Overview of Ladder Lotteries

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1 Introduction

An Amidakuji/Ladder Lottery is a custom in Japan which allows for a pseudo-random assignment of children to prizes. Usually done in Japanese schools, a teacher will draw N vertical lines, hereby known as lines, where N is the number of students in class. At the bottom of each line will be a unique prize. And at the top of each line will be the name of one of the students. [K. Yamanaka NakanoK. Yamanaka Nakano2017] The teacher will then draw 0 or more horizontal lines, hereby known as bars, connecting two adjacent lines. The more bars there are the more complicated (and fun) the Amidakuji is. No two endpoints of two bars can be touching. Each student then traces their line, and whenever they encounter an end point of a bar along their line, they must cross the bar and continue going down the adjacent line. The student continues tracing down the lines and crossing bars until they get to the end of the ladder lottery. The prize at the bottom of the ladder lottery is their prize. A7

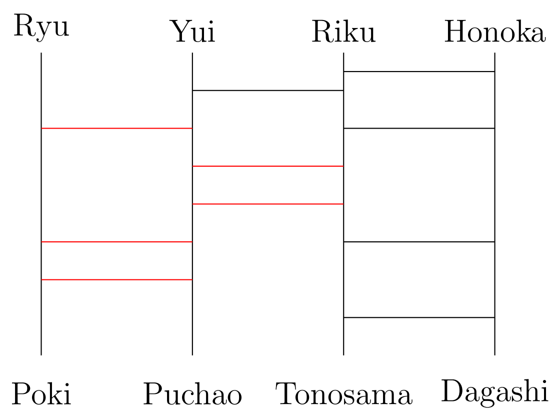

Figure 1: In the above ladder lottery Ryu gets Puchao, Yui gets Dagashi, Riku gets Tonosama and Honoka gets Poki. You can see that Ryu's path is marked by red bars.

2 Optimal Ladder Lotteries

An interesting property about ladder lotteries is that they can be derived from a *permutation*. A *permutation* is a unique ordering of objects. For the purposes on this paper, the objects of a permutation will be integers ranging from $[1 \leq N \leq \infty)$. When considering permutations of integers, each permutation contains a unique set of *optimal ladder lotteries*. An *optimal ladder lottery* is a ladder lottery with the minimal number of bars. The number of bars is minimal if and only if for each *inversion* in the permutation there is exactly one bar in the ladder lottery. An *inversion* is defined as follows - given two objects in a permutation, p and q, if $p > q$ and p is to the left of q then p and q form an *inversion*. In fact, every permutation has a *unique set of optimal ladder lotteries*.

When derived from a permutation, ladder lotteries closely resemble primitive sorting networks. Within a sorting network, a comparator replaces p_i and p_{i+1} with min(p_i, p_{i+1}) and max(p_i, p_{i+1}) respectively. A bar in a ladder lottery replaces p_i and p_{i+n} with min(p_i, p_{i+n}) and max(p_i, p_{i+n}) respectively. [K. Yamanaka NakanoK. Yamanaka Nakano2014]

A *set of optimal ladder lotteries* is the unique set of optimal ladder lotteries derived from a permutation, π, with N elements. For each optimal ladder in the set, the N elements in π are listed at the top of a ladder and each element is given its own line. [MUN. Yamanaka NakanoMUN. Yamanaka Nakano2010] At the bottom of a ladder is the *sorted permutation*, hereby known as the *identity permutation*. The identity permutation of size N is defined as follows - $I : (p_{1=1}, p_{2=2}, ...p_{N=N})$. Each ladder in the set has the minimal number of horizontal bars to sort π into the identity permutation. The minimal number of bars in the ladder is determined by the number of inversions in π. There is one bar for each inversion in π. For the remainder of this paper, only optimal ladder lotteries will be discussed. Therefore when the term ladder lottery is used, assume optimal ladder lottery.

