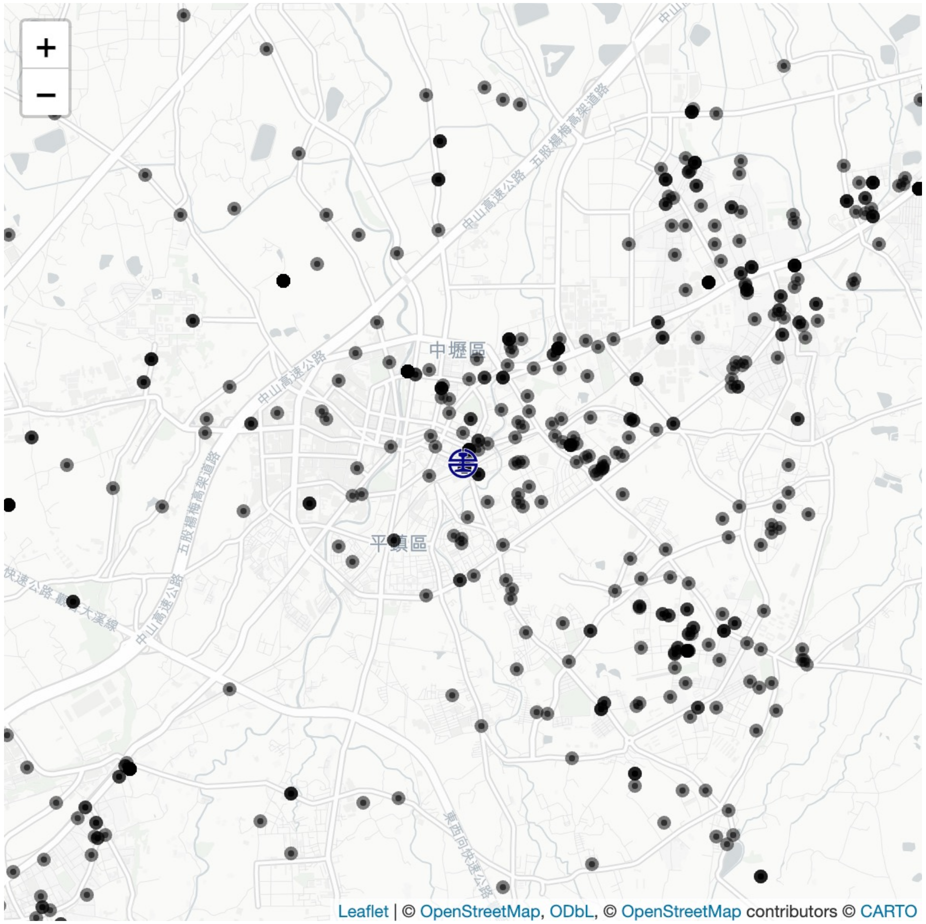
Introduction to spatial point process and its Fourier analysis

Qi-Wen Ding

Fundamentals

Point pattern data

Spatial **locations** of things or events

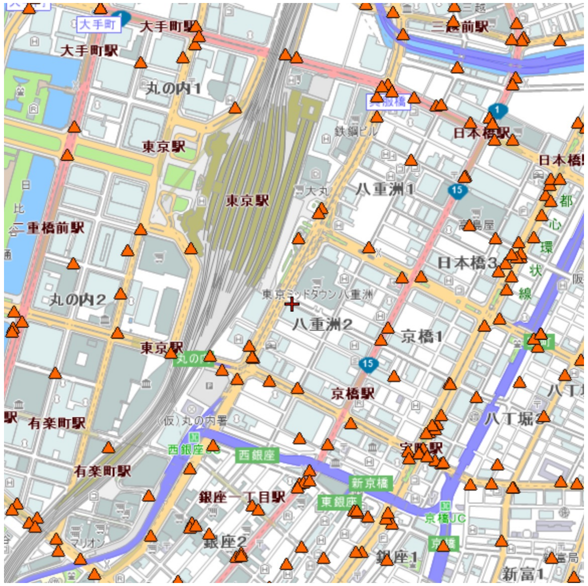


For a study area, all locations are observable!

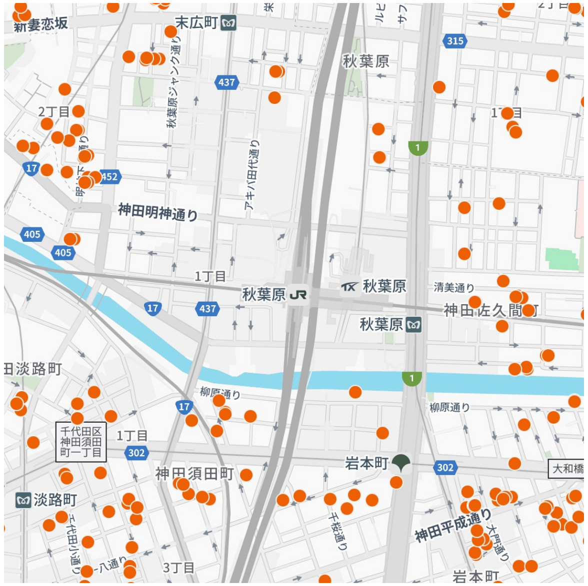
Locations without a point = observed but no event happened!

Point pattern data

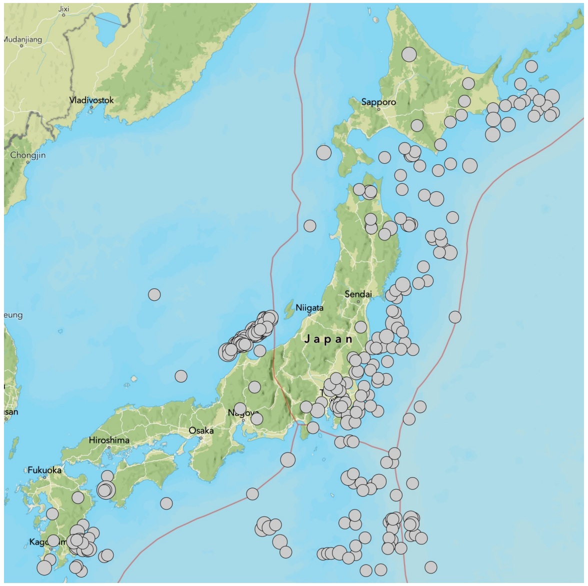
Spatial **locations** of things or events



Traffic **injuries** near Tokyo station (2021)



Rental housing spots near Akihabara (2025/6/7)

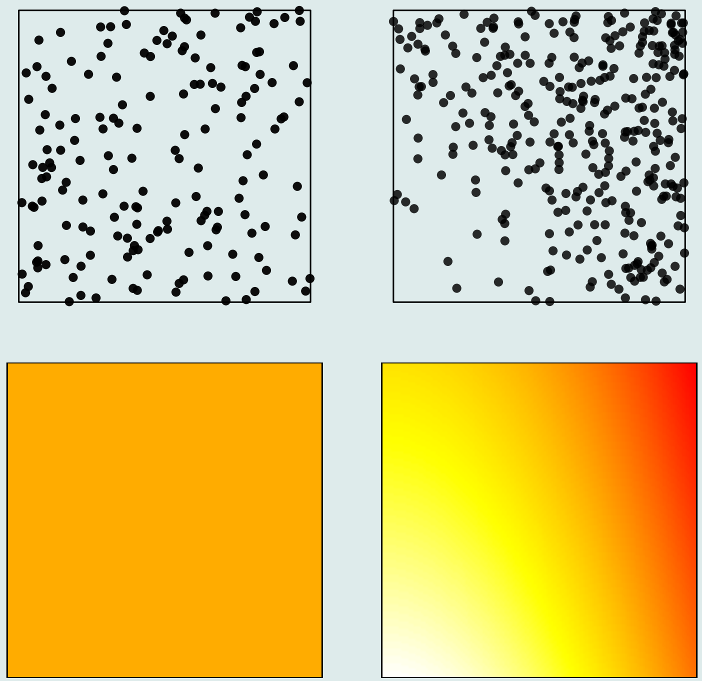


M4.5+ **earthquakes** in Japan (2024)

