Romberg Integration (RI) of an 1D Array

Here we consider an 1D array $f_i = f(x = i/m)$ for i=0,…,m, where the number of equal intervals m is an integer multiple of n=2^K, so that RI-K with the K^{th}-order (K=0,1,2,3,4) polynomial extrapolation in terms of h^2, where h=1/m denotes the step size, can be used to calculate $I \equiv \int_0^1 f(x)\mathrm{d}x \approx \frac{1}{m}\sum_{i=0}^{m} c_i f_i$ with c_i being the coefficient; this gives the exact result for a polynomial integrand of the order no greater than 2K+1.

In the k^{th}-step (k=1,…,K+1) of RI-K, the composite trapezoidal rule with a step size of $h_{K+1-k} = 2^{K+1-k}/m$ is used to calculate $I_{K+1-k} \equiv \frac{2^{K+1-k}}{m}\left(\frac{f_0+f_m}{2} + \sum_{j=1}^{m/2^{K+1-k}-1} f_{2^{K+1-k}j}\right)$. Specifically, RI-0 corresponds to the composite trapezoidal rule with a step size of 1/m, which calculates $I \approx I_0 = \frac{1}{m}\left(\frac{f_0+f_m}{2} + \sum_{j=1}^{m-1} f_j\right)$; RI-1 corresponds to the composite Simpson's 1/3 rule, which calculates $I_1 = \frac{2}{m}\left(\frac{f_0+f_m}{2} + \sum_{j=1}^{m/2-1} f_{2j}\right)$, I_0, and $I \approx \frac{4I_0 - I_1}{3}$ via linear extrapolation; RI-2 corresponds to the composite Boole's rule, which calculates $I_2 = \frac{4}{m}\left(\frac{f_0+f_m}{2} + \sum_{j=1}^{m/4-1} f_{4j}\right)$, I_1, I_0, and $I \approx \frac{64I_0 - 20I_1 + I_2}{45}$ via quadratic extrapolation; RI-3 calculates $I_3 = \frac{8}{m}\left(\frac{f_0+f_m}{2} + \sum_{j=1}^{m/8-1} f_{8j}\right)$, I_2, I_1, I_0, and $I \approx \frac{4096I_0 - 1344I_1 + 84I_2 - I_3}{2835}$ via cubic extrapolation; and RI-4 calculates $I_4 = \frac{16}{m}\left(\frac{f_0+f_m}{2} + \sum_{j=1}^{m/16-1} f_{16j}\right)$, I_3, I_2, I_1, I_0, and $I \approx \frac{1048576I_0 - 348160I_1 + 22848I_2 - 340I_3 + I_4}{722925}$ via quartic extrapolation.

Note that, for m>n, RI-K can be considered as ***composite*** since all the above extrapolation formulae are linear; we therefore take m=n below to show the final results: RI-1 gives $I \approx \frac{f_0 + 4f_1 + f_2}{3m}$; RI-2 gives $I \approx \frac{14f_0 + 64f_1 + 24f_2 + 64f_3 + 14f_4}{45m}$; RI-3 gives $I \approx \frac{868f_0 + 4096f_1 + 1408f_2 + 4096f_3 + 1744f_4 + 4096f_5 + 1408f_6 + 4096f_7 + 868f_8}{2835m}$; and RI-4 gives

$$I \approx \frac{\begin{pmatrix} 220472f_0 + 1048576f_1 + 352256f_2 + 1048576f_3 + 443648f_4 + 1048576f_5 \\ +352256f_6 + 1048576f_7 + 440928f_8 + 1048576f_9 + 352256f_{10} + 1048576f_{11} \\ +443648f_{12} + 1048576f_{13} + 352256f_{14} + 1048576f_{15} + 220472f_{16} \end{pmatrix}}{722925m}$$

. Also note that $\sum_{i=0}^{n} c_i = n$.