First-order logic

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Definition of Knowledge Representation

- Webster's dictionary
  - The fact or condition of knowing something with familiarity gained through experience or association.
- Artificial intelligence
  - first applied to formal languages and mathematical theorem proving, and formal logic.
- Knowledge representation is not a one-size-fits-all proposition.
- The computational objects, relationships, and inferences available to programmers are determined by the knowledge representation language they select.

Kinds of Knowledge

- Simple facts
- Complex relationships
- Mathematical formulas
- Rules
- Association between related concepts
- Inheritance hierarchies between classes of objects

First-order Logic (1)

- Propositional logic is not very useful in AI
  - Limited ability to represent real-world knowledge
- AI use predicate logic (first order logic, predicate calculus)
  - All the same concepts and rules of propositional logic
  - Symbolic representation
  - Use variable and function of variable in symbol statement
  - Quantifier
    - $\forall X$: for all X (for each X), universal quantifier
    - $\exists X$: there is an X (there exist an X), existential quantifier
    - $\forall x\ \exists y\ (P(x) \wedge t(x, y) \wedge r(x))$

First-order Logic (2)

- Natural language is ambiguous
  - "Everybody likes somebody."
    - For everybody, there is somebody they like,
    - or, there is somebody (a popular person) whom everyone likes?
  - "Somebody likes everybody."
    - Same problem: Depends on context, emphasis.

First-order Logic (3)

- Universes x1, y1, z1, … are objects.
  - x, y are variables
- Predicates P, Q, R, …(T or F)
  - P(x): mapping objects x to propositions (true/false)
  - P(x, y): Multi-argument predicates
- Functions
  - Mapping to another object
  - F(x)=y
- Quantifiers
  - $[\forall x\ P(x)]$:≡ "For all x's, $P(x)$."
  - $[\exists x\ P(x)]$:≡ "There is an x such that $P(x)$."

First-order Logic (4)

- Quantifier Equivalence Laws (DeMorgan's Law)
  - Definitions of quantifiers: If u.d.=a,b,c,…

    $\forall x\, P(x) \Leftrightarrow P(\text{a}) \wedge P(\text{b}) \wedge P(\text{c}) \wedge \ldots$
    $\exists x\, P(x) \Leftrightarrow P(\text{a}) \vee P(\text{b}) \vee P(\text{c}) \vee \ldots$
  - From those, we can prove the laws:
    $\forall x\, P(x) \Leftrightarrow \neg\exists x\, \neg P(x)$
    $\exists x\, P(x) \Leftrightarrow \neg\forall x\, \neg P(x)$
  - $\forall x\, \forall y\, P(x,y) \Leftrightarrow \forall y\, \forall x\, P(x,y)$

    $\exists x\, \exists y\, P(x,y) \Leftrightarrow \exists y\, \exists x\, P(x,y)$
    - And $\forall x\, (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\, P(x)) \wedge (\forall x\, Q(x))$
    - **Or** $\exists x\, (P(x) \vee Q(x)) \Leftrightarrow (\exists x\, P(x)) \vee (\exists x\, Q(x))$

First-order Logic (5)

- Notational Conventions
  - Quantifiers bind as loosely as needed (parenthesis)
    - $\forall x \ (P(x) \wedge \mathrm{Q}(x))$ 括號比較清楚
  - Consecutive quantifiers of the same type can be combined:
    - $\forall x \ \forall y \ \forall z \ P(x,y,z) \Leftrightarrow \forall x,y,z \ P(x,y,z)$
  - All quantified expressions can be reduced to the canonical *alternating* form
    - $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \ldots P(x_1, x_2, x_3, x_4, \ldots)$

First-order Logic (6)

- **Writing First Order Logic**
  - Nobody loves Jane

    $\forall x. \neg \text{loves}(x, \text{Jane})$

    $\neg\exists x. \text{loves}(x, \text{Jane})$
  - Everybody has a father

    $\forall x. \exists y. \text{father}(y, x)$
  - Everybody has a father and a mother