Figure 2: Two ladders for the permutation (4, 3, 2, 1). The left ladder is an optimal ladder and the right ladder is not. Therefore the left ladder belongs to the set of optimal ladder lotteries for the permutation (4, 3, 2, 1). The red bars in the right ladder are redundant, thus the right ladder is not optimal

2.1 Ladders and Adjacent Transpositions

A ladder lottery is a way of sorting a permutation, yet it can also be thought of as a decomposition of a permutation into *adjacent transpositions*. [KiyomiKiyomi2016] An *adjacent transposition* is simply a swap of two adjacent elements in a permutation. For example, given the permutation (1, 3, 4, 2), an adjacent transposition could be done on the following pairs of elements - (1, 3), (3, 4) or (4, 2). Each would result in a unique permutation. Simply put, given any arbitrary starting permutation, π, keep swapping adjacent inversions until the identity permutation is reached. An optimal ladder lottery from $\pi's$ optimal ladder set is a minimal sequence of adjacent transpositions such that π is sorted into the identity permutation; each ladder in the set represents a sequence of adjacent transpositions for sorting π into the identity permutation. For example, given the permutation (4, 3, 2, 1) there exists eight ladders in this permutation's optimal ladder set. Two of these ladders are the following:

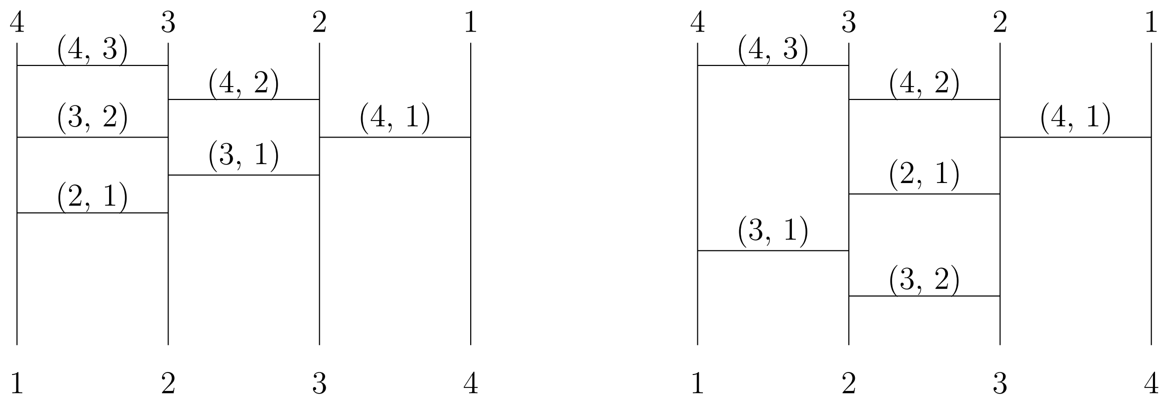

Figure 3: The left ladder is one of eight unique ladders from (4,3,2,1)'s optimal ladder set. The right ladder is another one of eight unique ladders form (4,3,2,1)'s optimal ladder set

From looking at the above ladders, going from top left to bottom right, the left ladder represents the sequence of adjacent transpositions ((4,3), (4,2), (4,1),(3,2),(3,1),(2,1)) whereas the right ladder represent the sequence of adjacent transpositions ((4, 3),(4, 2),(4, 1),(2, 1),(3, 1),(3, 2)). Notice how the length of the sequences are the same,because both lengths are equal to the minimal number of swaps to sort (4, 3, 2, 1) it is simply the order in which the adjacent transpositions occur in the sequence that makes the sequences different from each other.

Theorem 1. *Let π be an arbitrary permutation with N unique elements, then there is a sequence of adjacent transpositions of length $0 \leq N(N-1))/2$ such that π is sorted into the identity permutation.*

Proof. The rationale for the above theorem is as follows. The minimum number of inversions π can have is 0. Therefore π would already be sorted and there would be no adjacent transpositions. If π is in descending order, then the number of inversions in π is $N(N-1)/2$. Let element N be the maximum element in π, assuming π is descending. In order to put N in its correct position, it must undergo $N-1$ adjacent transpositions. For example, given permutation (5, 4, 3, 2, 1), element 5 must undergo four adjacent transpositions in order to be put in its correct spot, thus resulting in the permutation (4, 3, 2, 1, 5). The result is the first $N-1$ elements are descending, and the last $(N-(N-1))$ elements are sorted. Applying the same process to the left $N-1$ elements, element 4 would be swapped 3 times, resulting in (3, 2, 1, 4, 5) leaving to the first $N-2$ elements in descending order and the last $(N-(N-2))$ elements sorted. In general one can say, that given the descending permutation, applying adjacent transpositions starting with the leftmost element and working to the right, the first $N-i$ elements will be descending and the right $(N-(N-i))$ elements will be sorted where i is the index in π. In order for the entire descending π to be sorted, each element, p, must undergo $p-1$ adjacent transpositions. Thus, given a descending permutation of size N, the number of adjacent transpositions is the following summation.

