2. Digital Filter Design (A)

任何可以用來去除 noise 作用的 operation，皆被稱為 filter

甚至有部分的 operation，雖然主要功用不是用來去除 noise，但是可以用 $\underbrace{\text{FT + multiplication + IFT}}_{\parallel}$ 來表示，也被稱作是 filter

convolution, LTI system

Reference

[1] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

[2] D. G. Manolakis and V. K. Ingle, *Applied Digital Signal Processing*, Cambridge University Press, Cambridge, UK. 2011.

[3] B. A. Shenoi, *Introduction to Digital Signal Processing and Filter Design*, Wiley-Interscience, N. J., 2006.

[4] A. A. Khan, *Digital Signal Processing Fundamentals*, Da Vinci Engineering Press, Massachusetts, 2005.

[5] S. Winder, *Analog and Digital Filter Design*, 2nd Ed., Amsterdam, 2002.

◎ 2-A Classification for Filters

- filter
  - digital filter
    - IIR (infinite impulse response)
      - bilinear transform
      - impulse invariant
      - step invariant
      - others
    - FIR (finite impulse response)
      - 關心平均誤差
        (1) MSE (mean square error)
      - 最小化最大誤差
        (2) Minimax (minimize the maximal error)
      - (3) frequency sampling
      - (4) others
  - analog filter

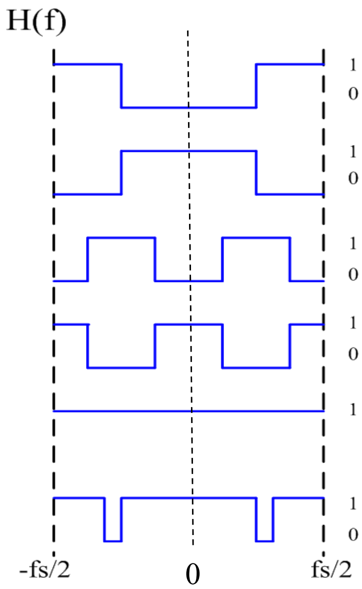
由 數位類比分類

由 response 是否有限長分類

由設計方法分類

Classification for filters (依型態分)

- pass-stop filter
  - high pass filter
  - low pass filter
  - band pass filter
  - band stop filter
  - all pass filter
  - Notch filter
- matched filter
- Wiener filter
- equalizer filter
- others
  - differentiation
  - integration
  - Hilbert transform
  - smoother
  - edge detection
  - Kalman filter
  - particle filter



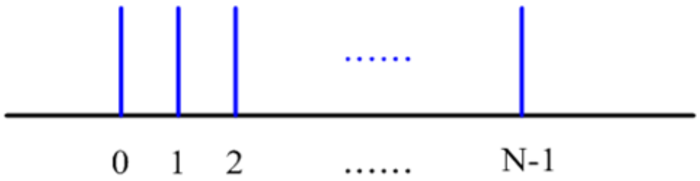
⦿ 2-B FIR Filter Design

FIR filter: impulse response is nonzero at finite number of points

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N$$

($h[n]$ has N points, N is a finite number)

$h[n]$ is causal



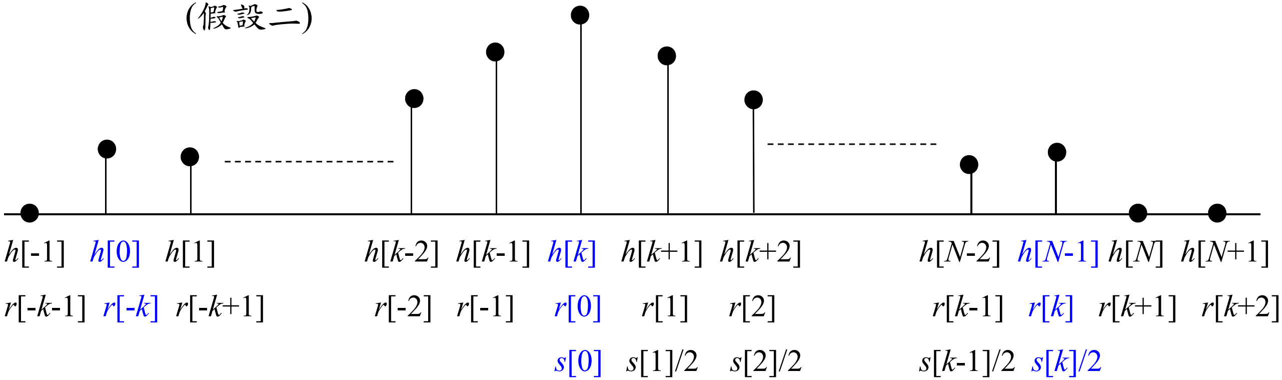
- FIR is more popular because its impulse response is finite.

(假設一)

Specially, when $h[n]$ is even symmetric $h[n] = h[N-1-n]$

and N is an odd number

(假設二)



(a) $r[n] = h[n+k]$, where $k = (N-1)/2$.

(b) $s[0] = r[0]$, $s[n] = 2r[n]$ for $0 < n \le k$.

Impulse Response of the FIR Filter:

$h[n]$ ($h[n] \neq 0$ for $0 \leq n \leq N-1$)
$r[n] = h[n+k]$, $k = (N-1)/2$ ($r[n] \neq 0$ for $-k \leq n \leq k$, see page 42)

Suppose that the filter is **even**, $r[n] = r[-n]$.

Set $s[0] = r[0]$, $s[n] = 2r[n]$ for $n \neq 0$.

