VI. Other Time Frequency Distributions

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 6, Prentice Hall, N.J., 1996.

Requirements for time-frequency analysis:

WDF STFT

(1) higher clarity $\xleftrightarrow{\text{tradeoff}}$ (2) avoid cross-term

(3) less computation time (4) good mathematical properties

VI-A Cohen's Class Distribution

VI-A-1 Ambiguity Function

模糊不清

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$$A_x(\tau,\eta)=\int_{-\infty}^{\infty}x(t+\tau/2)\cdot x^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt$$

(1) If $x(t)=\exp\left[-\alpha\pi(t-t_0)^2+j2\pi f_0 t\right]$

shifting, scaling, and modulation of a Gaussian function

$$A_x(\tau,\eta)=\int_{-\infty}^{\infty}e^{-\alpha\pi(t+\tau/2-t_0)^2+j2\pi f_0(t+\tau/2)}e^{-\alpha\pi(t-\tau/2-t_0)^2-j2\pi f_0(t-\tau/2)}\cdot e^{-j2\pi t\eta}\cdot dt$$

$$=\int_{-\infty}^{\infty}e^{-\alpha\pi\left[2(t-t_0)^2+\tau^2/2\right]+j2\pi f_0\tau}\cdot e^{-j2\pi t\eta}\cdot dt$$

$$=\int_{-\infty}^{\infty}e^{-\alpha\pi\left[2t^2+\tau^2/2\right]+j2\pi f_0\tau}\cdot e^{-j2\pi t\eta}e^{-j2\pi t_0\eta}\cdot dt$$

$$\boxed{A_x(\tau,\eta)=\sqrt{\frac{1}{2\alpha}}\exp\left[-\pi\left(\frac{\alpha\tau^2}{2}+\frac{\eta^2}{2\alpha}\right)\right]\exp\left[j2\pi\left(f_0\tau-t_0\eta\right)\right]}$$

$$|A_x(\tau,\eta)|=\sqrt{\frac{1}{2\alpha}}\,e^{-\pi\left(\frac{\alpha\tau^2}{2}+\frac{\eta^2}{2\alpha}\right)}$$

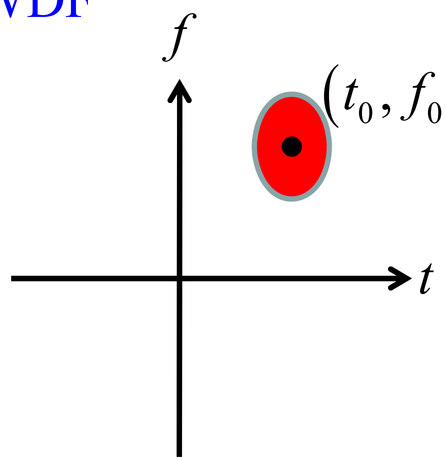
2D Gaussian function

和 α, t_0, f_0 無關

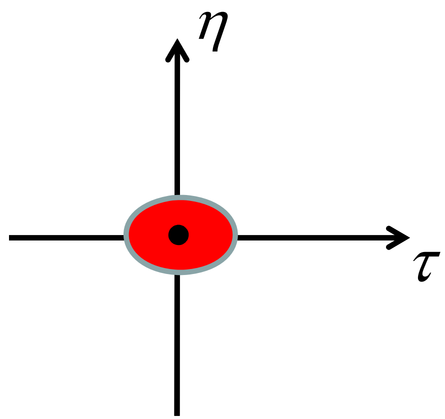
when $(\tau,\eta)=(0,0)$ $|A_x(\tau,\eta)|$ is maximal

WDF and AF for the signal with only 1 term

WDF



AF



The maximum is always at (0, 0) (independent of t_0, f_0, α)

$x(t+\frac{\tau}{2})\,x^*(t-\frac{\tau}{2})$

(2) If $x(t) = \underbrace{\exp\left[-\alpha_1\pi(t-t_1)^2 + j2\pi f_1 t\right]}_{x_1(t)} + \underbrace{\exp\left[-\alpha_2\pi(t-t_2)^2 + j2\pi f_2 t\right]}_{x_2(t)}$

auto terms

$$A_x(\tau,\eta) = \int_{-\infty}^{\infty} x_1(t+\tau/2)\cdot x_1^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt + \quad \Leftarrow A_{x_1}(\tau,\eta)$$

auto terms

$$\int_{-\infty}^{\infty} x_2(t+\tau/2)\cdot x_2^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt + \quad \Leftarrow A_{x_2}(\tau,\eta)$$

$$\int_{-\infty}^{\infty} x_1(t+\tau/2)\cdot x_2^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt + \quad \Leftarrow A_{x_1x_2}(\tau,\eta)$$

$$\int_{-\infty}^{\infty} x_2(t+\tau/2)\cdot x_1^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt + \quad \Leftarrow A_{x_2x_1}(\tau,\eta)$$

cross terms

$$\boxed{A_x(\tau,\eta) = A_{x_1}(\tau,\eta) + A_{x_2}(\tau,\eta) + A_{x_1x_2}(\tau,\eta) + A_{x_2x_1}(\tau,\eta)}$$

$$A_{x_1}(\tau,\eta) = \sqrt{\frac{1}{2\alpha_1}}\exp\left[-\pi\left(\frac{\alpha_1\tau^2}{2} + \frac{\eta^2}{2\alpha_1}\right)\right]\exp\left[j2\pi(f_1\tau - t_1\eta)\right]$$

$$A_{x_2}(\tau,\eta) = \sqrt{\frac{1}{2\alpha_2}}\exp\left[-\pi\left(\frac{\alpha_2\tau^2}{2} + \frac{\eta^2}{2\alpha_2}\right)\right]\exp\left[j2\pi(f_2\tau - t_2\eta)\right]$$

When $\alpha_1 = \alpha_2$ $|A_{x_1x_2}(\tau,\eta)|$ is maximal at $(\tau,\eta)=(t_d,f_d)$

$$\boxed{\begin{aligned}A_{x_1x_2}(\tau,\eta) &= \sqrt{\frac{1}{2\alpha_\mu}}\exp\left[-\pi\left(\alpha_\mu\frac{(\tau-t_d)^2}{2}+\frac{(\eta-f_d)^2}{2\alpha_\mu}\right)\right]\\ &\times\exp\left[j2\pi(f_\mu\tau - t_\mu\eta + f_d t_\mu)\right]\end{aligned}}$$

$$t_\mu = (t_1+t_2)/2,\quad f_\mu = (f_1+f_2)/2,\quad \alpha_\mu = (\alpha_1+\alpha_2)/2,$$

$$\underline{t_d = t_1 - t_2,}\quad \underline{f_d = f_1 - f_2,}\quad \alpha_d = \alpha_1 - \alpha_2$$

