XIV. Number Theoretic Transform (NTT)

◎ 14-A Definition

◆ **Number Theoretic Transform and Its Inverse**

比較：DFT ⇒ $F(k) = \sum_{n=0}^{N} f(n) e^{-j\frac{2\pi}{N}nk}$

$$F(k) = \sum_{n=0}^{N-1} f(n)\alpha^{nk} \pmod{M},\ k = 0, 1, 2\cdots, N-1$$

$e^{-j\frac{2\pi}{N}} \to \alpha$ $\quad f(n) = \frac{1}{N}\sum_{n=0}^{N-1} F(k) e^{j\frac{2\pi}{N}nk}$

$$f(n) = N^{-1}\sum_{k=0}^{N-1} F(k)\alpha^{-nk} \pmod{M} \quad ,n = 0, 1, 2\cdots, N-1$$

$$f(n) \underset{INTT}{\overset{NTT}{\Longleftrightarrow}} F(k)$$

$= N^{-1}\sum F(k)(\alpha^{-1})^{nk}$

Note：

(1) M is a prime number， (mod M): 是指除以 M 的餘數 (0 ~ $M-1$ 之間 integer) $\quad\alpha$ is some integer

(2) N is a factor of $M-1$ → 因數

(Note: when $N \neq 1$, N must be prime to M)

(3) N^{-1} is an integer that satisfies $(N^{-1})N \bmod M = 1$

(When $N = M-1$, $N^{-1} = M-1$)

ex: $M = 11$

$N = 5$

$N^{-1} = 9$

$\because 5 \cdot 9 \bmod 11 = 1$

(4) α is a root of unity of order N

$$\alpha^{N} = 1 \pmod{M}$$

$$\alpha^{k} \neq 1 \pmod{M},\ k = 1, 2, \cdots, N-1$$

When α satisfies the above equations and $N = M - 1$, we call α the "primitive root".

$$\boxed{\begin{array}{l} \alpha^{k} \neq 1 \pmod{M} \quad \text{for } k = 1, 2, \cdots, M-2 \\[1em] \alpha^{M-1} = 1 \pmod{M} \end{array}}$$

α^{-1} 的求法與 N^{-1} 相似

α^{-1} is an integer that satisfies $\left(\alpha^{-1}\right)\alpha \bmod M = 1$

if $M = 5$, $\alpha = 2$, $\alpha^{-1} = 3$

$\alpha = 3$, $\alpha^{-1} = 2$

$\alpha = 4$, $\alpha^{-1} = 4$

取乘數

Example 1:

3 9 27 81

if $\alpha = 3$ $\alpha^1 = 3$ $\alpha^2 = 4$ $\alpha^3 = 2$ $\alpha^4 = 1$

$M = 5$ $\alpha = 2$ $\alpha^1 = 2 \pmod 5$ $\alpha^2 = 4 \pmod 5$ $\alpha^3 = 3 \pmod 5$ $\alpha^4 = 1 \pmod 5$

When $N = 4$ $\alpha = 2$ $\alpha^{-1} = 3$

$\alpha^{nk} = \alpha^{nk-N} = \alpha^{((nk))_N}$

$((nk))_N$ = nk 除以 N 的餘數

α^{nk}

$$\begin{array}{c} \\ \begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{bmatrix} = \end{array} \begin{array}{c|cccc} & n=0 & 1 & 2 & 3 \\ \hline k=0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 & 3 \\ 2 & 1 & 4 & 1 & 4 \\ 3 & 1 & 3 & 4 & 2 \end{array} \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{bmatrix}$$

$\xRightarrow{\text{simple}}$ $\begin{bmatrix} \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 \\ \alpha^0 & \alpha^2 & \alpha^4 & \alpha^6 \\ \alpha^0 & \alpha^3 & \alpha^6 & \alpha^9 \end{bmatrix} = \begin{bmatrix} \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 \\ \alpha^0 & \alpha^2 & \alpha^0 & \alpha^2 \\ \alpha^0 & \alpha^3 & \alpha^2 & \alpha^1 \end{bmatrix}$

When $N = 2$ $\alpha \neq 2$ if $\alpha = 3$ $\alpha^2 = 4$ mod 5 (not suitable)

$(\alpha^{-1})^{nk} = (\alpha^{-1})^{((nk))_N}$ $(\because (\alpha^{-1})^N = 1)$

$$\begin{bmatrix} F[0] \\ F[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} f[0] \\ f[1] \end{bmatrix}$$

mod 5

$N = 2$, $\alpha = 4$

$N^{-1} = 3$

inverse

$\alpha^{-1} = 4$

if $\alpha = 4$, $\alpha^2 = 1$ mod 5

$3\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix}$

inverse mod 5, $N = 4$

$(4)^{-1}\begin{bmatrix} 3^0 & 3^0 & 3^0 & 3^0 \\ 3^0 & 3^1 & 3^2 & 3^3 \\ 3^0 & 3^2 & 3^4 & 3^6 \\ 3^0 & 3^3 & 3^6 & 3^9 \end{bmatrix} = 4\begin{bmatrix} 3^0 & 3^0 & 3^0 & 3^0 \\ 3^0 & 3^1 & 3^2 & 3^3 \\ 3^0 & 3^2 & 3^0 & 3^2 \\ 3^0 & 3^3 & 3^2 & 3^1 \end{bmatrix}$

$\Rightarrow 4\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 1 & 4 & 1 & 4 \\ 1 & 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 2 & 1 & 3 \\ 4 & 1 & 4 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

Example 2:

$M = 7$, $N = 6$: α cannot be 2 but can be 3.