Then, the discrete-time Fourier transform of the filter is

$$H(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi F n}$$ ($F = f\Delta_t$ is the **normalized frequency**)

See page 22

$$H(F) = e^{-j2\pi F k} R(F)$$

$$R(F) = \sum_{n=-k}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=-k}^{-1} r[n] e^{-j2\pi F n} + r[0] + \sum_{n=1}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^{k} r[-n] e^{j2\pi F n} + r[0] + \sum_{n=1}^{k} r[n] e^{-j2\pi F n}$$

$$\boxed{R(F) = \sum_{n=0}^{k} s[n] \cos(2\pi n F)}$$

⦿ 2-C Least MSE Form and Minimax Form FIR Filters

(1) least MSE (mean square error) form

(關心 平均 誤差)

$$\text{MSE} = f_s^{-1}\int_{-f_s/2}^{f_s/2}\left|H(f)-H_d(f)\right|^2 df \quad , \quad f_s\text{: sampling frequency}$$

$H(f)$: the spectrum of the filter we obtain

$H_d(f)$: the spectrum of the desired filter

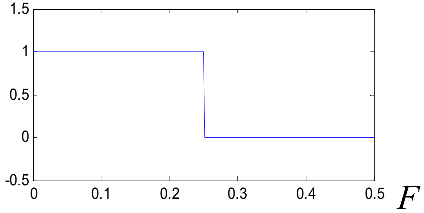
(2) mini-max (minimize the maximal error) form

(關心 最大 誤差)

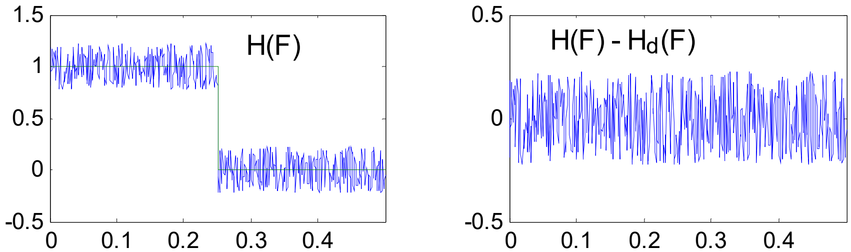
maximal error: $\underset{f}{Max}\left|H(f)-H_d(f)\right|$

The transition band is always ignored

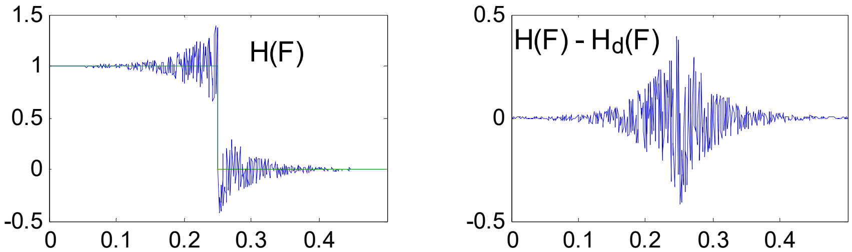
Example: desired output $H_d(F)$



(A) larger MSE, but smaller maximal error



(B) smaller MSE, but larger maximal error



⦿ 2-D Review: FIR Filter Design in the MSE Sense

$$R(F)=\sum_{n=0}^{k} s[n]\cos(2\pi nF)$$

$$MSE = f_s^{-1}\int_{-f_s/2}^{f_s/2}\left|R(f)-H_d(f)\right|^2 df = \int_{-1/2}^{1/2}\left|R(F)-H_d(F)\right|^2 dF$$

$F = f/f_s$

$$=\int_{-1/2}^{1/2}\left|\sum_{n=0}^{k} s[n]\cos(2\pi nF)-H_d(F)\right|^2 dF$$

$$=\int_{-1/2}^{1/2}\left(\sum_{n=0}^{k} s[n]\cos(2\pi nF)-H_d(F)\right)\left(\sum_{\tau=0}^{k} s[\tau]\cos(2\pi \tau F)-H_d(F)\right)dF$$

$$=\int_{-1/2}^{1/2}\sum_{n=0}^{k} s[n]\cos(2\pi nF)\sum_{\tau=0}^{k} s[\tau]\cos(2\pi \tau F)\,dF$$

$$-2\int_{-1/2}^{1/2}\sum_{n=0}^{k} s[n]\cos(2\pi nF)H_d(F)\,dF+\int_{-1/2}^{1/2}H_d^2(F)\,dF$$

$$\boxed{\frac{\partial MSE}{\partial s[n]} = 2\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi \tau F)\,dF - 2\int_{-1/2}^{1/2}H_d(F)\cos(2\pi nF)\,dF = 0}$$

$$\frac{\partial MSE}{\partial s[n]} = 2\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2} \cos(2\pi n F)\cos(2\pi \tau F)\,dF - 2\int_{-1/2}^{1/2} H_d(F)\cos(2\pi n F)\,dF = 0$$

From the facts that

$$\int_{-1/2}^{1/2} \cos(2\pi n F)\cos(2\pi \tau F)\,dF = 0 \qquad \text{when } n \neq \tau,$$

$$\int_{-1/2}^{1/2} \cos(2\pi n F)\cos(2\pi \tau F)\,dF = 1/2 \qquad \text{when } n = \tau,\ n \neq 0,$$

$$\int_{-1/2}^{1/2} \cos(2\pi n F)\cos(2\pi \tau F)\,dF = 1 \qquad \text{when } n = \tau,\ n = 0.$$

Therefore,

$$\frac{\partial MSE}{\partial s[0]} = 2s[0] - 2\int_{-1/2}^{1/2} H_d(F)\,dF = 0$$

$$\frac{\partial MSE}{\partial s[n]} = s[n] - 2\int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F)\,dF = 0 \qquad \text{for } n \neq 0.$$

Minimize MSE $\rightarrow$ Make $\frac{\partial MSE}{\partial s[n]} = 0$ for all n's

$$\therefore \boxed{s[0] = \int_{-1/2}^{1/2} H_d(F)\,dF}\,, \qquad \boxed{s[n] = 2\int_{-1/2}^{1/2} \cos(2\pi nF) H_d(F)\,dF}\,.$$

Finally, set $h[k] = s[0]$,

$h[k+n] = s[n]/2$, $h[k-n] = s[n]/2$ for $n = 1, 2, 3, \ldots, k$,

$h[n] = 0$ for $n < 0$ and $n \geq N$.

