Introduction to spatial point process and its Fourier analysis

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Fundamentals

Point pattern data

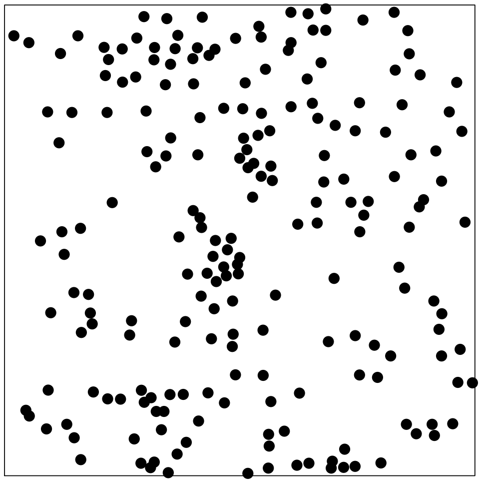
Spatial **locations** of things or events

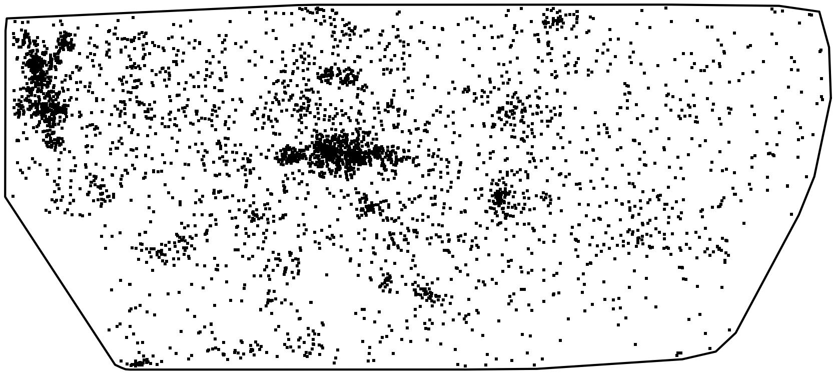
For a study area, all locations are observable!

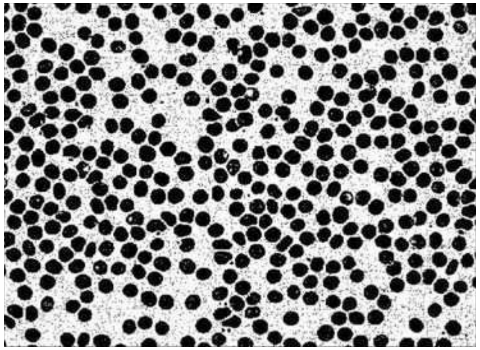
Locations without a point = observed but no event happened!

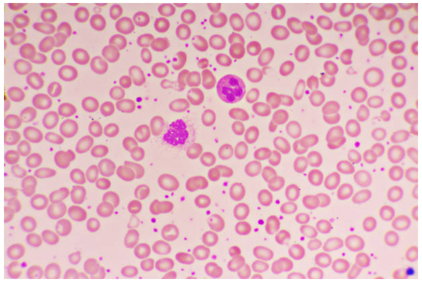
Point pattern data

Spatial **locations** of things or events









Japanese black pine seedlings

Sky positions of 4215 galaxies in the Shapley Supercluster

Red blood particles

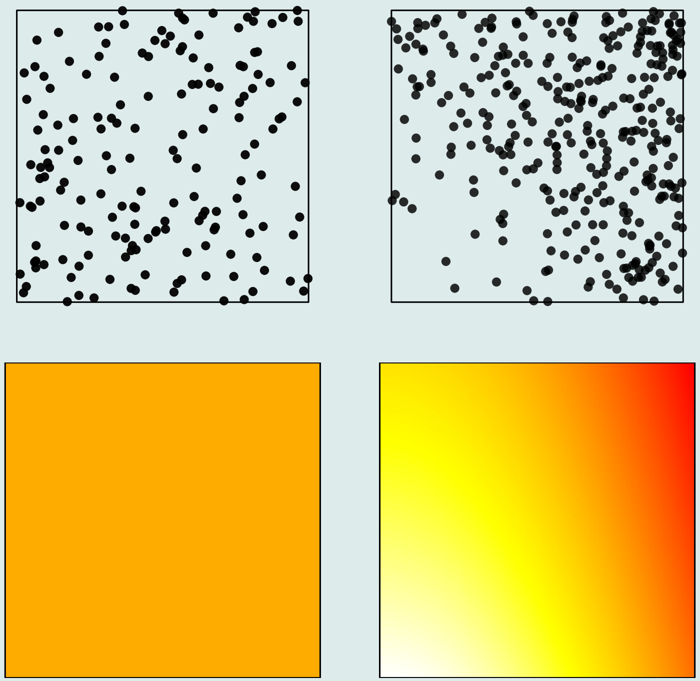
For a study area, all locations are observable!

Locations without a point = observed but no event happened!

Characteristics of point process

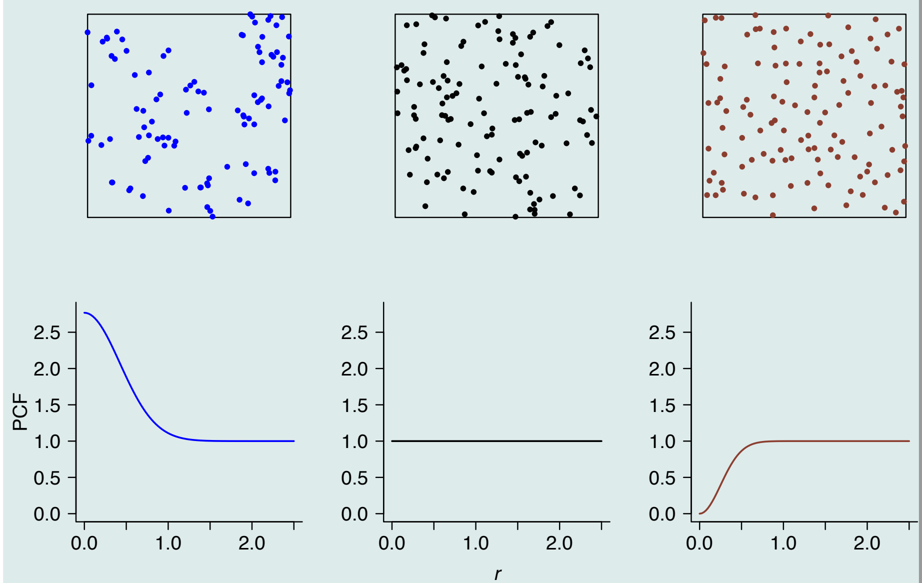
Intensity function: $\lambda(\boldsymbol{x})$

