Propositional Calculus

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Contents

Proposition Logic - Set

- Set Construction
  - *{set : range | condition • operation }*
  - *{Signature | Predicate • Term}*
- Examples of Sets
  - Set of colours: {green, blue, yellow}
  - Empty set: { } or φ
  - Alternate even numbers:
    - $\{ x : N \mid x \bmod 2 = 0 \cdot 2 * x \} = \{0,4,8,12,16,20,\ldots\}$
  - Squares of multiples of 4 (excluding zero):
    - $\{ x : Z \mid (x \bmod 4 = 0) \wedge (x > 0) \cdot x * x \}= \{16,64,144,256,\ldots\}$

Definition of Proposition Logic

- Description 宣告語句 - Proposition Logic
  - Binary-valued (true or false) about the world
  - Usage: The values of some features cannot be sensed directly, their values can be inferred from the values of other features
- Difficult to represent
  - General laws, "all blue boxes are pushable"
  - Negative information, "block *A* is not on the floor" (without saying where block *A* is)
  - Uncertain information, "either block *A* is on block *B* or block *A* is on block *C*"
  - "Help!" or "What I say is all false."

Using Constraints on Feature

- Consider: a robot that is able to lift a block
  - if that block is liftable and the robot's battery power source is adequate
  - If both are satisfied, then when the robot tries to life a block it is holding, its arm moves.
- Formulation
  - x1 (BATTERY_OK)
  - x2 (LIFTABLE)
  - x3 (MOVES)
  - constraint in the language of the propositional calculus
    - BATTERY_OK ∧ LIFTABLE ∧ MOVES

The Formal Language (1)

- Elements
  - *Atoms*
    - Strings of characters begin true or false, for example, P, Q, R, …, P1, P2, ON_A_B, and so on.
  - *Connectives*
    - $\vee$ (disjunction/or), $\wedge$ (conjunction/ and), $\Rightarrow$ (implies), $\neg$ (negation), $\equiv$ (equivalence).
  - *Well-formed formula* (wff), also called sentences
    - Any atom is a wff.
    - If P and Q are wffs, so are $P \vee Q$, $P \wedge Q$, $P \Rightarrow Q$, $\neg P$.

The Formal Language (2)

- *Literal*
  - atoms and a $\neg$ sign in front of them, e.g. P, $\neg$P
- Clause
  - Disjunction of literal, e.g. $\neg P \vee Q \vee R$
- *Antecedent* and *Consequent*
  - In $P \Rightarrow Q$,
  - P is called the antecedent, premise, or hypothesis of the implication.
  - Q is called the *consequent, or conclusion* of the implication.
- Contrapositive
  - $\neg P \Rightarrow Q$ is the contrapositive $\neg Q \Rightarrow P$

Logic Semantics (1)

- The semantics of propositional logic are an interpretation (**true/false**) of any expression in propositional logic
- The semantics of propositional logic are also called a Boolean valuation.
- A Boolean valuation is a mapping *v* from the set of propositional formulas to the set {T/F}
  - $v\ (True) = T,\ v\ (False) = F$
  - If $v\ (P) = T,\ v\ (Q) = F,\ v\ (R) = F$
  - $v\ ((P \Rightarrow Q) \wedge R) = v\ (T \Rightarrow F) \wedge v\ (F) = F \wedge F = F$

Logic Semantics (2)

- Interpretation
  - An association of atoms with propositions
  - Each atom has *values* – *True* or *False*.
- Given the values of atoms under some interpretation
  - use a truth table to compute a value for any wff.

Definitions of Proof (1)

- Proof, $\vdash$
  - The sequence of wffs $\{w_1, w_2, \ldots, w_n\}$ is called a *proof* of P from a set of wffs ($\Delta = \{w_1, w_2, \ldots, w_n\}$)
    - iff each w_i is either in Δ or can be inferred from a wff
  - $\Delta \vdash P$
    - A proof procedure is a way to calculate the above
  - Proof procedures are algorithms that perform "mechanical manipulations on strings of symbols.