$$\boxed{A_{x_2x_1}(\tau,\eta) = A^*_{x_1x_2}(-\tau,-\eta)}$$

Maximum is at $(\tau,\eta)=(-t_d,-f_d)$

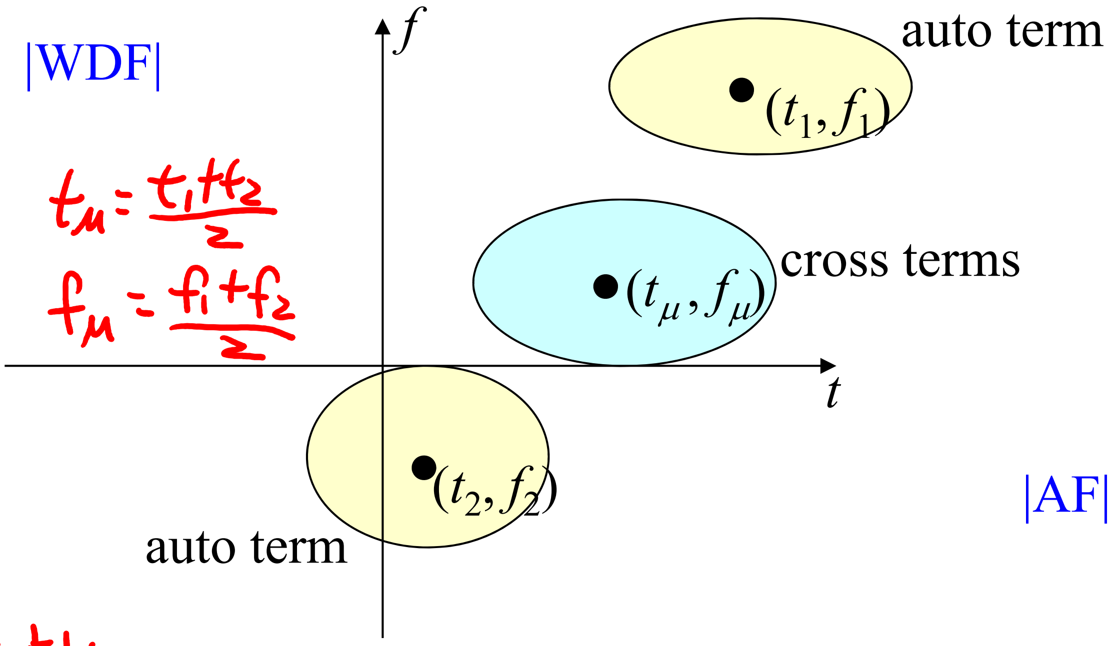
When $\alpha_1 \neq \alpha_2$

$$A_{x_1x_2}(\tau,\eta) = \sqrt{\frac{1}{2\alpha_\mu}}\exp\left[-\pi\frac{[(\eta-f_d)+j(\alpha_1t_1+\alpha_2t_2)-j\alpha_d\tau/2]^2}{2\alpha_\mu}\right]$$

$$\exp\left[-\pi\left(\alpha_1(t_1-\frac{\tau}{2})^2+\alpha_2(t_2+\frac{\tau}{2})^2\right)\right]\exp\left[j2\pi f_\mu\tau\right]$$

$$A_{x_2x_1}(\tau,\eta) = A^*_{x_1x_2}(-\tau,-\eta)$$

WDF and AF for the signal with 2 terms



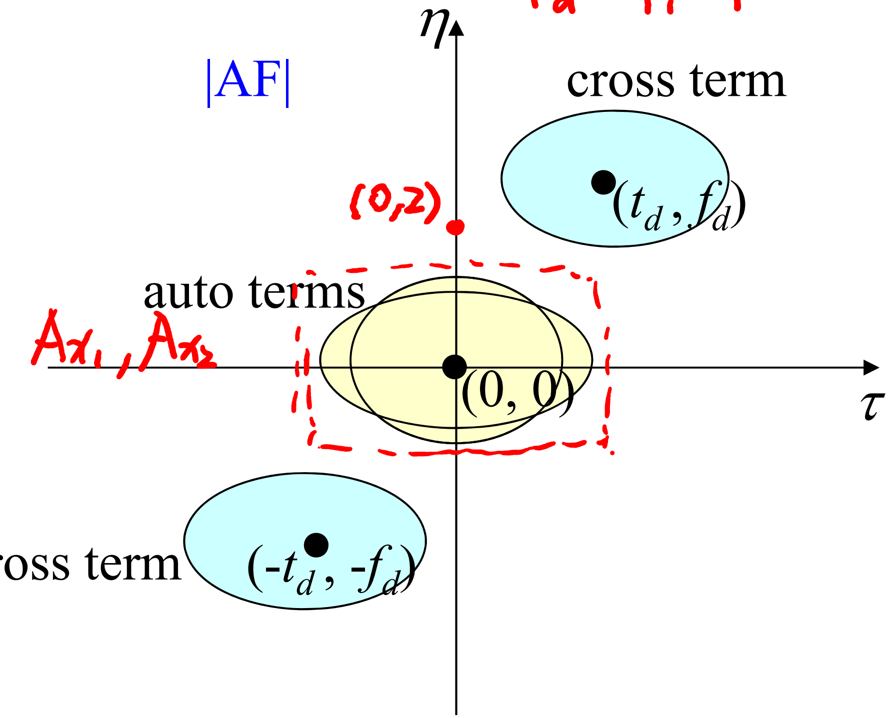
$t_\mu = \frac{t_1+t_2}{2}$

$f_\mu = \frac{f_1+f_2}{2}$

比較:
windowed WDF
only has τ-axis

$t_d = t_1 - t_2$

$f_d = f_1 - f_2$



For the ambiguity function

The auto term is always near to the origin

The cross-term is always far from the origin

VI-A-2 Definition of Cohen's Class Distribution

The Cohen's Class distribution is a further generalization of the Wigner distribution function

lowpass mask

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_x(\tau,\eta)\,\Phi(\tau,\eta)\exp\left(j2\pi(\eta t - \tau f)\right)d\eta\, d\tau$$

where $A_x(\tau,\eta) = \int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt$

is the ambiguity function (AF).

$$C_x^*(t,f) = \iint_{-\infty}^{\infty} A_x^*(\tau,\eta)\,\Phi^*(\tau,\eta)\exp(j2\pi(-\eta t+\tau f))d\tau\, d\eta$$

$\Phi(\eta, \tau) = 1 \rightarrow$ WDF

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(u+\tau/2)\,x^*(u-\tau/2)\,\phi(t-u,\tau)\,du\; e^{-j2\pi f\tau}d\tau$$

where $\phi(t,\tau) = \int_{-\infty}^{\infty} \Phi(\tau,\eta)\exp(j2\pi\eta t)d\eta$

complexity is near to 3 times of the WDF

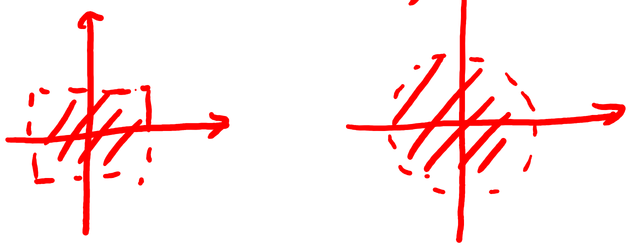
How does the Cohen's class distribution avoid the cross term?