$\propto \mathbb{P}$(Having a point at location $\boldsymbol{x}$)

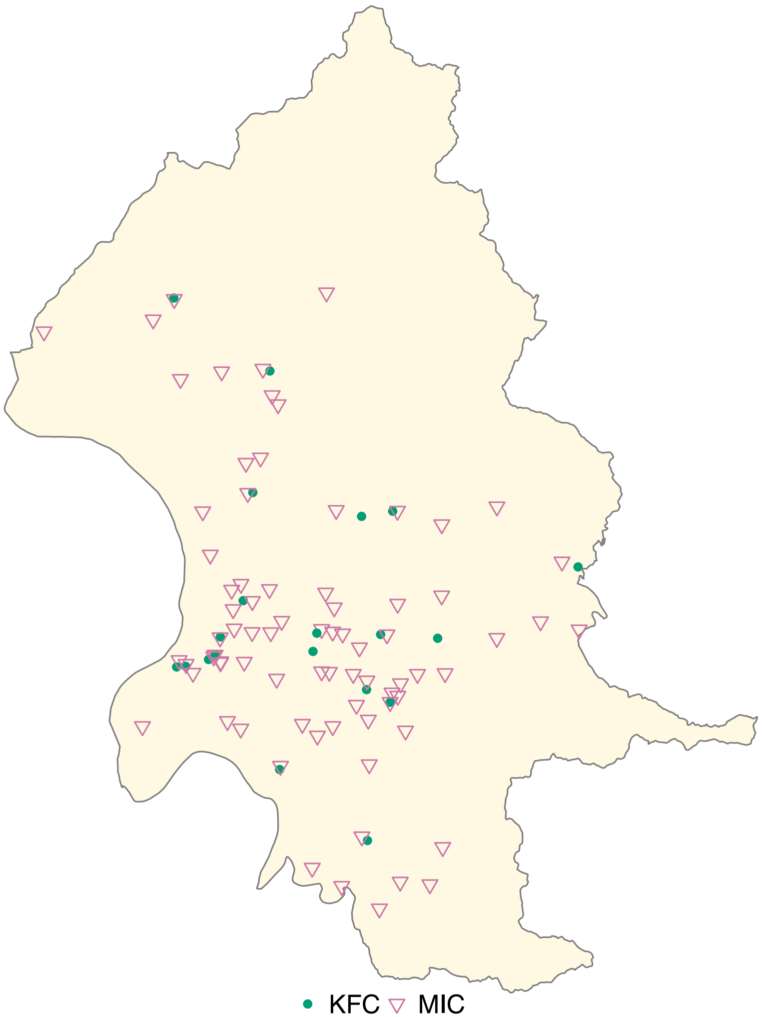


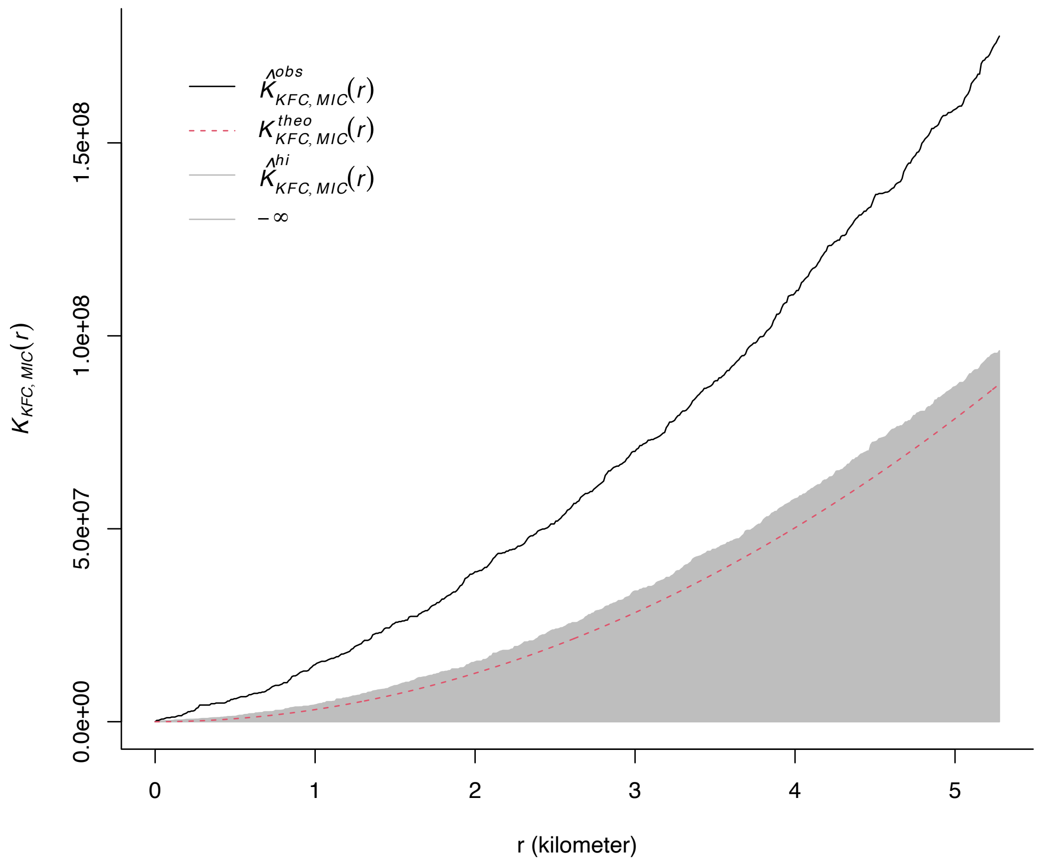
Pair correlation function: $g(r)$

How events interact at radius r

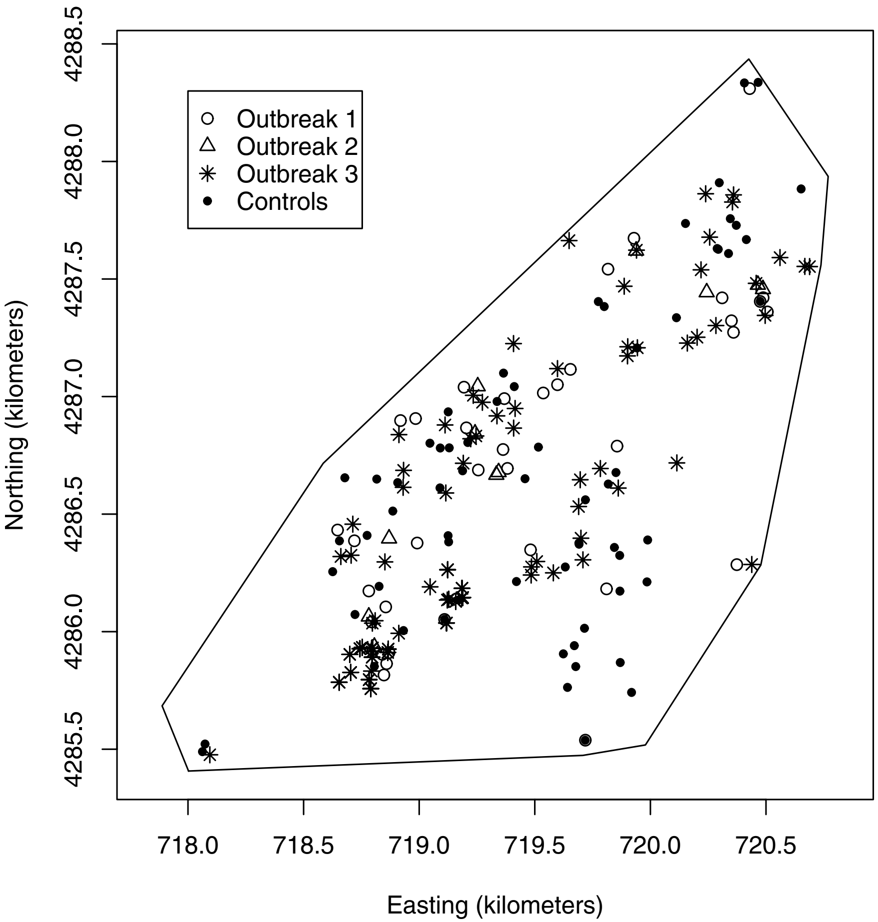


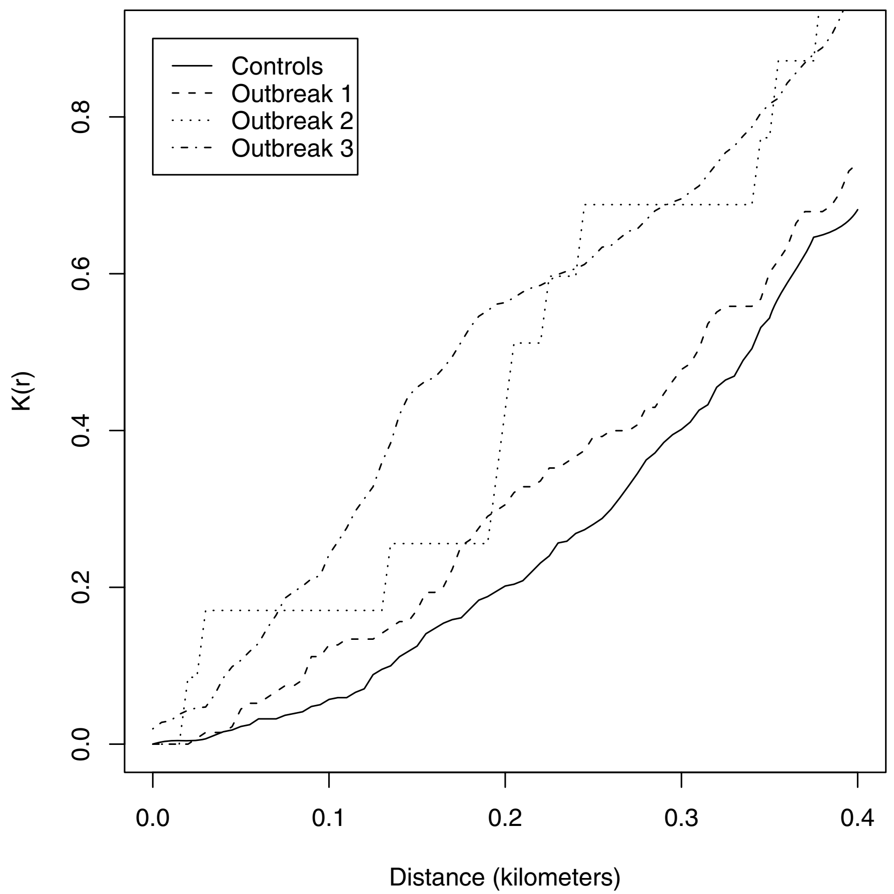
Example 1: Type interaction





Example 2: Source detection of infectious disease





Martínez-Beneito et al. (2006)

Fourier analysis of point processes

Time series vs. spatial point processes

Time series model

- White noise : $e_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$
- MA(1) : $X_t = e_t + \theta e_{t-1}$
- AR(1) : $X_t = \alpha X_{t-1} + e_t$

ACF

- $\gamma(k) = \sigma^2 \mathbb{1}\{k = 0\}$
- $\gamma(k) = \theta\sigma^2, k = \pm 1$
- $\gamma(k) = \alpha^{|k|}\sigma_X^2$