Then, $h[n]$ is the impulse response of the designed filter.

◉ 2-E FIR Filter Design in the Mini-Max Sense

It is also called "Remez-exchange algorithm"

or "Parks-McClellan algorithm"

References

[1] T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

[2] J. H. McClellan, T. W. Parks, and L. R. Rabiner "A computer program for designing optimum FIR linear phase digital filter", *IEEE Trans. Audio-Electroacoustics*, vol. 21, no. 6, Dec. 1973.

[3] F. Mintz and B. Liu, "Practical design rules for optimum FIR bandpass digital filter", *IEEE Trans. ASSP*, vol. 27, no. 2, Apr. 1979.

[4] E. Y. Remez, "General computational methods of Chebyshev approximation: The problems with linear real parameters," AEC-TR-4491. ERDA Div. Phys. Res., 1962.

Two constraints

Suppose that:

① Filter length = N, N is odd, $N = 2k+1$.

② Frequency response of the **desired filter**: $H_d(F)$ is an even function

(F is the normalized frequency)

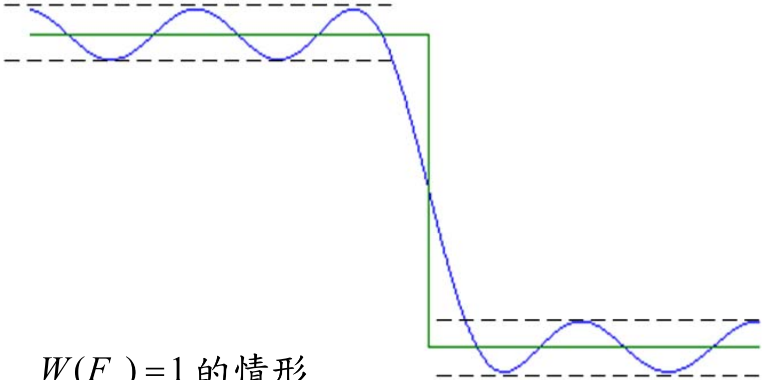
③ The weighting function is $W(F)$

用Mini-Max方法所設計出的filters，一定會滿足以下二個條件

(1) 有k+2個以上的extreme points → **Error 的 local maximal**

local minimum

(2) 在extreme points上，$W(F_m)\,|R(F_m)-H_d(F_m)|$ 是定值



$W(F_m)=1$ 的情形

證明可參考

T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

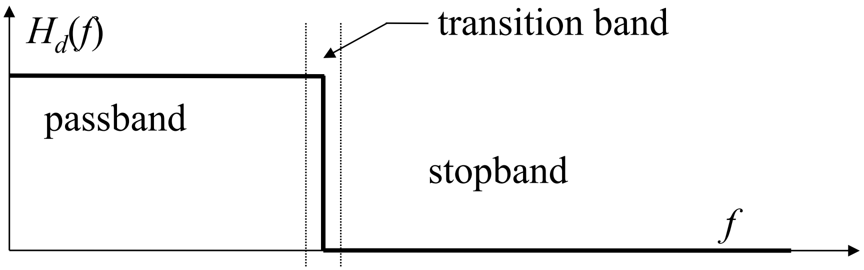
- Generalization for Mini-Max Sense by weight function

maximal error: $\underset{f}{Max}\left|H(f)-H_d(f)\right|$

weighted maximal error: $\underset{f}{Max}\left|W(f)[H(f)-H_d(f)]\right|$

where $W(f)$ is the weight function.

The weight function is designed according to <u>which band is more important</u>.



Q: How do we choose $W(f)$ When SNR ↑ ?

Example: If we treat the passband the same important as the stopband.
$W(f) = 1$ in the passband, $W(f) = 1$ in the stopband

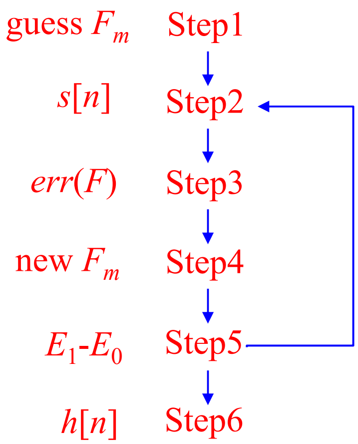
Q1: $W(f) = 1$ in the passband, $W(f) < 1$ in the stopband 代表什麼？

Q2: $W(f) < 1$ in the passband, $W(f) = 1$ in the stopband 代表什麼？

Q3: 如何用來壓縮特定區域 (如 transition band 附近) 的 error？

Q4: Weighting function 的概念可否用在 MSE sense ？

◎ 2-F Mini-Max Design Process



(Step 1): Choose arbitrary k+2 extreme frequencies in the range of

$0 \le F \le 0.5$, (denoted by F_0, F_1, F_2, ….., F_{k+1})

Note: (1) Exclude the transition band.

