

Introduction to spatial point process and its Fourier analysis

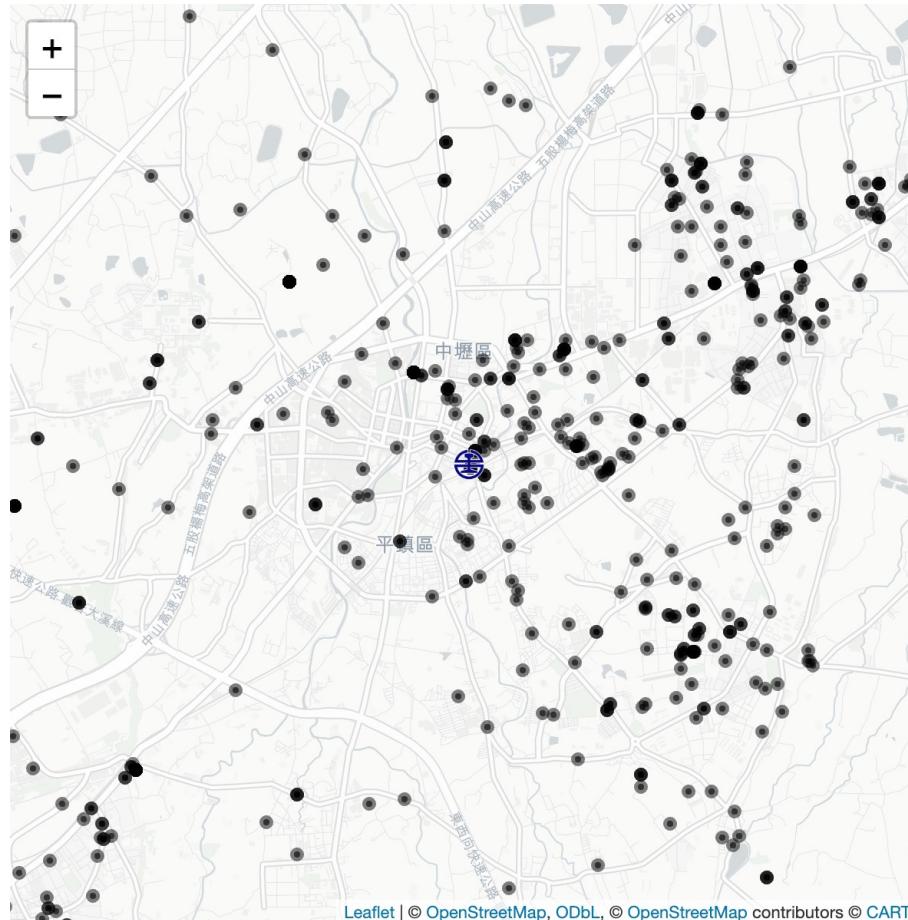
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Joint work with Junho Yang (AS) and Joonho Shin (Sungshin Women's U.)

Fundamentals

Point pattern data

Spatial **locations** of things or events

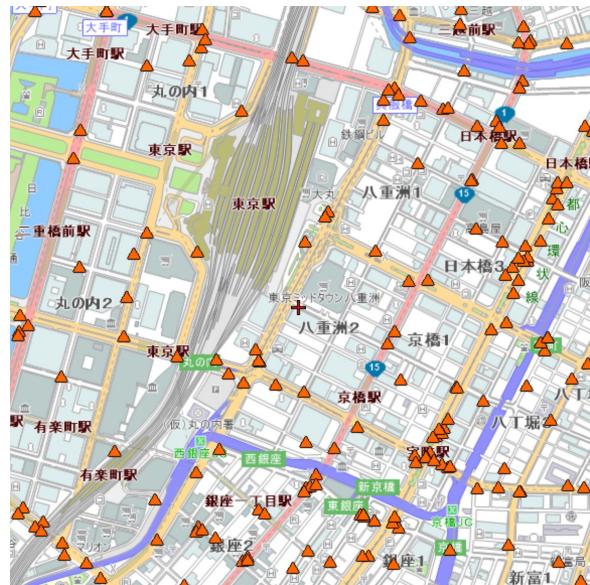


For a study area, **all locations are observable!**

Locations without a point = observed but no event happened!

Point pattern data

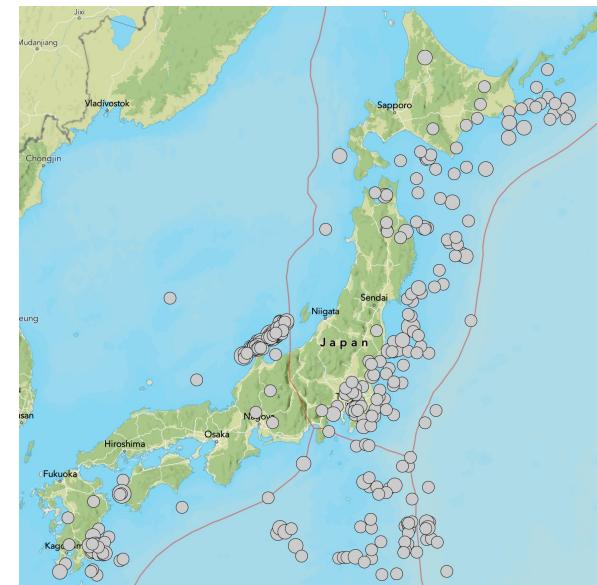
Spatial **locations** of things or events



Traffic injuries near Tokyo station (2021)



Rental housing spots near Akihabara (2025/6/7)

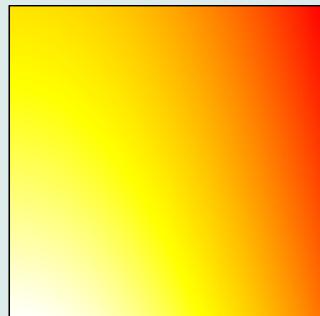
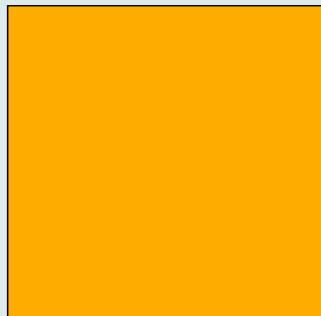
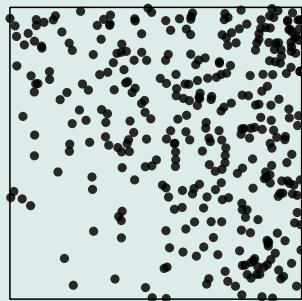
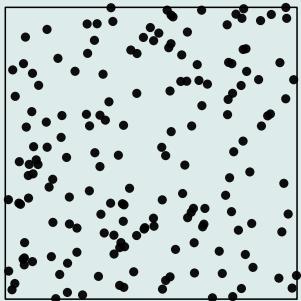


M4.5+ earthquakes in Japan (2024)

Characteristics of point process

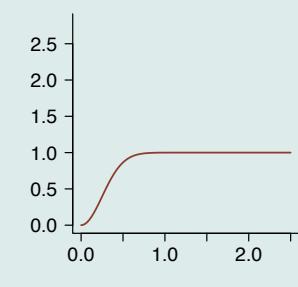
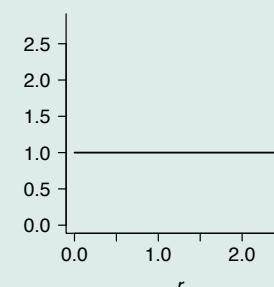
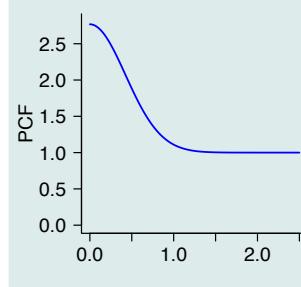
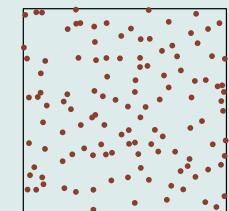
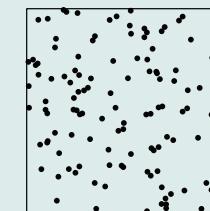
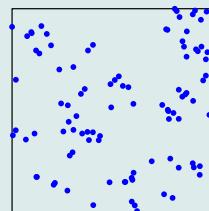
Intensity function: $\lambda(x)$