Chose $\Phi(\tau, \eta)$ low pass function.

many possible choices for $\Phi(\tau, \eta)$

$\Phi(\tau, \eta) \approx 1$ for small $|\eta|$, $|\tau|$

$\Phi(\tau, \eta) \approx 0$ for large $|\eta|$, $|\tau|$



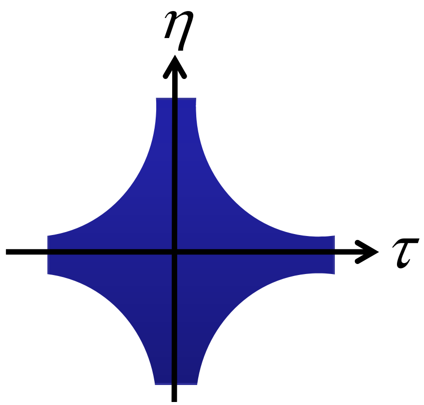
[Ref] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781-806, 1966.

[Ref] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, New York, 1995.

VI-A-3 Several Types of Cohen's Class Distribution

Choi-Williams Distribution (One of the Cohen's class distribution)

$$\Phi(\tau,\eta) = \exp\left[-\alpha(\eta\tau)^2\right]$$

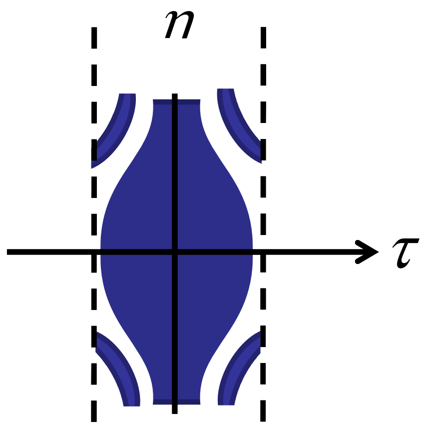


[Ref] H. Choi and W. J. Williams, "Improved time-frequency representation of multicomponent signals using exponential kernels," *IEEE. Trans. Acoustics, Speech, Signal Processing*, vol. 37, no. 6, pp. 862-871, June 1989.

Cone-Shape Distribution (One of the Cohen's class distribution)

$$\phi(t,\tau)=\frac{1}{|\tau|}\exp\left(-2\pi\alpha\tau^2\right)\Pi\left(\frac{t}{\tau}\right)$$

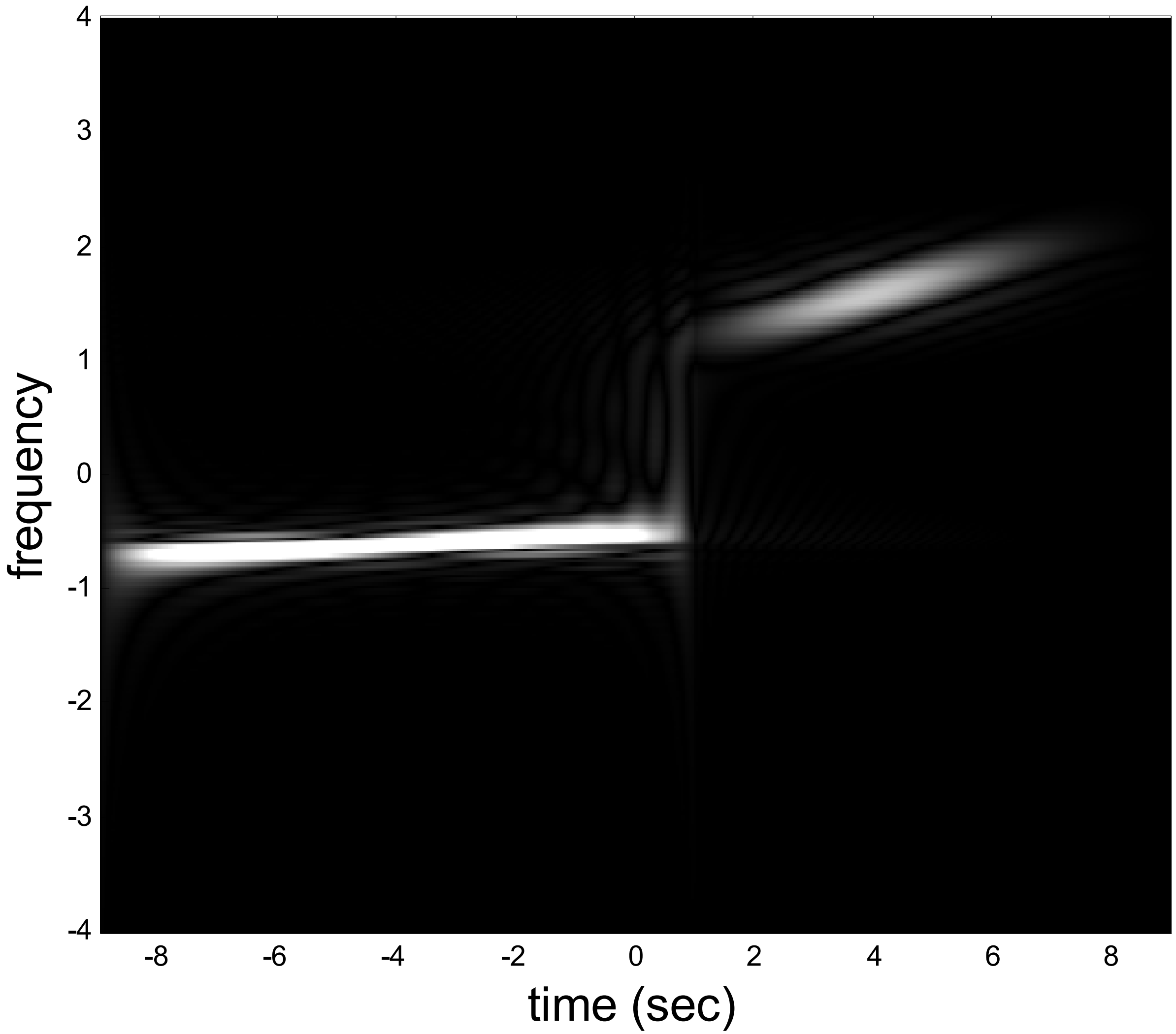
$$\Phi(\tau,\eta)=\sin c(\eta\tau)\exp\left(-2\pi\alpha\tau^2\right)$$



[Ref] Y. Zhao, L. E. Atlas, and R. J. Marks, "The use of cone-shape kernels for generalized time-frequency representations of nonstationary signals," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. 38, no. 7, pp. 1084-1091, July 1990.

Cone-Shape distribution for the example on pages 83, ~~125~~ 128

$\alpha = 1$



Distributions	$\Phi(\tau, \eta)$
Wigner	1
Cohen (circular)	$\Phi(\tau, \eta) = 1$ for $\sqrt{\eta^2+\tau^2} < r$ $\Phi(\tau, \eta) = 0$ otherwise
Cohen (rectangular)	$\Phi(\tau, \eta) = 1$ for $Max(\lvert\eta\rvert, \lvert\tau\rvert) < T$ $\Phi(\tau, \eta) = 0$ otherwise
Choi-Williams	$\exp\left[-\alpha(\eta\tau)^2\right]$
Cone-Shape	$\sin c(\eta\tau)\exp\left(-2\pi\alpha\tau^2\right)$
Page	$\exp\left(j\pi\eta\lvert\tau\rvert\right)$
Levin (Margenau-Hill)	$\cos(\pi\eta\tau)$
Born-Jordan	$\sin c(\eta\tau)$

註：感謝 2007年修課的王文阜同學

VI-A-4 Advantages and Disadvantages of Cohen's Class Distributions

The Cohen's class distribution may avoid the cross term and has higher clarity.

