{k+1}\right)=-\frac{(n-1)!}{(n-k-2)!\,(k+1)!}=\binom{n-1}{k+1}(-1)^{k+1}$$

Q.E.D for our lemma: for $M\le n-1$

$$\sum_{m=2}^{M}(-1)^m\binom{n}{m}-(n-1)=(-1)^M\binom{n-1}{M}$$

$\Rightarrow$ Let M be $n-1$: n is even by assuming

$$\sum_{m=2}^{n-1}(-1)^m\binom{n}{m}-(n-1)=-\binom{n-1}{n-1}=-1$$

Therefore, $\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}$

is even by assuming

$$= \sum_{m=2}^{n-1} \binom{n}{m} \cdot (-1)^m - (n-1) + (-1)^n \binom{n}{n}$$

$$= -1 + 1 = 0$$

Q.E.D.

Ps: Main idea: { Step 1: When n is odd.
Step 2: When n is even.

Step 2: { 2.1: Induction to calculate sum of first M items
2.2: Add the last item.

Question 7

17. A deck of 52 cards contains four aces. If the cards are shuffled and distributed in a random manner to four players so that each player receives 13 cards, what is the probability that all four aces will be received by the same player?

# of Sample Spaces:

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

Denote A: {all 4 aces received by same player}

A_1: {all 4 aces received by player 1}

A_2: {all 4 aces received by player 2}

A_3: {all 4 aces received by player 3}

A_4: {all 4 aces received by player 4}.

$A_1 \cup A_2 \cup A_3 \cup A_4 = A$, and $\forall\, i, j \in \{1, 2, 3, 4\}$ $i \neq j$

we have $A_i \neq A_j = \phi$

$\Rightarrow P(A) = \sum_{i=1}^{4} P(A_i)$

$\#$ of $A_1 = \binom{4}{4}\binom{48}{9}\binom{39}{13}\binom{26}{13}$

$\#$ of A_2, A_3, A_4 is same as A_1

$$\Rightarrow P(A) = 4 \times \frac{\binom{4}{4}\binom{48}{9}\binom{39}{13}\binom{26}{13}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}$$

Question 8

1. Three pollsters will canvas a neighborhood with 21 houses. Each pollster will visit seven of the houses. How many different assignments of pollsters to houses are possible?

Step 1: Divide 21 houses into 3 classes, with each class has 7 houses.

$$\frac{\binom{21}{7} \cdot \binom{14}{7} \cdot \binom{7}{7}}{P_{3,3}}$$

Step 2: arrange pollster to these 3 groups:

$$\frac{\binom{21}{7}\binom{14}{7}\binom{7}{7}}{P_{3,3}} \cdot P_{3,3} = \binom{21}{7}\binom{14}{7}\binom{7}{7}$$

Question 9

7. Suppose that a box contains r red balls, w white balls, and b blue balls. Suppose also that balls are drawn from the box one at a time, at random, without replacement. What is the probability that all r red balls will be obtained before any white balls are obtained?

We need to discuss different cases and add probabilities.

A: { all red balls are obtained before any white ones }.

Case 0:

A_0: { all red balls, and 0 blue balls are obtained }

Case 1:

A_1: { all red balls, and 1 blue balls are obtained }.

⋮

Case b:

A_b: { all red balls, and b blue balls are obtained }

for $\forall\ i, j \in \{0, 1, 2 \cdots b\}$, $i \neq j$, $A_i \cap A_j = \emptyset$, $\bigcup_{m=1}^{b} A_m = A$

$\Rightarrow P(A) = \sum_{m=1}^{b} P(A_m)$.

# of Sample space case m $\binom{r+b+w}{r+m}$

of A_m : $\binom{r}{r} \cdot \binom{b}{m}$ → m blue balls obtained

all r red balls obtained

$$\Rightarrow P(A) = \sum_{m=0}^{b} P(A_m) = \sum_{m=0}^{b} \frac{\binom{r}{r}\binom{b}{m}}{\binom{r+b+w}{r+m}}$$

Question 10

11. Let A_1, A_2, and A_3 be three arbitrary events. Show that the probability that exactly one of these three events will occur is

$$\Pr(A_1) + \Pr(A_2) + \Pr(A_3) - 2\Pr(A_1 \cap A_2) - 2\Pr(A_1 \cap A_3) - 2\Pr(A_2 \cap A_3) + 3\Pr(A_1 \cap A_2 \cap A_3).$$

$$P(A_1 \cap A_2^c \cap A_3^c) = P(A_1) - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \quad *$$

$$P(A_1^c \cap A_2 \cap A_3^c) = P(A_2) - P(A_1 \cap A_2) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \quad **$$

$$P(A_1^c \cap A_2^c \cap A_3) = P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \quad ***$$

$$* + ** + ***$$

$$\Rightarrow P(A_1 \cap A_2^c \cap A_3^c) + P(A_1^c \cap A_2 \cap A_3^c) + P(A_1^c \cap A_2^c \cap A_3)$$

$$= P(A_1) + P(A_2) + P(A_3) - 2P(A_1 \cap A_2) - 2P(A_1 \cap A_3) - 2P(A_2 \cap A_3) + 3P(A_1 \cap A_2 \cap A_3)$$

Q.E.D.