$$\sum_{x=1}^{n-1} x = N(N-1)/2 \tag{1}$$

This summation represents the maximum length of a sequence of adjacent transpositions used to sort an arbitrary permutation π of size N. □

Corollary 1.1. *Since an adjacent transposition sequence can be represented as a ladder lottery, where each adjacent transposition is a bar in a ladder lottery, then one can conclude that given any π of size N, where $N > 0$, and given any ladder, L_k, in π's optimal ladder set, $0 \leq$ the number of bars in L[k] $\leq N(N-1)/2$.*

2.2 Ladder Lotteries and their Relation to other Combinatorial Objects

Ladder Lotteries are closely related to a number of other combinatorial objects. Ladder Lotteries share a commonality with primitive sorting networks, pseudo-line arrangements and wiring diagrams. [Z.LiZ.Li2001] [**?**] [MUN. Yamanaka NakanoMUN. Yamanaka Nakano2010] [K. YamanakaK. Yamanaka2009] When an optimal ladder set is derived from a reverse permutation, $\pi = (N, N-1, N-2, ...1)$, the number of ladders in the set shares the same integer sequence as the number of primitive sorting networks, wiring diagrams and pseudo-line arrangement for the same permutation. There is currently no known closed form solution for this integer sequence. In their paper Counting *Primitive Sorting Networks by πDDs*, the authors have computed the value for this integer sequence when $N = 13$. Lastly, in 2009 Armstrong computed the value for $N = 15$ [ArmstrongArmstrong2009] which is currently the greatest value of N this sequence has been computed for. [https://oeis.org/search?q=2%2C+8%2C+62language=englihttps://oeis.org/se

3 Problems Related to Ladder Lotteries

The focus of this paper will be to cover three open problems related to Ladder Lotteries. The three problems are the following

1. When enumerating all permutations of size N, is there a corresponding Gray Code for enumerating all root ladders for each permutation of size N?

2. Given permutation of size N, is there a polynomial solution to determine a ladder with a minimal height.

3. The third problem has already been mentioned, but will be more generalized, which is given a permutation of size N, is there a closed form solution for determining the number of ladders in its optimal ladder set.

3.1 Solved Problem: Ladder Lottery Realization

In their paper *Ladder Lottery Realization*, Yamanka, Horiyama, Uno and Wasa pose a problem they termed the *Ladder Lottery Realization Problem.* A6 The question this problem poses is the following, given a target permutation, π of size N, and a multi-set of bars, is it possible to construct a ladder that uses every bar in the multi-set such that the ladder can sort the ascending permutation of size N into the target permutation? The authors prove that the problem is NP-complete. An example of the problem will now be provided.

Let $N = 6$
Let the target permutation, π be (4, 1, 6, 3, 5, 2).
Let the ascending permutation be (1, 2, 3, 4, 5, 6)
Let the multi-set, S = $\{(1,3)^2, (1,4), (2,3), (2,4)^3, (2,5)^3, (2,6), (3,4), (3,6), (5,6)^3\}$.
The exponent represents the number of times the bar must occur in the ladder. Assume exponent 1 if no exponent is given.
Problem: Is there a ladder such that every bar in the multi-set occurs exactly the number of times as its exponent in the multi-set and the ladder sorts the identity permutation into the target permutation?
The answer is yes, the following ladder is such a ladder.[Katsuhisa YamanakaKatsuhisa Yamanaka2018]