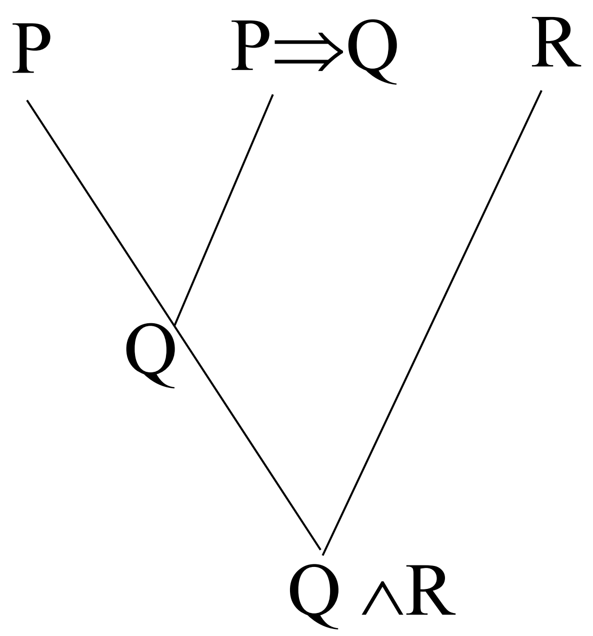
Δ	δ	delta	′dɛltə

Definitions of Proof (2)

- Theorem
  - $\Delta \vdash P$: If there is a proof of P from Δ, P is a theorem of the set Δ.
  - Denote the set of inference rules by the letter R.
    - P can be proved from Δ
    - $\Delta \vdash_R P$
- A theory is the set of theorems that can be proven by a proof procedure.

Proof Example

- Given a set, Δ, of wffs: $\{P, R, P \Rightarrow Q\}$ is a proof of $Q \wedge R$.



Propositional Truth Table (1)

- Let P and Q be wffs

P	Q	¬P	¬P∨Q	P⇒Q	P ∧ Q	P∨Q
T	T	F	T	T	T	T
T	F	F	F	F	F	T
F	T	T	T	T	F	T
F	F	T	T	T	F	F

An interpretation

Propositional Truth Table (2)

- If an model describes its world using n features (n atoms), then there are 2^n different value for its world.
- Given values for the n atoms, the model can be used the truth table to find the values of any wffs.
- Suppose the values of wffs in a set of wffs are given.
  - May be many interpretations that give each wff in a set of wffs the value *True* .

Equivalence (1)

- Two wffs are said to be *equivalent* iff their truth values are identical under *all* interpretations.
- DeMorgan's laws

  $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

  $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- Law of the contra positive

  $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$
- If $P \equiv Q$, then the following formula is valid

  $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Equivalence (2)

- Associative Laws

$$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$$

- Distributive Laws

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Equivalence (3)

- $P \wedge \neg P \equiv F$, $P \vee \neg P \equiv T$,
- $P \wedge P \equiv P$, $P \vee P \equiv P$,
- $P \wedge T \equiv P$, $P \wedge F \equiv F$,
- $P \vee T \equiv T$, $P \vee F \equiv P$,
- $P \wedge Q \equiv Q \wedge P$,
- $P \vee Q \equiv Q \vee P$,
- $P \equiv \neg\neg P$,
- $P \Rightarrow Q \equiv \neg P \vee Q$,
- $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Validity

- A wff is said to be *valid, or Tautologies*
  - It has value *True* under *all* interpretations of its constituent atoms.
    - for all possible truth values of the propositional letters used in the formula.
  - e.g.
    - $P \Rightarrow P$
    - $\neg (P \wedge \neg P)$
    - $Q \vee T$
    - $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$
    - $P \Rightarrow (Q \Rightarrow P)$