    $\forall x. \exists yz. \text{father}(y,x) \wedge \text{mother}(z,x)$
  - Whoever has a father, has a mother

    $$\forall x.[[\exists y. \text{father}(y,x)] \Rightarrow [\exists y. \text{mother}(y,x)]]$$

First-order Logic (7)

- **Writing First Order Logic**
  - Cats are mammals

    $\forall$ x. cat (x) $\Rightarrow$ mammal (x)
  - Jane is a tall surveyor (測量員)

    Tall (Jane) $\wedge$ surveyor (Jane)
  - A nephew is a sibling's (兄弟姊妹) son

    $\forall$x y. [nephew (x, y) $\Leftrightarrow\exists$z . [sibling (y, z) $\wedge$ son (x, z)]]
  - A maternal (母親那邊的) grandmother [functions: mgm, mother-of]

    $\forall$x y. x=mgm (y) $\Leftrightarrow\exists$z. x=mother-of (z) $\wedge$ z=mother-of (y)

First-order Logic (8)

- Some Number Theory Examples
  - *Let universal is numbers* 0, 1, 2, …
  - "A number x is *even*, $E(x)$, iff it is equal to 2 times some other number."
    $\forall x\ (E(x) \Leftrightarrow (\exists y\ (x=2y)))$
  - "A number is *prime*, $P(x)$, iff it's greater than 1 and it isn't the product of two non-unity numbers."
    $$\forall x \left(P(x) \Leftrightarrow \left(x>1 \wedge \neg\exists y,z\ \ x=yz \wedge y\neq 1 \wedge z\neq 1\right)\right)$$

First-order Logic (9)

- Goldbach's Conjecture (unproven)
  - "Every even number greater than 2 is the sum of two primes."
  - Using $E(x)$ and $P(x)$ from previous slide,
  - $\forall E(x{>}2){:}\ \exists P(p),P(q){:}\ p+q = x$
  - more explicit notation:
  - $\forall x\ [x{>}2 \wedge E(x)] \Rightarrow \exists p\ \exists q\ P(p) \wedge P(q) \wedge p+q = x.$
- Precisely defining the calculus concept of a *limit*, using quantifiers:

$$\lim_{x\to a} f(x) = L \Leftrightarrow (\forall\varepsilon{>}0{:}\exists\delta{>}0{:}\ \forall x(|x-a|<\delta\to(|f(x)-L|<\varepsilon)$$

First-order Logic (10)

- Natural language is ambiguous
  - For everybody, there is somebody they like,
    - $\forall x\ \exists y\ Likes(x,y)$
  - or, there is somebody (a popular person) whom everyone likes?
    - $\exists y\ \forall x\ Likes(x,y)$

First-order Logic (11)

- (relation) If R(x,y)="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\ R(x,y))=$ Everyone has *someone* to rely on.

$\exists y(\forall x\ R(x,y))=$ There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\ R(x,y))=$ There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\ R(x,y))=$ Everyone has *someone* who relies upon them.

$\forall x(\forall y\ R(x,y))=$ *Everyone* relies upon *everybody*, (including themselves)!

Substitution

- Given a variable *x*, a term *t* and a formula *P*
  - P [t/x]: be the formula obtained by replacing each free occurrence of variable *x* in *P* with *t*.
- Model for predicate logic consists of
  - a non-empty domain/universe of objects
  - a mapping, called an interpretation that associates the terms of the syntax with objects in a domain/universe

Interpretation (1)

- Interpretation I
  - U: domain of discourse, universe
  - Maps constant symbols to elements of U
  - Maps predicate symbols to relations on U (binary relation is a set of pairs)
  - Maps function symbols to functions on U
- Denotation of terms (naming)
  - I (Fred) if Fred is constant, then given
  - I (x) undefined

Interpretation (2)

- Syntax has the constant c, function f (unary), and two predicates P, and Q (both binary).
- The model,
  - choose the natural numbers domain
  - I (c) is 0
  - I (f) is suc, the successor function
  - I (P) is $<$
- The meaning of P (c, f (c)) in this model?

$$I\ (P(c, f(c))) = I\ (c) < I\ (f(c))$$
$$= 0 < suc(I(c)) = 0 < suc(0)$$
$$= 0 < 1$$

Which is true.