However, it requires more computation time and lacks of well mathematical properties.

Moreover, there is a tradeoff between the quality of the auto term and the ability of removing the cross terms.

VI-A-5 Implementation for the Cohen's Class Distribution

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_x(\tau,\eta)\Phi(\tau,\eta)\exp\left(j2\pi(\eta t - \tau f)\right)d\eta d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\Phi(\tau,\eta)e^{-j2\pi u\eta + j2\pi(\eta t-\tau f)}du\,d\eta\,d\tau$$

簡化法 1：不是所有的 $A_x(\eta,\tau)$ 的值都需要算出

If $\Phi(\tau,\eta)=0$ for $|\eta| > B$ or $|\tau| > C$

$$C_x(t,f) = \int_{-C}^{C}\int_{-B}^{B}\int_{-\infty}^{\infty} x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\Phi(\tau,\eta)e^{-j2\pi u\eta + j2\pi(\eta t-\tau f)}du\,d\eta\,d\tau$$

簡化法 2：注意，η 這個參數和input 及output 都無關

$$C_x(t,f)=\int_{-C}^{C}\int_{-\infty}^{\infty}x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot[\int_{-B}^{B}\Phi(\tau,\eta)e^{j2\pi\eta(t-u)}d\eta]e^{-j2\pi\tau f}dud\tau$$

$$=\int_{-C}^{C}\int_{-\infty}^{\infty}x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)\cdot\Psi(\tau,t-u)e^{-j2\pi\tau f}dud\tau$$

$$\Psi(\tau,t)=\int_{-B}^{B}\Phi(\tau,\eta)e^{j2\pi\eta t}d\eta$$

由於 $\Psi(\tau,t)$ 和 input 無關，可事先算出，所以只剩 2 個積分式

VI-B Modified Wigner Distribution Function

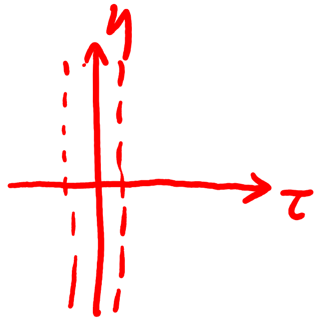
$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty} X(f+\eta/2)\cdot X^*(f-\eta/2)e^{j2\pi t\eta}\cdot d\eta$$

where $X(f) = FT[x(t)]$

Modified Form I

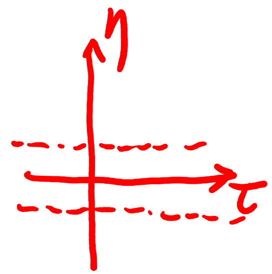
$$W_x(t,f)=\int_{-B}^{B} \underline{\underline{w(\tau)}}\, x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$



Modified Form II

$$W_x(t,f)=\int_{-B}^{B} \underline{\underline{w(\eta)}}\, X(f+\eta/2)\cdot X^*(f-\eta/2)e^{j2\pi t\eta}\cdot d\eta$$

(比較 page119)



Modified Form III (Pseudo *L*-Wigner Distribution)

$$W_x(t,f)=\int_{-\infty}^{\infty} w(\tau)\,x^L\left(t+\frac{\tau}{2L}\right)\cdot\overline{x^L\left(t-\frac{\tau}{2L}\right)}e^{-j2\pi\tau f}\cdot d\tau$$

增加 L 可以減少 cross term 的影響 (但是不會完全消除)

[Ref] L. J. Stankovic, S. Stankovic, and E. Fakultet, “An analysis of instantaneous frequency representation using time frequency distributions-generalized Wigner distribution,” *IEEE Trans. on Signal Processing,* pp. 549-552, vol. 43, no. 2, Feb. 1995

P.S.: 感謝2006年修課的林政豪同學

小:連乘

Modified Form IV (Polynomial Wigner Distribution Function)

$$W_x(t,f) = \int_{-\infty}^{\infty} \left[\prod_{l=1}^{q/2} x(t+d_l\tau)x^*(t-d_{-l}\tau) \right] e^{-j2\pi\tau f} d\tau$$

When $q = 2$ and $d_1 = d_{-1} = 0.5$, it becomes the original Wigner distribution function.

It can avoid the cross term when the order of phase of the exponential function is no larger than $q/2 + 1$.

when $q=2$ $q/2+1=2$
$q=4$ $q/2+1=3$
$q=6$ $q/2+1=4$

However, the cross term between two components cannot be removed.

If $k=\frac{q}{2}+1$, $q=2(k-1)$

[Ref] B. Boashash and P. O'Shea, "Polynomial Wigner-Ville distributions & their relationship to time-varying higher order spectra," *IEEE Trans. Signal Processing*, vol. 42, pp. 216–220, Jan. 1994.

[Ref] J. J. Ding, S. C. Pei, and Y. F. Chang, "Generalized polynomial Wigner spectrogram for high-resolution time-frequency analysis," *APSIPA ASC*, Kaohsiung, Taiwan, Oct. 2013.

d_l should be chosen properly such that

$$\prod_{l=1}^{q/2} x(t+d_l\tau)x^*(t-d_{-l}\tau) = \exp\left(j2\pi\sum_{n=1}^{q/2+1} na_n t^{n-1}\tau\right)$$

when $x(t) = \exp\left(j2\pi\sum_{n=1}^{q/2+1} a_n t^n\right)$

instantaneous frequency: $\sum_{n=1}^{q/2+1} n a_n t^{n-1}$

then

$$W_x(t,f) = \int_{-\infty}^{\infty} \exp\left(-j2\pi(f-\sum_{n=1}^{q/2+1} na_n t^{n-1})\tau\right)d\tau \cong \delta\left(f-\sum_{n=1}^{q/2+1} na_n t^{n-1}\right)$$

(from page 117 (3))

i.e., $W_x(t,f)$ is nonzero only at the instantaneous frequency

$$\prod_{l=1}^{q/2} x(t+d_l\tau)x^*(t-d_{-l}\tau) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} na_n t^{n-1}\tau \right)$$

$$x(t) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n \right)$$

when $q = 2$ $\quad x(t) = \exp\left(j2\pi(a_1 t + a_2 t^2) \right)$

$$x(t+d_1\tau)x^*(t-d_{-1}\tau) = \exp\left(j2\pi(a_1 + 2a_2 t)\tau \right)$$

$$a_2\left(t+d_1\tau\right)^2 + a_1\left(t+d_1\tau\right) - a_2\left(t-d_{-1}\tau\right)^2 - a_1\left(t-d_{-1}\tau\right) = 2a_2 t\tau + a_1\tau$$

$$2a_2(d_1+d_{-1})t\tau + a_2(d_1-d_{-1})\tau^2 + a_1(d_1+d_{-1})\tau = 2a_2 t\tau + a_1\tau$$

