XI. Hilbert Huang Transform (HHT)

Proposed by 黃鍔院士 (AD. 1998)

黃鍔院士的生平可參考

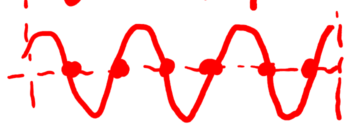
http://djj.ee.ntu.edu.tw/%E9%BB%83%E9%8D%94%E9%99%A2%E5%A3%AB.pdf

References

[1] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N. C. Yen, C. C. Tung, and H. H. Liu, "The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis," *Proc. R. Soc. Lond. A*, vol. 454, pp. 903-995, 1998.

[2] N. E. Huang and S. Shen, *Hilbert-Huang Transform and Its Applications*, World Scientific, Singapore, 2005.

2 zero crossings = 1 period



(PS: 謝謝 2007 年修課的趙逸群同學和王文阜同學)

$$\text{frequency} = \frac{3}{T} = \frac{\text{number of zero crossings}}{2T}$$

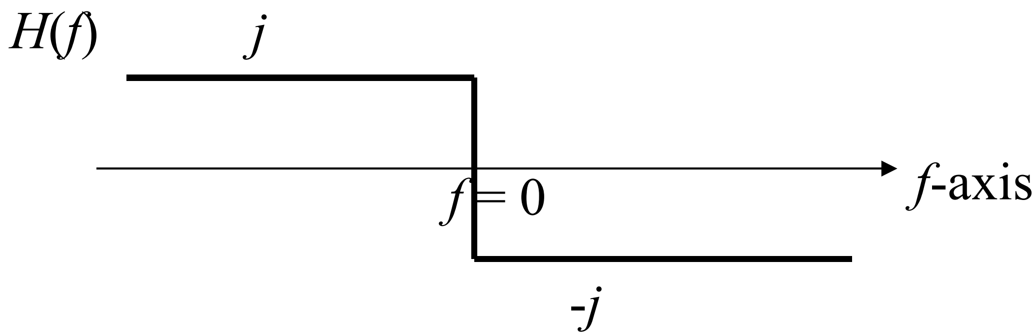
11-A The Origin of the Concept

另一種分析 instantaneous frequency 的方式： Hilbert transform

- Hilbert transform

$$x_H(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau$$

or $$x_H(t) = IFT\left\{FT\left[x(t)\right]H(f)\right\}$$



Applications of the Hilbert Transform

- analytic signal

$X_a(f) \qquad X(f)$

$$x_a(t) = x(t) + jx_H(t)$$

- edge detection

- another way to define the instantaneous frequency:

$$\text{instantaneous frequency} = \frac{1}{2\pi}\frac{d}{dt}\theta$$

$\tan(\theta) = \tan(\theta + \pi)$

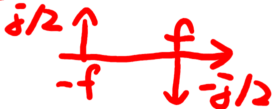
$$\text{where} \quad \theta = \tan^{-1}\frac{x_H(t)}{x(t)}$$

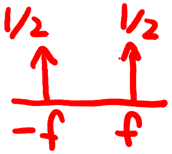
If $x(t) = \cos(2\pi f t)$

$\theta = \tan^{-1}\frac{\sin(2\pi f t)}{\cos(2\pi f t)} = 2\pi f t$

$\frac{1}{2\pi}\frac{d}{dt}\theta = f$

Example:





$$\cos(2\pi f t) \xrightarrow{\textit{Hilbert}} \sin(2\pi f t) \qquad \theta = 2\pi f t$$

$$\sin(2\pi f t) \xrightarrow{\textit{Hilbert}} -\cos(2\pi f t) \qquad \theta = 2\pi f t + \pi/2$$

If $x(t) = \sin(2\pi f t)$

$\theta = \tan^{-1}\frac{-\sin(2\pi f t + \frac{\pi}{2})}{-\cos(2\pi f t + \frac{\pi}{2})} = 2\pi f t + \frac{\pi}{2}$

$\frac{1}{2\pi}\frac{d\theta}{dt} = f$

Problem of using Hilbert transforms to determine the instantaneous frequency:

This method is only good for cosine and sine functions with single component.

Not suitable for (1) complex function

(2) non-sinusoid-like function

(3) multiple components

Moreover, θ has multiple solutions.

Example:

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) \xrightarrow{Hilbert} \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$$

$$\theta = \tan^{-1}\frac{\sin(2\pi f_1 t)+\sin(2\pi f_2 t)}{\cos(2\pi f_1 t)+\cos(2\pi f_2 t)} = \tan^{-1}\frac{\sin(\pi(f_1+f_2)t)\cancel{\cos(\pi(f_2-f_1)t)}}{\cos(\pi(f_1+f_2)t)\cancel{\cos(\pi(f_2-f_1)t)}}$$

$$= \pi(f_1+f_2)t$$

$$\frac{1}{2\pi}\frac{d\theta}{dt} = \frac{f_1+f_2}{2}$$

- Hilbert-Huang transform 的基本精神：

先將一個信號分成多個 sinusoid-like components + trend

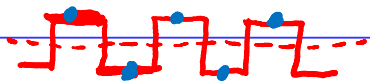
(和 Fourier analysis 不同的地方在於，這些 sinusoid-like components 的 period 和 amplitude 可以不是固定的)

再運用 Hilbert transform (或 STFT，number of zero crossings) 來分析每個 components 的 instantaneous frequency

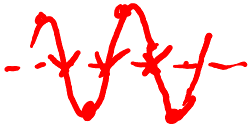
完全不需用到 Fourier transform

11-B Intrinsic Mode Function (IMF)

IMF



Amplitude and frequency can vary with time.



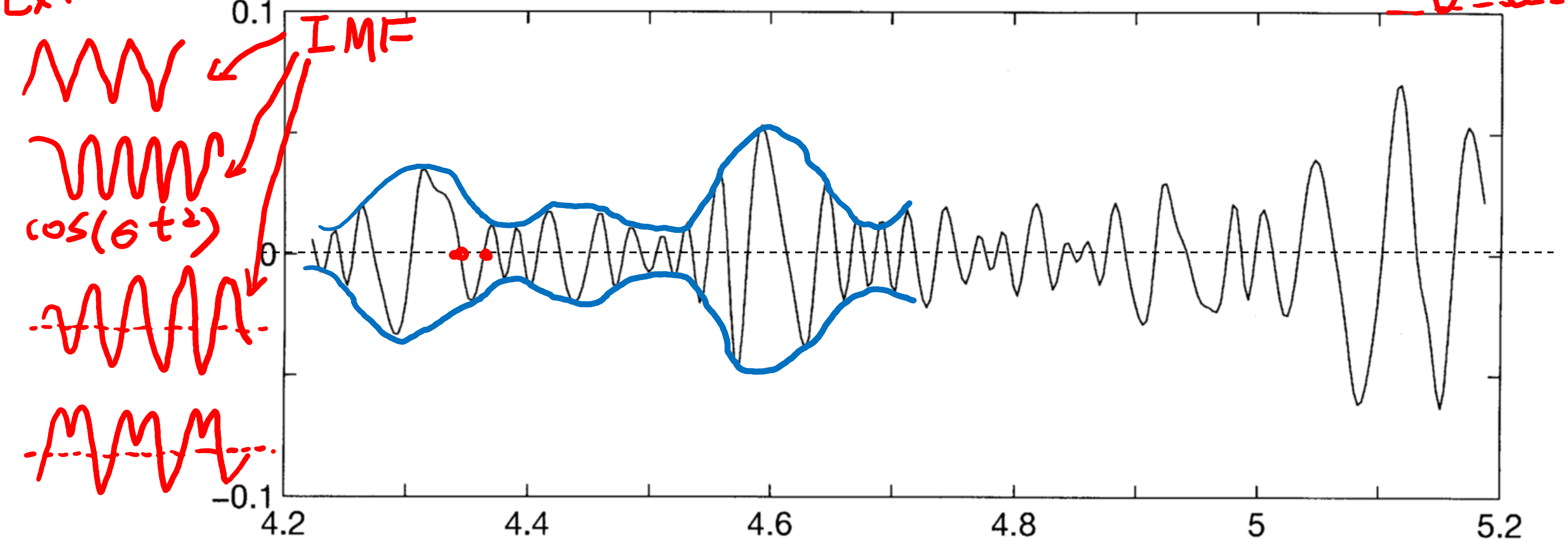
但要滿足

local maximums & local minimums

(1) The number of extremes and the number of zero-crossings must either equal or differ at most by one. i.e., local maximum > 0 must be satisfied
local minimum < 0

(2) At any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is near to zero.