Characteristics of point process

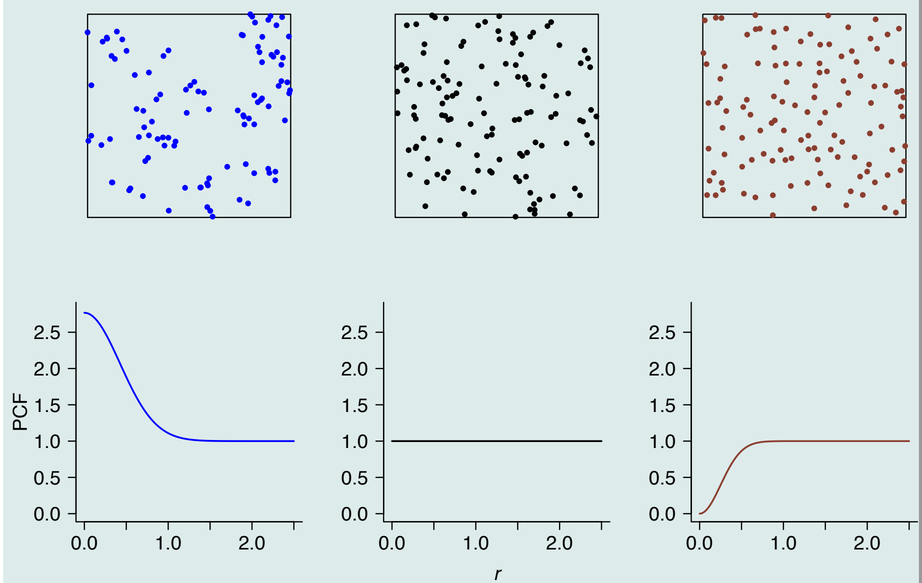
Intensity function: $\lambda(\boldsymbol{x})$

$\propto \mathbb{P}(\text{Having a point at location } \boldsymbol{x})$

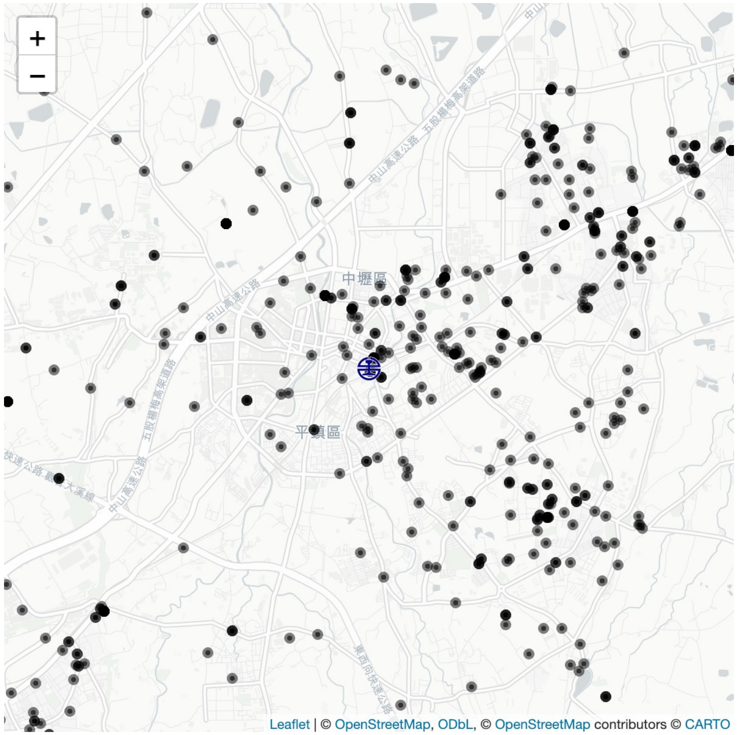


Pair correlation function: $g(r)$

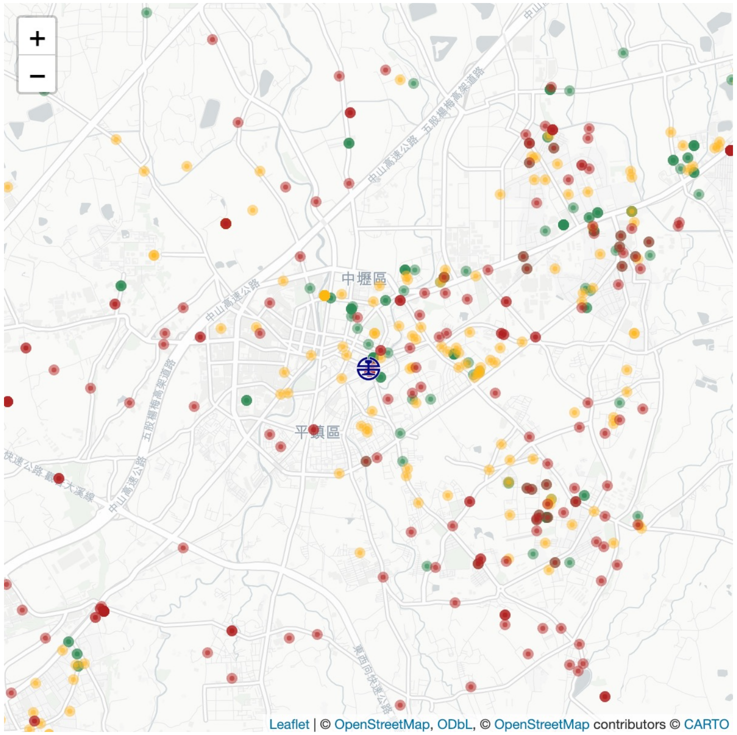
How events interact at radius r



From univariate to multivariate



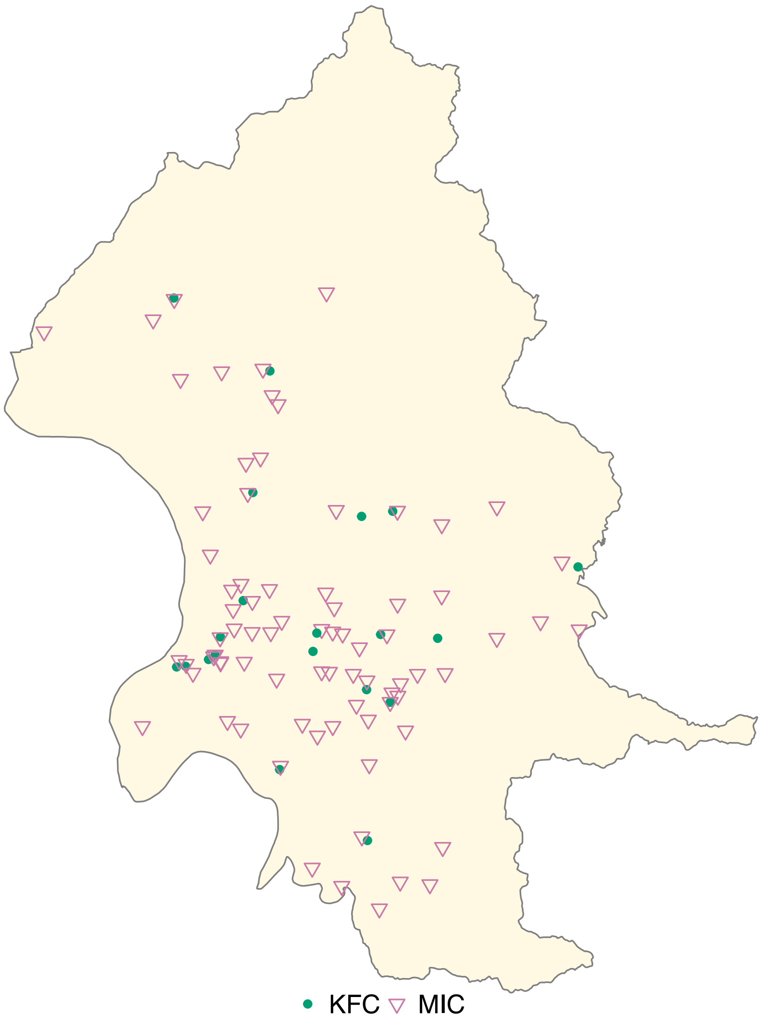
Burglary incident (2020-2023)