Spectral density

- $f(\omega) = \sigma^2/(2\pi)$
- $f(\omega) = \sigma^2(1 + \theta^2 + 2\theta\cos\omega)/(2\pi)$
- $f(\omega) = \dfrac{\sigma^2}{2\pi(1 - 2\alpha\cos\omega + \alpha^2)}$

Point process model

- Homogeneous Poisson process
- Thomas cluster process
- Determinantal point process

PCF

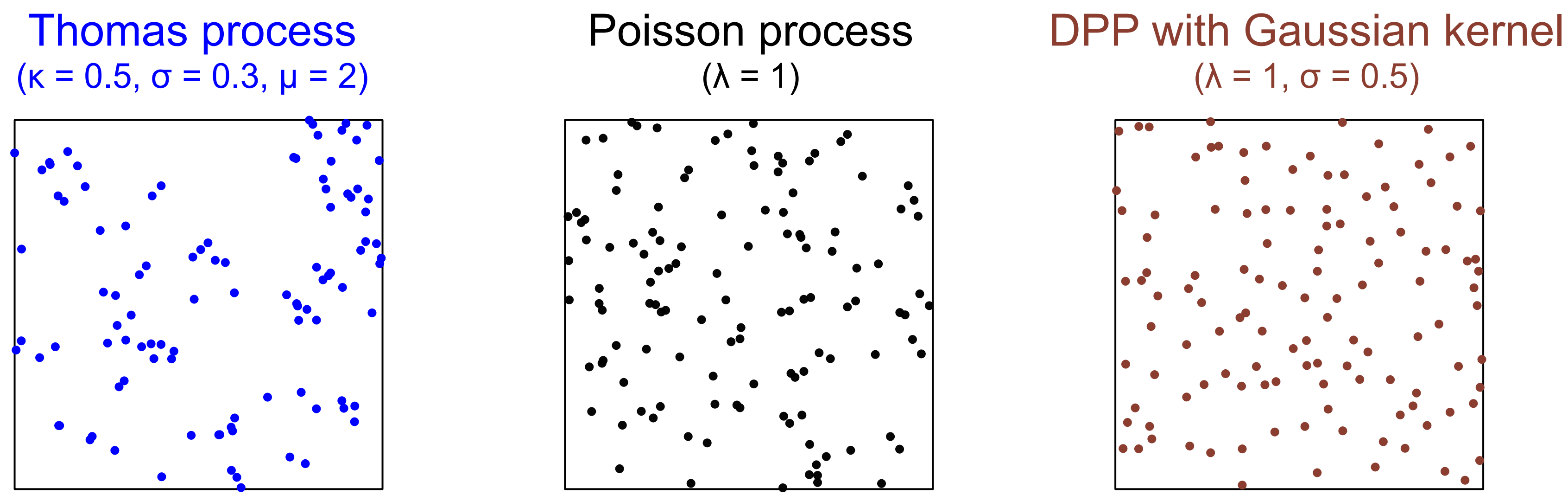
- $g(r) = 1$
- $g(r) = 1 + \dfrac{e^{-r^2/4\sigma^2}}{4\pi\kappa\sigma^2}$
- $g(r) = 1 - \left(e^{-r^2/\sigma^2}\right)^2$

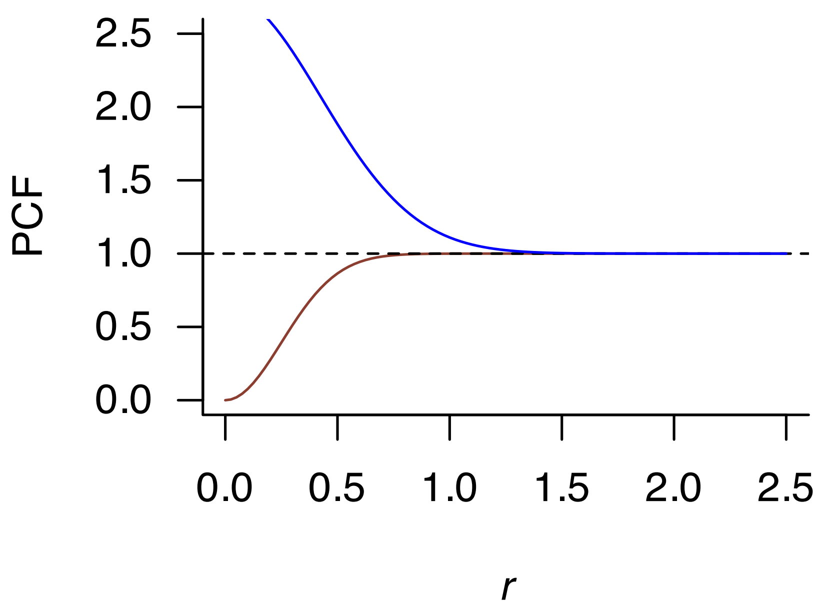
Spectral density

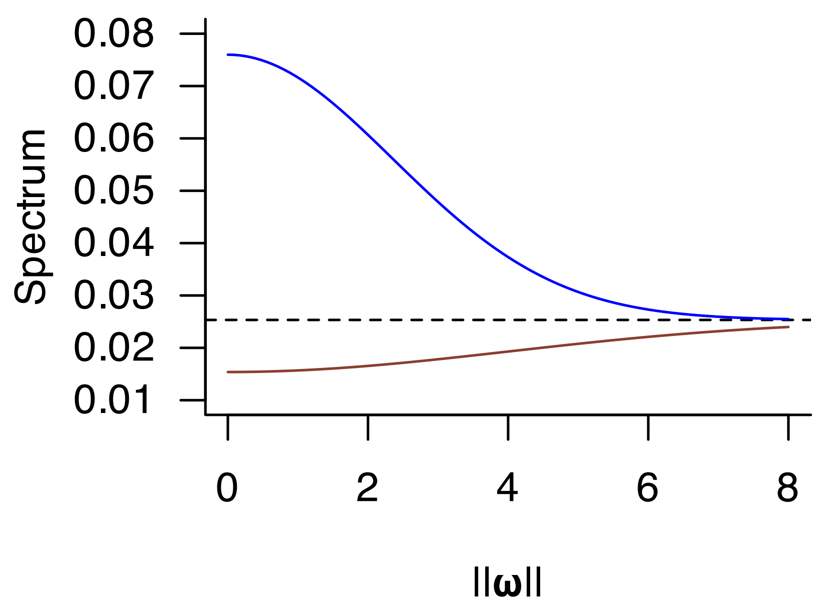
- $f(\boldsymbol{\omega}) = (2\pi)^{-d}\lambda$
- $f(\boldsymbol{\omega}) = (2\pi)^{-d}\kappa\mu\left[1 + \mu e^{-\sigma^2||\boldsymbol{\omega}||^2}\right]$
- $f(\boldsymbol{\omega}) = (2\pi)^{-d}\left[\lambda - \lambda^2\left(\dfrac{\pi\sigma^2}{2}\right)^{\frac{d}{2}} e^{\frac{-\sigma^2||\boldsymbol{\omega}||^2}{8}}\right]$