Satisfiability

- An interpretation *I* *satisfies* a wff *W*
  - if the *W* is assigned the *True* under that interpretation *I*.
  - there is a Boolean valuation *v*, $v(W) = T$
  - the *W* has a *satisfying assignment I*.
  - $I \models W$
- *Model, M*
  - An interpretation *M* that satisfies a wff *W*
  - $M \models W$
- *Inconsistent* or *Unsatisfiable*
  - When *no* interpretation satisfies a wff, the wff is inconsistent or unsatisfiable.
  - e.g. F or $P \wedge \neg P$

Entailment

- Entailment: $\Delta \models w$
  - a wff w, and a set of wffs $\Delta = \{w_1, w_2, \ldots, w_n\}$,
  - If $I \models \Delta$, then $I \models w$
  - Δ logically entails w,
  - w logically follows from Δ,
  - w is a logical consequence of Δ.
  - From premise $w_1, w_2, \ldots, w_n$, to conclude w
  - $(w_1, w_2, \ldots, w_n) \models w \equiv (w_1 \wedge w_2 \wedge \ldots \wedge w_n) \Rightarrow w$
- e.g.
  - $\{P\} \models P$
  - $\{P, P \Rightarrow Q\} \models Q$
  - $P \wedge Q \models P$

Metatheorems

- Theorems about the propositional calculus
- Deductive theorem

If $\{w_1, w_2, \ldots, w_n\} \models w$, then
$(w_1 \land w_2 \land \ldots \land w_n) \Rightarrow w$ is valid.

The PSAT Problem

- Propositional satisfiability (PSAT) problem
  - Finding a model for a formula. ($M \models W$)
  - *Clause:* A disjunction of literals
  - *Conjunctive Normal Form (CNF)*
    - A formula written as a conjunction of clauses
- Solving the CNF PSAT problem
  - to try all of the ways to assign True and False to the atoms in the formula.
  - If there are n atoms in the formula, there are 2^n different assignments.

Proof Procedures (1)

- Proof procedures for propositional logic
  - are alternate means to determine tautologies.
  - use the truth tables to determine tautologies.
- Forward proof
  - apply rules from premises to conclusions.
- Backward proof
  - apply rules from conclusions to premises.
- Many proof procedures for propositional logic.
  - Hilbert Systems (axiom systems)
  - Natural Deduction
  - Resolution

Proof Procedures (2)

- Prove $P \Rightarrow Q$ is true, ($P \Rightarrow Q \equiv \neg P \vee Q$)
  - trivial proof (逕證法): prove Q is true
  - vacuous proof (假前題法): prove $\neg P$ is true
  - direct proof (直接法): prove P, Q is true
  - indirect proof (間接法): prove $\{\neg Q \Rightarrow \neg P\}$

Hilbert Systems

- Also called axiom systems
  - use axioms and rules of inference (also called rules of derivation).
  - A proof is a finite sequence of formulas
    - each term is either an axiom or follows from earlier terms by one of the rules of inference.
  - Three axiom (schemes)
    - $P \Rightarrow (Q \Rightarrow P)$
    - $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))$
    - $(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)$
  - One rule of inference: (modus ponens) $\{P, P \Rightarrow Q\} \vdash Q$

Hilbert Systems

- $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H1} (A \Rightarrow C)$
  - 1. $(A \Rightarrow B)$ premise (1)
  - 2. $(B \Rightarrow C)$ premise (2)
  - 3. $(B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$ (2) and Ax-1 (3)
  - 4. $(A \Rightarrow (B \Rightarrow C))$ (2), (3), and MP (4)
  - 5. $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ (4) and Ax-2 (5)
  - 6. $(A \Rightarrow B) \Rightarrow (A \Rightarrow C)$ (4), (5), and MP (6)
  - 7. $(A \Rightarrow C)$ (1), (6), and MP (7)