$\propto \mathbb{P}(\text{Having a point at location } x)$

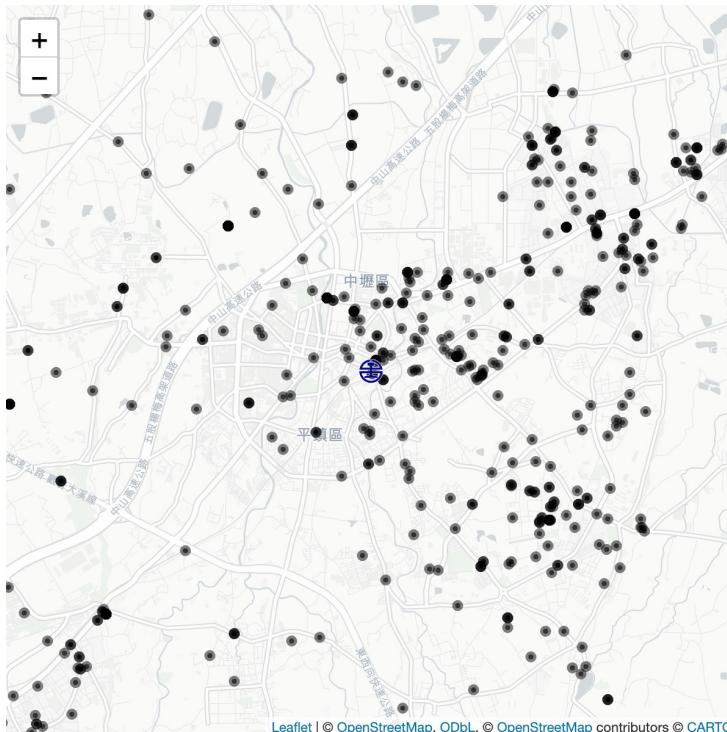


Pair correlation function: $g(r)$

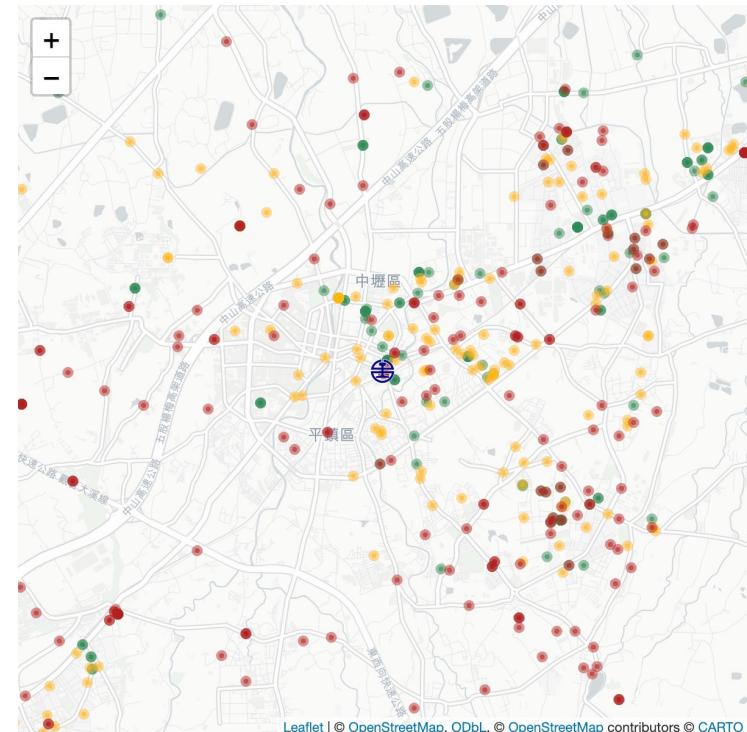
How events interact at radius r



From univariate to multivariate



● Burglary incident (2020-2023)



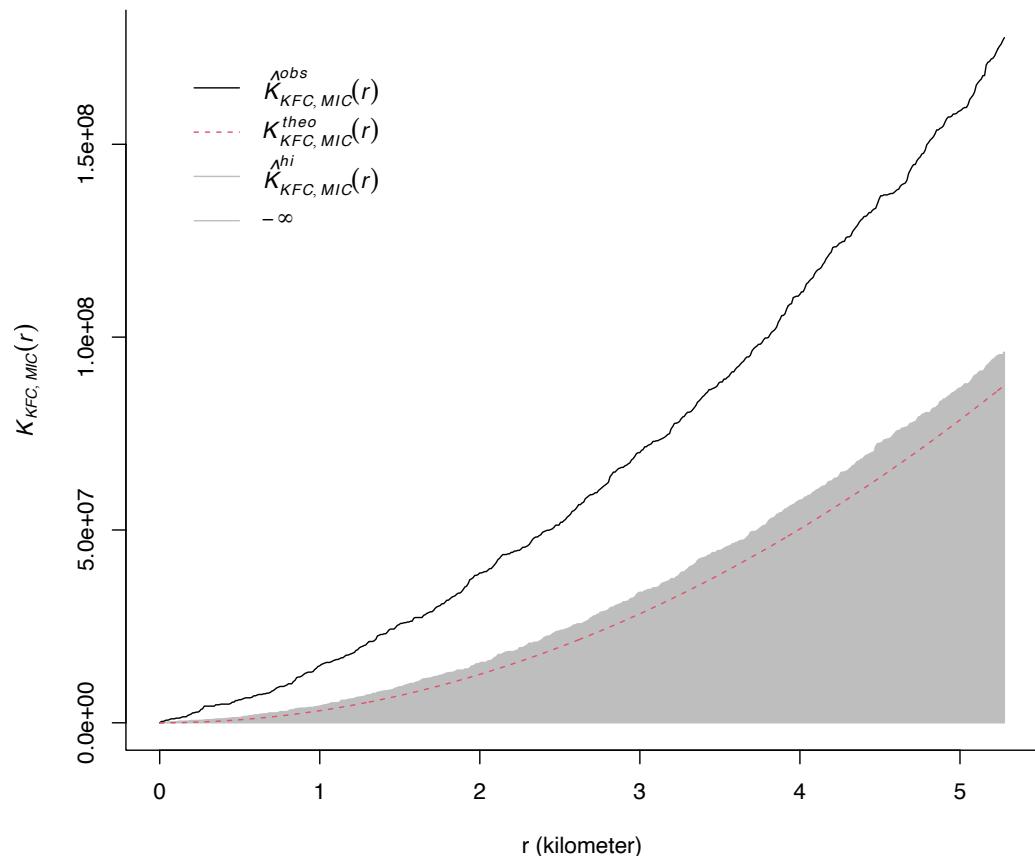
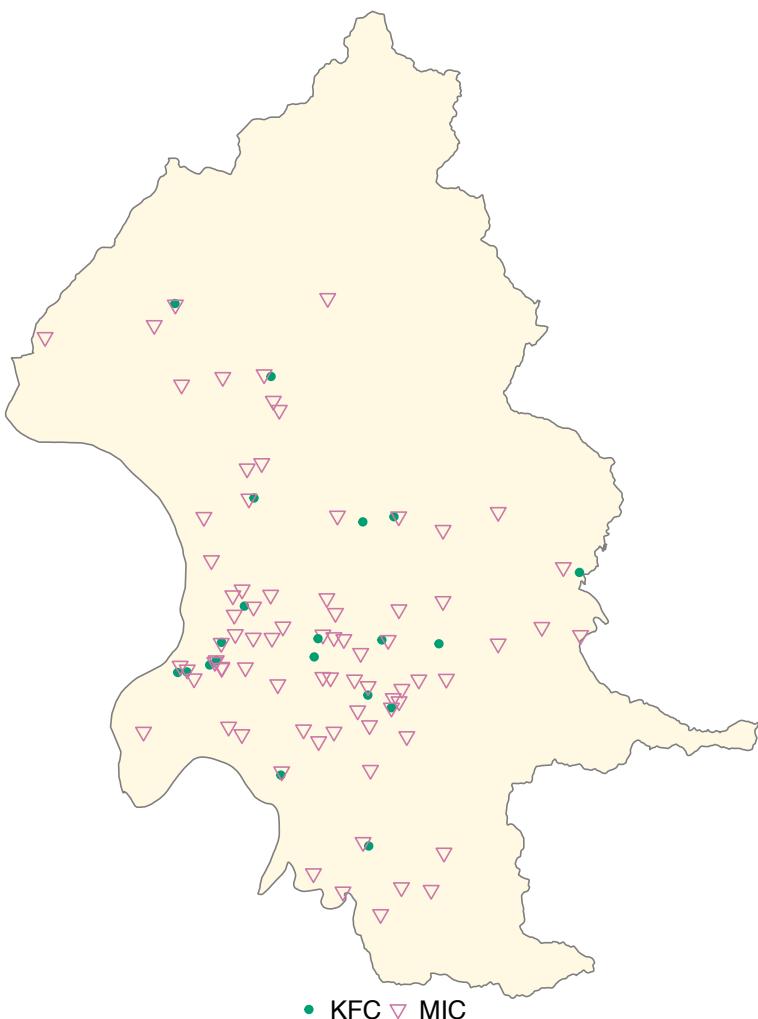
Type of burglary