For stationary and isotropic case, $g(\mathbf{x}_i, \mathbf{x}_j) = g(||\mathbf{x}_i - \mathbf{x}_j||) = g(r)$.
Parameter: $\boldsymbol{\theta}_{\text{Thomas}} = (\kappa, \mu, \sigma^2)^T$, $\boldsymbol{\theta}_{\text{Determinantal}} = (\lambda, \sigma^2)^T$.

PCF and spectral density







Frequency domain parameter estimation

Whittle-type likelihood

Let $\{X_{\boldsymbol{\theta}}\}$ be a family of 2^{nd}-order stationary point processes with parameter $\boldsymbol{\theta} \in \Theta$. The associated spectral density is denoted as $f_{\boldsymbol{\theta}}$. Then, we fit the model using the pseudo-likelihood

$$L(\boldsymbol{\theta}) = \sum_{\boldsymbol{\omega_k} \in D} \left\{ \frac{\hat{I}(\boldsymbol{\omega_k})}{f_{\boldsymbol{\theta}}(\boldsymbol{\omega_k})} + \log f_{\boldsymbol{\theta}}(\boldsymbol{\omega_k}) \right\}.$$

Intuition

- $\hat{I}(\boldsymbol{\omega})$ can be regarded as the **"truth"** since $\mathbb{E}[\hat{I}_n(\boldsymbol{\omega})] \to f(\boldsymbol{\omega})$.
- $f_{\boldsymbol{\theta}}(\boldsymbol{\omega})$ are the parameter family of spectral densities.
- $L(\boldsymbol{\theta})$ is the **spectral divergence between the truth and our guess**. ***The smaller the better!***

Proposed model parameter estimator

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta} \in \Theta} L(\boldsymbol{\theta})$$

Frequency domain parameter estimation

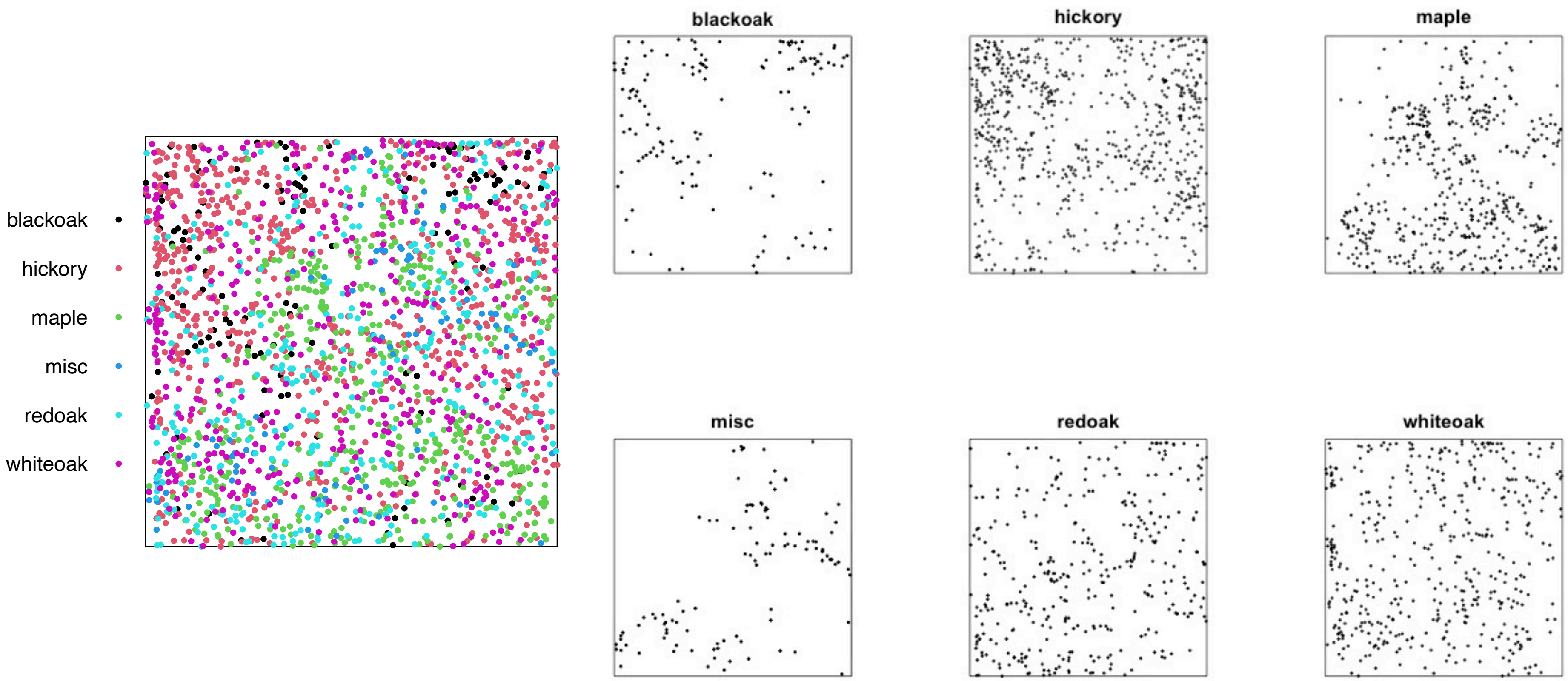
Model	Window	Parameter	Method		
			Ours	ML	MC
TCP	$[-5, 5]^2$	κ	–0.04 (0.11)	–0.02 (0.11)	–0.04 (0.10)
		μ	0.72 (3.52)	–0.24 (5.93)	–
		σ^2	0.02 (0.07)	–0.04 (0.22)	0.01 (0.10)
		Time (sec)	**0.74**	**0.38**	**0.07**
	$[-10, 10]^2$	κ	–0.02 (0.05)	–0.02 (0.05)	–0.01 (0.05)
		μ	0.60 (1.77)	0.33 (2.79)	–
		σ^2	0.01 (0.02)	–0.01 (0.11)	0.00 (0.06)
		Time (sec)	**2.38**	**5.67**	**0.23**
	$[-20, 20]^2$	κ	–0.01 (0.04)	0.00 (0.03)	0.00 (0.03)
		μ	0.25 (1.03)	0.15 (1.23)	–
		σ^2	0.01 (0.02)	0.00 (0.03)	0.00 (0.03)
		Time (sec)	**9.15**	**173.66**	**1.95**