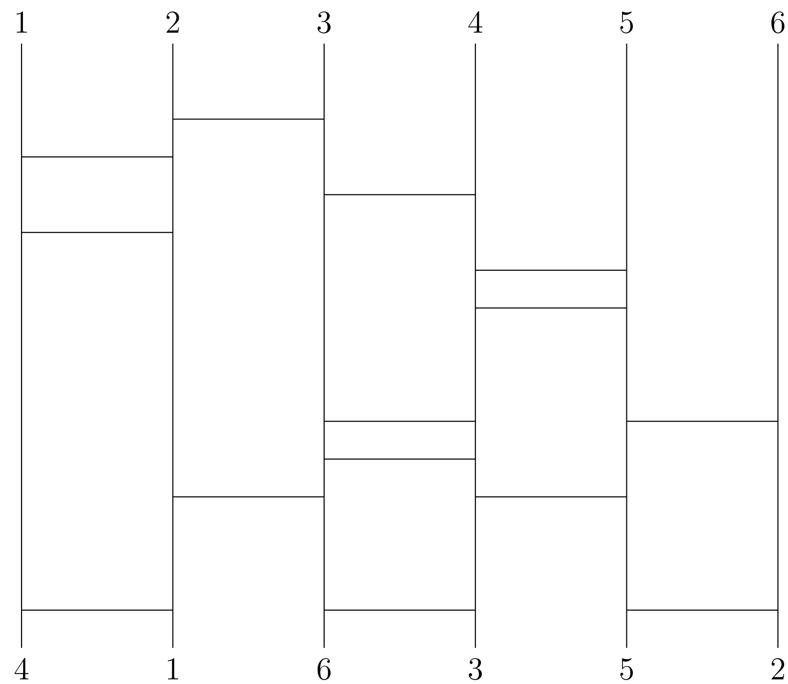

Figure 4: A ladder that is a solution to the above instance of the Ladder Lottery Realization Problem

3.2 Open Problems:

Some other open problems related to ladder lotteries are the following

1. Token Swapping Problem: Given a permutation and a set of allowed transpositions, where each transposition in the set can be done an infinite number of times, what is the minimum number of transpositions required to sort the permutation? [Katsuhisa YamanakaKatsuhisa Yamanaka2018]

4 Areas of Application

Outside of theoretical interest, ladder lotteries can be applied to the following areas. Firstly, there is potential to apply ladder lotteries to FPGAs (Field Programmable Gate Array). For example, assume an FPGA designer is looking to implements a sorting network. In order to save space, the designer could design a ladder lottery as sorting network such that the ladder had the minimal height; by minimal height, what is meant that the maximum number of heights can be placed at the same level in the ladder.

A second area of application is the following. Consider a permutation of N computers in a network. Each computer has a unique label between 1 and N. Next consider that a signal from each computer must travel through wires and cross wires such that its signal reaches the end of a wire and is in the position corresponding to its computer's label. In other words, once the signals travel through the wiring, they are sorted. If one thinks of the wires and cross-wires as a ladder lottery, then it becomes clear that there is a reliable way to sort the signals from the computer network.

References

[ArmstrongArmstrong2009] A24Armstrong, D. 2009. The sorting order on a Coxeter group The sorting order on a coxeter group. Journal of Combinatorial Theory1161285-1305.

[Katsuhisa YamanakaKatsuhisa Yamanaka2018] A6Katsuhisa Yamanaka, TUKW., Takashi Horiyama. 2018aug. Ladder-Lottery Realization Ladder-lottery realization. CCCG.

[KiyomiKiyomi2016] A1Kiyomi, M. 2016. Reverse Search; Enumeration Algorithms Reverse search; enumeration algorithms. Encyclopedia of Algorithms1840-1842. 10.1007/978-1-4939-2864-4$_7$31

[K. YamanakaK. Yamanaka2009] A13K. Yamanaka, SN. 2009. , Listing all plane graphs , listing all plane graphs. Graph Algorithms Appl135-18.

[https://oeis.org/search?q=2%2C+8%2C+62language=englihttps://oeis.org/search?q=2%2C+8%2C+
A25. Accessed: 2020-01-21

[K. Yamanaka NakanoK. Yamanaka Nakano2014] A22Yamanaka, K. Nakano, S. 2014. Efficient enumeration of all ladder lotteries with k bars Efficient enumeration of all ladder lotteries with k bars. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences97-A1163-1170.

[K. Yamanaka NakanoK. Yamanaka Nakano2017] A23Yamanaka, K. Nakano, S. 2017. Enumeration, counting, and random generation of ladder lotteries Enumeration, counting, and random generation of ladder lotteries. IEICE Transactions Information and Systems100-D(3)444- 451.

[MUN. Yamanaka NakanoMUN. Yamanaka Nakano2010] A7Yamanaka, MUN., Nakano. 2010. Efficient enumeration of all ladder lotteries and its application Efficient enumeration of all ladder lotteries and its application. Theoretical Computer Science4111714-1722.

[Z.LiZ.Li2001] A5Z.Li, S. 2001. Efficient generation of plane triangulations without repetitions Efficient generation of plane triangulations without repetitions. The 28th International Colloquium of Automata, Languages and ProgrammingLNCS2076433-443.