● Motorcycle

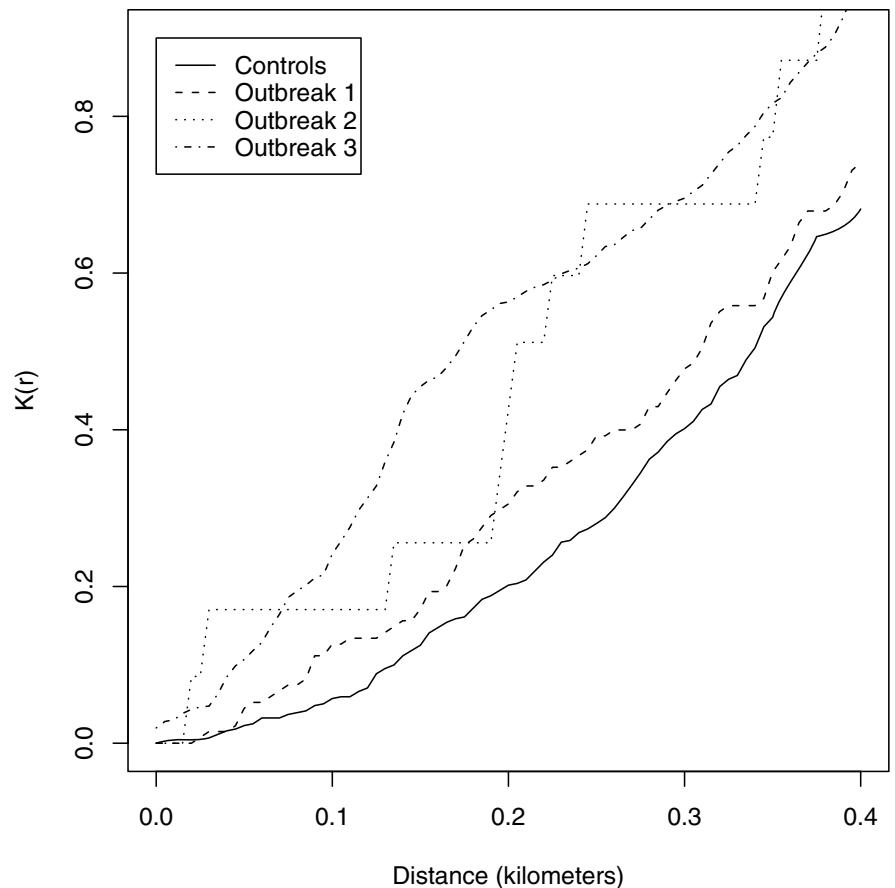
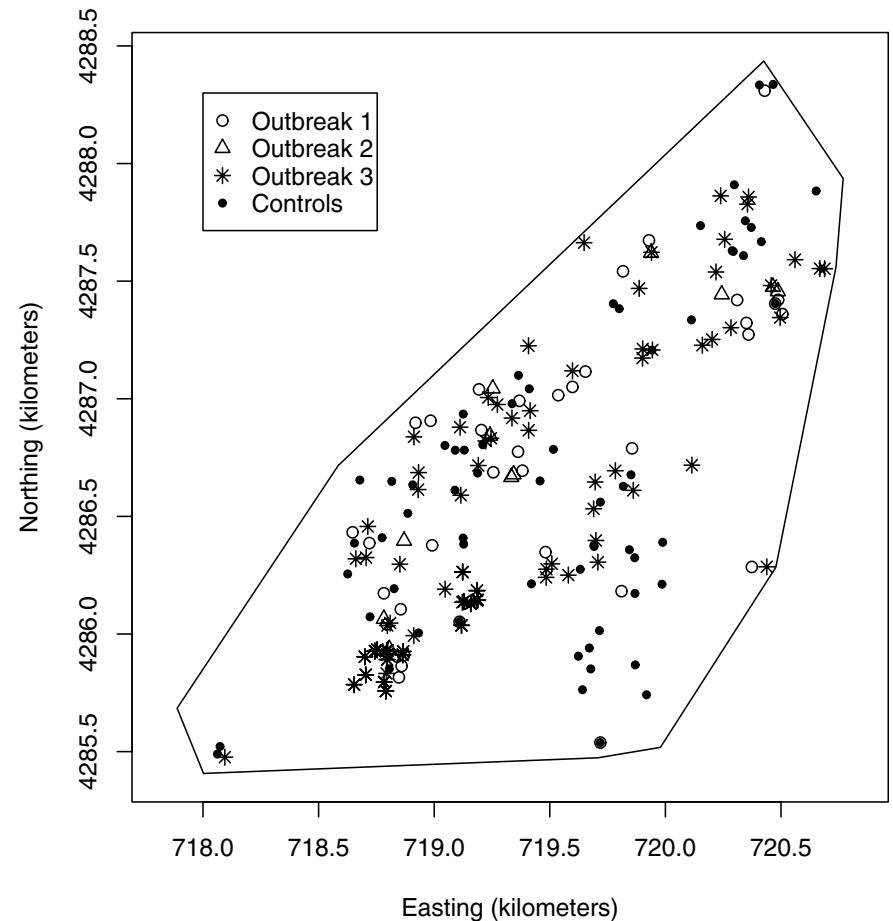
● Car

● Bicycle

Example 1: Type interaction



Example 2: Source detection of infectious disease⁸



Fourier analysis of point processes

Time series vs. spatial point processes

Time series model

- White noise : $e_t \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$
- MA(1) : $X_t = e_t + \theta e_{t-1}$
- AR(1) : $X_t = \alpha X_{t-1} + e_t$