Ex:



11-C Procedure of the Hilbert Huang Transform

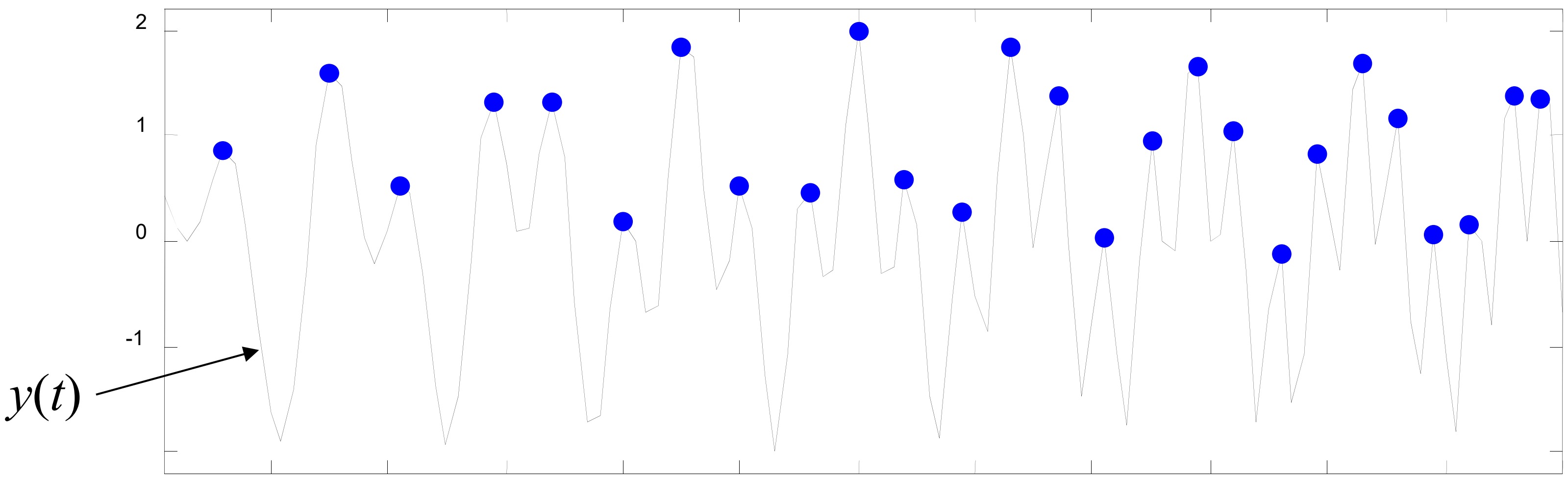
Steps 1~8 are called Empirical Mode Decomposition (EMD)

signal → IMFs + trend

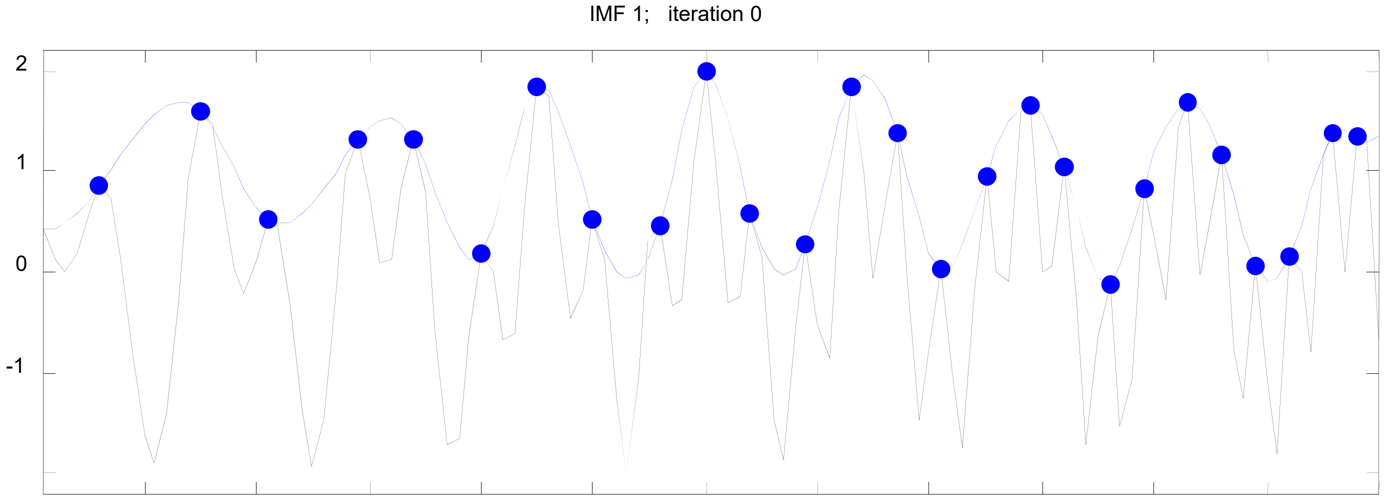
(Step 1) Initial: $y(t) = x(t)$, ($x(t)$ is the input) $n = 1$, $k = 1$