Interpretation (3)

- $\models_I P(t_1, .., t_n)$ iff $<I(t_1), .., I(t_n)> \in I(P)$
  - Brother(John, Joe)?
    - I(John) = [an element of U]
    - I(Joe) = [an element of U]
    - I(brother) = {<John, Joe>, <Tome, Kan>, ….}
    - $\models_I$ brother(John, Joe)

Interpretation (4)

- Example
  - Facts and rules
    - D. U ={□, ○, ◆, ▯}, Constants: Fred
    - Predicate: above, rectangle, square, circle
    - Function: Hat
    - I(Fred) = {◆}
    - I(above) ={<□, ○>,< ◆, ▯>}, I(square)={□}, I(rectangle)={□, ▯} I(circle) = {○}, I{oval} = {□, ○}
    - I(Hat)={<◆, □>,<○, ▯>}
  - Implies
    - $\models_{I}$ square(Fred) => square(◆)=> false
    - $\models_{I}$ above(Fred, Hat(Fred))=> above(◆, Hat(◆))=above (◆, □) => false
    - $\models_{I}$ ∃x. square(x) => $|\!=_{Ix/□}$ square(x) => square(□) => true (prove)

Interpretation (5)

- Exercise
  - For each of the following sentences, determine whether it is true or false in the interpretation I
  - $\forall x\ above(x, Fred)$
  - $\forall x\ above(x, Hat(x))$
  - $\forall x\ oval(x) \Rightarrow \exists y\ above(y,x)$
  - $Square(Hat(Hat(Fred)))$
  - $\forall x\ above(x, Fred) \Rightarrow square(x)$
  - $\exists x\ \forall\ y\ circle(y) \Rightarrow above(y, x)$

Valuations

- Definition. A valuation v, in an interpretation I, is a function from the terms to the domain
  - $D.U.$ is the set of Natural Numbers
  - g is the function +
  - h is the function suc
  - c (constant) is 3
  - y (variable) is 1

$$
\begin{aligned}
v(g(h(c), y)) &= v(h(c)) + v(y) \\
&= suc(v(c)) + 1 \\
&= suc(3) + 1 \\
&= 5
\end{aligned}
$$

Validity (Tautologies) (1)

- A predicate logic formula is satisfiable if there are an interpretation and valuation that satisfies the formula (i.e., in which the formula returns T).
- A predicate logic formula is logically valid (tautology) if it is true in every interpretation.
  - It must be satisfied by every valuation in every interpretation.
- A wff of predicate logic is a contradiction if it is false in every interpretation.
  - It must be false in every valuation in every interpretation.

Validity (Tautologies) (2)

- Free occurrence of a variable.
  - Any occurrence of an individual variable not within the scope of a quantifier on the same letter
- A wff is closed if it contains no free occurrences of any variable.
- A closed predicate logic formula, is satisfiable if there is an interpretation I in which the formula returns T.
- A closed predicate logic formula, A, is a tautology if it is T in every interpretation.
- A closed predicate logic formula is a contradiction if it is F in every interpretation.

Semantic Entailment

- Semantic entailment has the same meaning as propositional logic.

$$(\phi_1, \phi_2, \phi_3, \Psi)$$

means:

if $v(\phi_1)=T$ and $v(\phi_2)=T$ and $v(\phi_3)=T$ then $v(\Psi)=T$,

equivalent

$\phi_1 \wedge \phi_2 \wedge \phi_3 \Rightarrow \Psi$ is a tautology,

$$(\phi_1, \phi_2, \phi_3, \Psi) \equiv ((\phi_1 \wedge \phi_2 \wedge \phi_3) \Rightarrow \Psi)$$

Hilbert System

- An extension of the axiomatic system for propositional logic called **Axiom System (AS)** of First order language.
  - Consist any wff belonging to any one of the following six schemas:

(1) $P \Rightarrow (Q \Rightarrow P)$ (Repetition, Rep, Ax1)

(2) $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$ *(conditional distribution, ⇒Dist, Ax2)*

(3) $\neg\neg P \Rightarrow P$, $[(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)]$ *(contraposition, Contra, Ax3)*

(4) $\forall x\, P(x) \Rightarrow P[a/x]$, where $x \in V$, $a \in V(\Sigma)$ and a is free for x in P
(universal elimination, ∀O, Ax4)

(5) $\forall x(A \Rightarrow B) \Rightarrow (\forall x A \Rightarrow \forall x B)$,
$[\forall x(A \Rightarrow B(x)) \Rightarrow (A \Rightarrow \forall x\, B(x))$ *where* $x \in V$ *is not free in A]*
(universal distribution, ∀Dist, Ax5)

(6) $A \Rightarrow \forall x A$ (universal generalization, trivial quantification, TrivQ) (Gen, Ax6)

(7) $A, A \Rightarrow B \vdash B$ (modus ponens, MP)

Example

- Prove $\forall x\ \forall y\ A \vdash_{AS} \forall y\ \forall x\ A$

1	$\forall x.\forall y.A$	premise
2	$\forall x.\forall y.A \Rightarrow \forall y.A$	Ax4
3	$\forall y.A$	MP 1, 2
4	$\forall y.A \Rightarrow A$	Ax4
5	A	MP 3, 4
6	$\forall x.A$	Gen of 5
7	$\forall y.\forall x.A$	Gen of 6

Example

- Prove $\vdash A(a) \Rightarrow \exists x A(x)$ *[theorem 2]*
  - $1.\ \forall x \neg A(x) \Rightarrow \neg A(a)$ *Ax4*
  - $2.\ A(a) \Rightarrow \neg \forall x \neg A(x)$ *1 Ax3*
  - $3.\ A(a) \Rightarrow \exists x A(x)$ *3 Definition* $\exists$

Example

- Prove $\vdash \exists x \forall y A(x,y) \Rightarrow \forall y \exists x A(x,y)$.
  - 1. $A(a, b) \Rightarrow \exists x A(x,b)$ Theorem 2
  - 2. $\forall y A(a,y) \Rightarrow \forall y \exists x A(x,y)$ Gen 1
  - 3. $\neg \forall y \exists x A(x,y) \Rightarrow \neg \forall y A(a,y)$ 2 Ax3
  - 4. $\forall x(\neg \forall y \exists x A(x,y) \Rightarrow \neg \forall y A(x, y))$ Gen. 3
  - 5. $(\forall x(\neg \forall y \exists x A(x,y) \Rightarrow \neg \forall y A(x, y))) \Rightarrow (\neg \forall y \exists x A(x,y) \Rightarrow \forall x \neg \forall y A(x, y))$ Ax5
  - 6. $\neg \forall y \exists x A(x,y) \Rightarrow \forall x \neg \forall y A(x,y)$ MP 4, 5
  - 7. $\neg \forall x \neg \forall y A(x,y) \Rightarrow \forall y \exists x A(x,y)$ 6 Ax3
  - 8. $\exists x \forall y A(x,y) \Rightarrow \forall y \exists x A(x,y)$ Definition of $\exists$

Deduction Theorem

- **Theorem**
  - If $H \cup \{A\} \vdash_d$ B by a deduction containing no application of generalization to a variable that occurs free in *A*, then
  - $H \cup \{A\} \Rightarrow$ B
- **Corollary**
  - If A is closed and if $H \cup \{A\} \vdash_d$ B
  - then H $\vdash_d$ (A$\Rightarrow$ B)

Natural Deduction (1)

- Extend the set of rules we used for propositional logic with ones to handle quantifiers.