ACF

- $\gamma(k) = \sigma^2 \mathbb{1}\{k = 0\}$
- $\gamma(k) = \theta \sigma^2, k = \pm 1$
- $\gamma(k) = \alpha^{|k|} \sigma_X^2$

Spectral density

- $f(\omega) = \sigma^2 / (2\pi)$
- $f(\omega) = \sigma^2 (1 + \theta^2 + 2\theta \cos \omega) / (2\pi)$
- $f(\omega) = \frac{\sigma^2}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}$

Point process model

- Homogeneous Poisson process
- Thomas cluster process
- Determinantal point process

PCF

- $g(r) = 1$
- $g(r) = 1 + \frac{e^{-r^2/4\sigma^2}}{4\pi\kappa\sigma^2}$
- $g(r) = 1 - \left(e^{-r^2/\sigma^2}\right)^2$

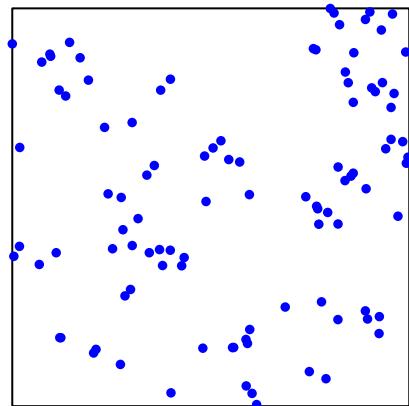
Spectral density

- $f(\omega) = (2\pi)^{-d} \lambda$
- $f(\omega) = (2\pi)^{-d} \kappa \mu \left[1 + \mu e^{-\sigma^2 \|\omega\|^2} \right]$
- $f(\omega) = (2\pi)^{-d} \left[\lambda - \lambda^2 \left(\frac{\pi\sigma^2}{2} \right)^{\frac{d}{2}} e^{-\frac{\sigma^2 \|\omega\|^2}{8}} \right]$

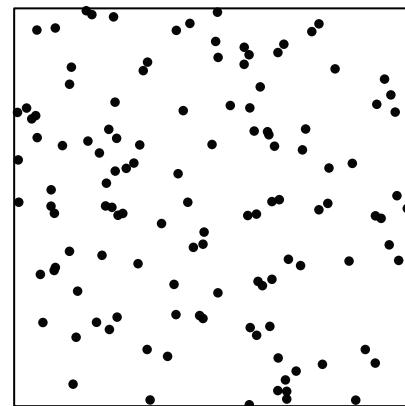
For stationary and isotropic case, $g(\mathbf{x}_i, \mathbf{x}_j) = g(\|\mathbf{x}_i - \mathbf{x}_j\|) = g(r)$.
Parameter: $\boldsymbol{\theta}_{\text{Thomas}} = (\kappa, \mu, \sigma^2)^T$, $\boldsymbol{\theta}_{\text{Determinantal}} = (\lambda, \sigma^2)^T$.

PCF and spectral density

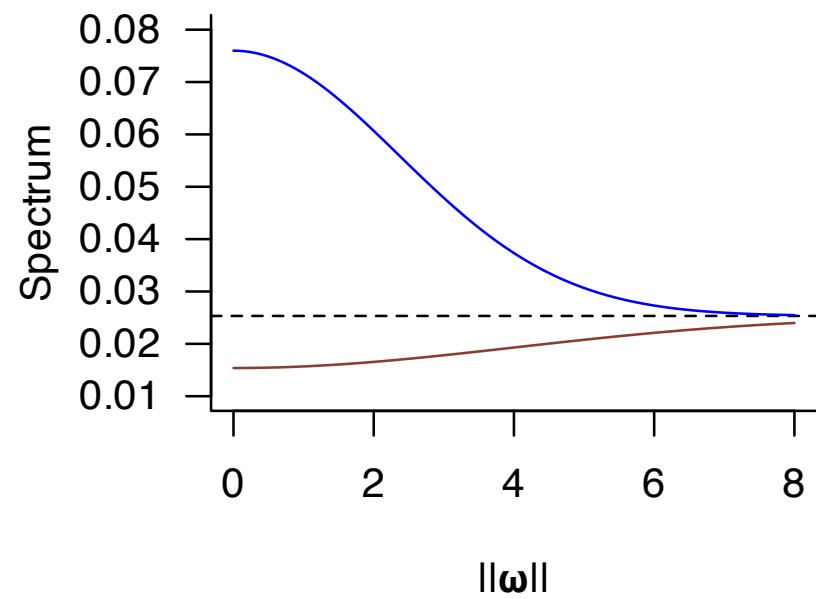
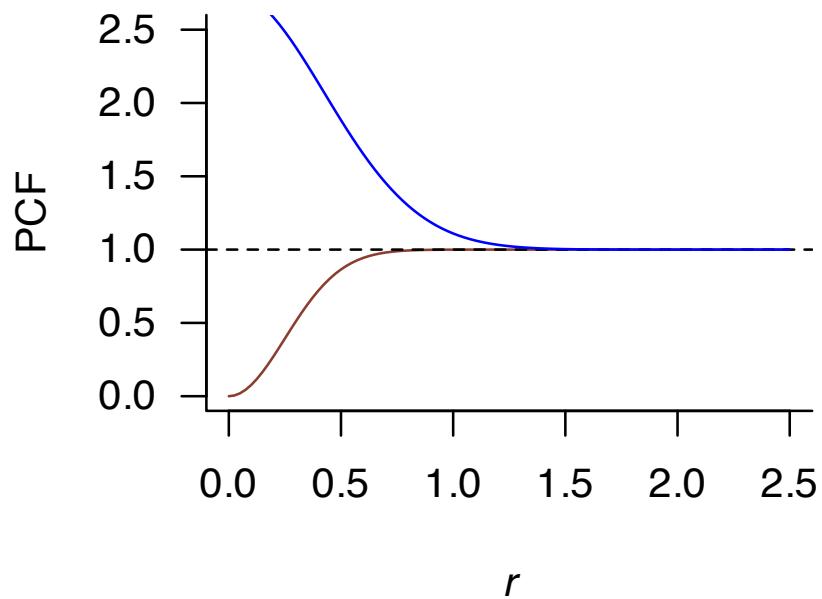
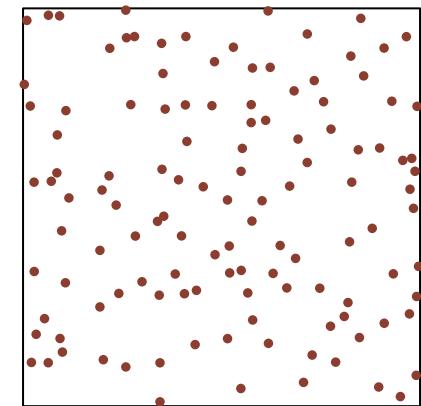
Thomas process
 $(\kappa = 0.5, \sigma = 0.3, \mu = 2)$



Poisson process
 $(\lambda = 1)$



DPP with Gaussian kernel
 $(\lambda = 1, \sigma = 0.5)$



Frequency domain parameter estimation

Whittle-type likelihood

Let $\{X_\theta\}$ be a family of 2nd-order stationary point processes with parameter $\theta \in \Theta$. The associated spectral density is denoted as f_θ . Then, we fit the model using the pseudo-likelihood

$$L(\theta) = \sum_{\omega_k \in D} \left\{ \frac{\hat{I}(\omega_k)}{f_\theta(\omega_k)} + \log f_\theta(\omega_k) \right\}.$$