(Step 2) Find the local peaks

$x(n\Delta t) > x((n+1)\Delta t)$
and $x(n\Delta t) > x((n-1)\Delta t)$



(Step 3) Connect local peaks



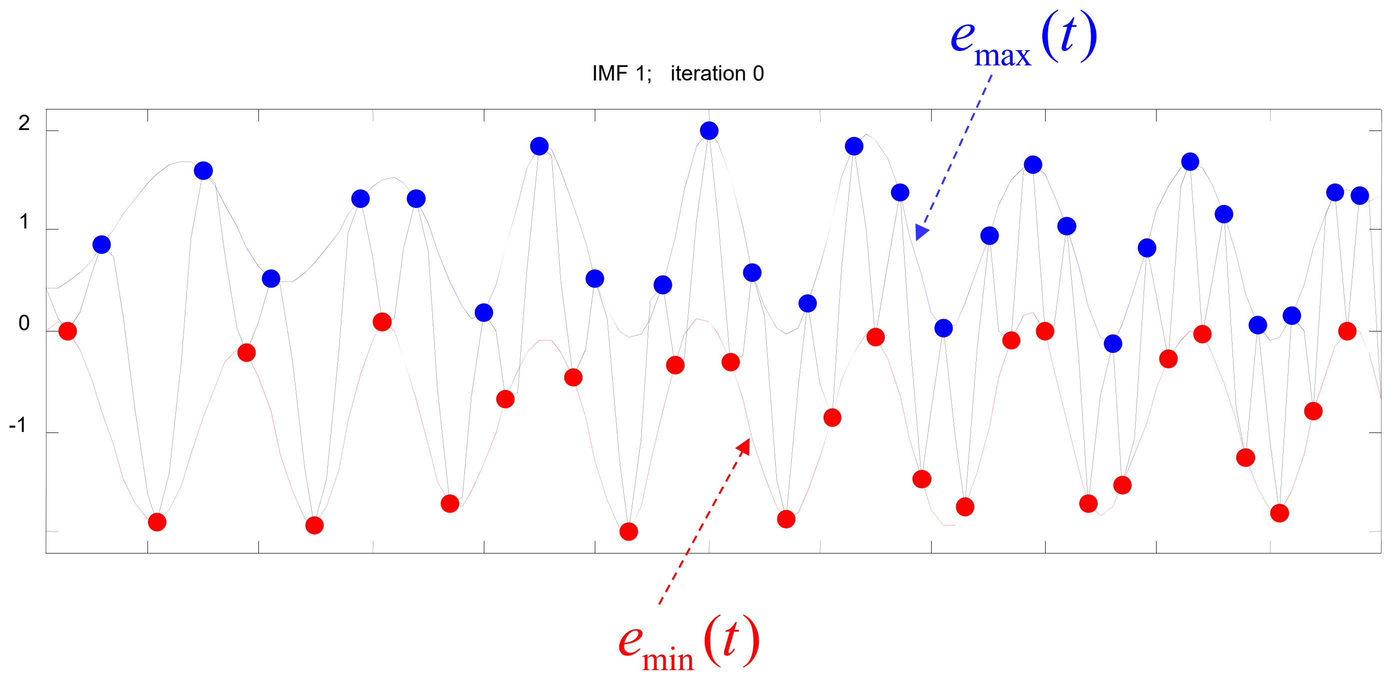
m = 3

通常使用 B-spline，尤其是 cubic B-spline 來連接

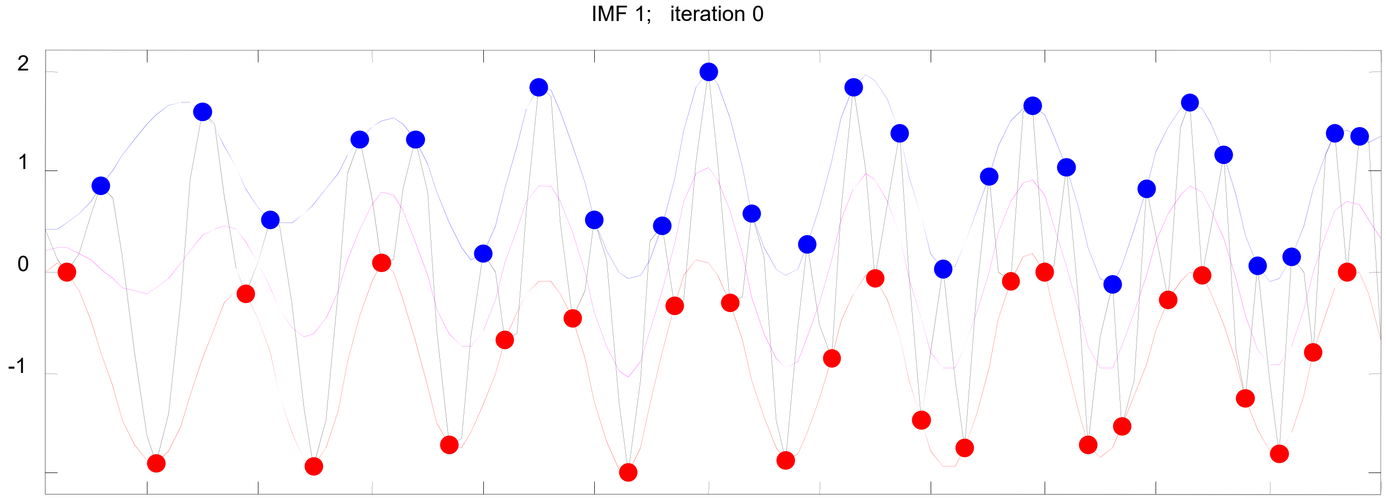
(參考附錄十一)

(Step 4) Find the local dips

(Step 5) Connect the local dips



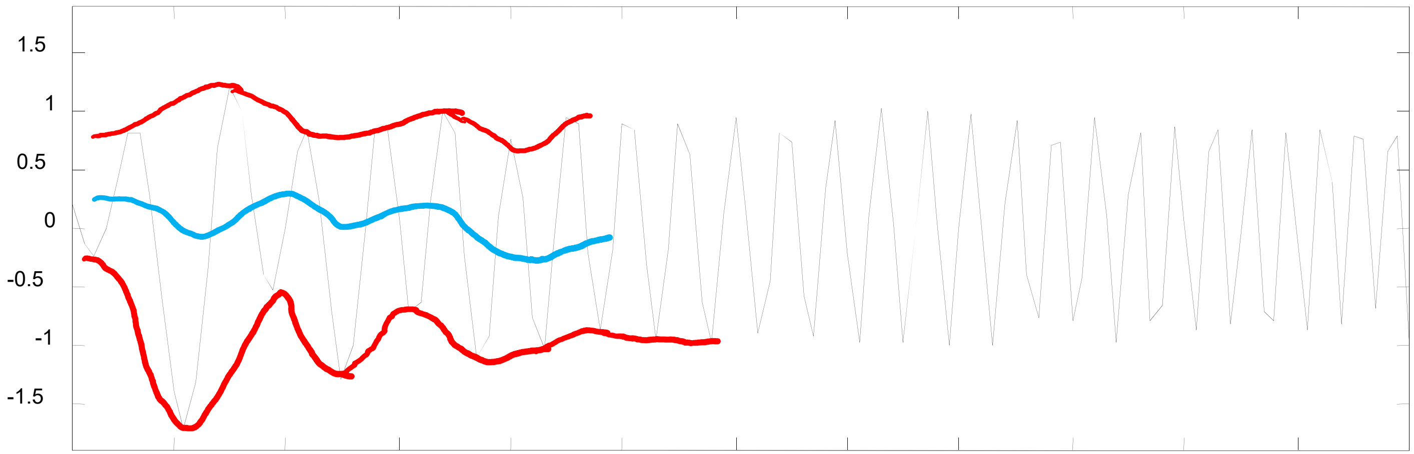
(Step 6-1) Compute the mean



$$z(t) = \frac{e_{\min}(t) + e_{\max}(t)}{2}$$

(pink line)

(Step 6-2) Compute the residue



$$h_k(t) = \underline{y(t)} - \underline{z(t)}$$

If threshod = 0.1 , $h_k(t)$ is not an IMF

If threshold: 0.5 , $h_k(t)$ is an IMF

(Step 7) Check whether $h_k(t)$ is an intrinsic mode function (IMF)

(1) 檢查是否 local maximums 皆大於 0
local minimums 皆小於 0

(2) 上封包： $u_1(t)$，　下封包： $u_0(t)$

檢查是否 $\left|\frac{u_1(t)+u_0(t)}{2}\right| < threshold$ for all t

If they are satisfied (or $k \geqq K$), set $c_n(t) = h_k(t)$ and continue to Step 8

$c_n(t)$ is the n^{th} IMF of $x(t)$.