Deduction Theorem

- In any Hilbert System
  - with at least Axiom Schemes 1 and 2, and
  - with Modus Ponens as the only rule of inference
  - $\{W, X\} \vdash Y \Leftrightarrow W \vdash (X \Rightarrow Y)$
- Show $(P \Rightarrow Q) \vdash P \Rightarrow (R \Rightarrow Q)$
  - Set out to show: $\{(P \Rightarrow Q), P\} \vdash (R \Rightarrow Q)$
  - (1) P premise
  - (2) $(P \Rightarrow Q)$ premise
  - (3) Q MP on 1 and 2
  - (4) $Q \Rightarrow (R \Rightarrow Q)$ Ax1
  - (5) $R \Rightarrow Q$ MP on 3 and 4
- Now we have proven $\{(P \Rightarrow Q), P\} \vdash (R \Rightarrow Q)$, using the deduction theorem, to conclude: $(P \Rightarrow Q) \vdash P \Rightarrow (R \Rightarrow Q)$

Natural Deduction (1)

- Natural deduction
  - is a form of forward proof.
  - is a collection of proof rules,
  - to infer formulas from other formulas,
  - eventually to get from a set of premises to a conclusion.
  - $p_1, p_2, p_3, \ldots \vdash q$
  - the argument with premises $p_1, p_2, p_3, \ldots$ and conclusion q is valid.
  - An inference rule is a primitive valid argument form.
  - Each inference rule enables the elimination or the introduction of a logical connective

Natural Deduction (2)

- Show: $P \wedge Q, R \vdash Q \wedge R$
  - (1) $P \wedge Q$ premise
  - (2) R premise
  - (3) Q elimination (1)
  - (4) $Q \wedge R$ introduction (2) (3)

Rule of Inference (1)

- Additional wffs can be produced from other ones (wffs: P and Q)
  - Modus ponens (deductive method, 演繹法)
    - $\{P, P \Rightarrow Q\} \vdash Q$
  - Modus tollens
    - $\{P \Rightarrow Q, \neg Q\} \vdash \neg P$
  - Introduction
    - $\{P, Q\} \vdash P \wedge Q$
  - Commutativity
    - $\{Q \wedge P\} \vdash P \wedge Q$
  - Elimination
    - $\{P \wedge Q\} \vdash P$, $\{P \wedge Q\} \vdash Q$

Rule of Inference (2)

- $\{P\} \vdash P \vee Q$
- $\{Q\} \vdash P \vee Q$
- $\{\neg Q, P \Rightarrow Q\} \vdash \neg P$
- $\{\neg P, P \vee Q\} \vdash Q$
- $\{P \Rightarrow Q, Q \Rightarrow R\} \vdash P \Rightarrow R$
- $\{P \Rightarrow Q, R \Rightarrow S\} \vdash P \wedge R \Rightarrow Q \wedge S$
- $\{P \vee Q, \neg P \vee Q\} \vdash Q$

Resolution Principle (1)

- A powerful inference rule, (Theorem Proving)
  - Prove by Resolution refutation $\Delta \vdash \omega$
  - Instead, we refute $\Delta \wedge \neg\omega$
    - Clause: $(P1 \vee P2 \vee \ldots \vee Pn)$
  - Steps
    - (1) Convert wff to set of clauses
    - (2) Convert $\neg\omega$ to a clause.
    - (3) Translate to CNF (Conjunctive Normal Form)
    - (4) iterative apply resolution principle to the clauses, and add result until
      - No more resolvents can be add
      - An empty clause (false) is produced

Resolution Principle (2)