$\alpha = 2$: $\alpha^1 = 2$ (mod 7) $\alpha^2 = 4$ (mod 7) $\alpha^3 = 1$ (mod 7)

$\alpha = 3$: $\alpha^1 = 3$ (mod 7) $\alpha^2 = 2$ (mod 7) $\alpha^3 = 6$ (mod 7)

$\alpha^4 = 4$ (mod 7) $\alpha^5 = 5$ (mod 7) $\alpha^6 = 1$ (mod 7)

Advantages of the NTT:

① no non-integer operations

② if "LUT" is applied, no multiplication and no addition are required.

③ preserve the convolution property.

④ N is not constraint to 2^k
约束

⑤ suitable for encryption (加密)

Disadvantages of the NTT:

① not suitable for frequency analysis.

② for convolution, the inputs should be integer $\max(\text{output}) - \min(\text{output}) \le M-1$

◉ 14-B 餘數的計算

(1) $x \pmod{M}$ 的值，必定為 $0 \sim M-1$ 之間

(2) $a+b \pmod{M} = \{a \pmod{M} + b \pmod{M}\} \pmod{M}$

例： $78 + 123 \pmod{5} = 3 + 3 \pmod{5} = 1$

(Proof): If $a = a_1M + a_2$ and $b = b_1M + b_2$, then

$$a + b = (a_1 + b_1)M + a_2 + b_2$$

(3) $a \times b \pmod{M} = \{a \pmod{M} \times b \pmod{M}\} \pmod{M}$

例： $78 \times 123 \pmod{5} = 3 \times 3 \pmod{5} = 4$

0 ~ M-1 之間 相加相乘

(Proof): If $a = a_1M + a_2$ and $b = b_1M + b_2$, then

$$a \times b = (a_1 b_1 M + a_1 b_2 + a_2 b_1)M + a_2 b_2$$

ex:

2234 × 112 mod 11

= 1 × 2 mod 11 = 2

在 Number Theory 當中

只有 M^2 個可能的加法，M^2 個可能的乘法

可事先將加法和乘法的結果存在記憶體當中

需要時再 “LUT”

LUT : lookup table

◉ 14-C Properties of Number Theoretic Transforms

P.1) Orthogonality Principle

$$S_N = \sum_{n=0}^{N-1} \alpha^{nk}\, \alpha^{-n\ell} = \sum_{n=0}^{N-1} \alpha^{n(k-\ell)} = N \cdot \delta_{k.\ell}$$

proof： for $k = \ell$, $\quad S_N = \sum_{n=0}^{N-1} \alpha^0 = N$

for $k \neq 0$, $\quad (\alpha^{k-\ell l} - 1)\, S_N = (\alpha^{k-\ell} - 1) \sum_{n=0}^{N-1} \alpha^{n(k-\ell)} = \alpha^{N(k-\ell)} - 1 = 1 - 1 = 0$

$$\because \alpha^{k-\ell} \neq 1 \qquad \therefore S_N = 0$$

P.2) The NTT and INTT are exact inverse

proof：

$$g(n) = \frac{1}{N} \sum_{k=0}^{N-1} F(k)\, \alpha^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{\ell=0}^{N-1} f(\ell)\, \alpha^{\ell k}\right) \alpha^{-nk}$$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} f(\ell) \sum_{k=0}^{N-1} \alpha^{(\ell-n)k} = \frac{1}{N} \sum_{\ell=0}^{N-1} f(\ell) \cdot N\delta_{\ell,n} = f(n)$$

P.3) Symmetry

$$f(n) = f(N-n) \quad \overset{\text{NTT}}{\Longleftrightarrow} \quad F(k) = F(N-k)$$

$$f(n) = -f(N-n) \quad \overset{\text{NTT}}{\Longleftrightarrow} \quad F(k) = -F(N-k)$$

P.4) INNT from NTT

$$f(n) = \frac{1}{N}\sum_{k=0}^{N-1} F(k)\,\alpha^{-nk} = \frac{1}{N}\sum_{(-k)=0}^{N-1} F(-k)\,\alpha^{nk} = NTT\ of\ \frac{1}{N}F(-k)$$

- • Algorithm for calculating the INNT from the NTT

(1) $F(-k)$: time reverse

$$F_0, F_1, F_2, \ldots, F_{N\text{-}1} \xrightarrow[\text{reverse}]{\text{time}} F_0, F_{N\text{-}1}, \ldots, F_2, F_1$$

(2) NTT[$F(-k)$]

(3) 乘上 $\dfrac{1}{N} = M-1$

P.5) Shift Theorem

$$f(n+\ell) \leftrightarrow F(k)\,\alpha^{-\ell k}$$

$$f(n)\,\alpha^{n\ell} \leftrightarrow F(k+\ell)$$

P.6) Circular Convolution (the same as that of the DFT)

If $f(n) \leftrightarrow F(k)$

$g(n) \leftrightarrow G(k)$

then $f(n) \otimes g(n) \leftrightarrow F(k)G(k)$

i.e., $f(n) \otimes g(n) = INTT\left\{NTT\left[f(n)\right]NTT\left[g(n)\right]\right\}$

$$f(n) \cdot g(n) \leftrightarrow \frac{1}{N} F(k) \otimes G(k)$$

P.7) Parseval's Theorem

$$N\sum_{n=0}^{N-1} f(n)\,f(-n) = \sum_{k=0}^{N-1} F^2(k)$$

$$N\sum_{n=0}^{N-1} f(n)^2 = \sum_{k=0}^{N-1} F(k)F(-k)$$

P.8) Linearity

$$a\ f(n) + b\ g(n) \leftrightarrow a\ F(k) + b\ G(k)$$

P.9) Reflection

If $\quad f(n) \leftrightarrow F(k) \quad$ then $\quad f(-n) \leftrightarrow F(-k)$

◉ 14-D Efficient FFT-Like Structures for Calculating NTTs

- If N (transform length) is a power of 2, then the radix-2 FFT butterfly algorithm can be used for efficient calculation for NTT.