(2) The extreme points cannot be all in the stop band.

Set E_1 (error) $\rightarrow \infty$

Extreme frequencies:

The locations where the error is maximal.

$$[R(F_0) - H_d(F_0)]W(F_0) = -e \qquad [R(F_1) - H_d(F_1)]W(F_1) = e$$

$$[R(F_2) - H_d(F_2)]W(F_2) = -e \qquad [R(F_3) - H_d(F_3)]W(F_3) = e$$

$$\vdots$$

$$[R(F_{k+1}) - H_d(F_{k+1})]W(F_{k+1}) = (-1)^{k+2} e$$

(參考 page 53)

(Step 2): From page 43, $[R(F_m) - H_d(F_m)]W(F_m) = (-1)^{m+1}e$ (where m = 0, 1, 2, ….. , k+1) can be written as

$$\boxed{\sum_{n=0}^{k} s[n]\cos\left(2\pi F_m n\right) + \left(-1\right)^m W^{-1}\left(F_m\right)e = H_d\left(F_m\right)}$$

where m = 0, 1, 2, ….. , k+1.

Expressed by the matrix form:

$$\begin{bmatrix} 1 & \cos(2\pi F_0) & \cos(4\pi F_0) & \cdots & \cos(2\pi k F_0) & 1/W(F_0) \\ 1 & \cos(2\pi F_1) & \cos(4\pi F_1) & \cdots & \cos(2\pi k F_1) & -1/W(F_1) \\ 1 & \cos(2\pi F_2) & \cos(4\pi F_2) & \cdots & \cos(2\pi k F_2) & 1/W(F_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(2\pi F_k) & \cos(4\pi F_k) & \cdots & \cos(2\pi k F_k) & (-1)^k/W(F_k) \\ 1 & \cos(2\pi F_{k+1}) & \cos(4\pi F_{k+1}) & \cdots & \cos(2\pi k F_{k+1}) & (-1)^{k+1}/W(F_{k+1}) \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ \vdots \\ s[k] \\ e \end{bmatrix} = \begin{bmatrix} H_d[F_0] \\ H_d[F_1] \\ H_d[F_2] \\ \vdots \\ H_d[F_k] \\ H_d[F_{k+1}] \end{bmatrix}$$

Square matrix

<u>Solve $s[0]$, $s[1]$, $s[2]$, ….. , $s[k]$ from the above matrix</u>

(performing the matrix inversion).

(Step 3): Compute err(F) for $0 \le F \le 0.5$, exclude the transition band.

$$err(F) = [R(F) - H_d(F)]W(F) = \{\sum_{n=0}^{k} s[n]\cos(2\pi n F) - H_d(F)\}W(F)$$

(Step 4): Find k+2 **local maximal (or minimal) points** of err(F)

local maximal point: if $q(\tau) > q(\tau + \Delta_F)$ and $q(\tau) > q(\tau - \Delta_F)$,

then τ is a local maximal of $q(x)$.

local minimal point: if $q(\tau) < q(\tau + \Delta_F)$ and $q(\tau) < q(\tau - \Delta_F)$,

then τ is a local minimal of $q(x)$.

Other rules: Page 60

Denote the local maximal (or minimal) points by $P_0, P_1, \ldots\ldots, P_k, P_{k+1}$

These k+2 extreme points could include the boundary points of the transition band

(Step 5):

Set E_0 = Max(|err(F)|).

$$\begin{cases} E_0: \text{現在的} Max\,|\,err(F)\,| \\ E_1: \text{前一次iteration的} Max\,|\,err(F)| \end{cases}$$

(Case a) If $E_1 - E_0 > \Delta$, or $E_1 - E_0 < 0$ (or the first iteration) $\rightarrow$

set $F_n = P_n$ and $E_1 = E_0$, return to Step 2.

(Case b) If $0 \le E_1 - E_0 \le \Delta$ $\rightarrow$ continue to Step 6.

(Step 6):

Set $h[k] = s[0]$,

$h[k+n] = s[n]/2$, $h[k-n] = s[n]/2$ for n = 1, 2, 3, …., k

(referred to page 42)

Then $h[n]$ is the impulse response of the designed filter.

◉ 2-G Mini-Max FIR Filter 設計時需注意的地方

(1) Extreme points 不要選在 transition band

Initial guess的extreme points只要注意別取在transition band裡，即能保證 converge，不同的guess會影響converge的速度但不影響結果

(2) E_1 (error of the previous iteration) < E_0 (present error) 時，亦不為收斂

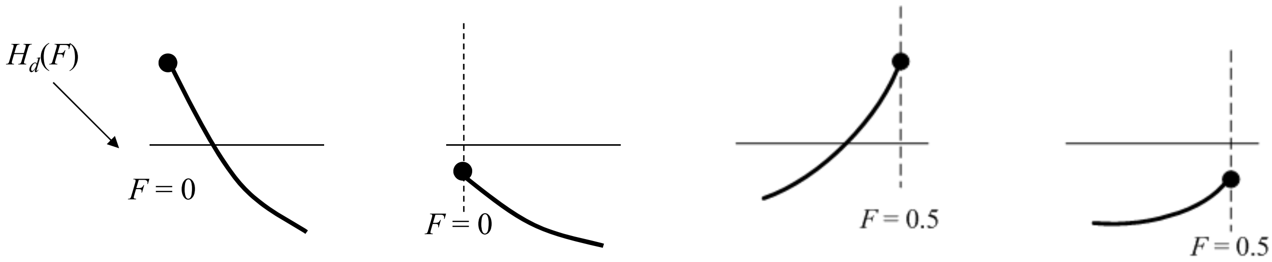
(3) 思考： page 56 中的 matrix operation，如何用一行的指令寫出？

(4) Extreme points 判斷的規則：

(a) The local peaks or local dips that are not at boundaries must be extreme points.

boundaries: $F = 0$, $F = 0.5$, 以及 transition band 的兩端

(b) For boundary points



Add a zero to the outside and conclude whether the point is a local maximum or a local minimum.