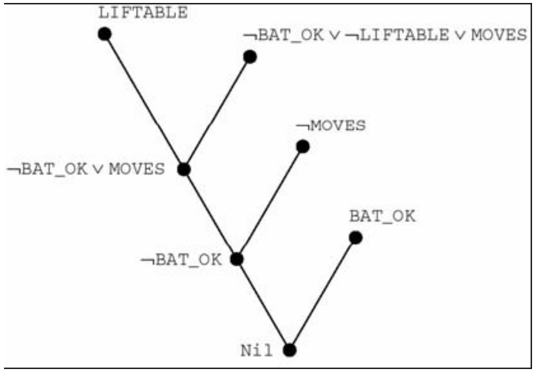
- (1) Convert wff to set of clauses
  - Delete $\Rightarrow$, $\Leftrightarrow$
    - $P \Rightarrow Q \equiv \neg P \vee Q$,
    - $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
  - Delete $\neg$
    - $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
    - $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
    - $\neg \neg P \equiv P$
- (2) Transfer wff into CNF
  - Put $\vee$ inside clause, put $\wedge$ outside clause
  - $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  - $(P \vee \neg R) \wedge (P \vee Q) \Rightarrow \{(P \vee \neg R), (P \vee Q)\}$

Resolution Principle (3)

- Resolution principles
  - 任找恰好分別各有正負literal的二個clauses，正負literals 相互抵消，合併其餘literals
    - $(P \vee Q) \wedge (R \vee \neg Q) \vdash (P \vee R)$: chaining
    - $R, \neg R \vee P \vdash P$: modus ponens
    - Chaining and modus ponens are special cases of resolution principle.
      - $P \vee Q \vee R \vee S, \neg P \vee Q \vee W \vdash Q \vee R \vee S \vee W$
  - 如發現二個單一literal的clauses，恰分別具正負號，即已產生矛盾現象(refutation)，則可推論W成立。

Resolution Principle (4)

- Prove **(BAT_OK, ¬MOVES, BAT_OK ∧ LIFTABLE ⇒ MOVES) ├ ¬ LIFTABLE**
- Given
  - 1. BAT_OK
  - 2. ¬MOVES
  - 3. BAT_OK ∧ LIFTABLE ⇒ MOVES



A refutation tree

- Convert into CNF
  - 4. ¬BAT_OK ∨ ¬LIFTABLE∨ MOVES ---Clause form of 3
- Negation of goal:
  - 5. LIFTABLE
- Perform resolution:
  - 6. ¬BAT_OK ∨ MOVES (from resolving 5 with 4)
  - 7. ¬BAT_OK (from 6, 2)
  - 8. Nil (from 7, 1)

Resolution Principle (5)

- Horn clauses
  - A Horn clause: a clause that has at most one positive literal.
  - Ex: P, $\neg P \vee Q$, $\neg P \vee \neg Q \vee R$, $\neg P \vee \neg R$
  - Three types of Horn clauses.
    - A single atom: called a "fact"
    - An implication: called a "rule"
    - A set of negative literals: called "goal"
  - There are linear-time deduction algorithms for propositional Horn clauses.

Consistency

- A proof procedure is consistent
  - if it is not possible to prove both w and $\neg w$
  - not both $\vdash w$ and $\vdash \neg w$.

Soundness and Completeness

- for any set of wffs Δ, and wff w,
  - $(\Delta \vdash_R w) \Rightarrow (\Delta \models w)$
  - the set of inference rules, *R*, is *sound*
  - we can find a proof of *w* from Δ, *w* logically follows from Δ..
- for any set of wffs Δ, and wff *w*,
  - $(\Delta \models w) \Rightarrow (\Delta \vdash_R w)$,
  - that *R* is *complete*.
  - Eventually/whenever $\Delta \models w$, there exist (complete search) a proof of *w* from Δ using the set of inference rules.
- Inference rules, R, are sound and complete
  - $(\Delta \vdash_R w) \Rightarrow (\Delta \models w)$, and $(\Delta \models w) \Rightarrow (\Delta \vdash_R w)$,

Specifying with Propositions

- Consider the following text:
  - The system is in alert state only when it is waiting for an intruder and is on practice alert.
    - $alert \Leftrightarrow waiting \wedge practice$
  - If the system is in teaching mode and is on practice alert, then it is in alert state.
    - $teaching \wedge practice \Rightarrow alert$
  - The system will be in teaching mode and on practice alert and not waiting for an intruder.
    - $teaching \wedge practice \wedge \neg alert$