  Decimation-in-time NTT

  Decimation-in-frequency NTT

- The prime factor algorithm can also be applied for NTTs.

$$F(k)=\sum_{n=0}^{N-1} f(n)\,\alpha^{nk}=\sum_{r=0}^{\frac{N}{2}-1} f(2r)\,\alpha^{2rk}+\sum_{r=0}^{\frac{N}{2}-1} f(2r+1)\,\alpha^{(2r+1)k}$$

$$=\sum_{r=0}^{\frac{N}{2}-1} f(2r)\,(\alpha^2)^{rk}+\alpha^k\sum_{r=0}^{\frac{N}{2}-1} f(2r+1)\,(\alpha^2)^{rk}$$

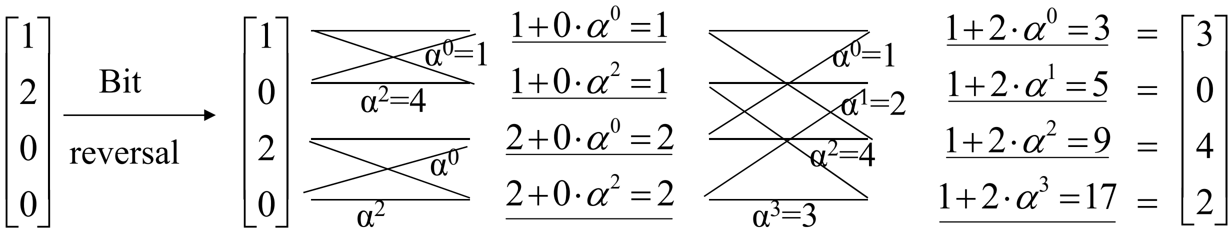
$$=\begin{cases} G(k)+\alpha^k\,H(k) & ,0\le k\le \dfrac{N}{2}-1 \\ G(k-\dfrac{N}{2})+\alpha^k\,H(k-\dfrac{N}{2}) & ,\dfrac{N}{2}\le k\le N \end{cases}$$

where $G(k)=\sum_{r=0}^{N/2-1} f(2r)\,(\alpha^2)^{rk}$ $\quad H(k)=\sum_{r=0}^{N/2-1} f(2r+1)\,(\alpha^2)^{rk}$

One *N*-point NTT $\longrightarrow$ Two (*N*/2)-point NTTs

plus twiddle factors

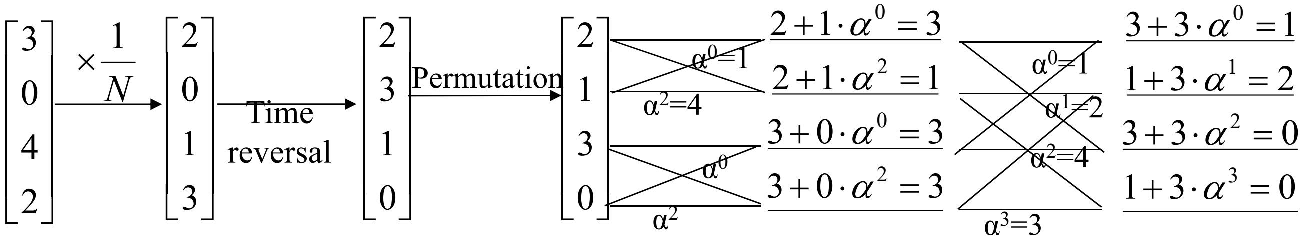
Original sequence	$f(n) = (1, 2, 0, 0)$	$N = 4, M = 5$
Permutation	$(1, 0, 2, 0)$	
After the 1st stage	$(1, 1, 2, 2)$	
After the 2nd stage	$F(k) = (3, 0, 4, 2)$	



Inverse NTT by Forward NTT :

$$F(-k)\times\frac{1}{N} \quad (4^{-1}=4)$$

1) $1/N$
2) Time reversal
3) permutation
4) After first stage
5) After 2nd stage



⦿ 14-E Convolution by NTT

假設 $x[n] = 0$ for $n < 0$ and $n \geq K$, $h[n] = 0$ for $n < 0$ and $n \geq H$

要計算 $x[n] * h[n] = z[n]$

且 $z[n]$ 的值可能的範圍是 $0 \leq z[n] < A$ (more general, $A_1 \leq z[n] < A_1 + T$)

(1) 選擇 M (the prime number for the modulus operator), 滿足

(a) M is a prime number, (b) $M \geq \max(H+K, A)$

(2) 選擇 N (NTT 的點數), 滿足

(a) N is a factor of $M-1$, (b) $N \geq H+K-1$

(3) 添 0: $x_1[n] = x[n]$ for $n = 0, 1, \ldots\ldots, K-1$,
$x_1[n] = 0$ for $n = K, K+1, \ldots\ldots., N-1$

$h_1[n] = h[n]$ for $n = 0, 1, \ldots\ldots, H-1$,
$h_1[n] = 0$ for $n = H, H+1, \ldots\ldots., N-1$

(4) $X_1[m] = \text{NTT}_{N,M}\{x_1[n]\}$, $H_1[m] = \text{NTT}_{N,M}\{h_1[n]\}$

$\text{NTT}_{N,M}$ 指 N-point 的 DFT (mod M)