$\Rightarrow$ $d_1 + d_{-1} = 1 \qquad d_1 - d_{-1} = 0$

$\Rightarrow$ $d_1 = d_{-1} = 1/2$

$q/2+1=3$

When $q=4$ $\qquad x(t)=\exp\left(j2\pi(a_1t+a_2t^2+a_3t^3)\right)$

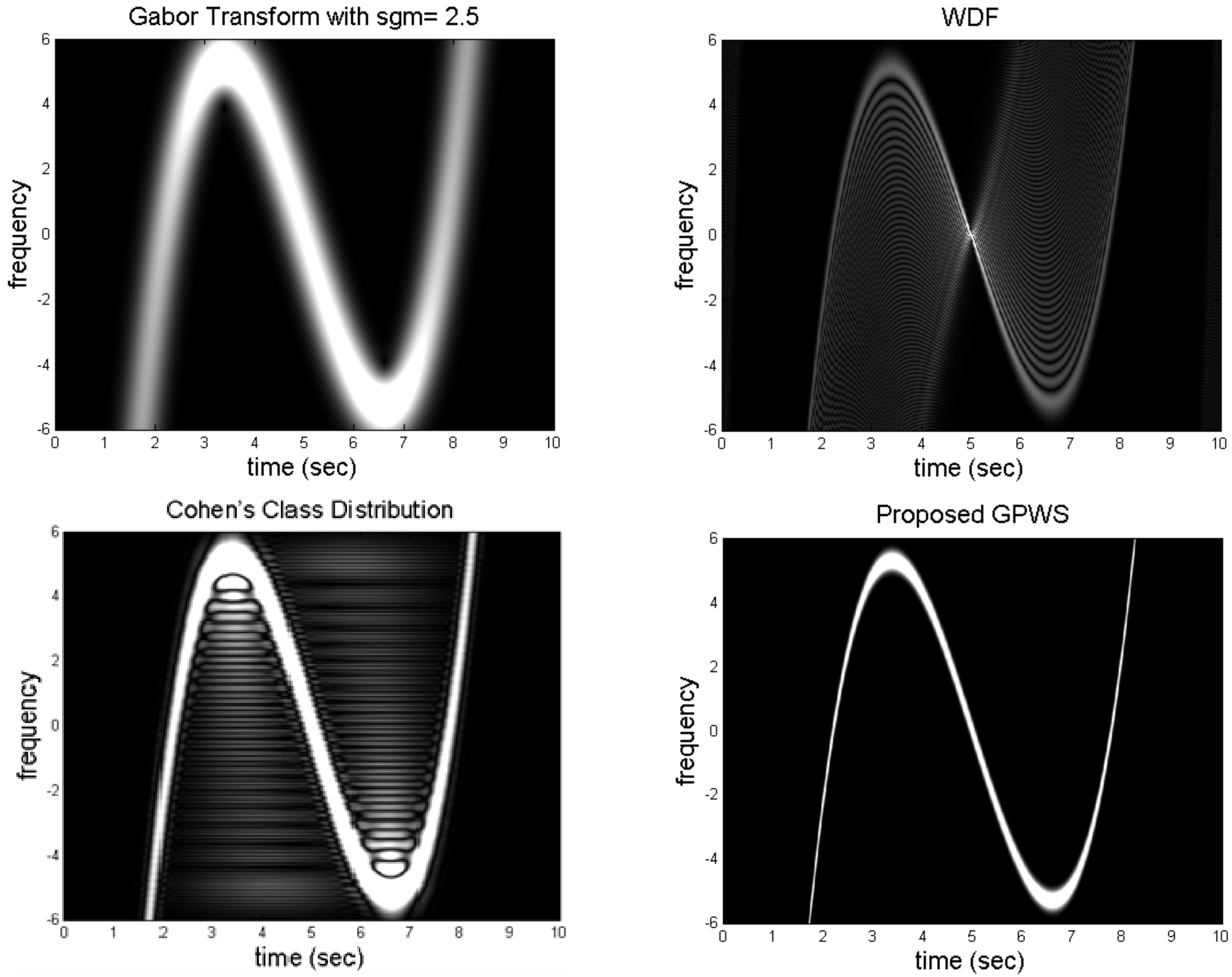
$$\prod_{l=1}^{2}x(t+d_l\tau)x^*(t-d_{-l}\tau)=\exp\left(j2\pi\sum_{n=1}^{3}na_nt^{n-1}\tau\right)$$

$$x(t+d_1\tau)x^*(t-d_{-1}\tau)x(t+d_2\tau)x^*(t-d_{-2}\tau)=\exp\left(j2\pi\sum_{n=1}^{3}na_nt^{n-1}\tau\right)$$

$$\begin{aligned}&a_3\left(t+d_1\tau\right)^3+a_2\left(t+d_1\tau\right)^2+a_1\left(t+d_1\tau\right)\\&+a_3\left(t+d_2\tau\right)^3+a_2\left(t+d_2\tau\right)^2+a_1\left(t+d_2\tau\right)\\&-a_3\left(t-d_{-1}\tau\right)^3-a_2\left(t-d_{-1}\tau\right)^2-a_1\left(t-d_{-1}\tau\right)\\&-a_3\left(t-d_{-2}\tau\right)^3-a_2\left(t-d_{-2}\tau\right)^2-a_1\left(t-d_{-2}\tau\right)\\&=3a_3t^2\tau+2a_2t\tau+a_1\tau\end{aligned}$$

$$\Rightarrow\begin{cases}d_1+d_2+d_{-1}+d_{-2}=1\\d_1^2+d_2^2-d_{-1}^2-d_{-2}^2=0\\d_1^3+d_2^3+d_{-1}^3+d_{-2}^3=0\end{cases}$$