Universal Quantifi cation

forall-elimination

$$\frac{\forall x.P}{P[t/x]}\forall\mathrm{e}$$

forall-introduction

$$\frac{\left[\begin{array}{l} x_0 \\ \quad\vdots \\ \quad P[x_0/x] \end{array}\right.}{\forall x.P}\forall\mathrm{i}$$

x_0 must be arbitrary, meaning it doesn't appear outside the subproof. t must be free for x in P.

Natural Deduction (2)

- Existential Quantification
  - exists-introduction

$$\frac{P[\frac{t}{x}]}{\exists x.P}\ \exists\mathrm{i}$$

$$\frac{\exists x.P \quad \left[\begin{array}{l} x_0 \quad P[x_0/x] \quad \textit{assumption} \\ \quad \vdots \\ \quad Q \end{array}\right.}{Q}\ \exists e$$

  - x_0 必須是任意的，t 必須是free for x in P.
  - Informally
    - 若已知針對某些值，predicate是 True，則使用其中某一個任意值，可導出某一formula正確，則可推論此formula是正確的。

Natural Deduction (3)

- Show $\forall x.\ P(x) \Rightarrow Q(x),\ \forall x.\ P(x) \vdash_{ND} \forall x.\ Q(x)$

1		$\forall x.\ P(x) \Rightarrow Q(x)$	premise
2		$\forall x.\ P(x)$	premise
	3	x_0	
	4	$P(x_0) \Rightarrow Q(x_0)$	$\forall$e 1
	5	$P(x0)$	$\forall$e 2
	6	$Q(x0)$	$\Rightarrow$ e 4, 5
7		$\forall x.\ Q(x)$	$\forall$i 3 – 6

Exercise

- Show (Natural Deduction)
  - (1) $P(a), \forall x P(x) \Rightarrow \neg Q(x) \vdash \neg Q(x)$
  - (2) $\neg \forall x P(x) \vdash \exists x \neg P(x)$

Resolution Principle (1)

- Resolution
  - An algorithm for proving facts true or false by virtue of contradiction.
  - To prove a theorem X is true, to show that the negation of X is not true.
- Example
  - Assume: Tweety has feathers.
    - Feather (Tweety) - true
  - Assume: Everything that has feathers is a bird.
    - $\forall$ x Feather (x) $\Rightarrow$ Bird (x)
  - Prove: Tweety is a bird.
  - not bird (Tweety)   - Add a negation assumption

Resolution Principle (2)

- Resolution example
  - Definitions:
    - $H(x) :\equiv$ "x is human";
    - $M(x) :\equiv$ "x is mortal";
    - $G(x) :\equiv$ "x is a god"
  - Premises:
    - $\forall x\ H(x) \Rightarrow M(x)$ ("Humans are mortal")
    - $\forall x\ G(x) \Rightarrow \neg M(x)$ ("Gods are immortal").
  - Show that No human is a god.
    - $\neg \exists x\ (H(x) \wedge G(x))$

Resolution Principle (3)

- Resolution example derivation
  - $\forall x\ H(x) \Rightarrow M(x)$ and $\forall x\ G(x) \Rightarrow \neg\, M(x)$
  - $\forall x \neg\, M(x) \Rightarrow \neg\, H(x)$ **[Contra positive]**
  - $\forall x\ [G(x) \Rightarrow \neg\, M(x)] \wedge [\neg M(x) \Rightarrow \neg H(x)]$
  - $\forall x\ G(x) \Rightarrow \neg\, H(x)$ **[Transitivity of $\Rightarrow$]**
  - $\forall x \neg\, G(x) \vee \neg\, H(x)$ **[Definition of $\Rightarrow$]**
  - $\forall x \neg(G(x) \wedge H(x))$ **[DeMorgan's law]**
  - $\neg\, \exists x\ G(x) \wedge H(x)$ **[An equivalence law]**

Resolution Principle (4)

- $p(f(x)) \vdash p(x)$
  - $1.\ p(f(x)),\ \neg p(x)$
  - $2.\ p(x')\ [f(x) \leftarrow x']$
  - $3.\ \neg p(x')\ \ [x \leftarrow x']$
  - Nil [2, 3]