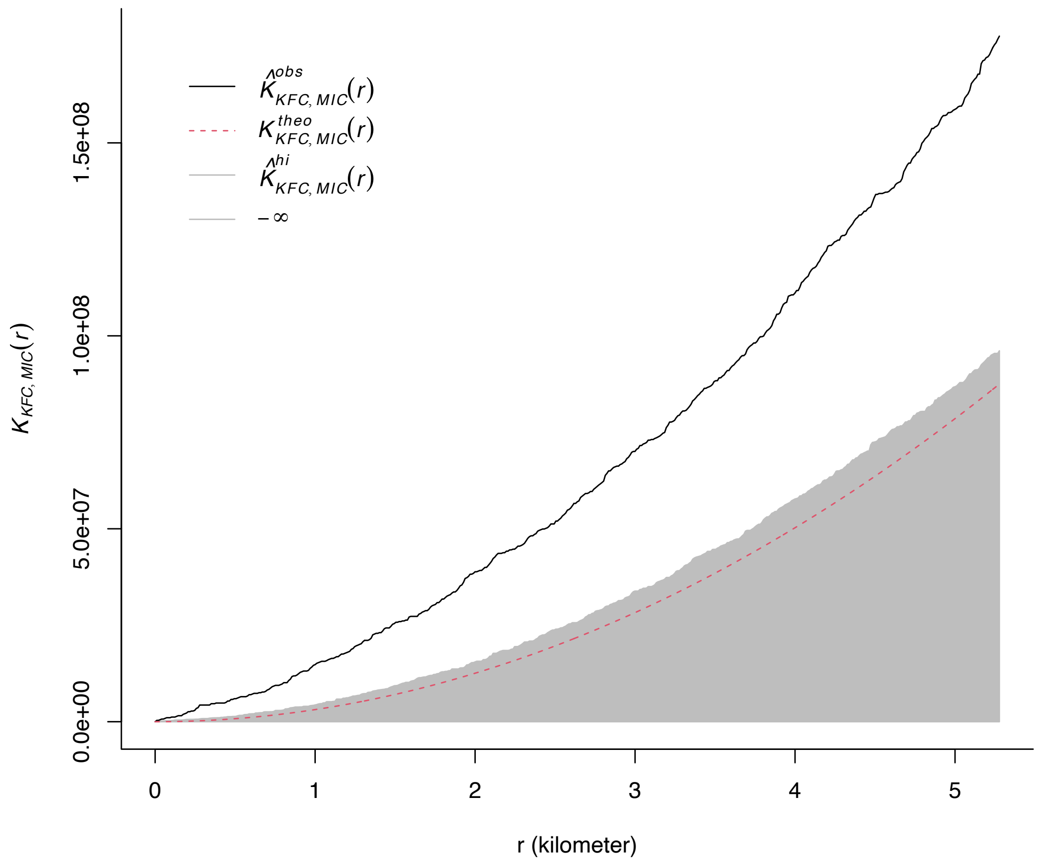


Type of burglary

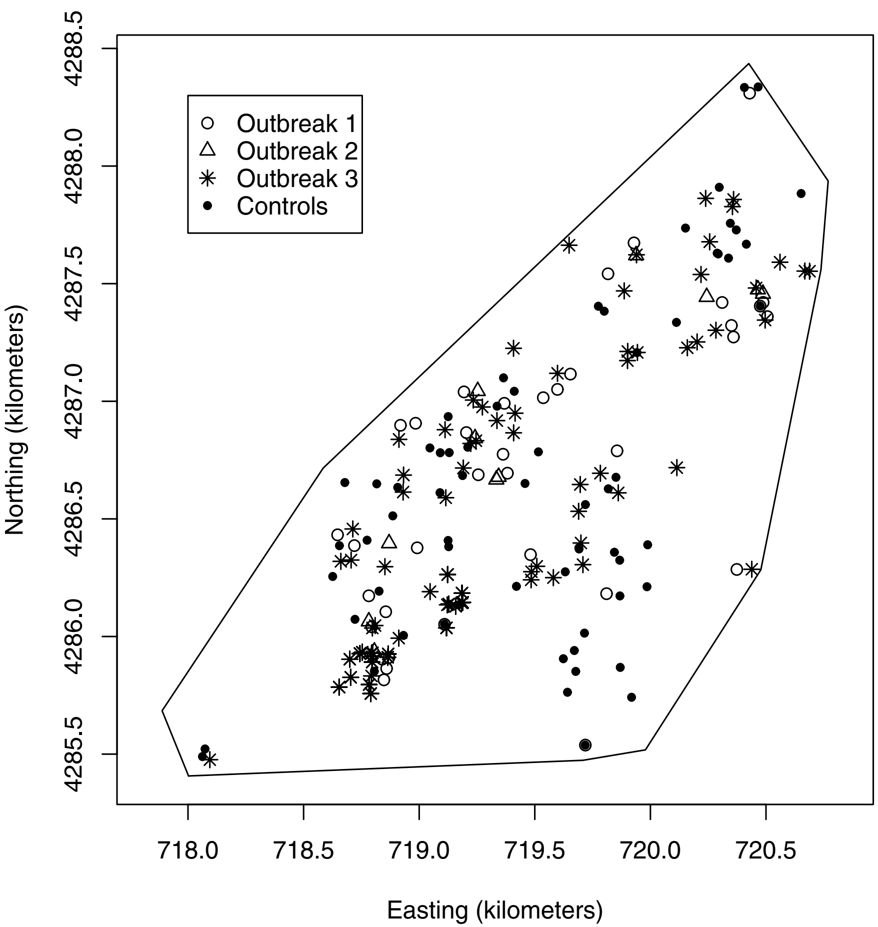
Motorcycle Car Bicycle

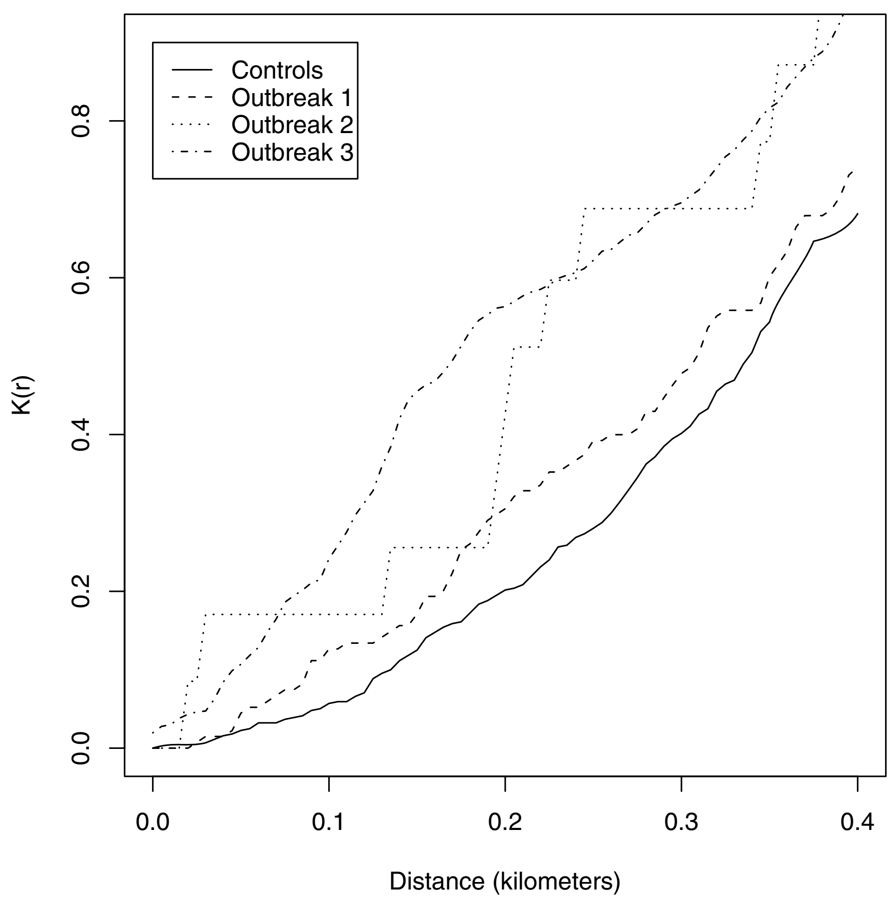
Example 1: Type interaction





Example 2: Source detection of infectious disease





Martínez-Beneito et al. (2006)