(5) $Z_1[m] = X_1[m]H_1[m]$, $z_1[n] = \text{INTT}_{N,M}\{Z_1[m]\}$,

(6) $z[n] = z_1[n]$ for $n = 0, 1, \ldots., H+K-1$

(移去 $n = H+K, H+K+1, \ldots\ldots N-1$ 的點)

(More general, if we have estimated the range of $z[n]$ should be $A_1 \le z[n] < A_1 + T$, then

$z[n] = ((z_1[n] - A_1))_M + A_1$

適用於 (1) $x[n]$，$h[n]$皆為整數　　$x[n] * h[n]$

(2) $\text{Max}(z[n]) - \min(z[n]) < M$ 的情形。

$M-1$　　0

Consider the convolution of (1, 2, 3, 0) * (1, 2, 3, 4)

Choose $M = 17$, $N = 8$，結果為：

1	2	3	4		
	2	4	6	8	
		3	6	9	12
1	4	10	16	17	12

by NTT ($M=17$) → 1 4 10 16 0 12

($M=41$) → 1 4 10 16 17 12

- Max(z[n]) − min(z[n]) 的估測方法

假設 $x_1 \le x[n] \le x_2$, $\quad z[n] = x[n] * h[n] = \sum_{m=0}^{H-1} h[m]x[n-m]$

則 $\quad Max\left(z[n]\right) - \min\left(z[n]\right) = \left(x_2 + x_1\right)\sum_{n=0}^{H-1} \left|h[n]\right|$

(Proof): $\quad Max\left(z[n]\right) = \sum_{m=0}^{H-1} h_1[m]x_2 + \sum_{m=0}^{H-1} h_2[m]x_1$

where $h_1[m] = h[m]$ when $h[m] > 0$, $\quad h_1[m] = 0$ otherwise

$h_2[m] = h[m]$ when $h[m] < 0$, $\quad h_2[m] = 0$ otherwise

$$\min\left(z[n]\right) = \sum_{m=0}^{H-1} h_1[m]x_1 + \sum_{m=0}^{H-1} h_2[m]x_2$$

$$Max\left(z[n]\right) - \min\left(z[n]\right) = \sum_{m=0}^{H-1} h_1[m](x_2 - x_1) + \sum_{m=0}^{H-1} h_2[m](x_1 - x_2)$$

$$= (x_2 - x_1)\left\{\sum_{m=0}^{H-1} h_1[m] - \sum_{m=0}^{H-1} h_2[m]\right\} = (x_2 - x_1)\sum_{m=0}^{H-1} \left|h[m]\right|$$

◉ 14-F Special Numbers

- **Fermat Number** : $M = 2^{2^p} + 1$
- • $P = 0, 1, 2, 3, 4, 5, \ldots..$

  $M = 3, 5, 17, 257, 65537, \ldots$

Mersenne Number : $M = 2^p + 1$

$P = 1, 2, 3, 5, 7, 13, 17, 19$

$M = 1, 3, 7, 31, 127, 8191, \ldots..$

If $M = 2^p - 1$ is a prime number, p must be a prime number.

However, if p is a prime number, $M = 2^p - 1$ may not be a prime number.

The modulus operations for Mersenne and Fermat prime numbers are very easy for implementation.

$2^k \pm 1$

Example: 25 mod 7

$$\begin{array}{r} 11 \\ 100a\overline{)11001} \\ \underline{100a} \\ 1011 \\ \underline{100a} \\ 12 \\ \downarrow \\ 100 \end{array} \qquad a = -1$$

◉ 14-G Complex Number Theoretic Transform (CNT)

The integer field Z_M can be extended to complex integer field

If the following equation does not have a sol. in Z_M

$$x^2 = -1 \pmod{M}$$ 無解

This means (-1) does not have a square root

When $M = 4k + 1$, there is a solution for $x^2 = -1 \pmod{M}$.

When $M = 4k + 3$, there is no solution for $x^2 = -1 \pmod{M}$.

For example, when $M = 13$, $8^2 = -1 \pmod{13}$.

$2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 3$, $2^5 = 6$, $2^6 = 12 = -1$,

$2^7 = 11$, $2^8 = 9$, $2^9 = 5$, $2^{10} = 10$, $2^{11} = 7$, $2^{12} = 1$

When $M = 11$, there is no solution for $x^2 = -1 \pmod{M}$.

If there is no solution for $x^2 = -1 \pmod{M}$, we can define an imaginary number i such that

$$i^2 = -1 \pmod{M}$$

Then, "i" will play a similar role over finite field Z_M such that plays over the complex field.

$$(a + i\,b) \pm (c + i\,d) = (a \pm c) + i\,(b \pm d)$$

$$\begin{aligned}(a + i\,b) \cdot (c + i\,d) &= ac + i^2 bd + i\,bc + i\,ad \\ &= (ac - bd) + i\,(bc + ad)\end{aligned}$$

◉ 14-H Applications of the NTT

NTT 適合作 convolution

但是有不少的限制

新的應用： encryption (密碼學) $x[n] \xrightarrow[M,\alpha]{NTT} x_1[n] \xrightarrow[M_1,\alpha_1]{NTT} x_2[n] \xrightarrow[\cdots]{} \cdots$

☆ CDMA

References:

(1) R. C. Agavard and C. S. Burrus, “Number theoretic transforms to implement fast digital convolution,” *Proc. IEEE*, vol. 63, no. 4, pp. 550-560, Apr. 1975.

(2) T. S. Reed & T. K. Truoay, ”The use of finite field to compute convolution,” *IEEE Trans. Info. Theory*, vol. IT-21, pp.208-213, March 1975

(3) E. Vegh and L. M. Leibowitz, “Fast complex convolution in finite rings,” *IEEE Trans ASSP,* vol. 24, no. 4, pp. 343-344, Aug. 1976.