(5) 有時，會找到多於 k+2 個 extreme points, 該如何選 $P_0, P_1, \ldots\ldots, P_k, P_{k+1}$

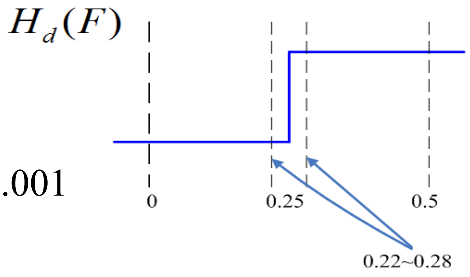
(a) 優先選擇不在 boundaries 的 extreme points

(b) 其次選擇 boundary extreme points 當中 |err(F)| 較大的，

直到湊足 k+2 個 extreme points 為止

◉ 2-H Examples for Mini-Max FIR Filter Design

- **Example 1:** Design a 9-length highpass filter in the mini-max sense



ideal filter: $H_d(F) = 0$ for $0 \le F < 0.25$,

$H_d(F) = 1$ for $0.25 < F \le 0.5$,

transition band: $0.22 < F < 0.28$ $\Delta = 0.001$

weighting function: $W(F) = 0.25$ for $0 \le F \le 0.22$,

$W(F) = 1$ for $0.28 \le F \le 0.5$,

(**Step 1**) Since $N = 9$, $k = (N\text{-}1)/2 = 4$, $k+2 = 6$,

$\rightarrow$ **Choose 6 extreme frequencies**

(e.g., $F_0 = 0$, $F_1 = 0.1$, $F_2 = 0.2$, $F_3 = 0.3$, $F_4 = 0.4$, $F_5 = 0.5$)

$$[R(F_n) - H_d(F_n)]W(F_n) = (-1)^{n+1}e, \qquad n = 0, 1, 2, 3, 4, 5.$$

(Step 2)

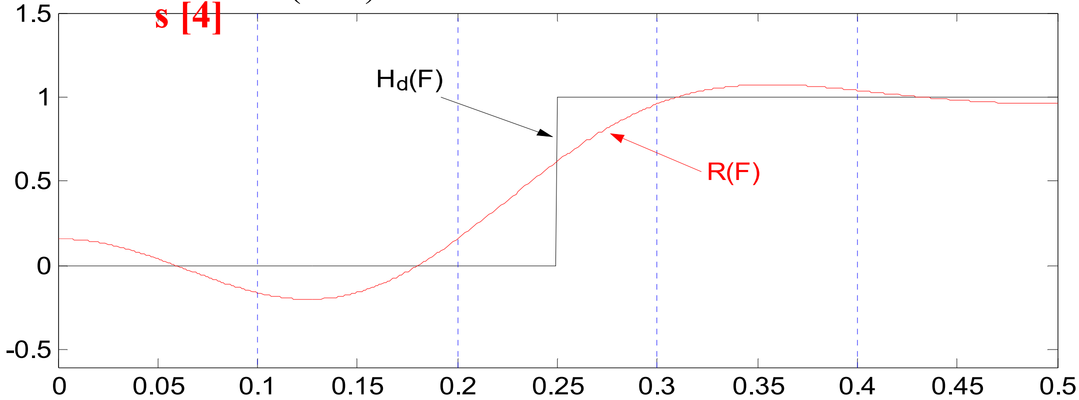
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5120 \\ -0.6472 \\ -0.0297 \\ 0.2472 \\ 0.0777 \\ -0.040 \end{bmatrix}$$

s [0] s [1] s [2] s [3]

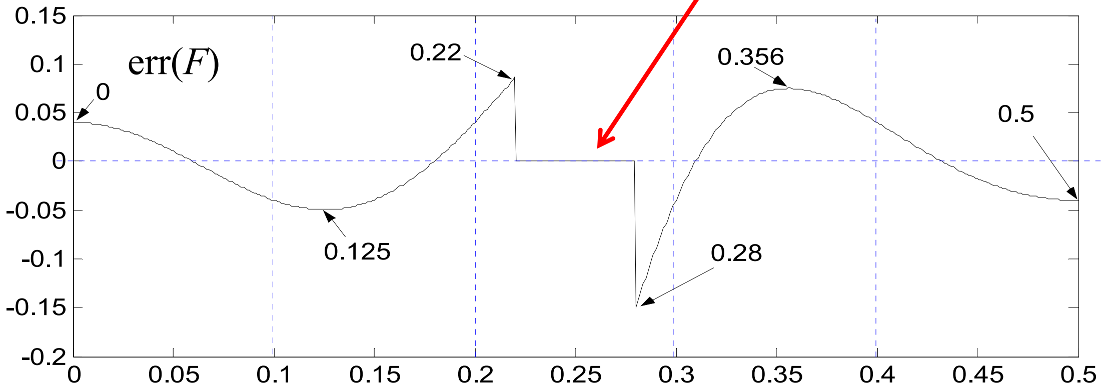
$$R(F) = 0.5120 - 0.6472\cos(2\pi F) - 0.0297\cos(4\pi F) + 0.2472\cos(6\pi F) + 0.0777\cos(8\pi F)$$

s [4]



(Step 3) $\mathbf{err}(F) = [R(F) - H_d(F)]W(F)$

$W[F] = 0$ for $0.22 < F < 0.28$



(Step 4) Extreme points:

$F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$

(Step 5) E_0 **=** $\text{Max}[|\text{err}(F)|] = \underline{0.1501}$, return to Step 2.