Detecting Contradictions

- From the previous system:
  - alert $\Leftrightarrow$ waiting $\wedge$ practice
  - teaching $\wedge$ practice $\Rightarrow$ alert
  - teaching $\wedge$ practice $\wedge$ $\neg$ alert
- The 2 and 3 propositions yield a contradiction
  - alert $\wedge$ $\neg$alert …contradiction!
- The first proposition yields another contradiction:
  - teaching $\wedge$ practice $\wedge$ $\neg$(waiting $\wedge$ practice)
  - Simplifies to
  - alert $\wedge$ $\neg$waiting …contradiction
  - because alert requires waiting!

Proposition Example (1)

- (1) If the train arrives late and there are no taxis at the station, then John is late for his meeting.
- (2) John is not late for his meeting.
- (3) The train did arrive late.
- (4) Therefore, there were taxis at the station.
  - P: the train arrives late
  - Q: there are taxis at the station
  - R: John is late for his meeting
  - (1) $P \land \neg Q \Rightarrow R$
  - (2) $\neg R$
  - (3) P
  - Prove (4) $\{P \land \neg Q \Rightarrow R, \neg R, P\} \vdash Q$

Proposition Example (2)

- Prove (4) $\{P \land \neg Q \Rightarrow R, \neg R, P\} \vdash Q$
  - Assert $\neg Q$
  - $P \land \neg Q \Rightarrow R$,
  - $\neg Q$
  - R
  - But $\neg R$ contradiction
  - Assert false, Q

Not Specify with Propositions

- Fido is a dog, dogs like bones so Fido likes bones.
  - Propositional analysis of this sentence yields:
    - Fido is a dog (propos. 1) --- P
    - Dogs like bones (propos. 2) --- Q
    - Fido likes bones (propos. 3) --- R
  - Cannot derive R from P and Q using only propositional calculus
  - Predicate Calculus
    - Dog (Fido)=true;
    - Dog (lecturer)=false

Exercise I

- (1) $\{(P \wedge Q) \wedge R, S \wedge T\} \vdash (Q \wedge S)$
- (2) $\{P, \neg\neg(Q \wedge R)\} \vdash (\neg\neg P \wedge R)$
- (3) $\{\neg P \Rightarrow Q, \neg Q)\} \vdash (P)$
- (4) $\{(P \wedge Q) \Rightarrow R\} \vdash (P \Rightarrow (Q \Rightarrow R))$
- (5) $\{P \wedge (Q \Rightarrow R)\} \vdash (P \wedge Q \Rightarrow R)$
- (6) $\{P \Rightarrow Q, P \Rightarrow \neg Q\} \vdash (\neg P)$

Exercise II

- Knowledge Base
  - $GasInTank \land FuelLineOK \Rightarrow GasInEngine$
  - $GasInEngine \land GoodSpark \Rightarrow EngineRuns$
  - $PowerToPlugs \land PlugsClean \Rightarrow GoodSpark$
  - $BatteryCharged \land CablesOK \Rightarrow PowerToPlugs$
- Observed:
  - $\neg$ EngineRuns,
  - GasInTank, PlugsClean, BatteryCharged
- Prove:
  - $\neg FuelLineOK \lor \neg CablesOK$

Exercise II

- Knowledge Base and Observations:
  - (¬ GasInTank ∨ ¬ FuelLineOK ∨ GasInEngine)
  - (¬ GasInEngine ∨ ¬ GoodSpark ∨ EngineRuns)
  - (¬ PowerToPlugs ∨ ¬ PlugsClean ∨ GoodSpark)
  - (¬ BatteryCharged ∨ ¬ CablesOK ∨ PowerToPlugs)
  - (¬EngineRuns)
  - (GasInTank)
  - (PlugsClean)
  - (BatteryCharged)
- Negation of Conclusion:
  - (FuelLineOK)
  - (CablesOK)