$$x(t) = \exp(j(t-5)^4 - j5\pi(t-5)^2)$$

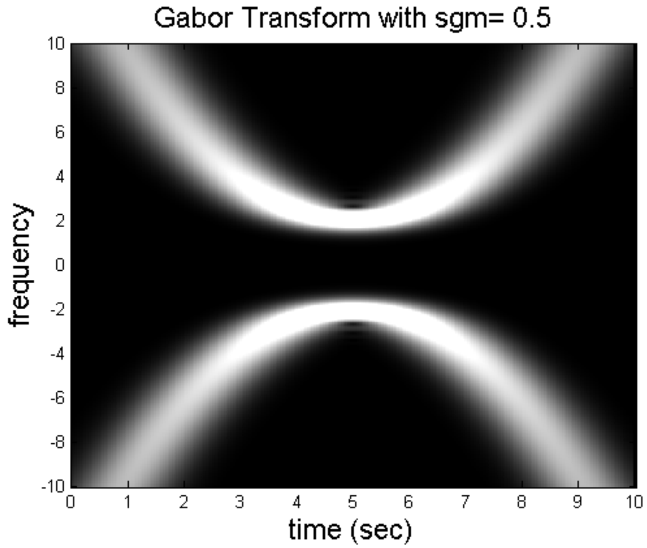


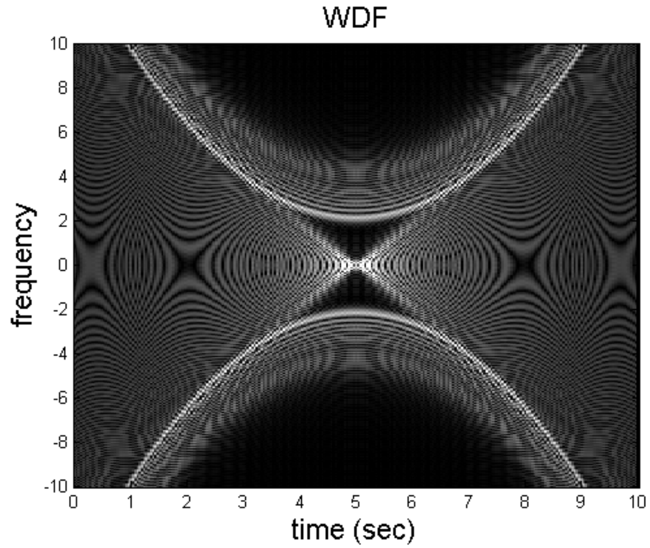
$q = ?$

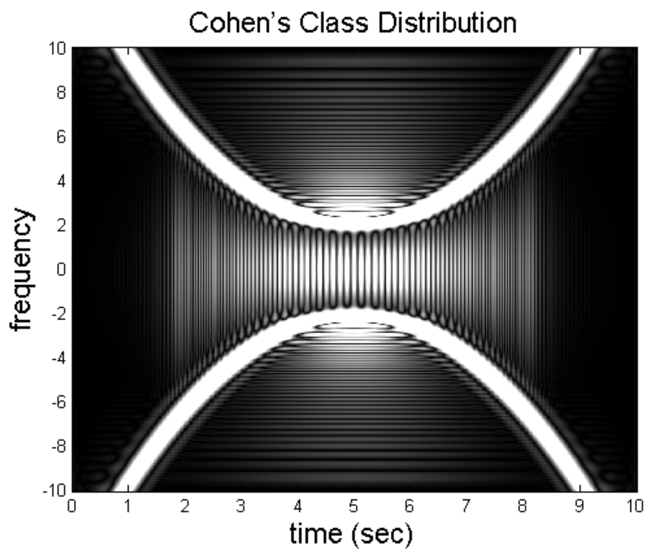
q=6

$q \geq 4$ $\quad$ $q = 2(k-1)$

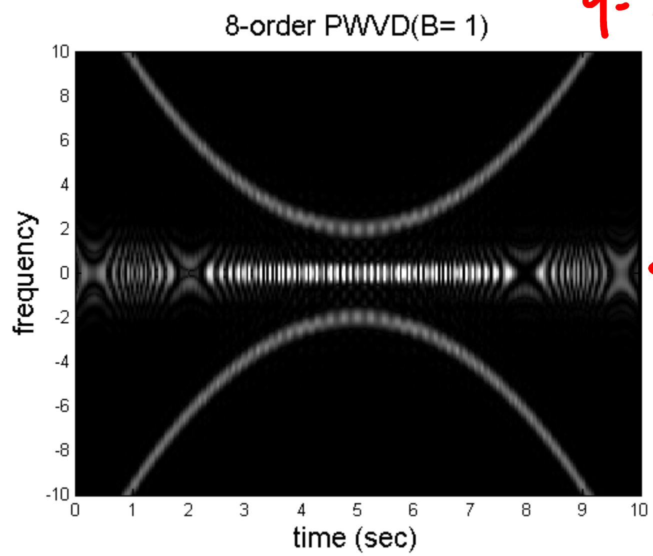
$$x(t) = 2\cos((t-5)^3 + 4\pi t)$$







$q = 8$



← cross term (for multiple components)

VI-C Gabor-Wigner Transform

[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Advantages:

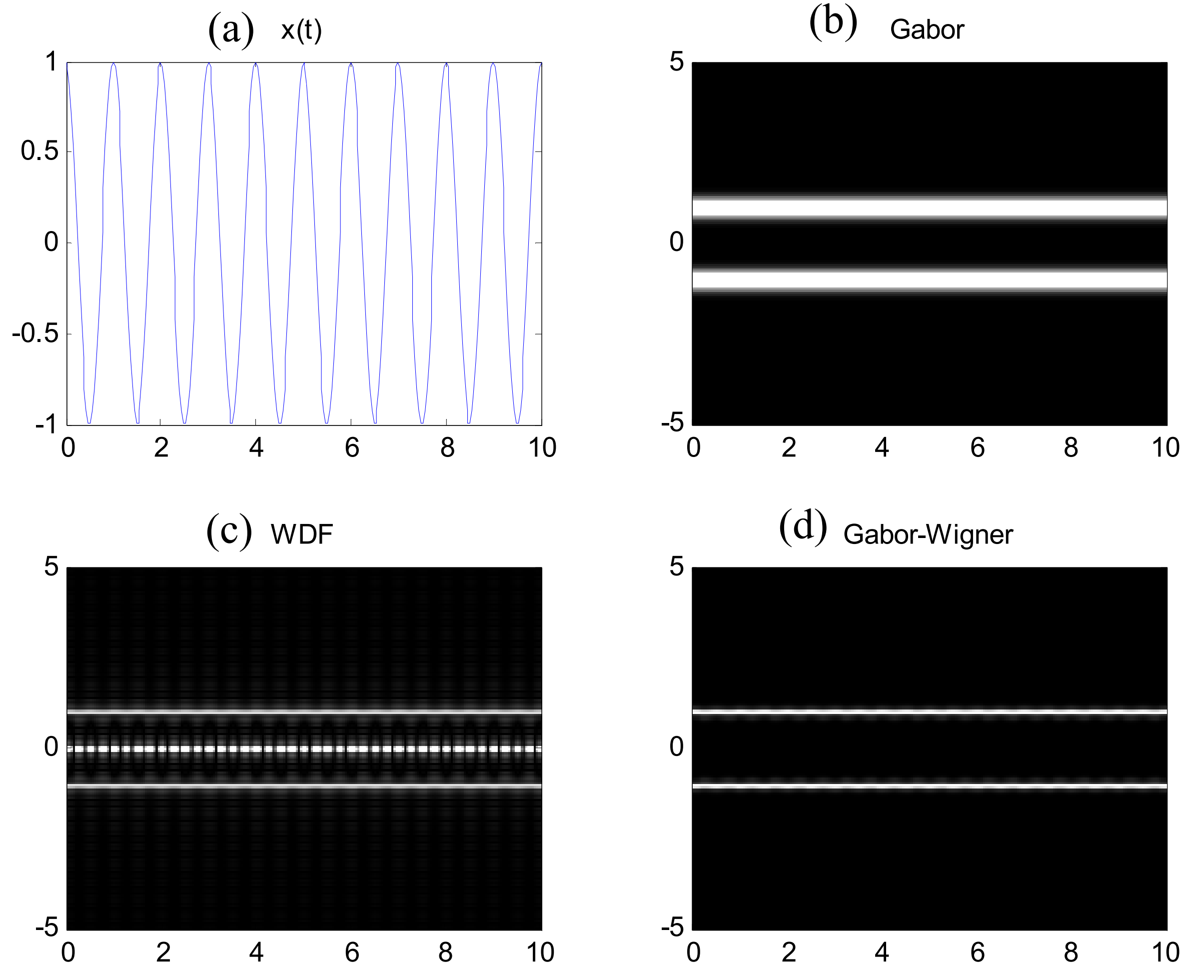
combine the advantage of the WDF and the Gabor transform