If not, set $y(t) = h_k(t)$,
$k = k + 1$, and repeat Steps 2~6

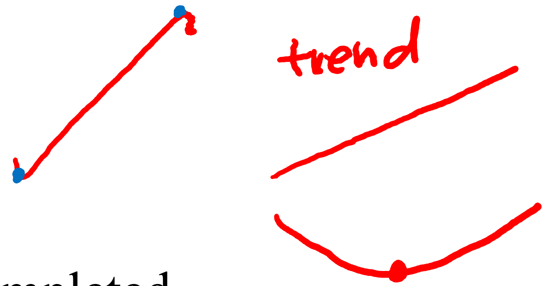
(為了避免無止盡的迴圈，可以定 k 的上限 K)

(Step 8) Calculate $x_0(t) = x(t) - \sum_{s=1}^{n} c_s(t)$ existing IMFs

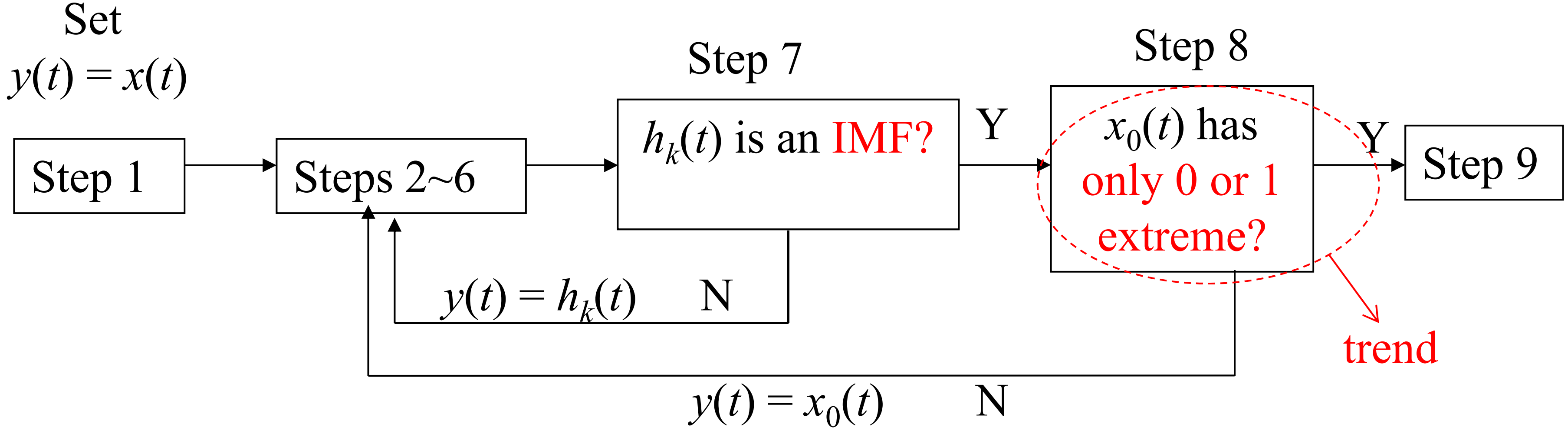
and check whether $x_0(t)$ is a function with no more than one extreme point.

If not, set $n = n+1$, $y(t) = x_0(t)$

and repeat Steps 2~7



If so, the empirical mode decomposition is completed.



$$x(t) = x_0(t) + \sum_{s=1}^{n} c_s(t)$$

(Step 9) Find the instantaneous frequency for each IMF $c_s(t)$ ($s = 1, 2, \ldots, n$).

Method 1: Using the Hilbert transform

Method 2: Calculating the STFT for $c_s(t)$.

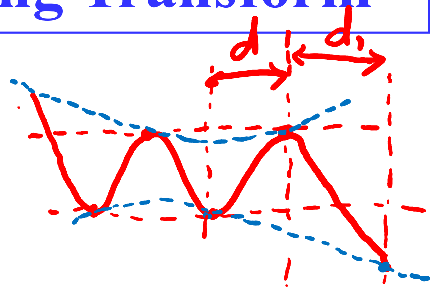
Method 3: Furthermore, we can also calculate the instantaneous frequency from the number of zero-crossings directly.

2 zero-crossings = 1 period

instantaneous frequency $F_s(t)$ of $c_s(t)$

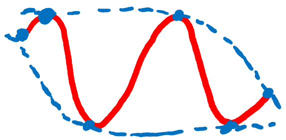
$$= \frac{\text{the number of zero-crossings of } c_s(t) \text{ between } t-B \text{ and } t+B}{4B}$$

Technique Problems of the Hilbert Huang Transform



(A) 邊界處理的問題：

目前尚未有一致的方法，可行的方式有



(1) 只使用非邊界的 extreme points

(2) 將最左、最右的點當成是 extreme points

(3) 預測邊界之外的 extreme points 的位置和大小

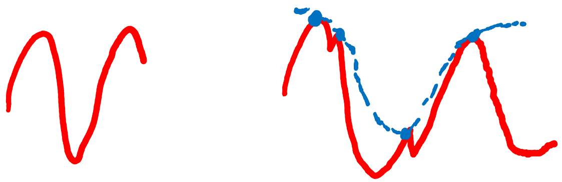
(4) 用邊界和最近的 extreme point 的距離來判斷是否邊界要當成 extreme points