Intuition

- $\hat{I}(\omega)$ can be regarded as the “truth” since $\mathbb{E}[\hat{I}_n(\omega)] \rightarrow f(\omega)$.
- $f_\theta(\omega)$ are the parameter family of spectral densities.
- $L(\theta)$ is the **spectral divergence between the truth and our guess. The smaller the better!**

Proposed model parameter estimator

$$\hat{\theta} = \arg \min_{\theta \in \Theta} L(\theta)$$

Frequency domain parameter estimation

Model	Window	Parameter	Method	
			Ours	ML
TCP	[-5, 5] ²	κ	-0.04 (0.11)	-0.02 (0.11)
		μ	0.72 (3.52)	-0.24 (5.93)
		σ^2	0.02 (0.07)	-0.04 (0.22)
	[-10, 10] ²	Time (sec)	0.74	0.38
		κ	-0.02 (0.05)	-0.02 (0.05)
		μ	0.60 (1.77)	0.33 (2.79)
	[-20, 20] ²	σ^2	0.01 (0.02)	-0.01 (0.11)
		Time (sec)	2.38	5.67
		κ	-0.01 (0.04)	0.00 (0.03)
		μ	0.25 (1.03)	0.15 (1.23)
		σ^2	0.01 (0.02)	0.00 (0.03)
		Time (sec)	9.15	173.66

The bias and the standard errors (in parentheses) of the estimated parameters based on three different approaches for the Thomas clustered process (TCP)

Spectral analysis

Time series model

- White noise : $e_t \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$
- MA(1) : $X_t = e_t + \theta e_{t-1}$
- AR(1) : $X_t = \alpha X_{t-1} + e_t$

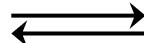
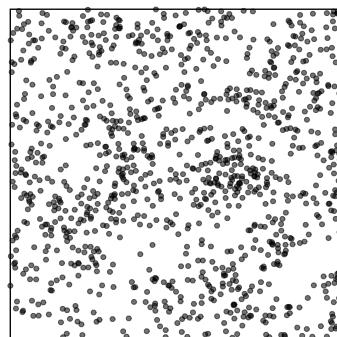
ACF

- $\gamma(k) = \sigma^2 \mathbb{1}\{k = 0\}$
- $\gamma(k) = \theta \sigma^2, k = \pm 1$
- $\gamma(k) = \alpha^{|k|} \sigma_X^2$

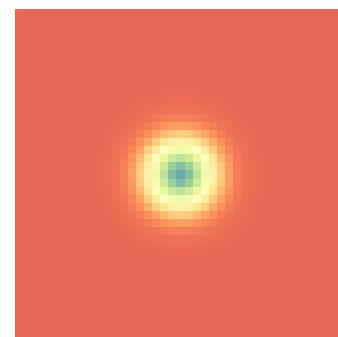
Spectrum

- $f(\omega) = \sigma^2 / (2\pi)$
- $f(\omega) = \sigma^2 (1 + \theta^2 + 2\theta \cos \omega) / (2\pi)$
- $f(\omega) = \frac{\sigma^2}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}$

2nd-order stationary
point process

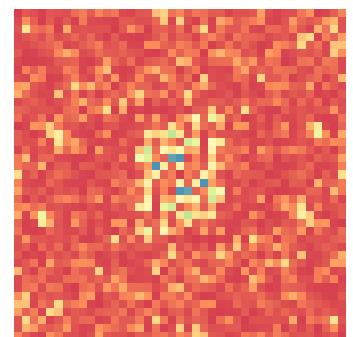


Spectrum
 $f(\omega)$



Its estimator is

Periodogram
 $\hat{I}(\omega)$



Spectral analysis

Time series model

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ACF

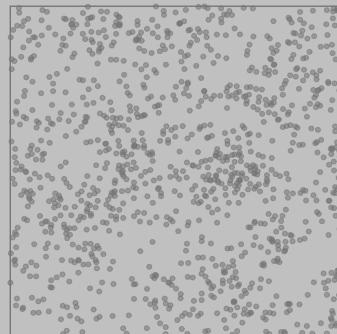
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Spectrum

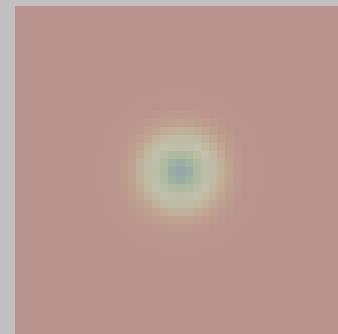
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- $f(\omega) = \frac{\sigma^2}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)}$

Does NOT work for inhomogeneous point processes!

2nd-order stationary
point process

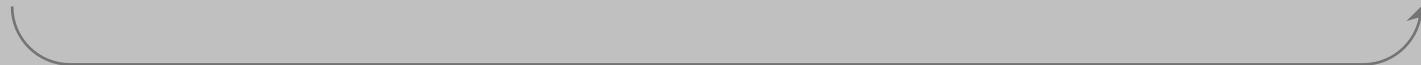
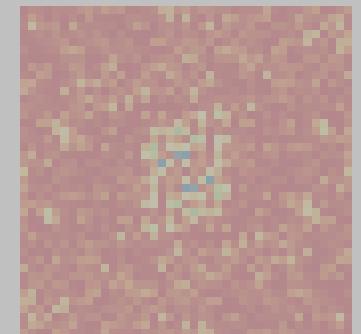