The bias and the standard errors (in parentheses) of the estimated parameters based on three different approaches for the Thomas clustered process (TCP)

Future directions

Multivariate point pattern

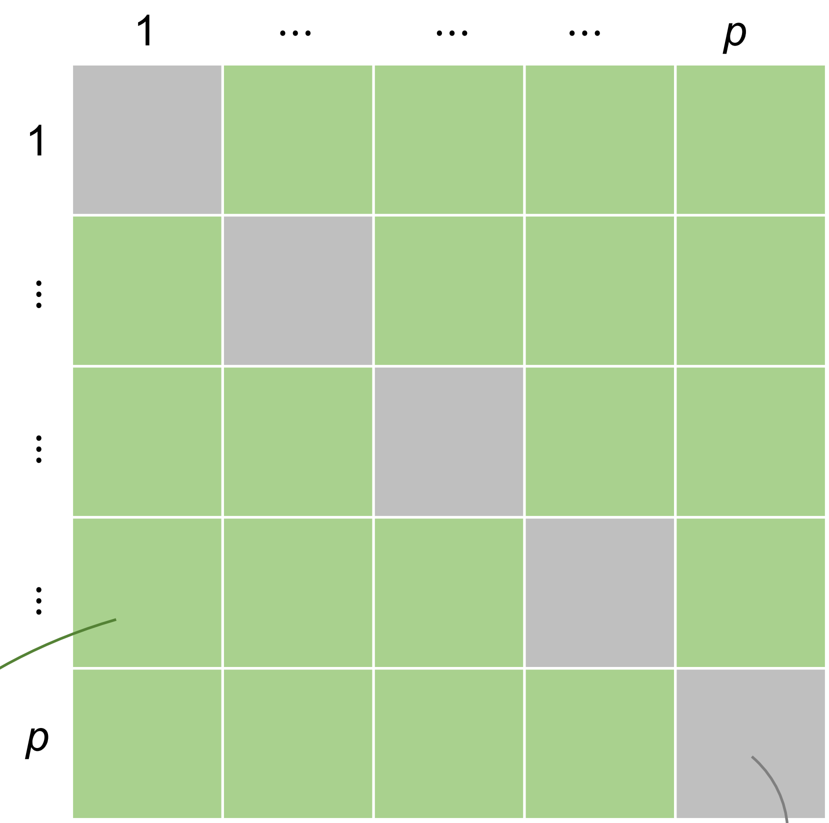
Multivariate = each point belongs to some **category**



Location of trees in Lansing woods, **marked** by their **species**

Multitype point pattern

Consider a p-variate process: $i, j \in \mathcal{V} = \{1, 2, \ldots, p\}$



Spectral coherence:

$$R_{ij}(\boldsymbol{\omega}) = \frac{f_{ij}(\boldsymbol{\omega})}{[f_{ii}(\boldsymbol{\omega}) f_{jj}(\boldsymbol{\omega})]^{\frac{1}{2}}}$$

Cross-spectrum:

$$f_{ij}(\boldsymbol{\omega}) = (2\pi)^{-d} \int_{\mathbb{R}^d} C_{ij}(\boldsymbol{x}_1 - \boldsymbol{x}_2) e^{-i\boldsymbol{x}^\intercal \boldsymbol{\omega}} \, \mathrm{d}\boldsymbol{x}$$

Marginal spectrum:

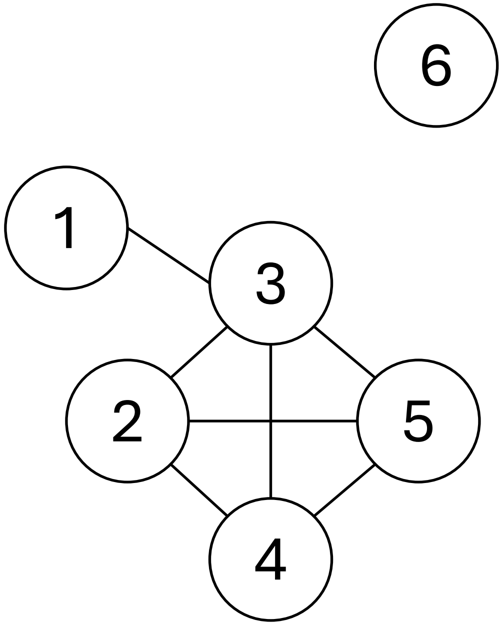
$$f_{ii}(\boldsymbol{\omega}) = (2\pi)^{-d} \int_{\mathbb{R}^d} C_{ii}(\boldsymbol{x}_1 - \boldsymbol{x}_2) e^{-i\boldsymbol{x}^\intercal \boldsymbol{\omega}} \, \mathrm{d}\boldsymbol{x}$$