Fourier analysis of point processes

Time series vs. spatial point processes

Time series model

- White noise : $e_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$
- MA(1) : $X_t = e_t + \theta e_{t-1}$
- AR(1) : $X_t = \alpha X_{t-1} + e_t$

ACF

- $\gamma(k) = \sigma^2 \mathbb{1}\{k = 0\}$
- $\gamma(k) = \theta\sigma^2, k = \pm 1$
- $\gamma(k) = \alpha^{|k|}\sigma_X^2$

Spectral density

- $f(\omega) = \sigma^2/(2\pi)$
- $f(\omega) = \sigma^2(1 + \theta^2 + 2\theta\cos\omega)/(2\pi)$
- $f(\omega) = \dfrac{\sigma^2}{2\pi(1 - 2\alpha\cos\omega + \alpha^2)}$

Point process model

- Homogeneous Poisson process
- Thomas cluster process
- Determinantal point process

PCF

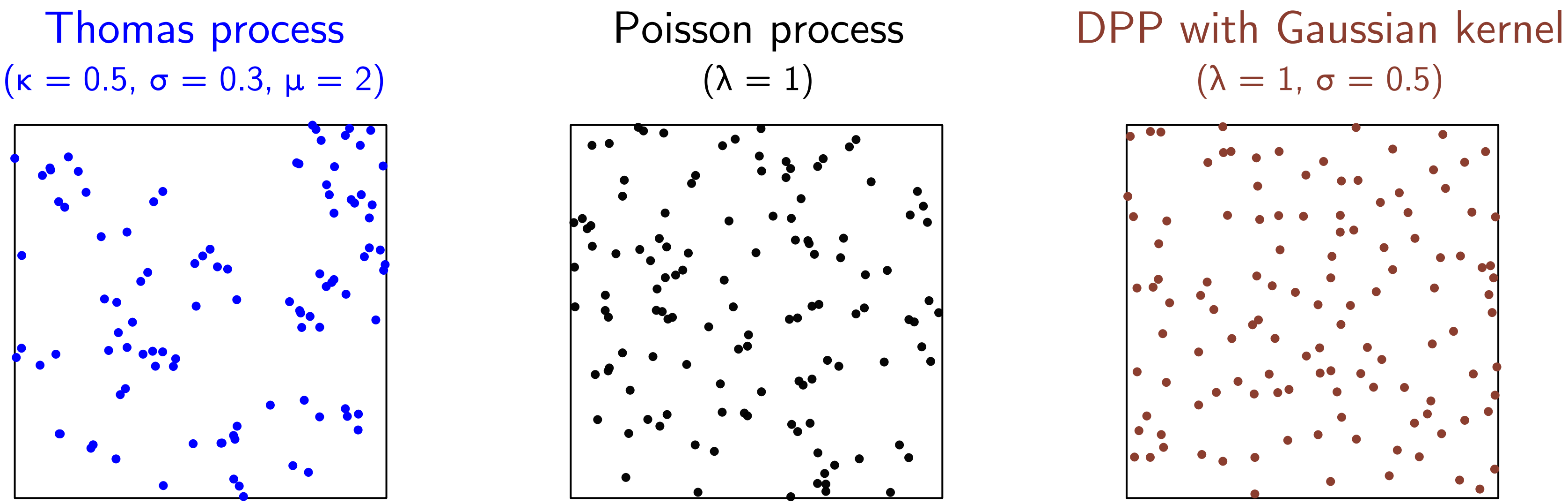
- $g(r) = 1$
- $g(r) = 1 + \dfrac{e^{-r^2/4\sigma^2}}{4\pi\kappa\sigma^2}$
- $g(r) = 1 - \left(e^{-r^2/\sigma^2}\right)^2$

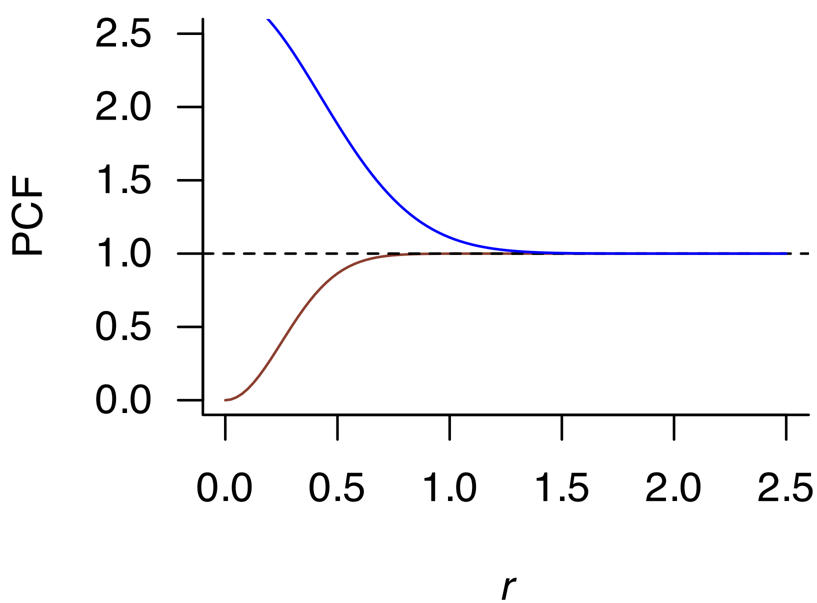
Spectral density

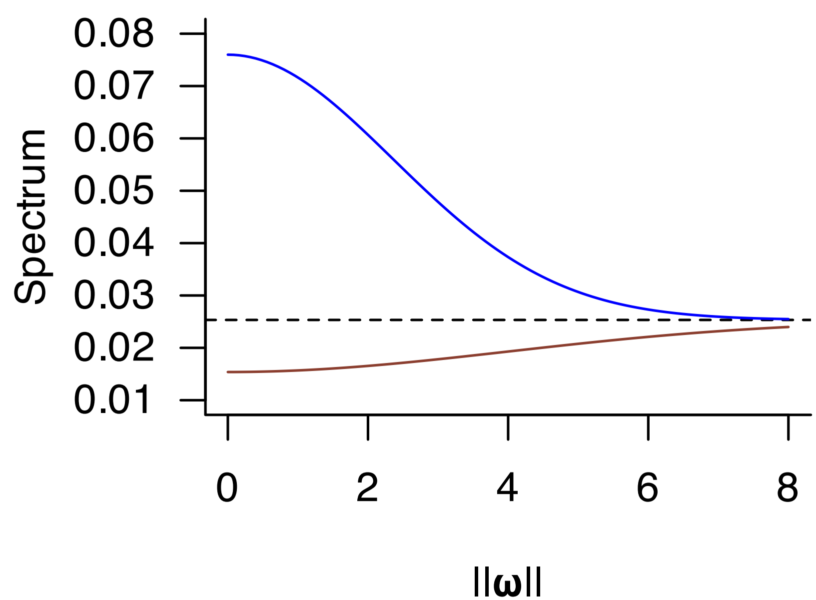
- $f(\boldsymbol{\omega}) = (2\pi)^{-d}\lambda$
- $f(\boldsymbol{\omega}) = (2\pi)^{-d}\kappa\mu\left[1 + \mu e^{-\sigma^2||\boldsymbol{\omega}||^2}\right]$
- $f(\boldsymbol{\omega}) = (2\pi)^{-d}\left[\lambda - \lambda^2\left(\dfrac{\pi\sigma^2}{2}\right)^{\frac{d}{2}} e^{\frac{-\sigma^2||\boldsymbol{\omega}||^2}{8}}\right]$

For stationary and isotropic case, $g(\mathbf{x}_i, \mathbf{x}_j) = g(||\mathbf{x}_i - \mathbf{x}_j||) = g(r)$.
Parameter: $\boldsymbol{\theta}_{\text{Thomas}} = (\kappa, \mu, \sigma^2)^T$, $\boldsymbol{\theta}_{\text{Determinantal}} = (\lambda, \sigma^2)^T$.