$\frac{d_1}{d} > \frac{1}{2}$ boundary point is an extreme

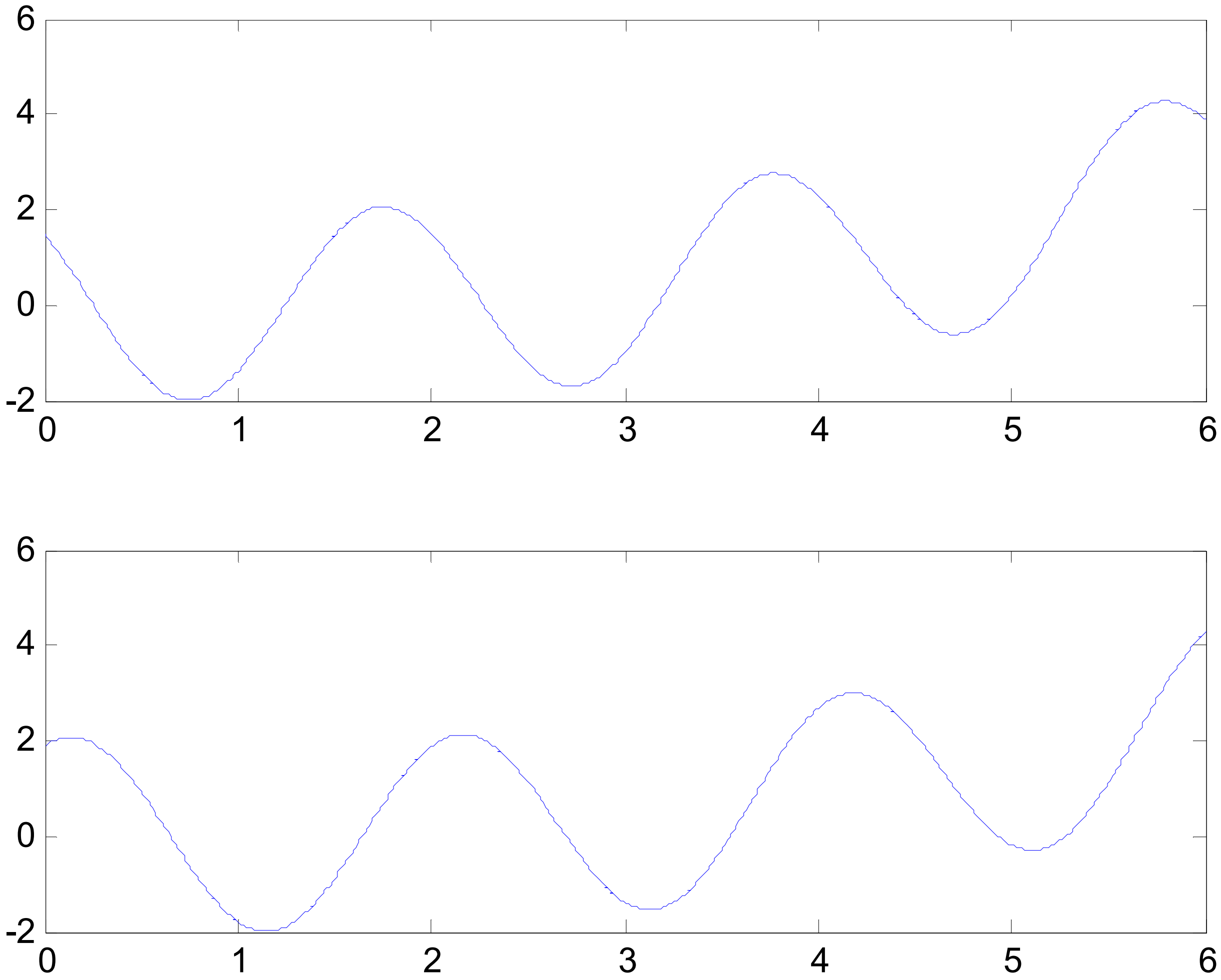
$< \frac{1}{2}$ not an extreme

(B) Noise 的問題：

先用 pre-filter 來處理



最左、最右的點是否要當成是 extreme points



11-D Example

trend

low frequency part

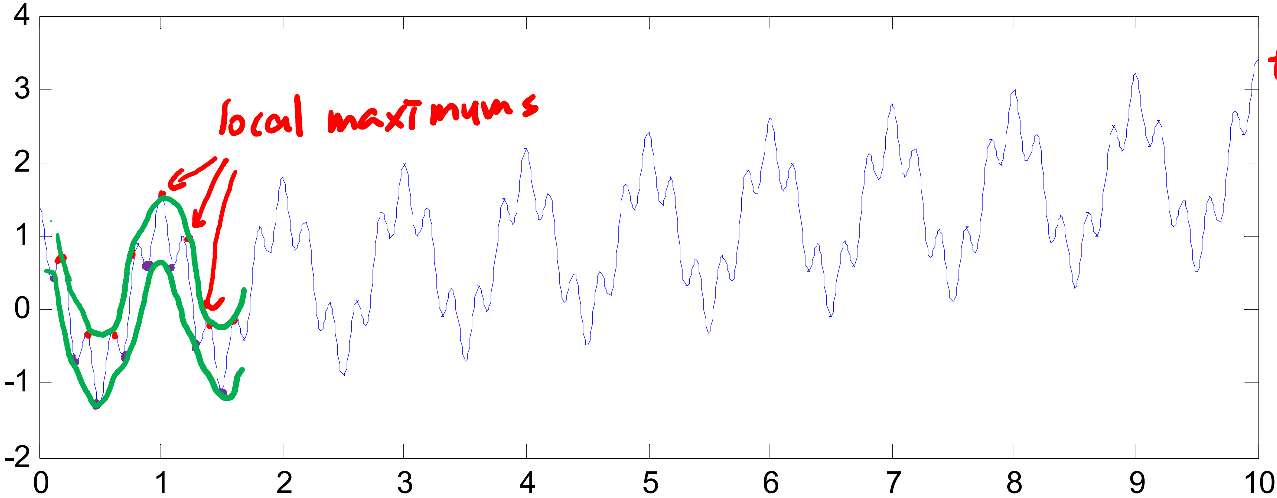
high frequency part

Example 1 $x(t) = 0.2t + \cos(2\pi t) + 0.4\cos(10\pi t)$

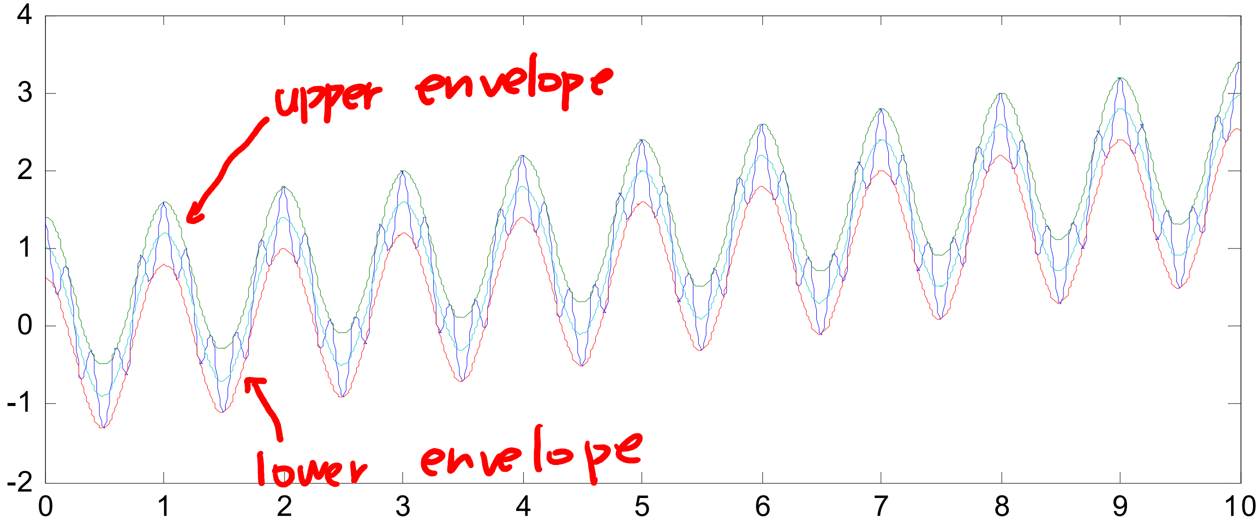
$1 \xrightarrow{FT} \delta(f)$

$t \xrightarrow{FT} \dfrac{\delta'(f)}{-j2\pi}$

$t^n \xrightarrow{FT} \dfrac{\delta^{(n)}(f)}{(-j2\pi)^n}$



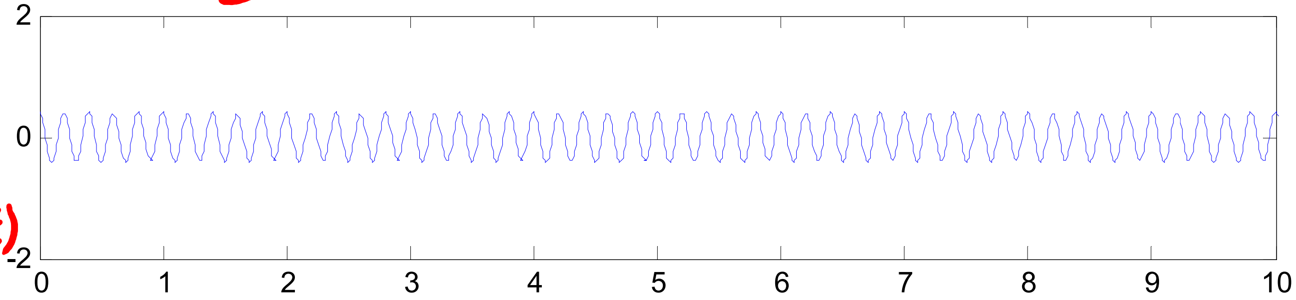
After Step 6



$$x(t) - \frac{\text{upper envelope} + \text{lower envelope}}{2}$$

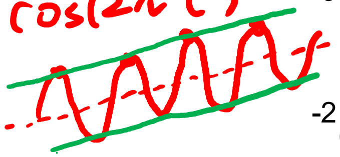
IMF1

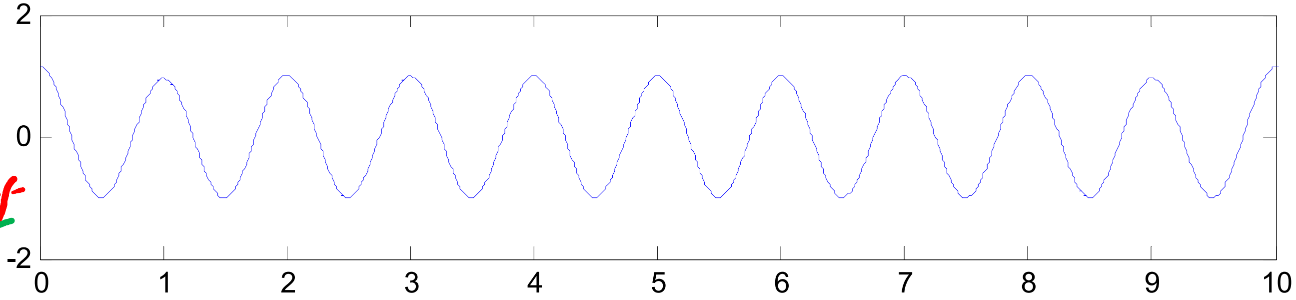
$0.4\cos(10\pi t)$



IMF2

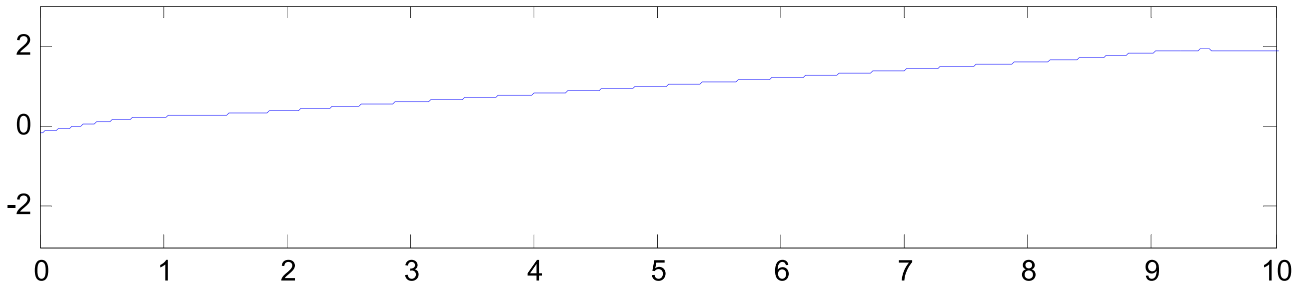
$\cos(2\pi t)$





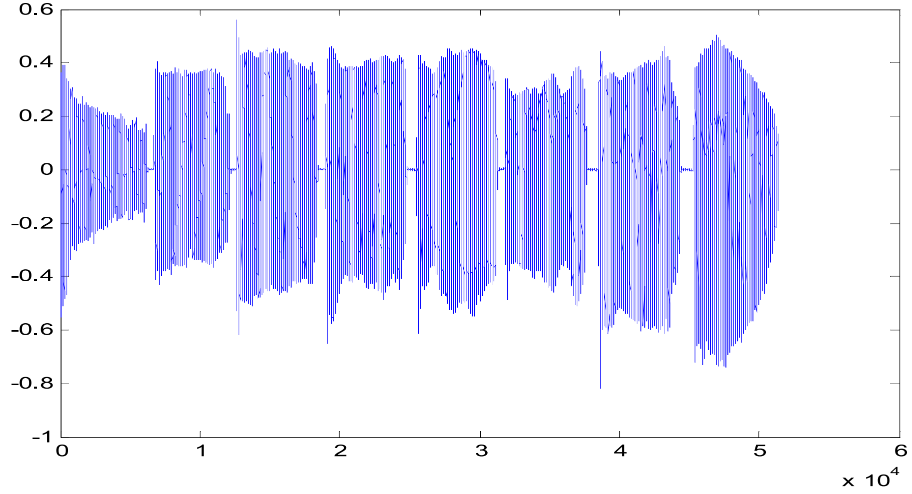
$x_0(t)$

$0.2t$

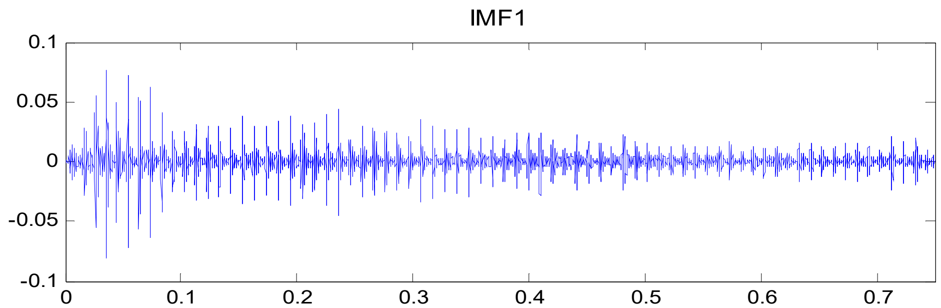


Example 2

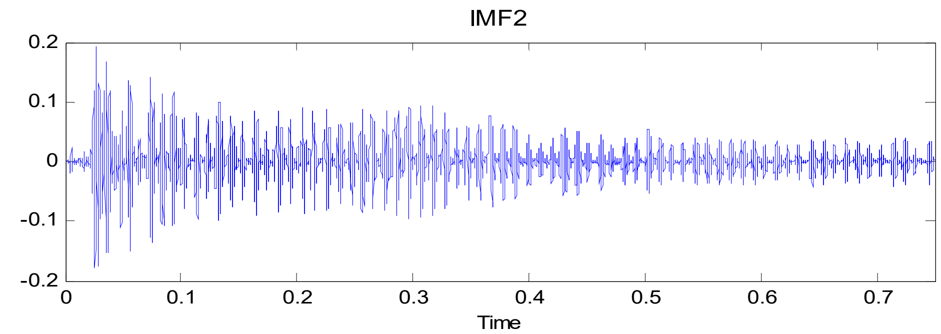
hum signal