Second iteration

(Step 2) Using $F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$

$\rightarrow s[0] = 0.5018, s[1] = -0.6341, s[2] = -0.0194, s[3] = 0.3355,$
$s[4] = 0.1385$

(Step 3) $\text{err}(F) = [R(F) - H_d(F)]W(F)$,

(Step 4) extreme points : 0, 0.132, 0.22, 0.28, 0.336, 0.5

(Step 5) E_0 **=** $\text{Max}[|\text{err}(F)|] = \underline{0.0951}$, return to Step 2.

Third iteration

(Step 2), **(Step 3)**, **(Step 4)**, peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5) E_0 **=** $\underline{0.0821}$, return to Step 2.

Fourth iteration

(Step 2), **(Step 3)** , **(Step 4)**, peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

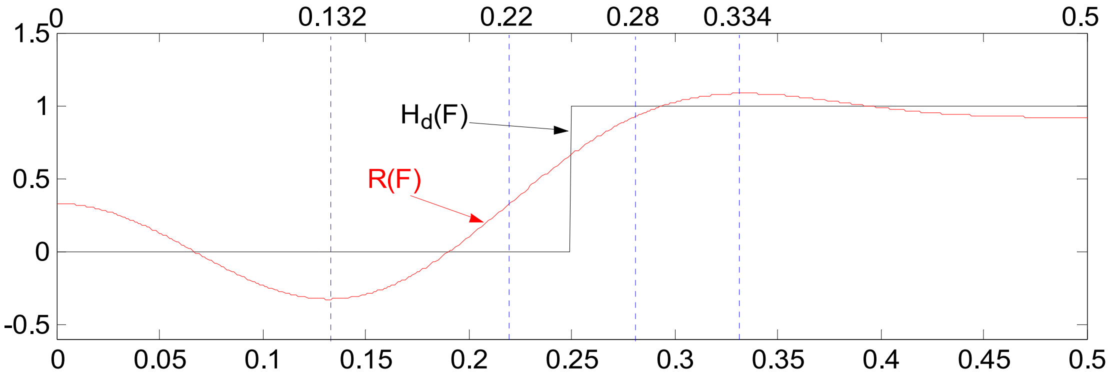
(Step 5) E_0 **=** $\underline{0.0820}$, $E_1 - E_0 = 0.0001 \leq \Delta$, continues to Step 6.

(Step 6) From $s[0] = 0.4990$, $s[1] = -0.6267$, $s[2] = -0.0203$, $s[3] = 0.3316$, $s[4] = 0.1442$

$h[4] = s[0] = 0.4990$, $h[3] = h[5] = s[1]/2 = -0.3134$,

$h[2] = h[6] = s[2]/2 = -0.0101$, $h[1] = h[7] = s[3]/2 = 0.1658$,

$h[0] = h[8] = s[4]/2 = 0.0721$.



- **Example 2:** Design a 7-length digital filter in the mini-max sense

ideal filter: $H_d(F) = 1$ for $0 \le F < 0.24$,

$H_d(F) = 0$ for $0.24 < F \le 0.5$,

transition band: $0.21 < F < 0.27$

weighting function: $W(F) = 1$ for $0 \le F \le 0.21$,

$W(F) = 0.5$ for $0.27 \le F \le 0.5$,

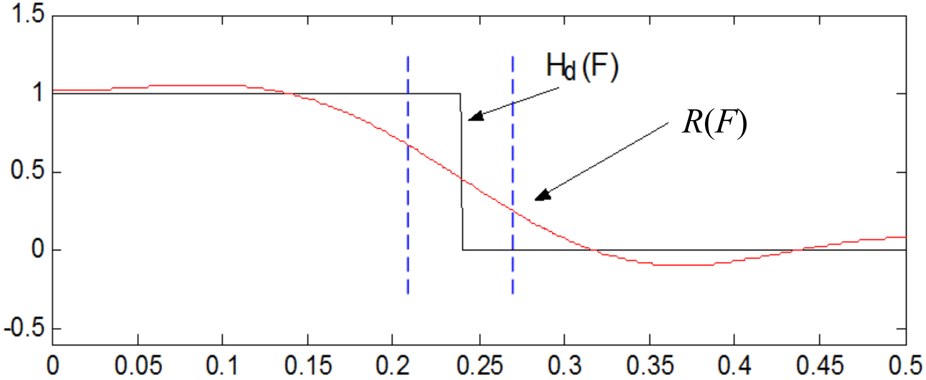
(**Step 1**) Since $N = 7$, $k = (N\text{-}1)/2 = 3$, $k+2 = 5$,

$\rightarrow$ Choose 5 extreme frequencies

(e.g., $F_0 = 0.05$, $F_1 = 0.15$, $F_2 = 0.3$, $F_3 = 0.4$, $F_4 = 0.5$)

(Step 2)
$$\begin{bmatrix}1 & 0.9511 & 0.8090 & 0.5878 & 1\\ 1 & 0.5878 & -0.309 & -0.9511 & -1\\ 1 & -0.309 & -0.809 & 0.809 & 2\\ 1 & -0.809 & 0.309 & 0.309 & -2\\ 1 & -1 & 1 & -1 & 2\end{bmatrix}\begin{bmatrix}s[0]\\ s[1]\\ s[2]\\ s[3]\\ e\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\\ 0\\ 0\end{bmatrix}$$