PCF and spectral density







Frequency domain parameter estimation

Whittle-type likelihood

Let $\{X_{\boldsymbol{\theta}}\}$ be a family of 2nd-order stationary point processes with parameter $\boldsymbol{\theta} \in \Theta$. The associated spectral density is denoted as $f_{\boldsymbol{\theta}}$. Then, we fit the model using the pseudo-likelihood

$$L(\boldsymbol{\theta}) = \sum_{\boldsymbol{\omega_k} \in D} \left\{ \frac{\hat{I}(\boldsymbol{\omega_k})}{f_{\boldsymbol{\theta}}(\boldsymbol{\omega_k})} + \log f_{\boldsymbol{\theta}}(\boldsymbol{\omega_k}) \right\}.$$

Intuition

- $\hat{I}(\boldsymbol{\omega})$ can be regarded as the **"truth"** since $\mathbb{E}[\hat{I}_n(\boldsymbol{\omega})] \to f(\boldsymbol{\omega})$.
- $f_{\boldsymbol{\theta}}(\boldsymbol{\omega})$ are the parameter family of spectral densities.
- $L(\boldsymbol{\theta})$ is the **spectral divergence between the truth and our guess**. ***The smaller the better!***

Proposed model parameter estimator

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta} \in \Theta} L(\boldsymbol{\theta})$$

Frequency domain parameter estimation

Model	Window	Parameter	Method	
			Ours	ML
TCP	$[-5, 5]^2$	κ	–0.04 (0.11)	–0.02 (0.11)
		μ	0.72 (3.52)	–0.24 (5.93)
		σ^2	0.02 (0.07)	–0.04 (0.22)
		Time (sec)	**0.74**	**0.38**
	$[-10, 10]^2$	κ	–0.02 (0.05)	–0.02 (0.05)
		μ	0.60 (1.77)	0.33 (2.79)
		σ^2	0.01 (0.02)	–0.01 (0.11)
		Time (sec)	**2.38**	**5.67**
	$[-20, 20]^2$	κ	–0.01 (0.04)	0.00 (0.03)
		μ	0.25 (1.03)	0.15 (1.23)
		σ^2	0.01 (0.02)	0.00 (0.03)
		Time (sec)	**9.15**	**173.66**

The bias and the standard errors (in parentheses) of the estimated parameters based on three different approaches for the Thomas clustered process (TCP)

Spectral analysis

Time series model

- White noise : $e_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$
- MA(1) : $X_t = e_t + \theta e_{t-1}$
- AR(1) : $X_t = \alpha X_{t-1} + e_t$

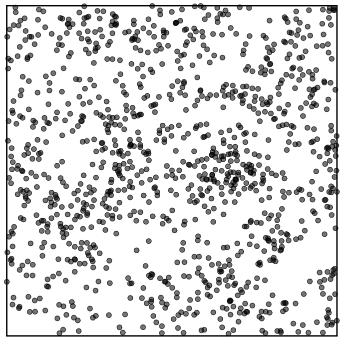
ACF

- $\gamma(k) = \sigma^2 \mathbb{1}\{k = 0\}$
- $\gamma(k) = \theta\sigma^2, k = \pm 1$
- $\gamma(k) = \alpha^{|k|}\sigma_X^2$

Spectrum

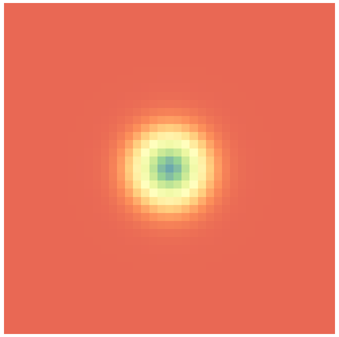
- $f(\omega) = \sigma^2/(2\pi)$
- $f(\omega) = \sigma^2(1 + \theta^2 + 2\theta\cos\omega)/(2\pi)$
- $f(\omega) = \dfrac{\sigma^2}{2\pi(1 - 2\alpha\cos\omega + \alpha^2)}$

2nd-order stationary point process



Spectrum

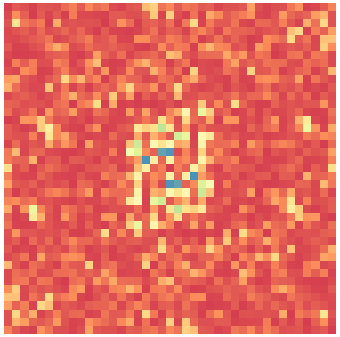
$f(\omega)$



Its estimator is

Periodogram

$\hat{I}(\omega)$



Spectral analysis

Time series model

- White noise : $e_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$
- MA(1) : $X_t = e_t + \theta e_{t-1}$
- AR(1) : $X_t = \alpha X_{t-1} + e_t$

ACF

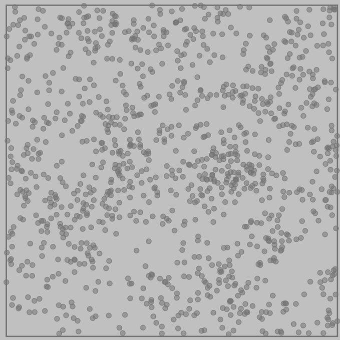
- $\gamma(k) = \sigma^2 \mathbb{1}\{k = 0\}$
- $\gamma(k) = \theta\sigma^2, k = \pm 1$
- $\gamma(k) = \alpha^{|k|}\sigma_X^2$

Spectrum

- $f(\omega) = \sigma^2/(2\pi)$
- $f(\omega) = \sigma^2(1 + \theta^2 + 2\theta\cos\omega)/(2\pi)$
- $f(\omega) = \dfrac{\sigma^2}{2\pi(1 - 2\alpha\cos\omega + \alpha^2)}$

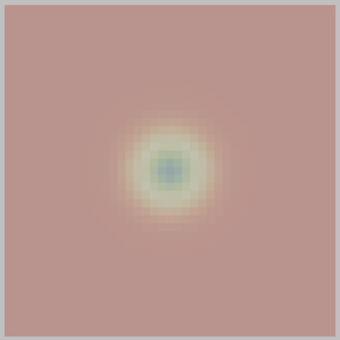
Does NOT work for inhomogeneous point processes!