Spectrum
 $f(\omega)$



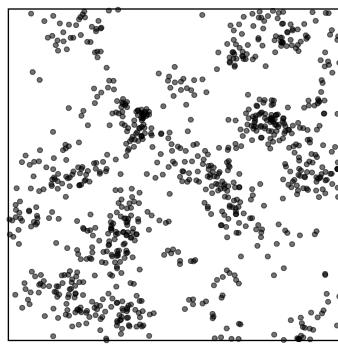
Periodogram
 $\hat{I}(\omega)$

Its estimator is



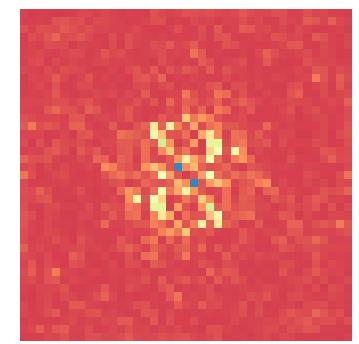
Spectral analysis under inhomogeneity

Inhomogeneous
point process

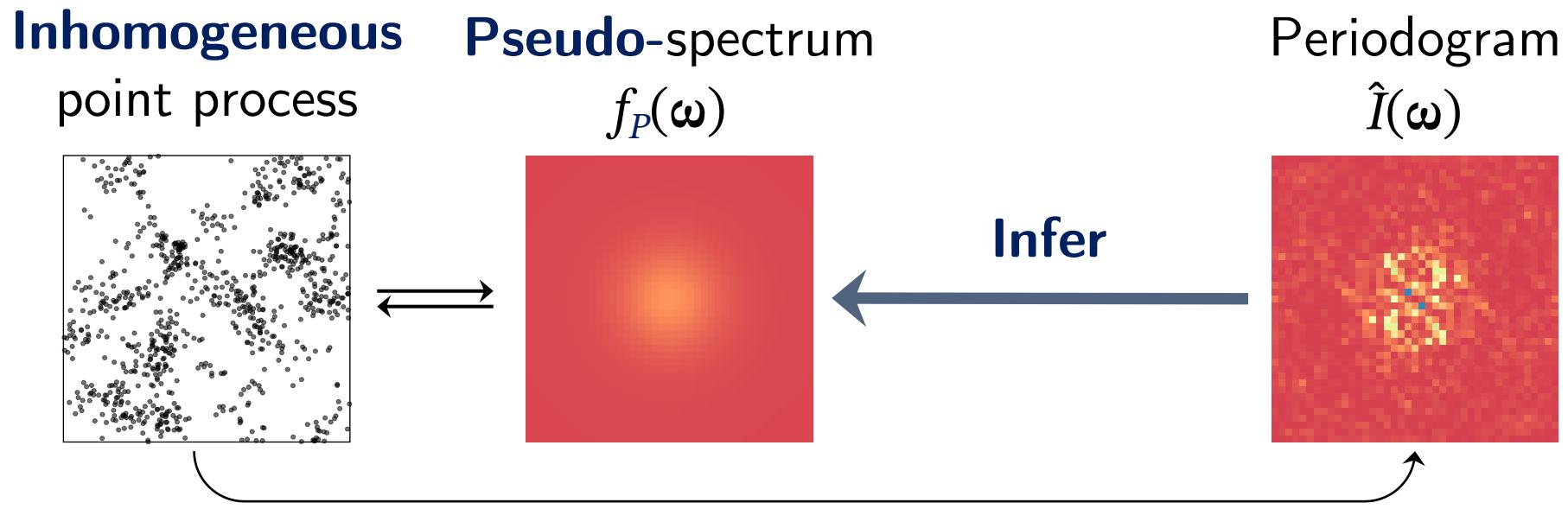


???

Periodogram
 $\hat{I}(\omega)$



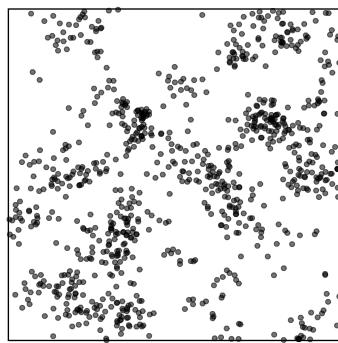
Spectral analysis under inhomogeneity



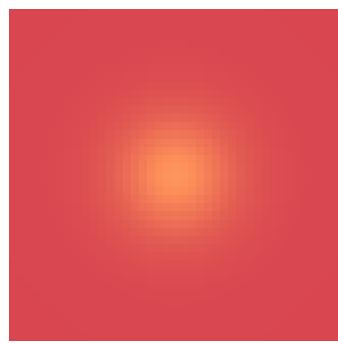
$$f_P(\omega) := \lim_{n \rightarrow \infty} \mathbb{E}[\hat{I}_n(\omega)], \quad \forall \omega \in \mathbb{R}^d.$$