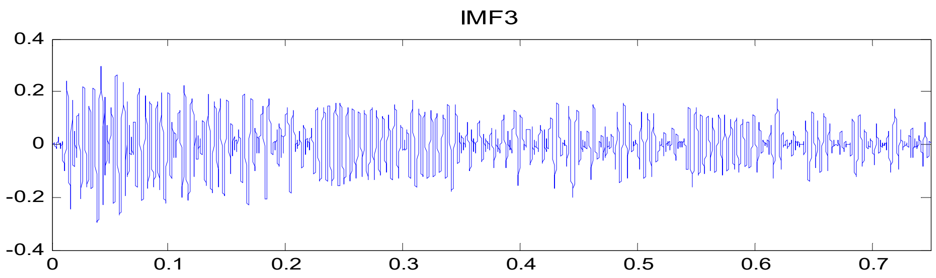
IMF1



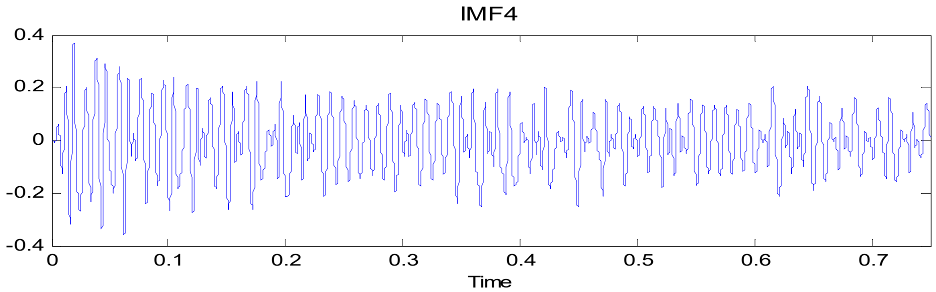
IMF2



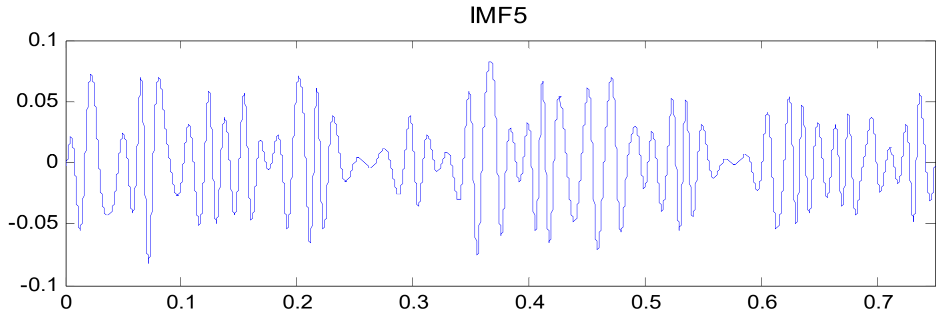
IMF3



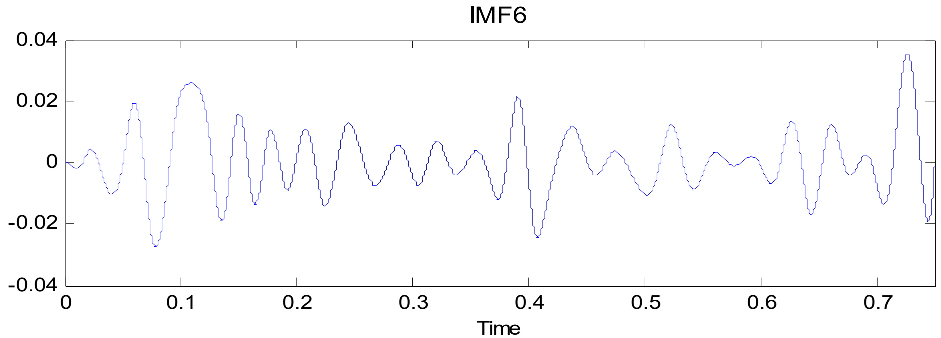
IMF4



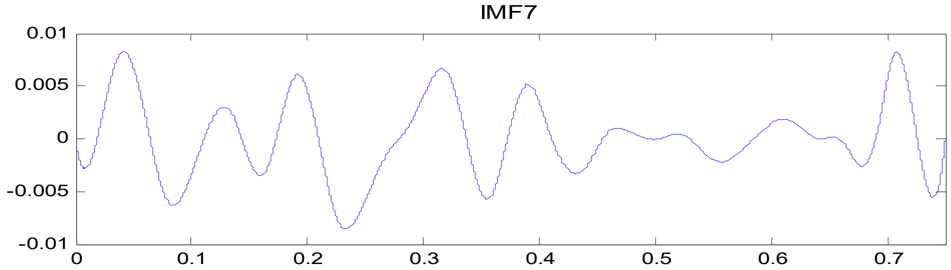
IMF5



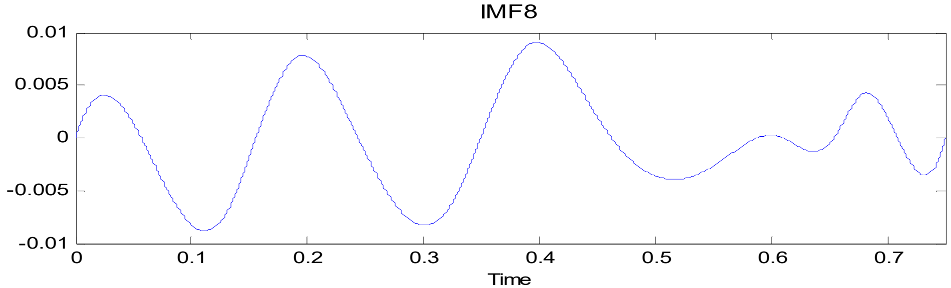
IMF6



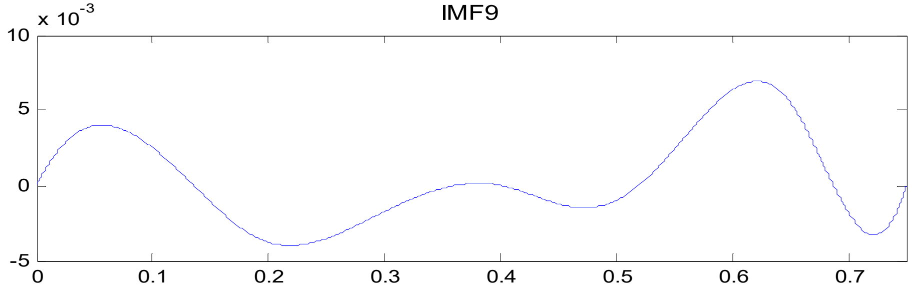
IMF7



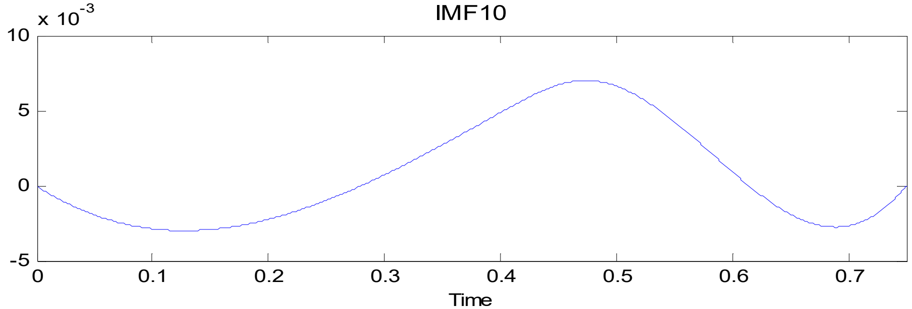
IMF8



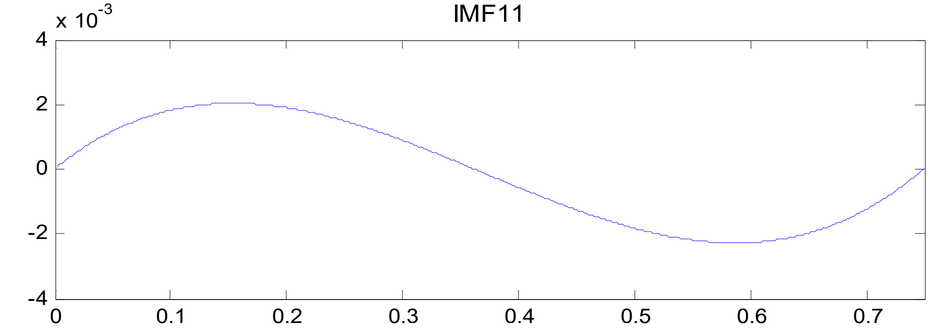
IMF9



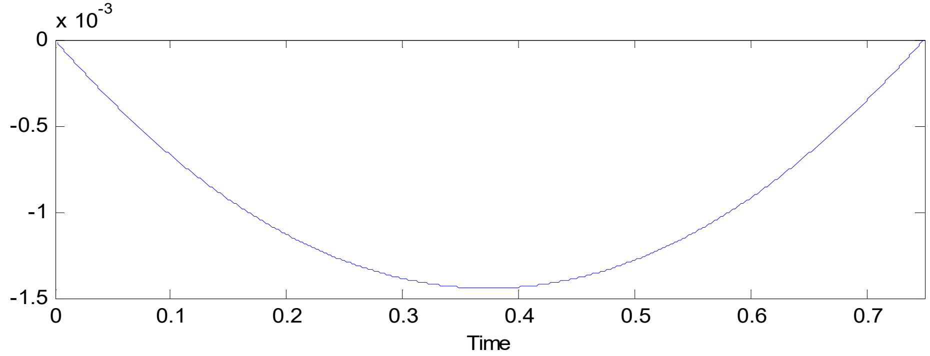
IMF10



IMF11


IMF11
x 10^-3
4
2
0
-2
-4
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7

$x_0(t)$


x 10^-3
0
-0.5
-1
-1.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
Time

11-E Comparison

(1) 避免了複雜的數學理論分析

(2) 可以找到一個 function 的「趨勢」

(3) 和其他的時頻分析一樣，可以分析頻率會隨著時間而改變的信號

(4) 適合於

- Climate analysis
- Economical data
- Geology

Acoustics

Music signal

- Conclusion

當信號含有「趨勢」

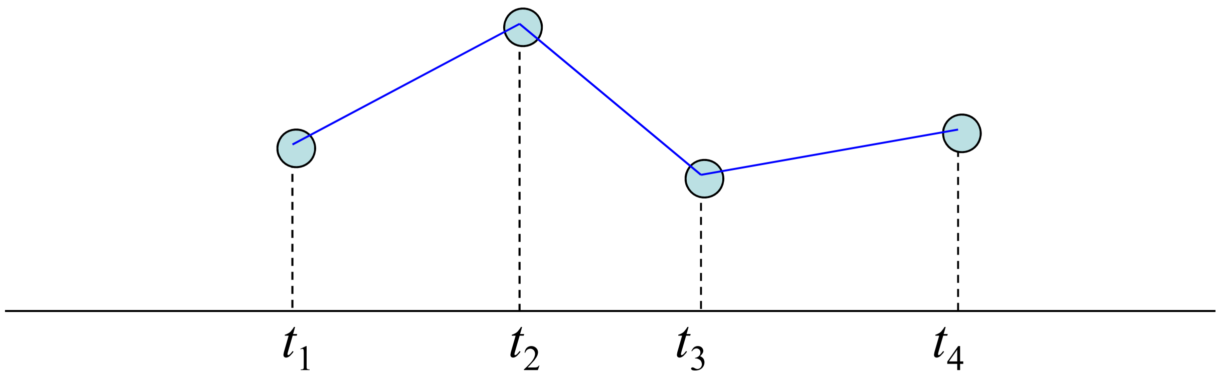
或是由少數幾個 sinusoid functions 所組合而成，而且這些sinusoid functions 的 amplitudes 相差懸殊時，可以用 HHT 來分析