(4) J. H. McClellan and C. M. Rader, *Number Theory in Digital Signal Processing*, Prentice-Hall, New Jersey, 1979.

(5) 華羅庚, “數論導引,” 凡異出版社, 1997。

XIV. Orthogonal Transform and Multiplexing

◉ 14-A Orthogonal and Dual Orthogonal

Any $M \times N$ discrete linear transform can be expressed as the matrix form:

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[M-1] \end{bmatrix}}_{\mathbf{Y}=} = \underbrace{\begin{bmatrix} \phi_0^*[0] & \phi_0^*[1] & \phi_0^*[2] & \cdots & \phi_0^*[N-1] \\ \phi_1^*[0] & \phi_1^*[1] & \phi_1^*[2] & \cdots & \phi_1^*[N-1] \\ \phi_2^*[0] & \phi_2^*[1] & \phi_2^*[2] & \cdots & \phi_2^*[N-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{M-1}^*[0] & \phi_{M-1}^*[1] & \phi_{M-1}^*[2] & \cdots & \phi_{M-1}^*[N-1] \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\mathbf{X}}$$

$$y[m] = \langle x[n], \phi_m[n] \rangle = \sum_{n=0}^{N-1} x[n]\phi_m^*[n]$$

inner product

Orthogonal: $\langle \phi_k[n], \phi_h[n] \rangle = \sum_{n=0}^{N-1} \phi_k[n]\phi_h^*[n] = 0$ **when *k* ≠ *h***

orthogonal transforms 的例子：

- discrete Fourier transform
- discrete cosine, sine, Hartley transforms
- Walsh Transform, Haar Transform
- discrete Legendre transform
- discrete orthogonal polynomial transforms

  Hahn, Meixner, Krawtchouk, Charlier

為什麼在信號處理上，我們經常用 orthogonal transform?

Orthogonal transform 最大的好處何在？

If $\phi_1[n]$ $\phi_2[n]$, ... $\phi_n[n]$ are orthogonal

ie; $\langle \phi_m[n], \phi_k[n] \rangle = 0$ for $m \neq k$

modulation

$c[n] = \sum_k a_k \phi_k[n]$ a_k: the data to be transmitted

demodulation

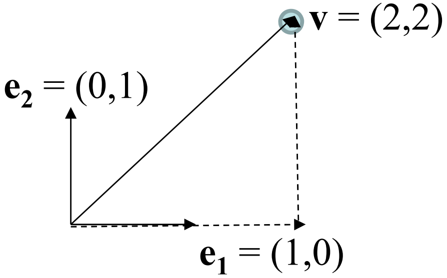
$\langle c[n], \phi_m[n] \rangle = \sum_k a_k \langle \phi_k[n], \phi_m[n] \rangle = a_k \langle \phi_m[n], \phi_m[n] \rangle$

$a_k = \dfrac{\langle c[n], \phi_m[n] \rangle}{\langle \phi_m[n], \phi_m[n] \rangle}$

$\langle x[n], y[n] \rangle$

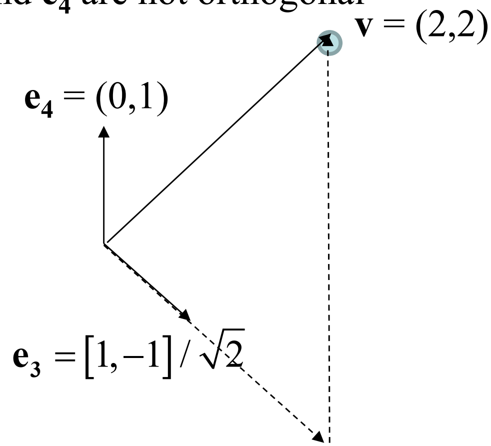
x[n] 跟 y[n] 做內積

$\mathbf{e_1}$ and $\mathbf{e_2}$ are orthogonal



$$\mathbf{v} = 2\mathbf{e_1} + 2\mathbf{e_2}$$

$\mathbf{e_3}$ and $\mathbf{e_4}$ are not orthogonal



$$\mathbf{v} = 2\sqrt{2}\mathbf{e_3} + 4\mathbf{e_4}$$

- If partial terms are used for reconstruction

for orthogonal case,

perfect reconstruction: $x[n] = \sum_{m=0}^{N-1} C_m^{-1} y[m] \phi_m[n]$

partial reconstruction: $x_K[n] = \sum_{m=0}^{K-1} C_m^{-1} y[m] \phi_m[n] \qquad K < N$

reconstruction error of partial reconstruction

$$\begin{aligned}
\left\| x[n] - x_K[n] \right\|^2 &= \sum_{n=0}^{N-1} \left\| \sum_{m=K}^{N-1} C_m^{-1} y[m] \phi_m[n] \right\|^2 \\
&= \sum_{n=0}^{N-1} \sum_{m=K}^{N-1} C_m^{-1} y[m] \phi_m[n] \sum_{m_1=K}^{N-1} C_{m_1}^{-1} y^*[m_1] \phi_{m_1}^*[n] \\
&= \sum_{m=K}^{N-1} \sum_{m_1=K}^{N-1} C_m^{-1} y[m] C_{m_1}^{-1} y^*[m_1] \sum_{n=0}^{N-1} \phi_m[n] \phi_{m_1}^*[n] \\
&= \sum_{m=K}^{N-1} \sum_{m_1=K}^{N-1} C_m^{-1} y[m] C_{m_1}^{-1} y^*[m_1] C_m \delta[m - m_1] = \sum_{m=K}^{N-1} C_m^{-1} \left| y[m] \right|^2
\end{aligned}$$