Estimating the pseudo-spectrum

Inhomogeneous
point process

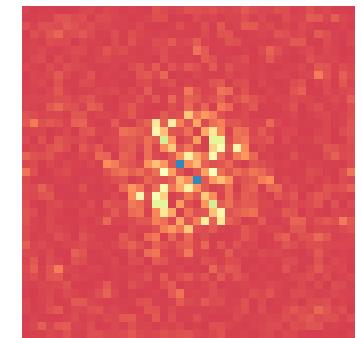


Pseudo-spectrum
 $f_{\text{P}}(\omega)$



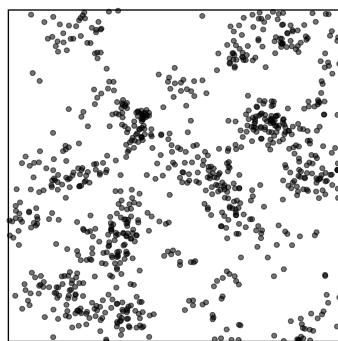
Estimators: ----- ↘

- ① Periodogram
 $\hat{I}(\omega)$

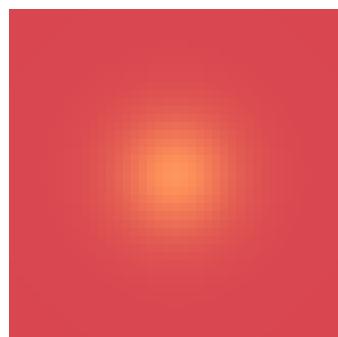


Estimating the pseudo-spectrum

Inhomogeneous
point process

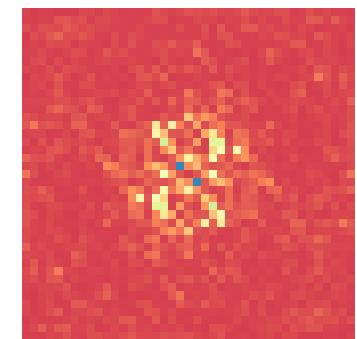


Pseudo-spectrum
 $f_P(\omega)$

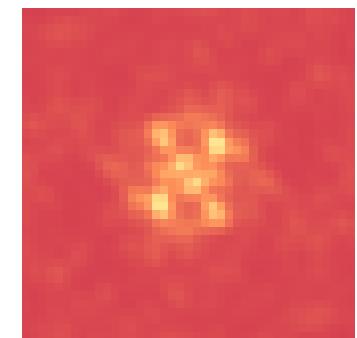


Estimators: ----- ↘

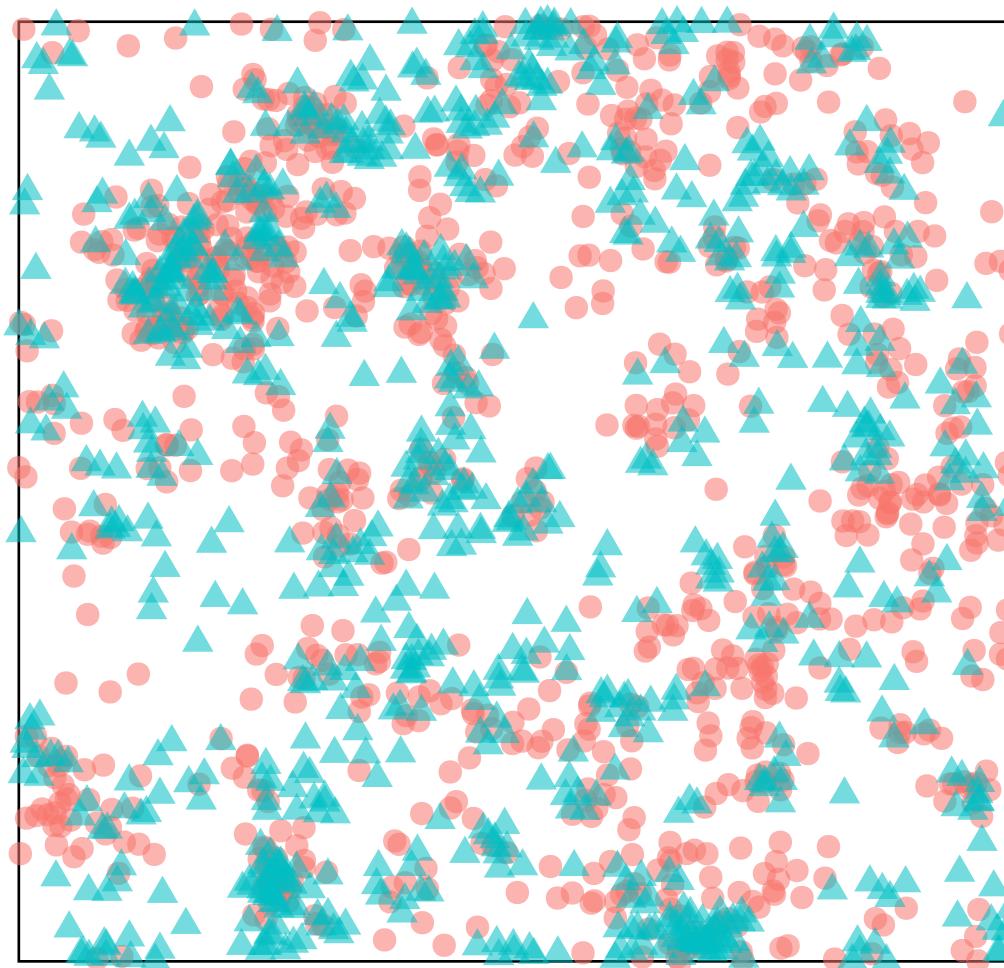
- ① Periodogram
 $\hat{I}(\omega)$



- ② Kernel estimator
 $\hat{f}_P(\omega)$



Simulation: Data generating process

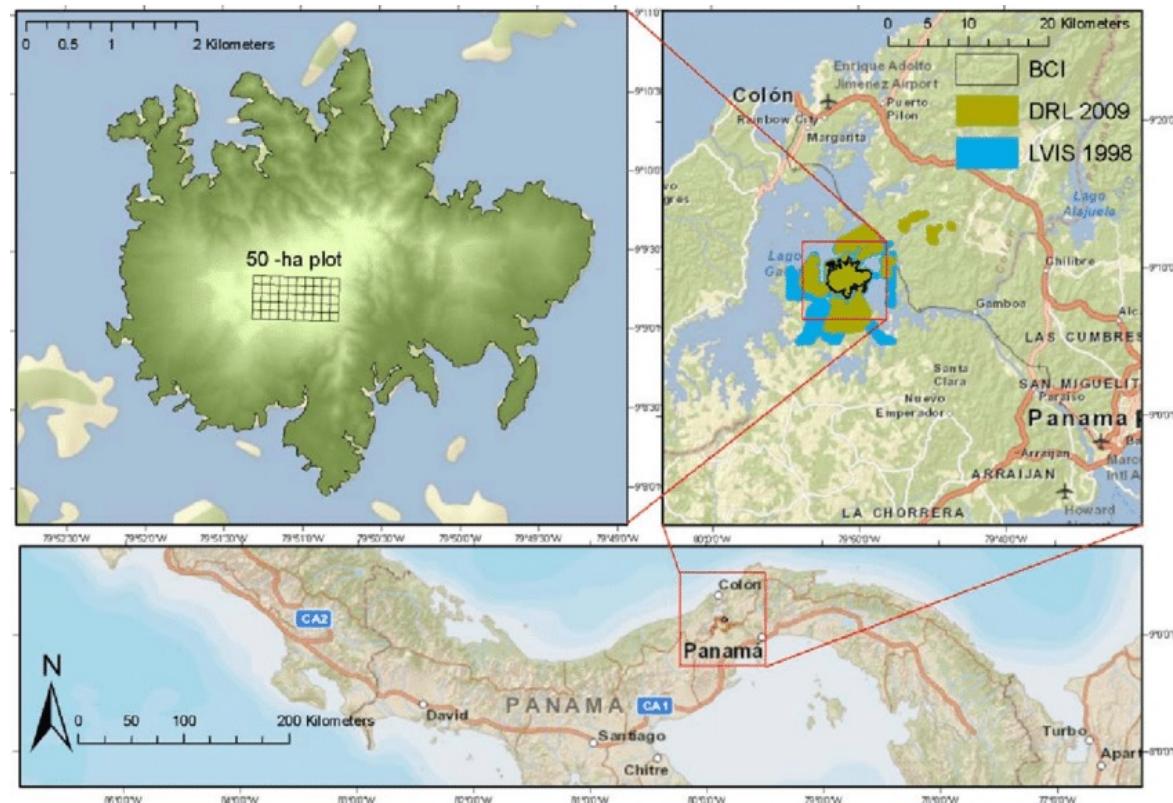


Bivariate point process with marginal and joint clustering interactions

Simulation: Result

Pseudo-spectrum	Window	$\hat{I}(\omega)$		$\hat{f}_P(\omega)$	
		Bias	MSE	Bias	MSE
Marginal	$[-5, 5]^2$	0.03	1.49	0.02	0.49
	$[-10, 10]^2$	0.01	1.09	0.00	0.17
	$[-20, 20]^2$	0.00	1.02	0.00	0.13
Cross	$[-5, 5]^2$	0.06	28.14	0.01	3.76
	$[-10, 10]^2$	0.05	22.61	0.01	2.45
	$[-20, 20]^2$	0.05	20.18	0.01	2.11

Barro Colorado Island (BCI) data



207,718 alive trees

310 species

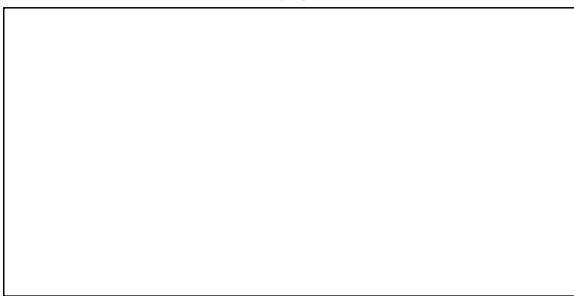
7 censuses



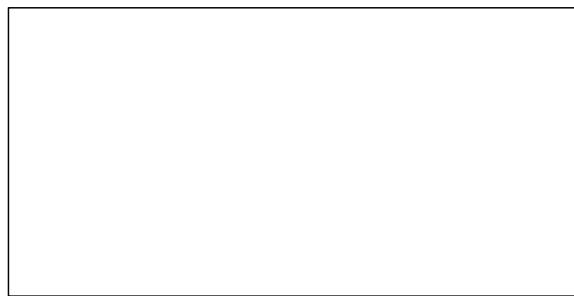
Figure 1 in Fricker et al. (2012)