$\rightarrow$ $s[0] = 0.4638$, $s[1] = 0.6327$, $s[2] = 0.0809$, $s[3] = -0.1608$, $e = -0.0364$



After Step 2,

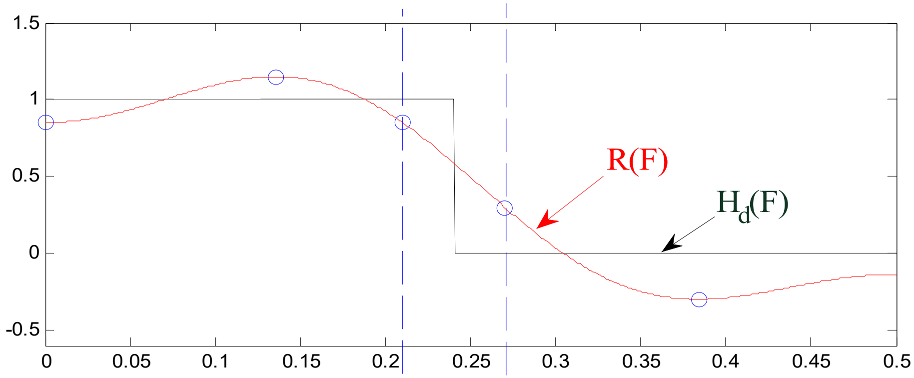
(Step 3) $\mathrm{err}(F) = [0.4638 + 0.6327\cos(2\pi F) + 0.0809\cos(4\pi F)$
$- 0.1608\cos(6\pi F) - H_d(F)]W(F)$

(Step 4) extreme points: 0.089, 0.21, 0.27, 0.369, 0.5.

(Step 5) E_0 **=** Max[|err(F)|] = 0.3396, return to Step 2.

Iteration	1	2	3	4	5	6	7
Max[|err(F)|]	0.3396	0.2371	0.3090	0.1944	0.1523	0.1493	0.1493

After 7 times of iteration



$s[0] = 0.4243$, $s[1] = 0.7559$, $s[2] = -0.0676$, $s[3] = -0.2619$, $e = 0.1493$

(Step 6):

$h[3] = 0.4243$, $h[2] = h[4] = s[1]/2 = 0.3780$,
$h[1] = h[5] = s[2]/2 = -0.0338$,
$h[0] = h[6] = s[3]/2 = -0.1309$, $h[n] = 0$ for $n < 0$ and $n > 6$

附錄二：Spectrum Analysis for Sampled Signals

(學信號處理的人一定要會的基本常識)

已知 $x[n]$ 是由一個 continuous signal $y(t)$ 取樣而得

$$x[n] = y(n\Delta_t)$$

DFT: $X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N}$ FT: $Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t} y(t)\,dt$

Q: $x[n]$ 的 DFT 和 $y(t)$ 的 Fourier transform 之間有什麼關係？

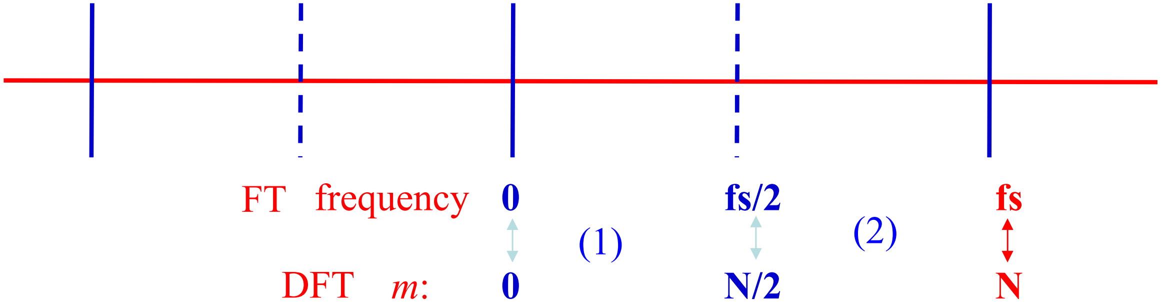
Basic rule：把間隔由 1 換成 f_s/N where $f_s = 1/\Delta_t$

$$f = m\frac{f_s}{N}$$

(Very important)

(1) $X[m]\Delta_t \cong Y\left(m\dfrac{f_s}{N}\right)$ $\quad f_s = 1/\Delta_t$ $\quad$ for $m \leq N/2$

(2) $X[m]\Delta_t \cong Y\left((m-N)\dfrac{f_s}{N}\right) = Y\left(m\dfrac{f_s}{N} - f_s\right)$ $\quad$ for $m > N/2$



If the sampling frequency is f_s, the FT output has the period of f_s

The DFT output has the period of N

Proof： $Y(f)=\int_{-\infty}^{\infty} e^{-j2\pi f t} y(t)\,dt$

用 $t=n\Delta_t,\ \ f=m\Delta_f$ 代入

$$Y(m\Delta_f) \cong \sum_n e^{-j2\pi m\Delta_f n\Delta_t} y(n\Delta_t)\Delta_t = \Delta_t \sum_n e^{-j2\pi m\Delta_f n\Delta_t} x[n]$$