Spatial dependence graph model (SDGM)

SDGM (Eckardt, 2016)

Let $\mathbf{N}_{\mathcal{V}}$ be a multivariate spatial counting process. A SDGM is an **undirected graphical model** $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where any $v_i \in \mathcal{V}(\mathcal{G})$ encodes a component of $\mathbf{N}_{\mathcal{V}}$ and $\mathcal{E}(\mathcal{G}) = \{(v_i, v_j): R_{ij|\mathcal{V}\setminus\{i,j\}}(\boldsymbol{\omega}) \neq 0\}$ such that

$$\{N_i\} \perp\!\!\!\perp \{N_j\} | \{N_{\mathcal{V}\setminus\{i,j\}}\} \Leftrightarrow (v_i, v_j) \notin \mathcal{E}(\mathcal{G}).$$



For $\{N_i: i \in I \subset \mathbf{N}_{\mathcal{V}}\}$, $\{N_j: j \in J \subset \mathbf{N}_{\mathcal{V}}\}$, and $\{N_h: h \in H \subset \mathbf{N}_{\mathcal{V}}\}$,

$$\begin{aligned} N_I \perp\!\!\!\perp N_J | N_H &\Leftrightarrow f_{I,J|H}(\boldsymbol{\omega}) = 0, \forall \boldsymbol{\omega} \\ &\Leftrightarrow R_{I,J|H}(\boldsymbol{\omega}) = 0, \forall \boldsymbol{\omega} \\ &\Leftrightarrow \text{Missing edges between } N_I \text{ and } N_J \end{aligned}$$

Barro Colorado Island (BCI) data

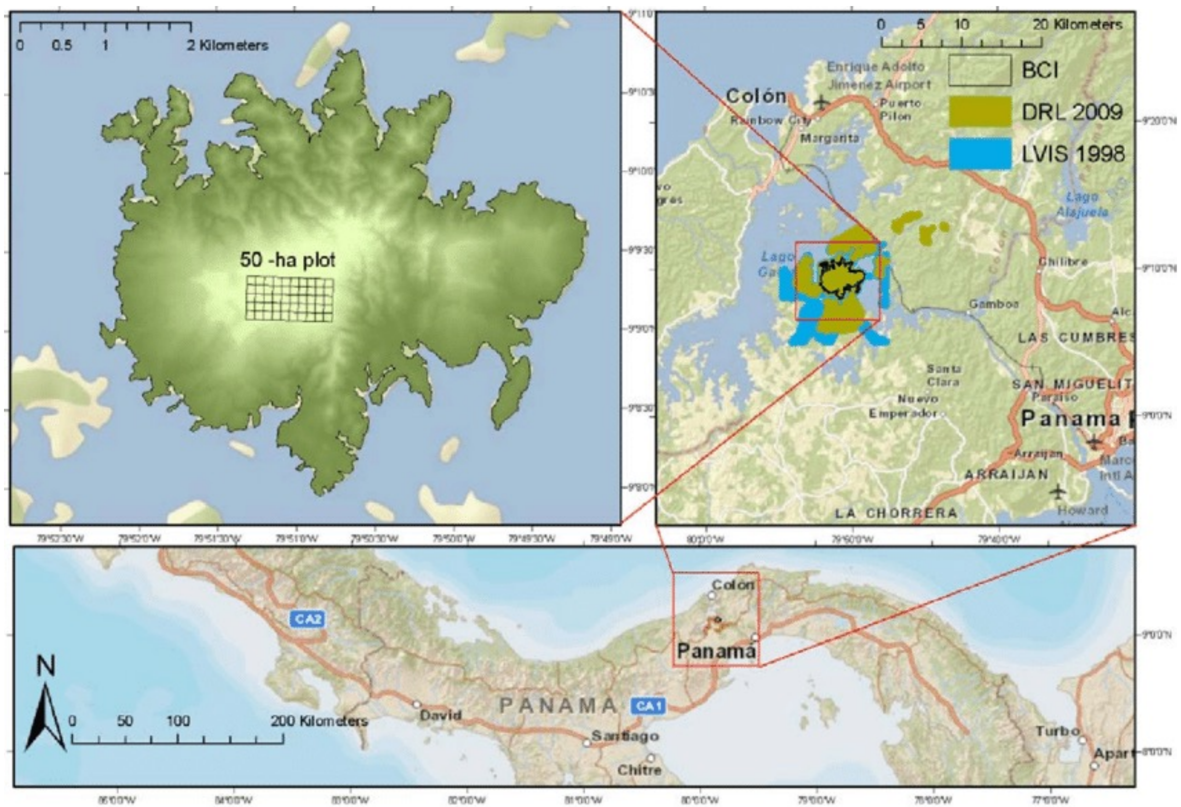


Figure 1 in Fricker et al. (2012)

207,718 alive trees

310 species

7 censuses

BCI Plot Census
November 26, 2015 by Forest Global Earth Observatory (ForestGEO).