2nd-order stationary point process



Spectrum

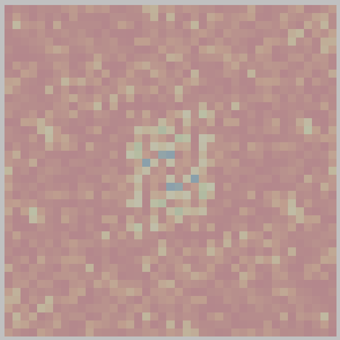
$f(\omega)$



Its estimator is

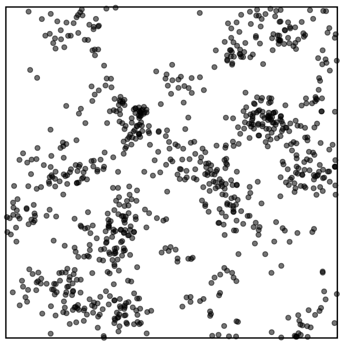
Periodogram

$\hat{I}(\omega)$



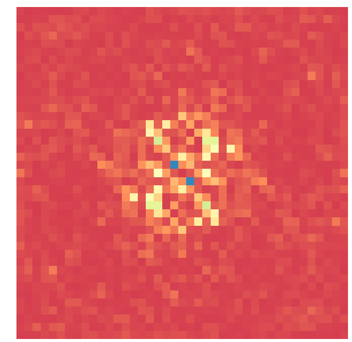
Spectral analysis under inhomogeneity

Inhomogeneous
point process

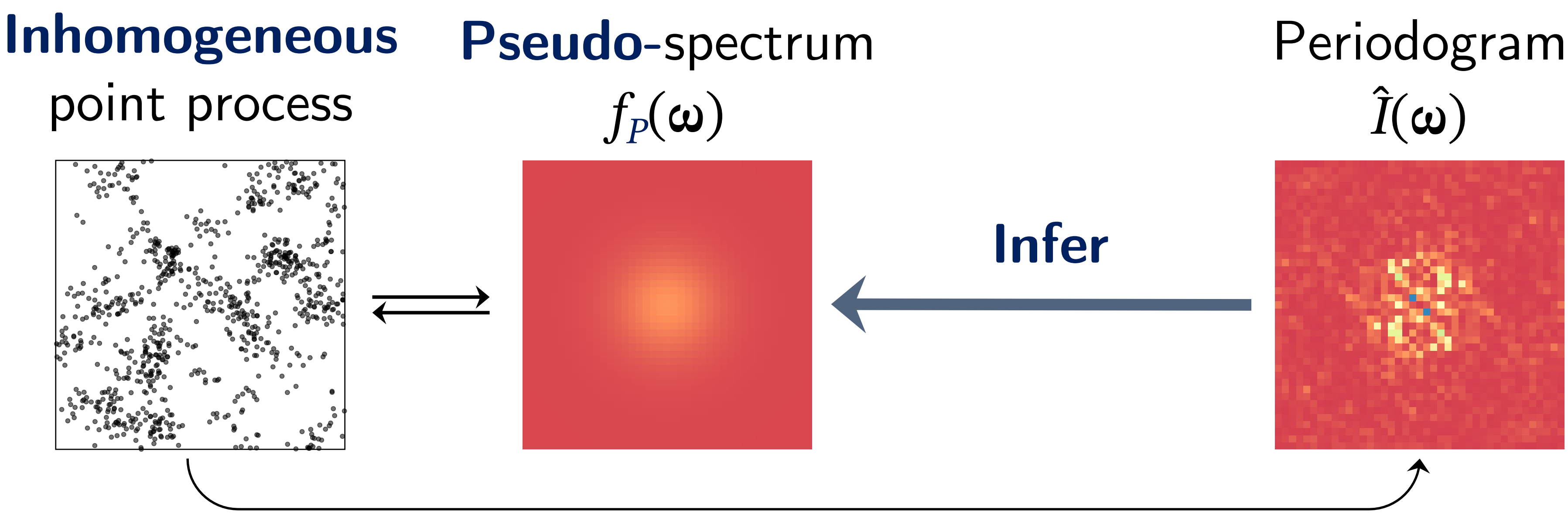


???

Periodogram
$\hat{I}(\omega)$



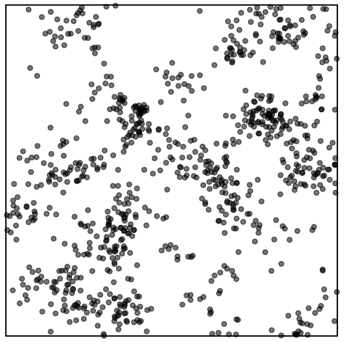
Spectral analysis under inhomogeneity



$$f_P(\boldsymbol{\omega}) := \lim_{n\to\infty} \mathbb{E}[\hat{I}_n(\boldsymbol{\omega})], \quad \forall \boldsymbol{\omega} \in \mathbb{R}^d.$$

Estimating the pseudo-spectrum

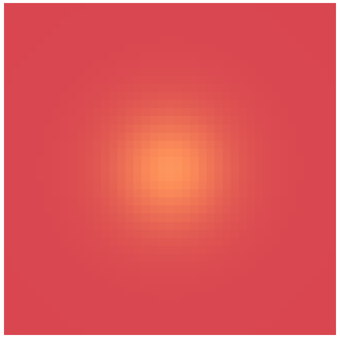
Inhomogeneous point process



$\rightleftarrows$

Pseudo-spectrum

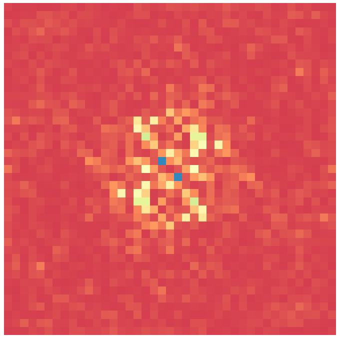
$f_P(\omega)$



Estimators:

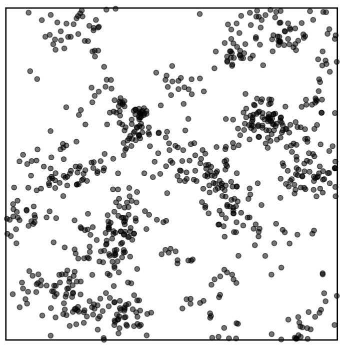
① Periodogram

$\hat{I}(\omega)$

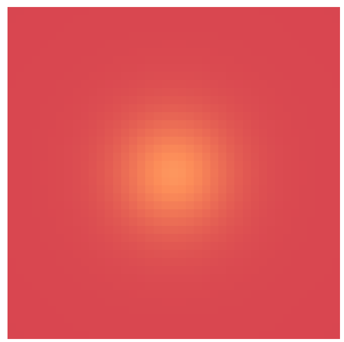


Estimating the pseudo-spectrum

Inhomogeneous point process

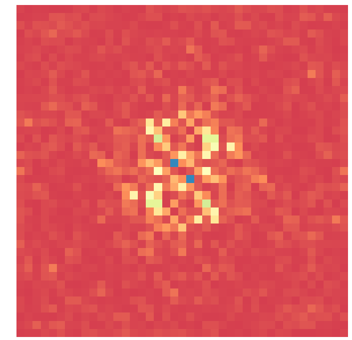


Pseudo-spectrum $f_P(\omega)$

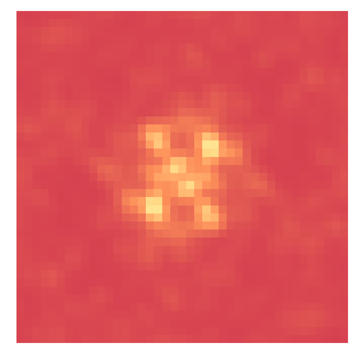


Estimators:

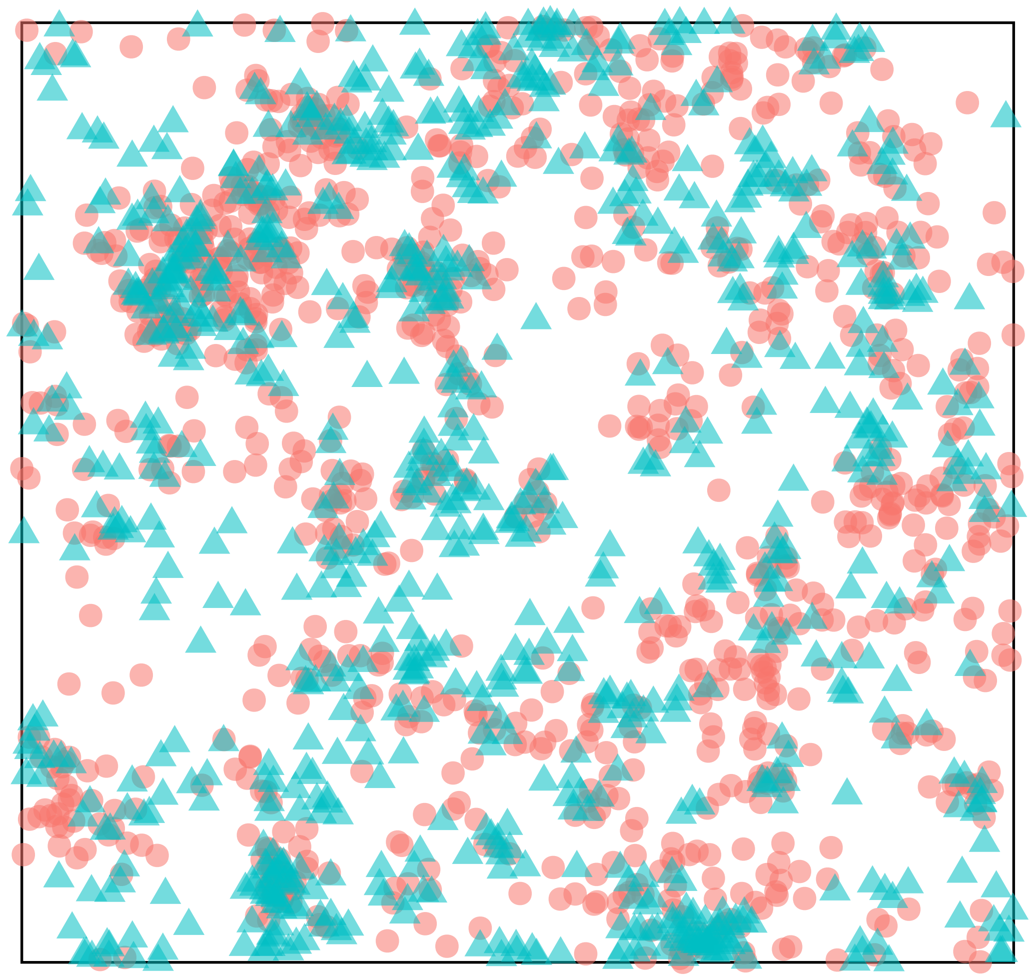
① Periodogram $\hat{I}(\omega)$



② Kernel estimator $\hat{f}_P(\omega)$



Simulation: Data generating process



Bivariate point process with marginal and joint clustering interactions

Simulation: Result

Pseudo-spectrum	Window	$\hat{I}(\omega)$		$\hat{f}_P(\omega)$	
		Bias	MSE	Bias	MSE
Marginal	$[-5, 5]^2$	0.03	1.49	0.02	0.49
	$[-10, 10]^2$	0.01	1.09	0.00	0.17
	$[-20, 20]^2$	0.00	1.02	0.00	0.13
Cross	$[-5, 5]^2$	0.06	28.14	0.01	3.76
	$[-10, 10]^2$	0.05	22.61	0.01	2.45
	$[-20, 20]^2$	0.05	20.18	0.01	2.11

Barro Colorado Island (BCI) data

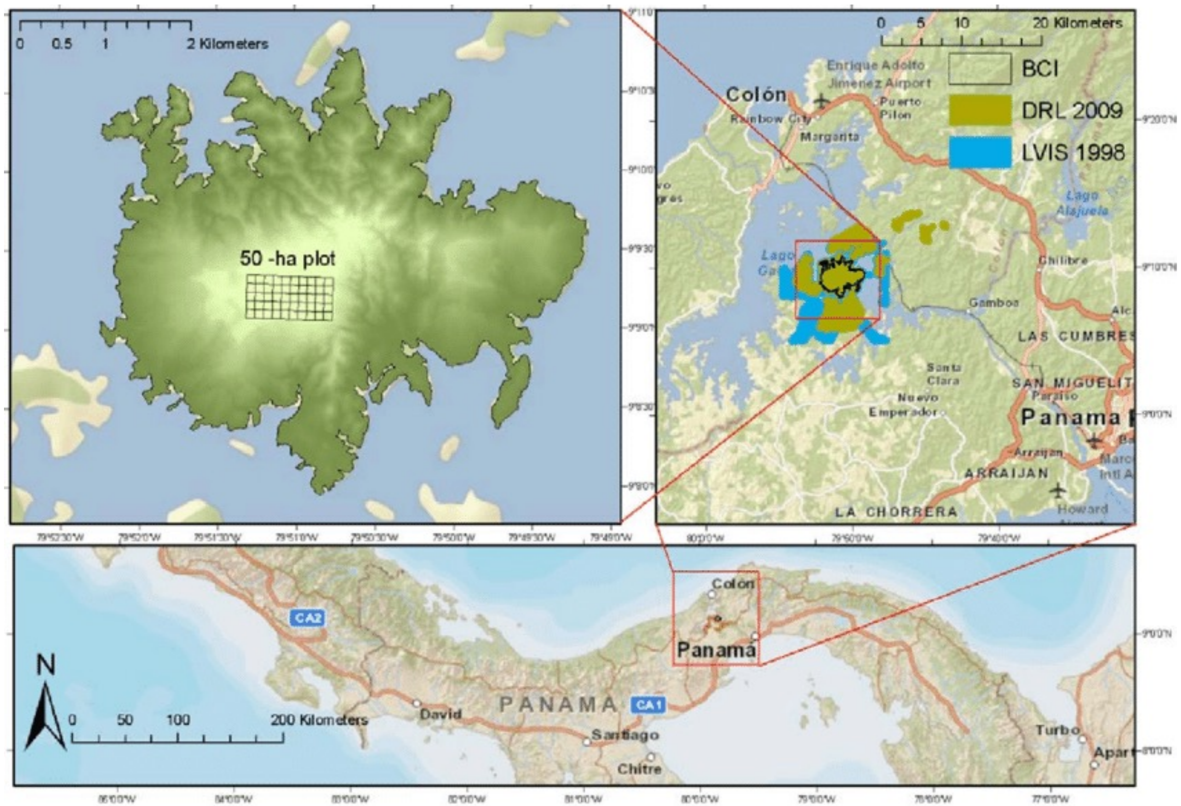


Figure 1 in Fricker et al. (2012)

207,718 alive trees

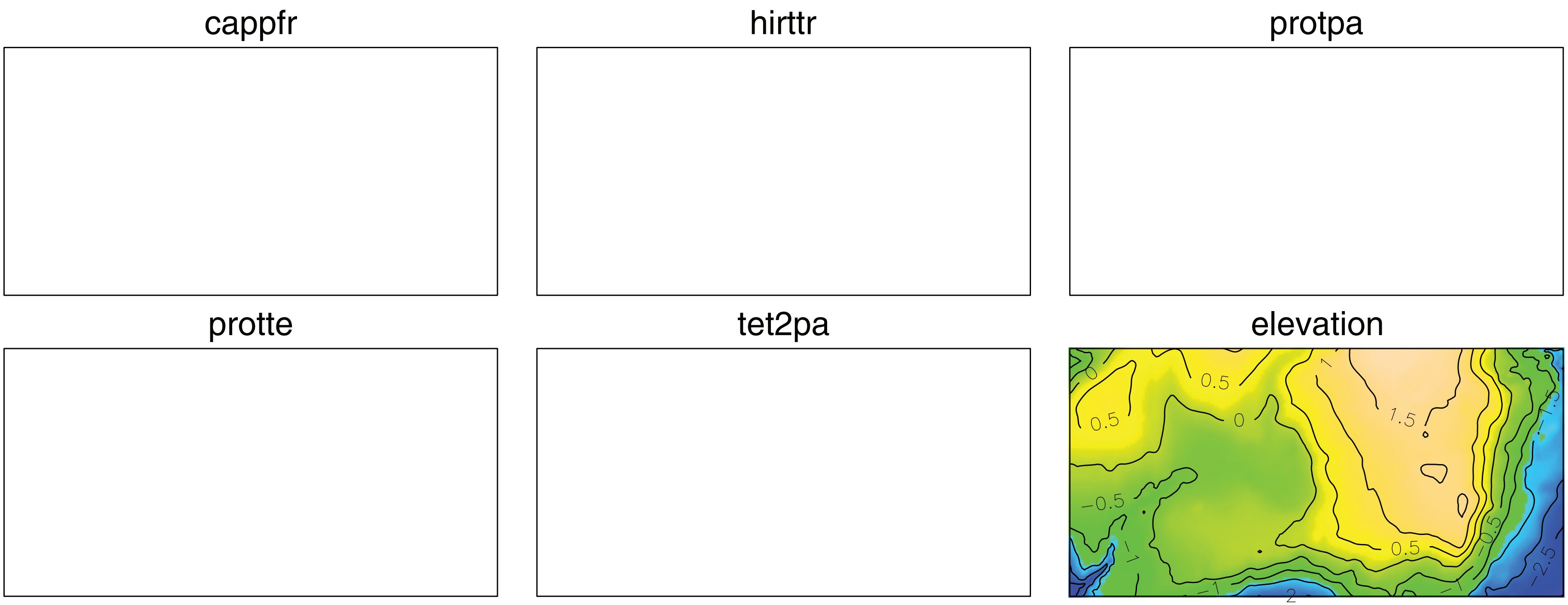
310 species

7 censuses

BCI Plot Census
November 26, 2015 by Forest Global Earth Observatory (ForestGEO).