由於 $C_m^{-1} \left| y[m] \right|^2$ 一定是正的，可以保證 <u>K 越大, reconstruction error 越小</u>

For non-orthogonal case,

perfect reconstruction: $x[n] = \sum_{m=0}^{N-1} B[n,m]\,y[m]$ $\mathbf{B} = \mathbf{A}^{-1}$

partial reconstruction: $x_K[n] = \sum_{m=0}^{K-1} B[n,m]\,y[m]$ $K < N$

reconstruction error of partial reconstruction

$$\left\| x[n] - x_K[n] \right\|^2 = \sum_{n=0}^{N-1} \left\| \sum_{m=K}^{N-1} B[n,m]\,y[m] \right\|^2$$

$$= \sum_{n=0}^{N-1} \sum_{m=K}^{N-1} B[n,m]\,y[m] \sum_{m_1=K}^{N-1} B^*[n,m_1]\,y^*[m_1]$$

$$= \sum_{m=K}^{N-1} \sum_{m_1=K}^{N-1} y[m]\,y^*[m_1] \sum_{n=0}^{N-1} B[n,m]\,B^*[n,m_1]$$

由於 $y[m]y^*[m_1]\sum_{n=0}^{N-1} B[n,m]B^*[n,m_1]$ 不一定是正的，

無法保證 K 越大, reconstruction error 越小

◎ 14-B Frequency and Time Division Multiplexing

傳統 Digital Modulation and Multiplexing：使用 Fourier transform

- **Frequency-Division Multiplexing (FDM)**

$$z(t) = \sum_{n=0}^{N-1} X_n \exp\left(j 2\pi f_n t\right)$$

$X_n = 0$ or 1

$\int_{-\infty}^{\infty} e^{-j2\pi f_n t} \left(e^{-j2\pi f_m t}\right)^* dt = 0$ if $m \neq n$

X_n can also be set to be -1 or 1

orthogonal for $t \in [-\infty, \infty]$

When (1) $t \in [0, T]$ $\boxed{(2)\ f_n = n/T}$ constraint for f_n

$$z(t) = \sum_{n=0}^{N-1} X_n \exp\left(j \frac{2\pi n t}{T}\right)$$

it becomes the **orthogonal frequency-division multiplexing (OFDM) in the continuous case.**

Furthermore, if the time-axis is also sampled

$$\boxed{t = mT/N, \qquad m = 0,\ 1,\ 2,\ \ldots\ldots,\ N{-}1}$$

$t \in [0,T]$
sampling for t-axis

$$z\left(m\frac{T}{N}\right) = \sum_{n=0}^{N-1} X_n \exp\left(j\frac{2\pi nm}{N}\right)$$

跟 DFT 差 -j

(OFDM in the discrete case)

then the OFDM is equivalent to the transform matrix of the inverse discrete Fourier transform (IDFT), which is one of the discrete orthogonal transform.

Modulation: $Y_m = z\left(m\frac{T}{N}\right) = \sum_{m=0}^{N-1} A[m,n]X_n$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & e^{j\frac{4\pi}{N}} & \cdots & e^{j\frac{2(N-1)\pi}{N}} \\ 1 & e^{j\frac{4\pi}{N}} & e^{j\frac{8\pi}{N}} & \cdots & e^{j\frac{4(N-1)\pi}{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\frac{2(N-1)\pi}{N}} & e^{j\frac{4(N-1)\pi}{N}} & \cdots & e^{j\frac{2(N-1)(N-1)\pi}{N}} \end{bmatrix}$$

Modulation: $$Y_m = \sum_{m=0}^{N-1} A[m,n] X_n$$

(IDFT)

Demodulation: $$X_n = \frac{1}{N} \sum_{m=0}^{N-1} A^*[m,n] Y_m$$

(DFT)

$$\begin{bmatrix} i & i \\ \vdots & i \\ \vdots & \vdots \\ & \vdots \end{bmatrix}\begin{bmatrix} x \end{bmatrix}$$

Example: $N = 8$

$X_n = [1, 0, 1, 1, 0, 0, 1, 1]$ $\quad (n = 0 \sim 7)$

• Time-Division Multiplexing (TDM)

$$z(0) = X_0, \quad z\left(\frac{T}{N}\right) = X_1, \quad z\left(2\frac{T}{N}\right) = X_2, \quad \cdots\cdots, \quad z\left((N-1)\frac{T}{N}\right) = X_{N-1}$$

$$y(m) = z\left(m\frac{T}{N}\right) = \sum_{m=0}^{N-1} A[m,n]X_n$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$ (also a discrete orthogonal transform)

思考：

既然 time-division multiplexing 那麼簡單

那為什麼要使用 frequency-division multiplexing
和 orthogonal frequency-division multiplexing (OFDM)?

◉ 14-C Code Division Multiple Access (CDMA)

Any orthogonal transform

除了 frequency-division multiplexing 和 time-division multiplexing，是否還有其他 multiplexing 的方式？

IDFT (for OFDM)

identity

使用其他的 orthogonal transforms

即 code division multiple access (CDMA)

CDMA is an important topic in spread spectrum communication

參考資料

[1] M. A. Abu-Rgheff, *Introduction to CDMA Wireless Communications*, Academic, London, 2007

[2] 邱國書, 陳立民譯, "CDMA 展頻通訊原理", 五南, 台北, 2002.