附錄十一 Interpolation and the B-Spline

Suppose that the sampling points are t_1, t_2, t_3, …, t_N
and we have known the values of $x(t)$ at these sampling points.

There are several ways for interpolation.

(1) The simplest way: Using the straight lines (i.e., linear interpolation)



(2) Lagrange interpolation

$$x(t) = \sum_{n=1}^{N} \frac{\prod_{\substack{j=1 \\ j \neq n}}^{N} (t - t_j)}{\prod_{\substack{j=1 \\ j \neq n}}^{N} t_n - t_j} x(t_n)$$

$\prod$ 指的是連乘符號，

$$\prod_{j=1}^{N} h_j = h_1 h_2 h_3 \cdots\cdots h_N$$

(3) Polynomial interpolation

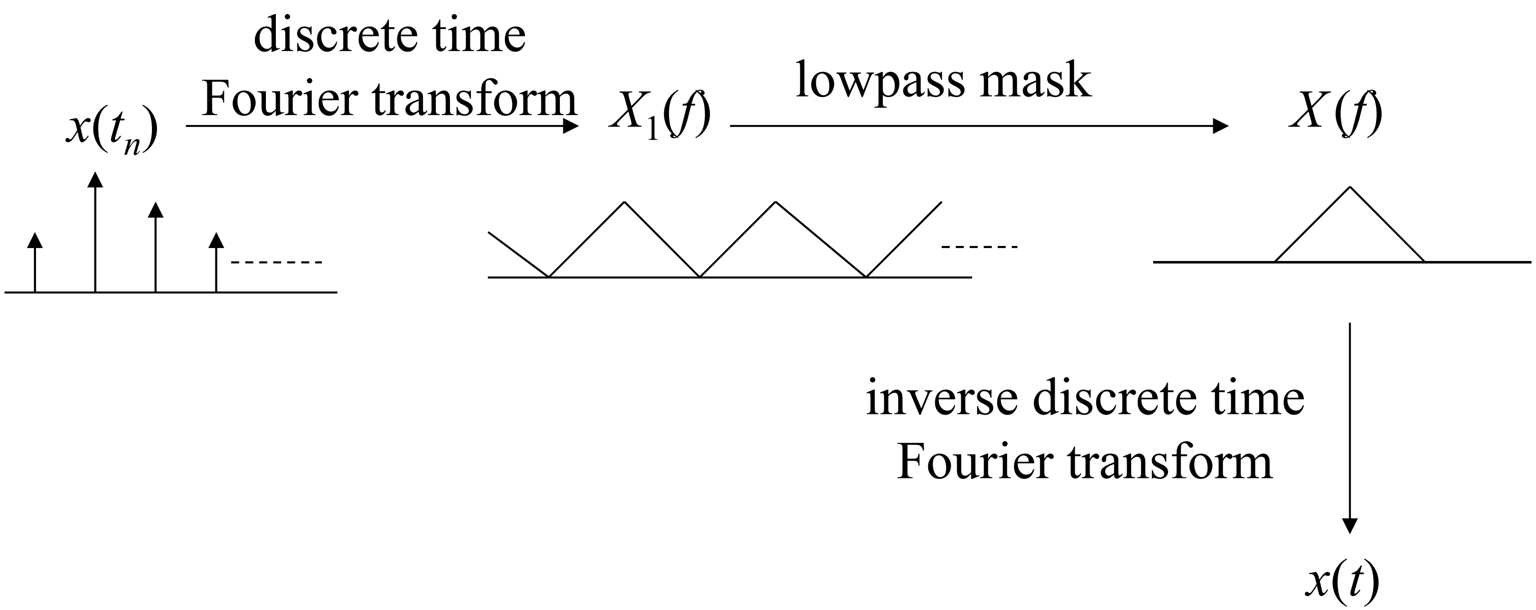
$$x(t) = \sum_{n=1}^{N} a_n t^{n-1},$$ solve a_1, a_2, a_3, ……, $a_{N\text{-}1}$ from

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{N-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{N-1} \\ 1 & t_3 & t_3^2 & \cdots & t_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^{N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ x(t_3) \\ \vdots \\ x(t_N) \end{bmatrix}$$

(4) Lowpass Filter Interpolation