Barro Colorado Island (BCI) data

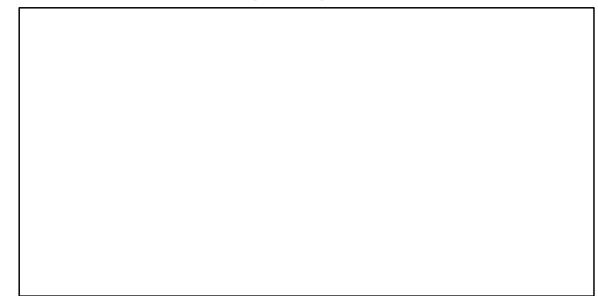
cappfr



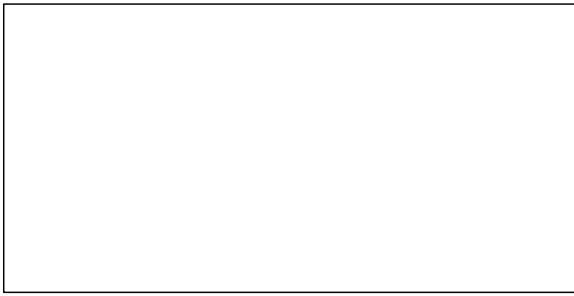
hirttr



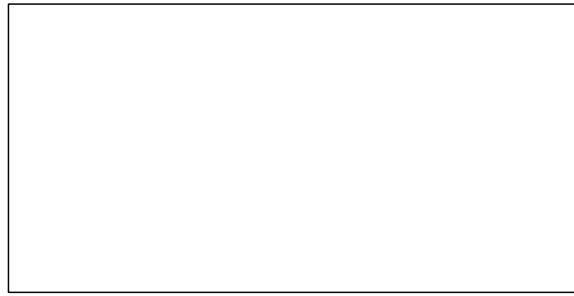
protpa



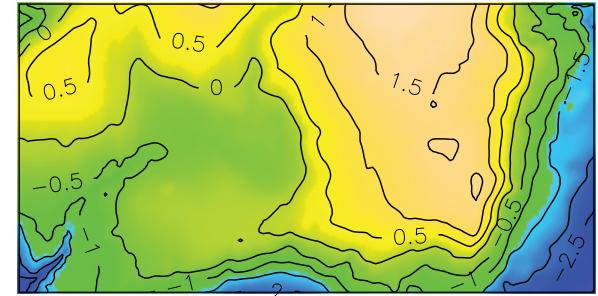
protte



tet2pa



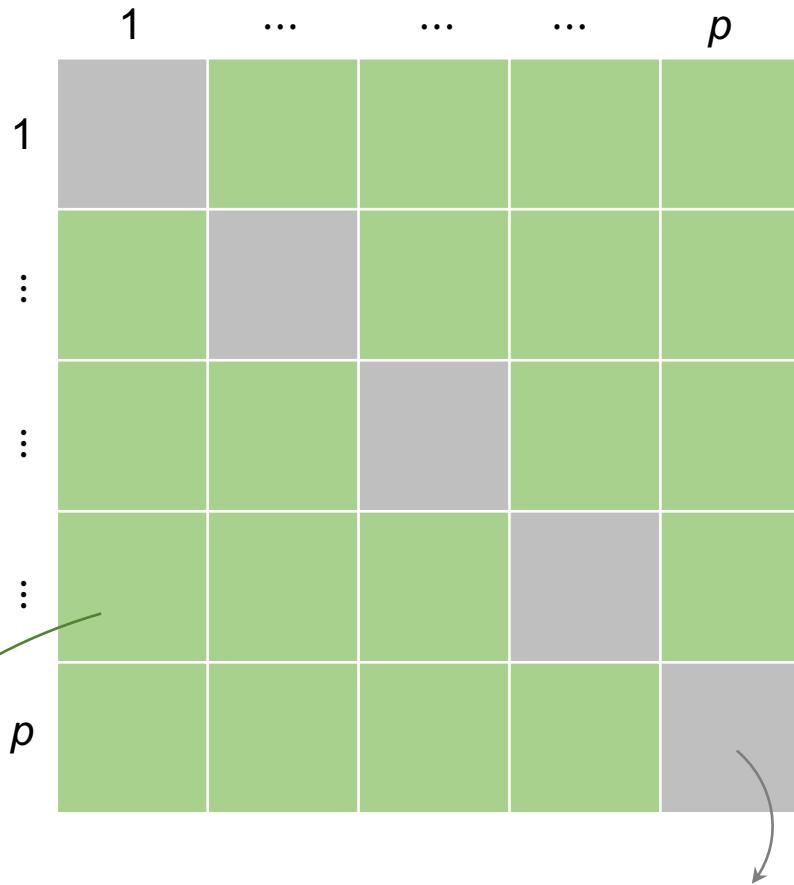
elevation



Point patterns of five species in the BCI dataset and image of elevation in the study region.

Multivariate point pattern

Consider a p -variate process: $i, j \in \mathcal{V} = \{1, 2, \dots, p\}$



Spectral coherence:

$$R_{ij}(\boldsymbol{\omega}) = \frac{f_{ij}(\boldsymbol{\omega})}{[f_{ii}(\boldsymbol{\omega})f_{jj}(\boldsymbol{\omega})]^{\frac{1}{2}}}$$

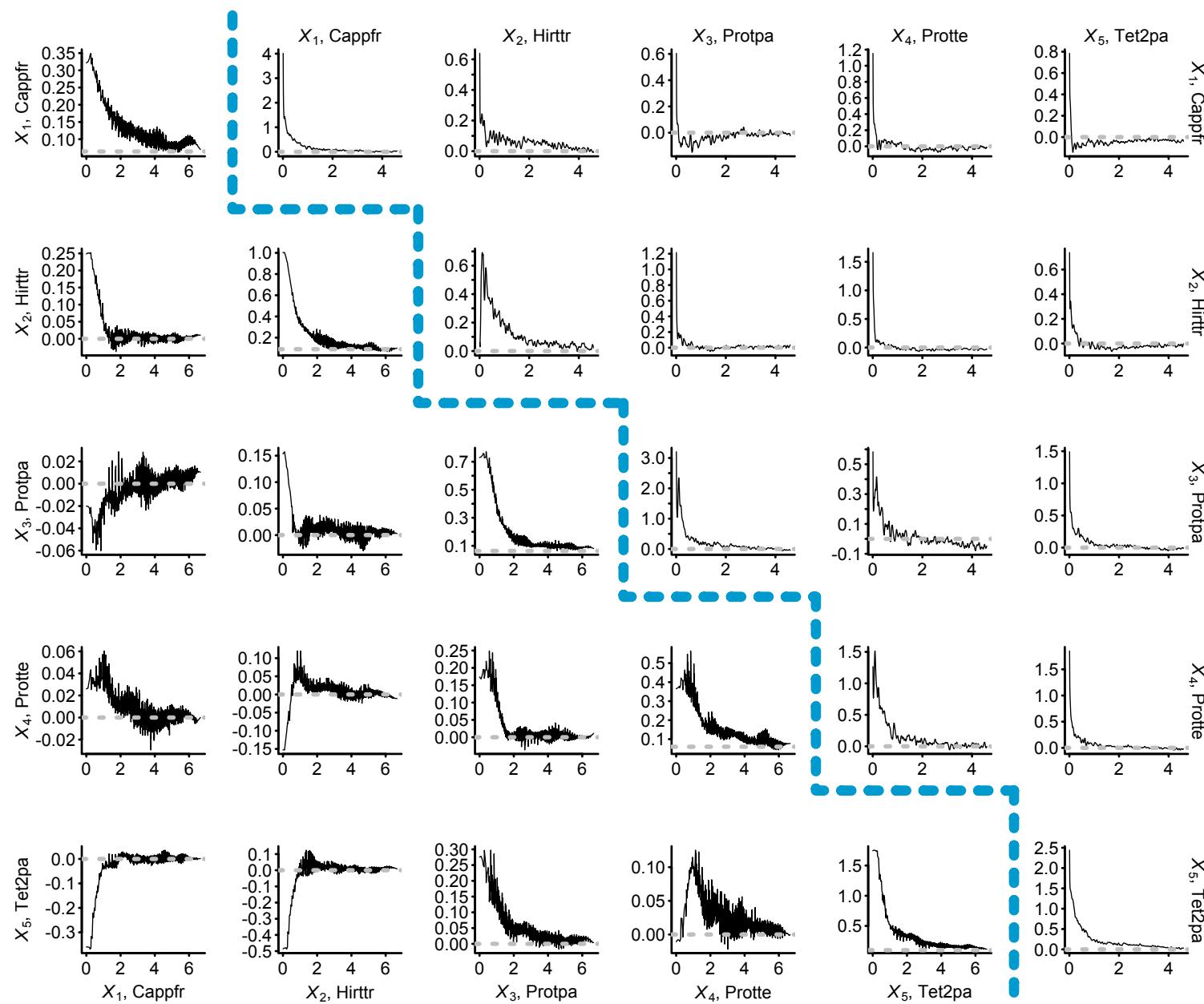
Cross-spectrum:

$$f_{ij}(\boldsymbol{\omega}) = (2\pi)^{-d} \int_{\mathbb{R}^d} C_{ij}(\mathbf{x}_1 - \mathbf{x}_2) e^{-i\mathbf{x}^\top \boldsymbol{\omega}} d\mathbf{x}$$

Marginal spectrum:

$$f_{ii}(\boldsymbol{\omega}) = (2\pi)^{-d} \int_{\mathbb{R}^d} C_{ii}(\mathbf{x}_1 - \mathbf{x}_2) e^{-i\mathbf{x}^\top \boldsymbol{\omega}} d\mathbf{x}$$

Pseudo-spectra



Coherence analysis