當 $\Delta_t\Delta_f = \dfrac{1}{N}$ i.e., $\Delta_f = \dfrac{1}{N\Delta_t} = \dfrac{f_s}{N}$

$$Y\left(m\frac{f_s}{N}\right) \cong \Delta_t \sum_n e^{-j2\pi\frac{mn}{N}} x[n]$$

$$= \Delta_t DFT\{x[n]\}$$

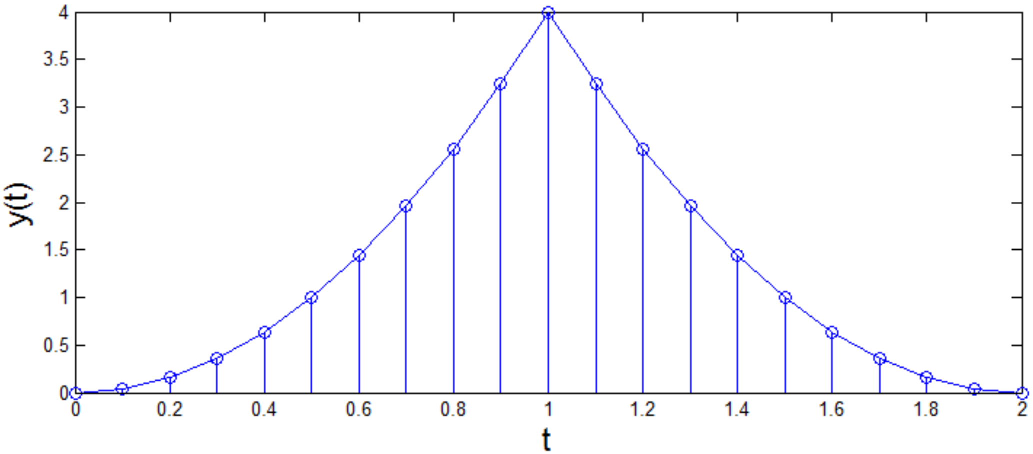
Example：已知

$$y(t) = (2t)^2 \quad \text{for } 0 \le t \le 1 \qquad\qquad y(t) = (4-2t)^2 \quad \text{for } 1 \le t \le 2$$

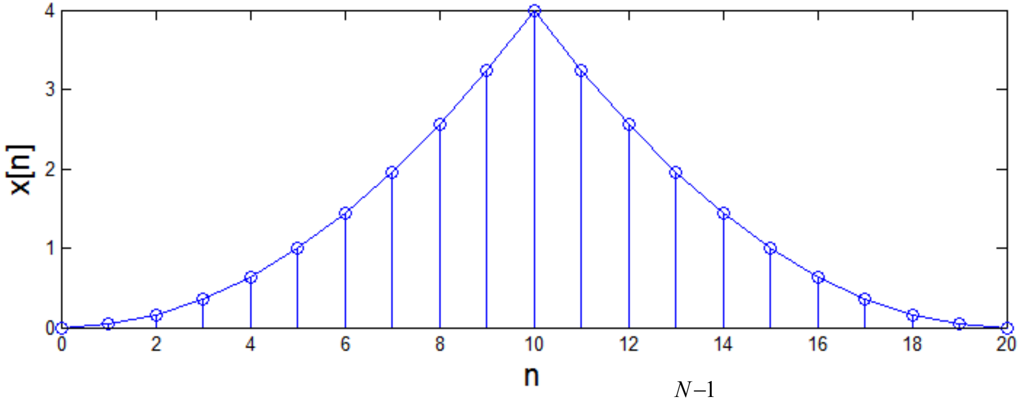
取樣間隔：$\Delta_t = 0.1$

$x[n] = y(n\,\Delta_t)$ for $0 \le n \le 20$

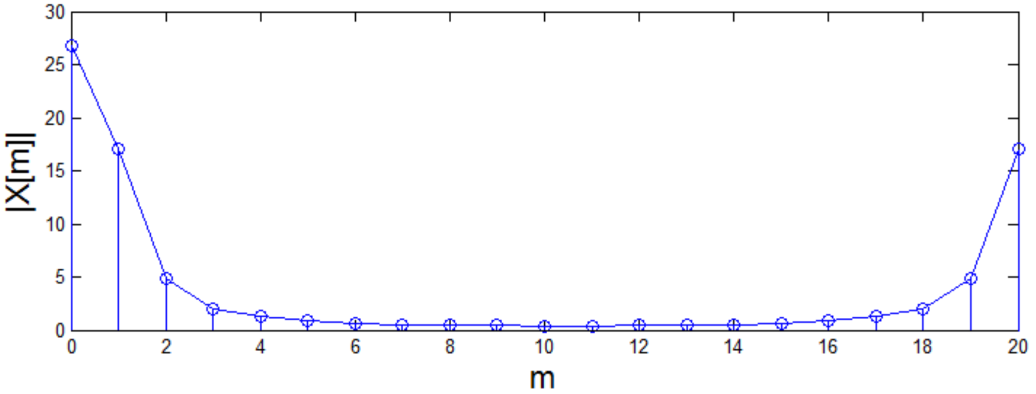
如何用 DFT 來正確的畫出 $y(t)$ 的頻譜？



$x[n] = y(n\,\Delta_t)$ for $0 \le n \le 20$



(Step 1) Perform the DFT for $x[n]$ $$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N} \qquad N = 21$$



(Step 2-1) $$Y\left(m\frac{f_s}{N}\right) \cong X[m]\Delta_t \qquad \text{for } m \le N/2$$

(Step 2-2) $$Y\left((m-N)\frac{f_s}{N}\right) \cong X[m]\Delta_t \qquad \text{for } m > N/2$$

In this example, $\frac{f_s}{N} = \frac{1}{N\Delta_t} = \frac{1}{21 \cdot 0.1} = 0.4762$