https://datadryad.org/stash/dataset/doi:10.15146/5xcp-0d46

Barro Colorado Island (BCI) data

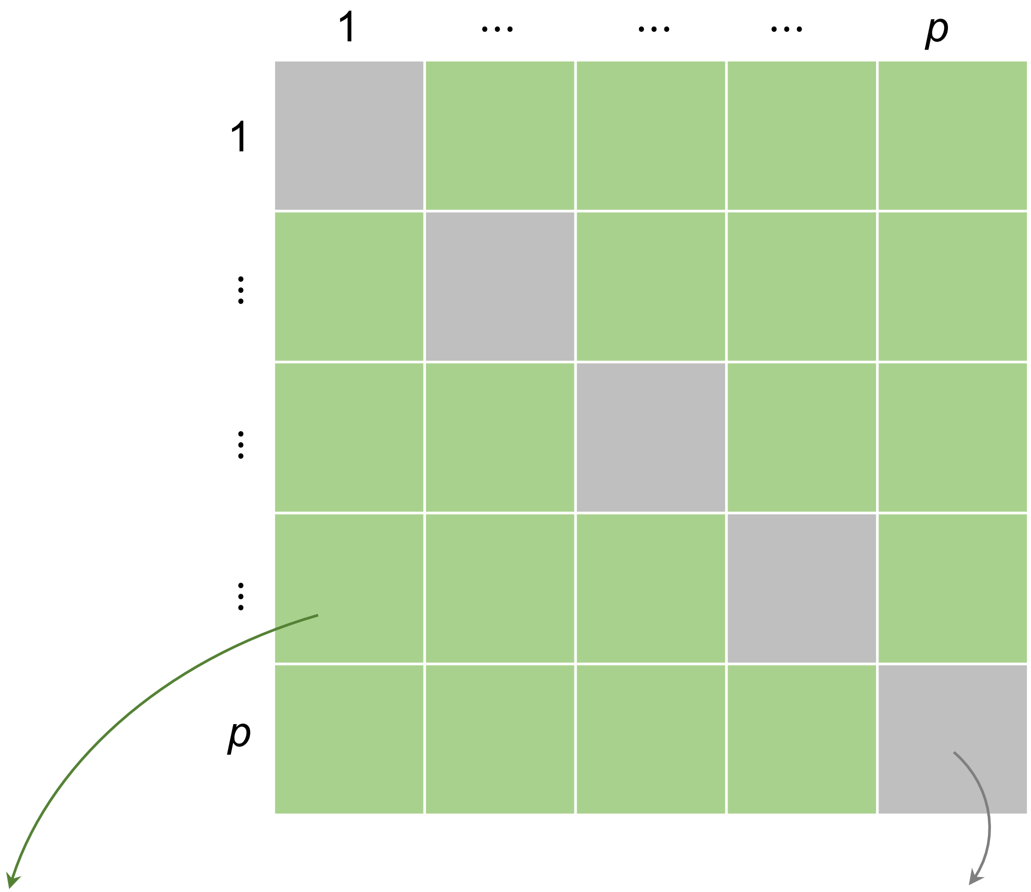


Point patterns of five species in the BCI dataset and image of elevation in the study region.

Figure from Jalilian et al. (2015)

Multivariate point pattern

Consider a p-variate process: $i, j \in \mathcal{V} = \{1, 2, \ldots, p\}$



Spectral coherence:

$$R_{ij}(\boldsymbol{\omega}) = \frac{f_{ij}(\boldsymbol{\omega})}{[f_{ii}(\boldsymbol{\omega}) f_{jj}(\boldsymbol{\omega})]^{\frac{1}{2}}}$$

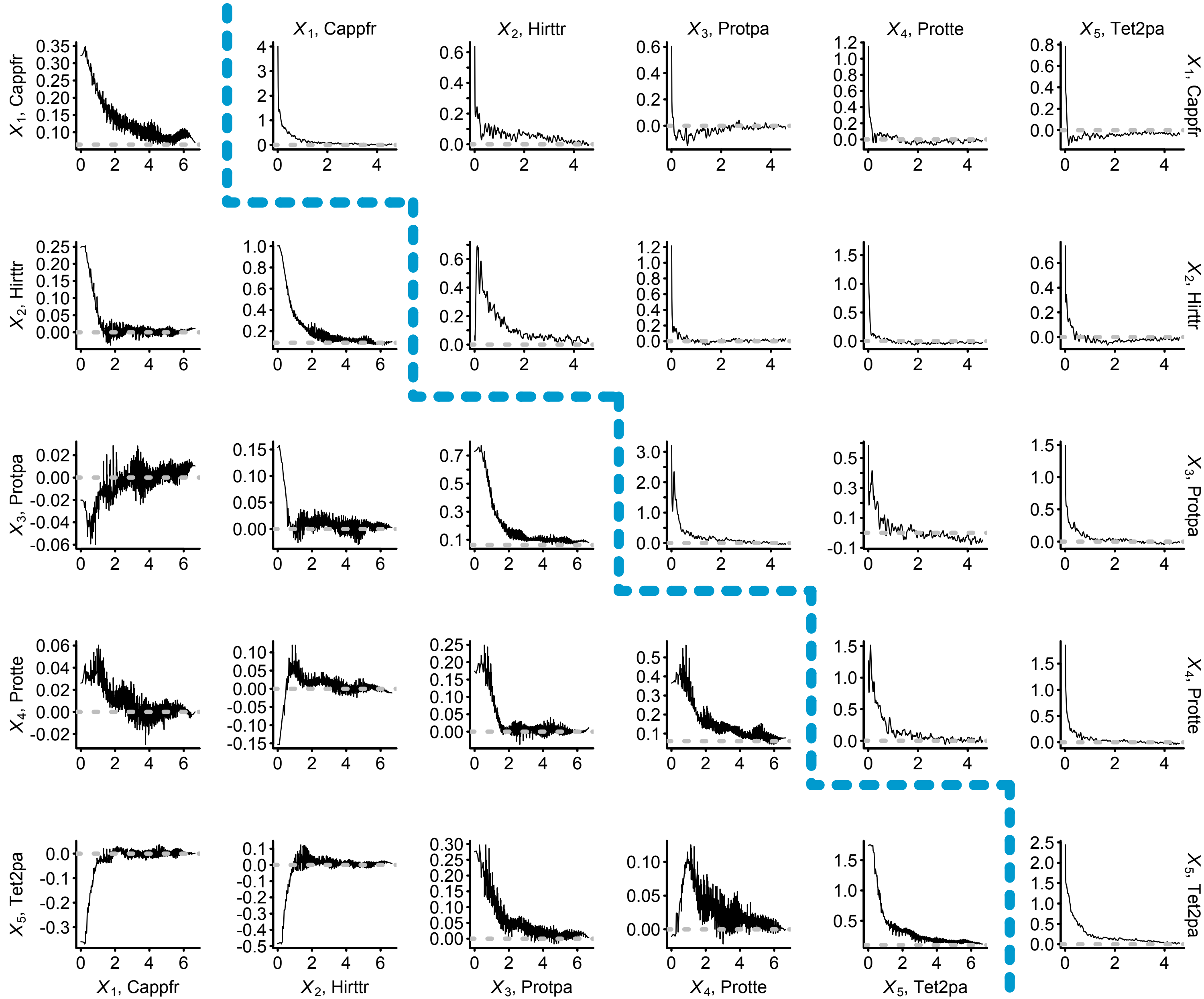
Cross-spectrum:

$$f_{ij}(\boldsymbol{\omega}) = (2\pi)^{-d} \int_{\mathbb{R}^d} C_{ij}(\boldsymbol{x}_1 - \boldsymbol{x}_2) e^{-i\boldsymbol{x}^\intercal \boldsymbol{\omega}} \, \mathrm{d}\boldsymbol{x}$$

Marginal spectrum:

$$f_{ii}(\boldsymbol{\omega}) = (2\pi)^{-d} \int_{\mathbb{R}^d} C_{ii}(\boldsymbol{x}_1 - \boldsymbol{x}_2) e^{-i\boldsymbol{x}^\intercal \boldsymbol{\omega}} \, \mathrm{d}\boldsymbol{x}$$

Pseudo-spectra



Coherence analysis