advantage of the WDF $\rightarrow$ higher clarity

advantage of the Gabor transform $\rightarrow$ no cross-term

$$D_x(t,f) = G_x^2(t,f)W_x(t,f)$$

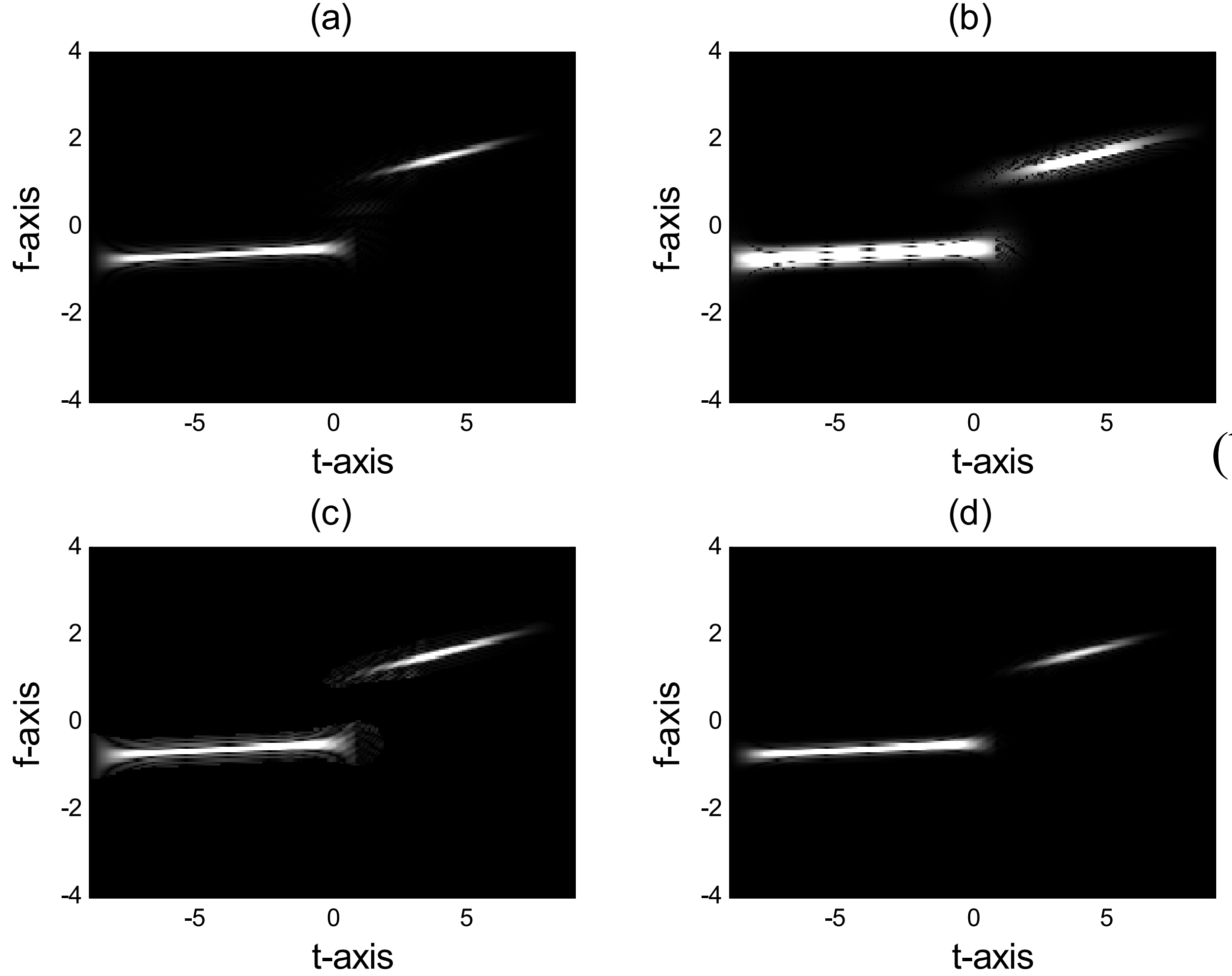
$$x(t) = \cos(2\pi t)$$



(a) $D_x(t,f) = G_x(t,f)W_x(t,f)$ (b) $D_x(t,\omega) = \min\left(|G_x(t,f)|^2, |W_x(t,f)|\right)$

(c) $D_x(t,f) = W_x(t,f) \times \{|G_x(t,f)| > 0.25\}$

(d) $D_x(t,f) = G_x^{2.6}(t,f)W_x^{0.7}(t,f)$



(b)、(c) are real

思考：

(1) Which type of the Gabor-Wigner transform is better?

(2) Can we further generalize the results?

Implementation of the Gabor-Wigner Transform ：簡化技巧

(1) When $G_x(t,f) \approx 0$, $D_x\left(t,f\right) = G_x^{\alpha}(t,f)W_x^{\beta}(t,f) \approx 0$

先算 $G_x(t,f)$

$W_x(t,f)$ 只需算 $G_x(t,f)$ 不近似於 0 的地方

(2) When $x(t)$ is real，對 Gabor transform 而言

$X(f) = X^*(-f)$ if $x(t)$ is real, where $X(f) = FT[x(t)]$

附錄六： Fourier Transform 常用的性質

$$X(f)=FT[x(t)]=\int_{-\infty}^{\infty}x(t)\exp(-j2\pi f t)\,dt$$

(1) Recovery (inverse Fourier transform)	$x(t)=\int_{-\infty}^{\infty}X(f)\exp(j2\pi f t)\,df$
(2) Integration	$x(0)=\int_{-\infty}^{\infty}X(f)\,df$
(3) Modulation	$FT\left[x(t)e^{j2\pi f_0 t}\right]=X(f-f_0)$
(4) Time Shifting	$FT\left[x(t-t_0)\right]=X(f)e^{-j2\pi f t_0}$
(5) Scaling	$FT\left[x(at)\right]=\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
(6) Time Reverse	$FT\left[x(-t)\right]=X(-f)$

(7) Real / Imaginary Input	If $x(t)$ is real, then $X(f) = X^*(-f)$; If $x(t)$ is pure imaginary, then $X(f) = -X^*(-f)$
(8) Even / Odd Input	If $x(t) = x(-t)$, then $X(f) = X(-f)$; If $x(t) = -x(-t)$, then $X(f) = -X(-f)$;
(9) Conjugation	$FT\left[x^*(t)\right] = X^*(-f)$
(10) Differentiation	$FT\left[x'(t)\right] = j2\pi f\, X(f)$
(11) Multiplication by t	$FT\left[tx(t)\right] = \frac{j}{2\pi} X'(f)$
(12) Division by t	$FT\left[\frac{x(t)}{t}\right] = -j2\pi \int_{-\infty}^{f} X(\mu)\, d\mu$
(13) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f) \rvert^2 df$
(14) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$