CDMA 最常使用的 orthogonal transform 為 Walsh transform

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \begin{array}{l} \text{channel 1} \\ \text{channel 2} \\ \text{channel 3} \\ \text{channel 4} \\ \text{channel 5} \\ \text{channel 6} \\ \text{channel 7} \\ \text{channel 8} \end{array}$$

channel 1

當有兩組人在同一個房間裡交談 (A 和B交談)，(C 和D交談) ，

如何才能夠彼此不互相干擾？

(1) Different Time

(2) Different Tone

(3) Different Language

CDMA 分為：

(1) Orthogonal Type (2) Pseudorandom Sequence Type

Orthogonal Type 的例子： 兩組資料 [1, 0, 1] [1, 1, 0]

(1) 將 0 變為 −1 [1, −1, 1] [1, 1, −1]

(2) 1, −1, 1 modulated by [1, 1, 1, 1, 1, 1, 1, 1] (channel 1)

→ [1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1]

1, 1, −1 modulated by [1, 1, 1, 1, -1, -1, -1, -1] (channel 2)

→ [1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1]

(3) 相合

[2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, -2, -2, 0, 0, 0, 0, 2, 2, 2, 2]

output

$\frac{6}{8} = 0.75$ $\frac{-6}{8} = -0.75$ $\frac{4}{8} = 0.5 \Rightarrow 1$

demodulation

0 2 2 2 2 0 0 0 0 0 0 0 -2 -2 -2 -2 0 0 0 0 2 2 2

[2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, -2, -2, 0, 0, 0, 0, 2, 2, 2, 2]

[1, 1, 1, 1, 1, 1, 1, 1] [1, 1, 1, 1, 1, 1, 1,1]

[1, 1, 1, 1, 1, 1, 1, 1]

內積 = 8 $\frac{8}{8} = 1$ $\frac{-8}{8} = -1$ $\frac{8}{8} = 1$

[1, -1, 1]

[1, 0, 1]

$$\frac{\langle c[n], a_n[n] \rangle}{8}$$

2nd Channel

[1 1 1 1 -1 -1 -1 -1] [1 1 1 1 -1 -1 -1 -1] [1 1 1 1 -1 -1 -1 -1]

$\frac{8}{8} = 1$ $\frac{8}{8} = 1$ $\frac{-8}{8} = -1$

[1 1 -1] $\xrightarrow[\text{modulation}]{}$ [1 1 0]

$\frac{4}{8} = 0.5 \rightarrow 1$ $\frac{6}{8} = 0.75 \rightarrow 1$ $\frac{-4}{8} = -0.5 \rightarrow -1$

Use Walsh transform 容錯率高

注意：

(1) 使用 N-point Walsh transform 時，總共可以有N 個 channels

(2) 除了 Walsh transform 以外，其他的 orthogonal transform 也可以使用

(3) 使用 Walsh transform 的好處

• Orthogonal Transform 共通的問題: 需要同步 synchronization

$\mathbf{R_1}$ = [1, 1, 1, 1, 1, 1, 1, 1]

$\mathbf{R_2}$ = [1, 1, 1, 1, –1, –1, –1, –1]

$\mathbf{R_5}$ = [1, –1, –1, 1, 1, –1, –1, 1]

$\mathbf{R_8}$ = [1, –1, 1, –1, 1, –1, 1, –1]

但是某些 basis, 就算不同步也近似 orthogonal

$<\mathbf{R_1}[n], \mathbf{R_1}[n]> = 8$, $<\mathbf{R_1}[n], \mathbf{R_k}[n]> = 0$ if $k \neq 1$

$<\mathbf{R_1}[n], \mathbf{R_k}[n-1]> = 2$ or 0 if $k \neq 1$.

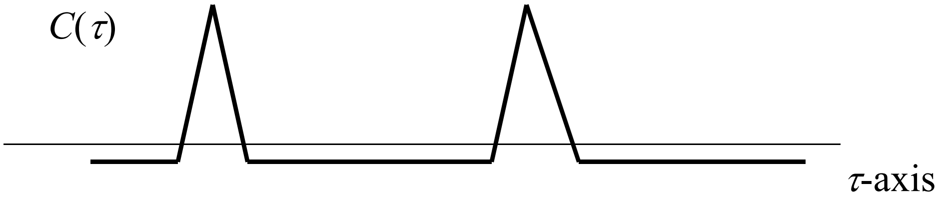
這裡的shift為circular shift

Pseudorandom Sequence Type

不為 orthogonal，capacity 較少

但是不需要同步 (asynchronous)

Pseudorandom Sequence 之間的 correlation



$b_1 p(t+\tau_1) + b_2 p(t+\tau_2)$

recovered: $\int \left(b_1 p(t+\tau_1) + b_2 p(t+\tau_2)\right) p(t+\tau_1)\, dt = b_1 C\left(0\right) + b_2 C\left(\tau_2 - \tau_1\right) \approx b_1$

(若 $C(0) = 1$, $C(\tau_2 - \tau_1) \approx 0$)

τ_1, τ_2 不必一致

CDMA 的優點：

(1) 運算量相對於 frequency division multiplexing 減少很多

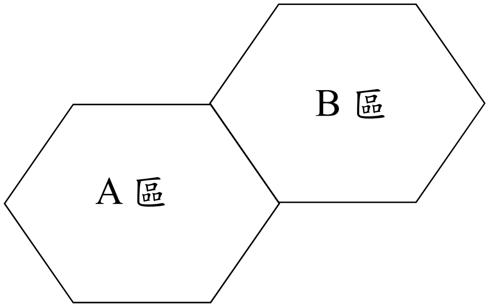
(2) 可以減少 noise 及 interference的影響

(3) 可以應用在保密和安全傳輸上

(4) 就算只接收部分的信號，也有可能把原來的信號 recover 回來

(5) 相鄰的區域的干擾問題可以減少

相鄰的區域，使用差距最大的「語言」，則干擾最少



假設 A 區使用的 orthogonal basis 為 $\phi_k[n]$, $k = 0, 1, 2, \ldots, N-1$

B 區使用的 orthogonal basis 為 $\mu_h[n]$, $h = 0, 1, 2, \ldots, N-1$

設法使 $\max\left(\left|\frac{\langle \phi_k[n], \mu_h[n] \rangle}{\langle \phi_k[n], \phi_h[n] \rangle}\right|\right)$ 為最小