https://datadryad.org/stash/dataset/doi:10.15146/5xcp-0d46

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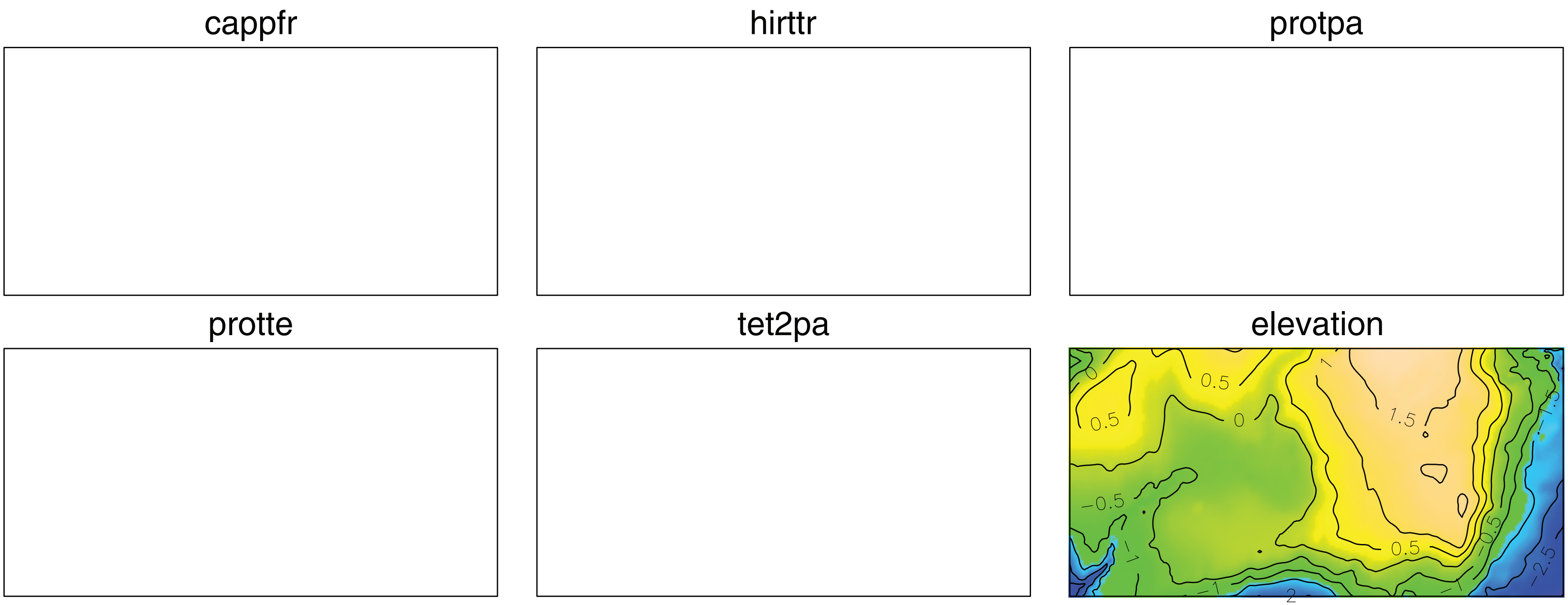
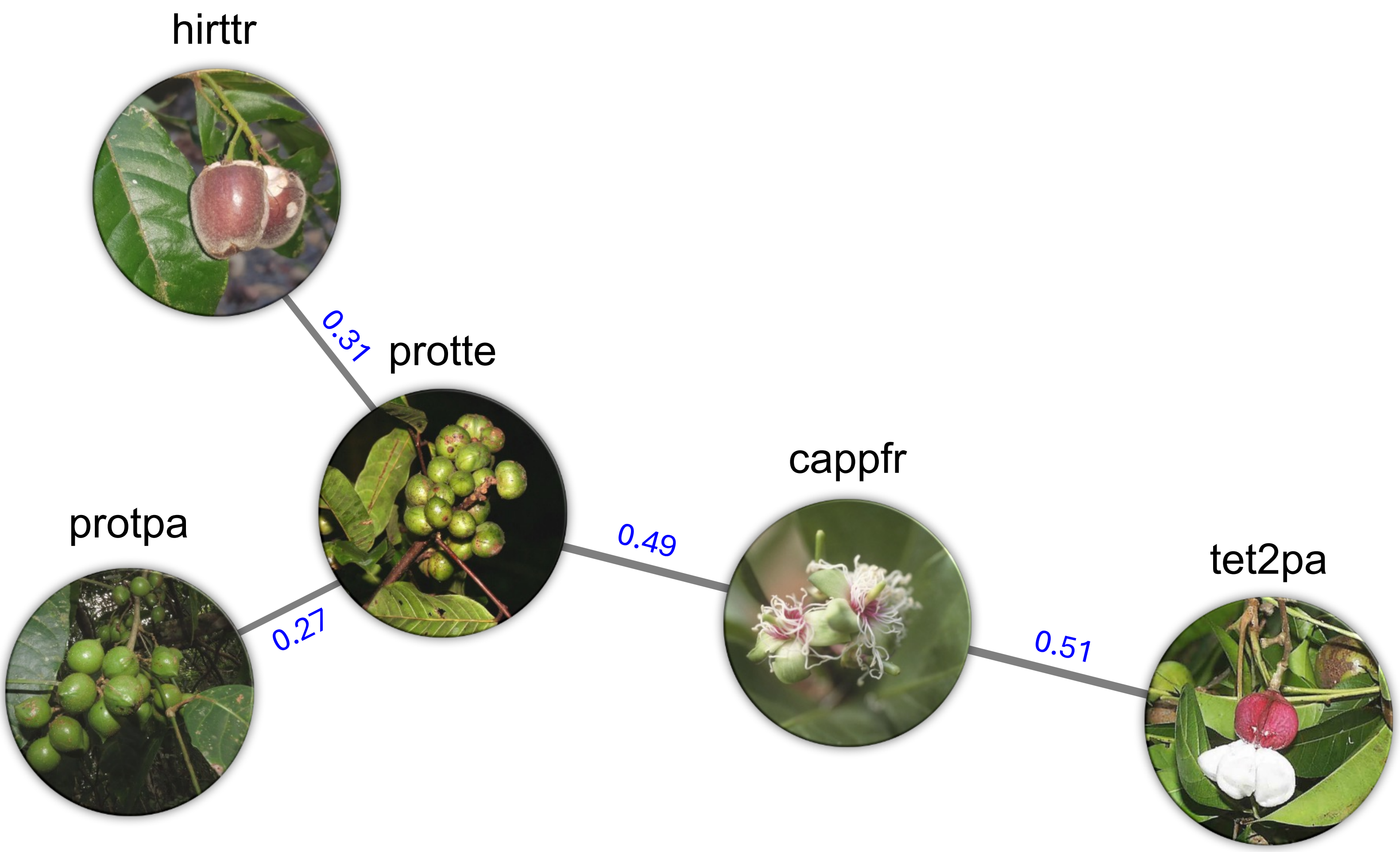


Point patterns of five species in the BCI dataset and image of elevation in the study region.

Figure from Jalilian et al. (2015)

BCI data: Graphical model

Threshold = 0.2



Value = max squared partial coherence