適用於 sampling interval 為固定的情形 $t_{n+1} - t_n = \Delta_t$ for all n

$$x(t) = \sum_{n=1}^{N} x(t_n)\,\mathrm{sinc}\left(\frac{t - t_n}{\Delta_t}\right)$$



(5) B-Spline Interpolation

B-spline 簡稱為 spline

$$B_{n,0}(t) = 1 \qquad \text{for } t_n < t < t_{n+1}$$

$$B_{n,0}(t) = 0 \qquad \text{otherwise}$$

$$B_{n,m}(t) = \frac{t - t_n}{t_{n+m} - t_n} B_{n,m-1}(t) + \frac{t_{n+m+1} - t}{t_{n+m+1} - t_{n+1}} B_{n+1,m-1}(t)$$

$$\boxed{x(t) = \sum_{n=1}^{N} x(t_n) B_{n,m}(t)}$$

$m = 1$: linear B-spline

$m = 2$: quadratic B-spline

$m = 3$: cubic B-spline (通常使用) $x(t)$, $x'(t)$, $x''(t)$ are continuous

In Matlab, the command "spline" can be used for spline interpolation

(Note: In the command, the cubic B-spline is used)

☆ Example:

Generating a sine-like spline curve and samples it over a finer mesh:

```
x = 0:1:10;          % original sampling points
y = sin(x);
xx = 0:0.1:10;       % new sampling points
yy = spline(x,y,xx);
plot(x,y,'o',xx,yy)
```

for page 319
x: local maximum locations
y: values
yy: upper envelope