Resolution Principle (5)

- Prove that:
  - If Everybody loves somebody and Everybody loves every lover
  - then Everybody loves everybody.
- Formalize the assumptions and the goal conclusion
  - L(x; y) 'x loves y'
  - Everybody loves somebody: $\forall x\ \exists y L(x, y)$
  - Everybody loves every lover: $\forall x(\exists y L(x, y) \Rightarrow \forall z L(z, x))$
  - Everybody loves everybody: $\forall x\ \forall y\ L(x, y)$
- Transform the set
  - $\{\forall x\ \exists y L(x, y), \forall x(\exists y L(x, y) \Rightarrow \forall z L(z, x)), \neg (\forall x\ \forall y\ L(x, y))\}$
  - to clausal form: $\{L(x, f(x))\}, \{\neg L(x_1, y), L(z, x_1)\}, \{\neg L(a, b)\}$

Resolution Principle (6)

- Apply Res. repeatedly to the resulting set of clauses {L(x, f (x))}, {¬L(x1, y), L(z, x1)}, {¬L(a, b)}:
  - 1. Unify L(z, x1) and L(a, b) with MGU [b/x1, a/z] and resolve {¬L(b, y), L(a, b)} and {¬L(a, b)} to obtain {¬L(b, y)}.
  - 2. Unify L(x, f (x)) and L(b, y) with MGU [b/x, f(b)/y] and resolve {L(b, f (b))} and {¬L(b, f (b))} to obtain Nil.
  - **MGU = most general unifier**

Resolution Principle (7)

- Lines 1–7 contain a set of clauses. The resolution refutation in lines8–15
  - 1. $\{\neg p(x), q(x), r(x,f(x))\}$
  - 2. $\{\neg p(x), q(x), r'(f(x))\}$
  - 3. $\{p'(a)\}$
  - 4. $\{p(a)\}$
  - 5. $\{\neg r(a, y), p'(y)\}$
  - 6. $\{\neg p'(x), \neg q(x)\}$
  - 7. $\{\neg p'(x), \neg r'(x)\}$
  - 8. $\{\neg q(a)\}$ $x \leftarrow a$ 3, 6
  - 9. $\{q(a), r'(f(a))\}$ $x \leftarrow a$ 2, 4
  - 10. $\{r'(f(a))\}$ 8, 9
  - 11. $\{q(a), r(a, f(a))\}$ $x \leftarrow a$ 1, 4
  - 12. $\{r(a,f(a))\}$ 8, 11
  - 13. $\{p'(f(a))\}$ $y \leftarrow f(a)$ 5, 12
  - 14. $\{\neg r'(f(a))\}$ $x \leftarrow f(a)$ 7, 13
  - 15. $\{\}$ 10, 14

Proof Exercise 1

- Statement
  - $\forall x\, \exists y\, (P(x, y) \wedge Q(x, y))$
  - $\forall x\, (\exists y\, (P(x, y) \wedge \exists z\, Q(x, z)) \Rightarrow \exists w\, R(x, w))$
- Prove
  - $\forall x\, \exists w\, R(x, y)$

Proof Exercise 2

- Statement

  All people will die

  Socrate is a people

- Prove
  - Socrate will die
- First order logic

  $\forall x \; people(x) \Rightarrow will\text{-}die(x)$

  $people(Socrate)$

  $will\text{-}die(Socrate)$

Proof Exercise 3

- Statement

  A father is a people has son

  Johnson is a son of John

  John is the father of Johnson
- Prove
  - John is the father of Johnson
- First order logic

  $\exists x$ people(x) $\wedge$ son(y,x) $\Rightarrow$ father(x)

  son(Johnson, John)

  people(John)

  father(John, Johnson)

Proof Exercise 4

- Facts or Rules
  - Ken是男人
  - Ken是臺灣人
  - Ken生於西元1800年
  - 所有男人都難免一死
  - 沒有難免一死的生物活超過150年
  - 現在是西元1983年
- Problem
  - Ken現在還活著嗎？
- Solution