$k = 0, 1, 2, \ldots, N-1$, $h = 0, 1, 2, \ldots, N-1$

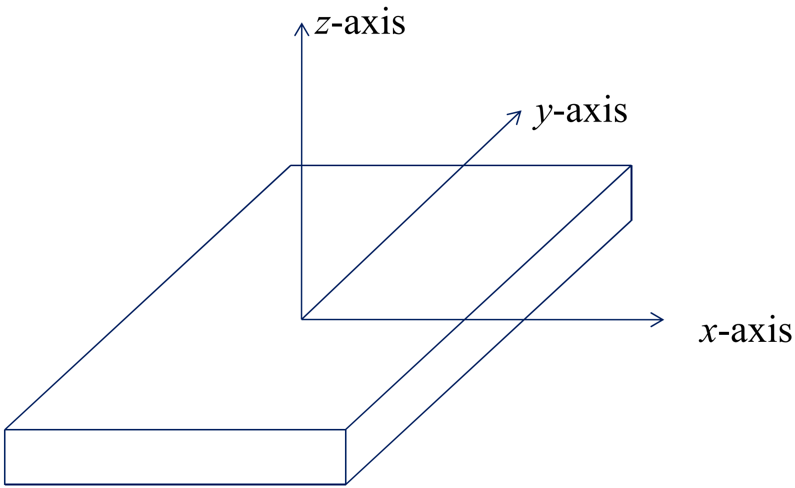
附錄十四 3-D Accelerometer 的簡介

3-D Accelerometer: 三軸加速器，或稱作加速規

許多儀器(甚至包括智慧型手機) 都有配置三軸加速器

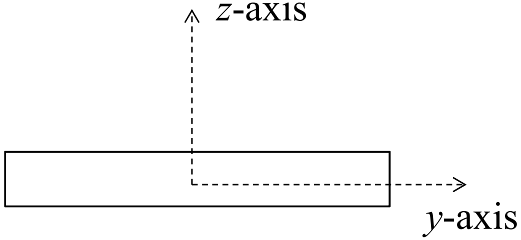
可以用來判別一個人的姿勢和動作

註：Gyrator (陀螺儀) 可以用來量測物體旋轉之方向，可補 3-D Accelerometer 之不足，許多儀器 (包括智慧型手機) 也內建陀螺儀之裝置， 3-D Accelerometer Signal Processing 和 gyrator signal processing 經常並用



根據 x, y, z 三個軸的加速度的變化，來判斷姿勢和動作

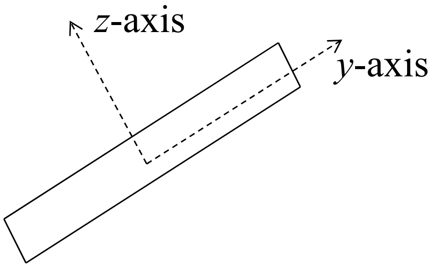
平放且靜止時，z-axis 的加速度為 $-g = -9.8$



y: 0

z: -9.8

tilted by θ



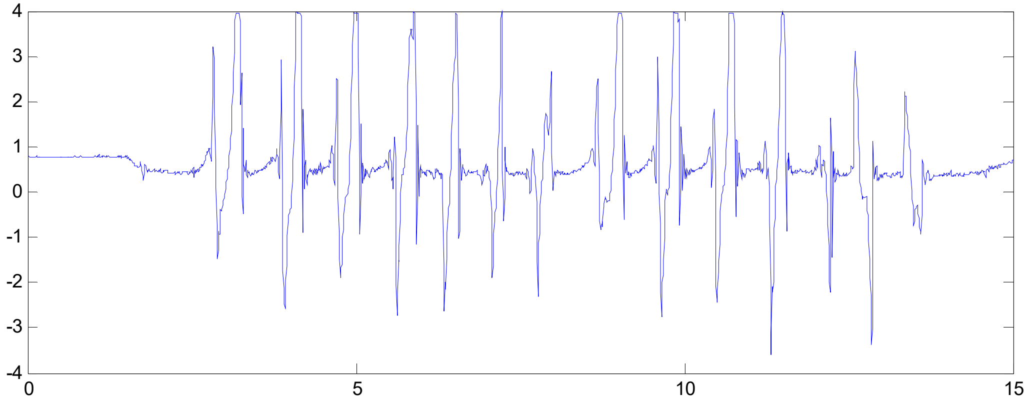
y: $-9.8\sin\theta$

z: $-9.8\cos\theta$

可藉由加速規傾斜的角度，來判斷姿勢和動作

例子：若將加速規放在腳上................

走路時，沿著其中一個軸的加速度變化



應用： 動作辨別

運動 (訓練，計步器)

醫療復健，如 Parkinson 患者照顧，傷患復原情形

其他 (如動物的動作，機器的運轉情形的偵測)

3-D Accelerometer Signal Processing 是訊號處理的重要課題之一

一方面固然是因為應用多，另一方面， 3-D Accelerometer Signal 容易受 noise 之干擾，要如何藉由 3-D Accelerometer Signal 來還原動作以及移動速度，仍是個挑戰

祝各位同學暑假愉快！

各位同學在研究上或工作上，有任何和 digital signal processing 或 time frequency analysis 方面的問題，歡迎找我來一起討論。