(15) Linearity	$FT\left[ax(t)+by(t)\right]=aX(f)+bY(f)$
(16) Convolution	If $z(t)=x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau,$ then $Z(f)=X(f)Y(f)$
(17) Multiplication	If $z(t)=x(t)y(t)$, then $Z(f)=X(f)*Y(f)=\int_{-\infty}^{\infty}X(\mu)Y(f-\mu)d\mu$
(18) Correlation	If $z(t)=\int_{-\infty}^{\infty}x(\tau)y^{*}(\tau-t)d\tau,$ then $Z(f)=X(f)Y^{*}(f)$
(19) Two Times of Fourier Transforms	$FT\left\{FT\left[x(t)\right]\right\}=x(-t)$
(20) Four Times of Fourier Transforms	$FT\left[FT\left(FT\left\{FT\left[x(t)\right]\right\}\right)\right]=x(t)$

VII. Other Time Frequency Distributions (II)

The trend of time-frequency analysis in recent years:

(1) S transform and its generalization

(2) Time-variant signal expansion (Compressive sensing)

(3) Improvement for the Hilbert-Huang transform

VII-A S Transform

(Modification from the Gabor transform)

no cross term
(∵ linear)

$$S_x(t,f) = |f|\int_{-\infty}^{\infty} x(\tau)\exp\left[-\pi(t-\tau)^2 f^2\right]\exp(-j2\pi f\tau)\,d\tau$$

Gabor $\exp(-\pi(t-\tau)^2)$ scaled Gabor $\exp(-\pi\sigma(t-\tau)^2)$

closely related to the wavelet transform

advantages and disadvantages

$f\uparrow$ window width $\downarrow$ time resolution $\uparrow$ frequency resolution $\downarrow$

$f\downarrow$ $\uparrow$ $\downarrow$ $\uparrow$

[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

S transform 和 Gabor transform 相似。

但是 Gaussian window 的寬度會隨著 f 而改變

$$w(t) = \exp\left[-\pi t^2\right] \qquad w(t) = |f|\exp\left[-\pi t^2 f^2\right]$$

低頻： worse time resolution, better frequency resolution

高頻： better time resolution, worse frequency resolution

for vocal signal

frequency ratio

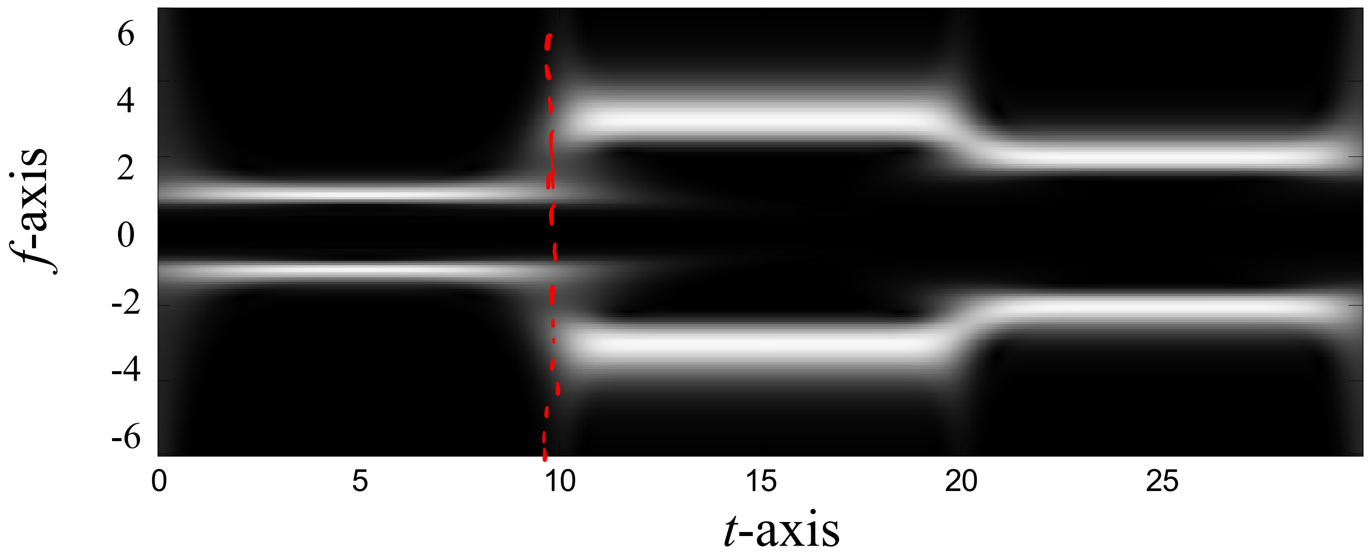
100:200=1:2

1000:1100=1:1.1

$261.63 \cdot 2^{\frac{k}{12}}$

k: 和 Do 相差半音數

The result of the S transform (compared with page 81)

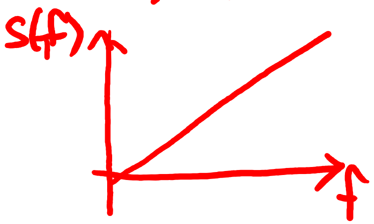


- General form

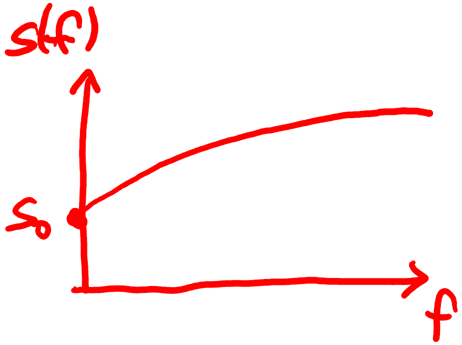
$$S_x(t,f)=|s(f)|\int_{-\infty}^{\infty}x(\tau)\exp\left[-\pi(t-\tau)^2 s^2(f)\right]\exp(-j2\pi f\tau)\,d\tau$$

$s(f)$ increases with f

original
S transform
$s(f) = f$



modified



ex: $s(f) = s_0 + \sqrt[\alpha]{|f|}$

C. R. Pinnegar and L. Mansinha, "The S-transform with windows of arbitrary and varying shape," *Geophysics*, vol. 68, pp. 381-385, 2003.

Fast algorithm of the S transform

When f is fixed, the S transform can be expressed as a convolution form:

$$S_x(t,f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 s^2(f)\right] \exp(-j2\pi f \tau) d\tau$$

$$\Downarrow$$

$$S_x(t,f) = |s(f)| \left(x(t)\exp(-j2\pi ft) \underset{\substack{convolution \\ along\ t-axis}}{*} \exp\left[-\pi t^2 s^2(f)\right] \right)$$

(for every fixed f)

Remember: $g(t) * h(t) = \int g(\tau) h(t-\tau) d\tau$

Q: Can we use the FFT-based method on page 99 to implement the S transform?

VII-B Generalized Spectrogram

$|X(t,f)|^2 = X(t,f)\,X^*(t,f)$

[Ref] P. Boggiatto, G. De Donno, and A. Oliaro, “Two window spectrogram and their integrals," *Advances and Applications*, vol. 205, pp. 251-268, 2009.

Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^*_{x,w_2}(t,f)$

w_1 can be different from w_2

$$G_{x,w_1}(t,f) = \int_{-\infty}^{\infty} w_1(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$G_{x,w_2}(t,f) = \int_{-\infty}^{\infty} w_2(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

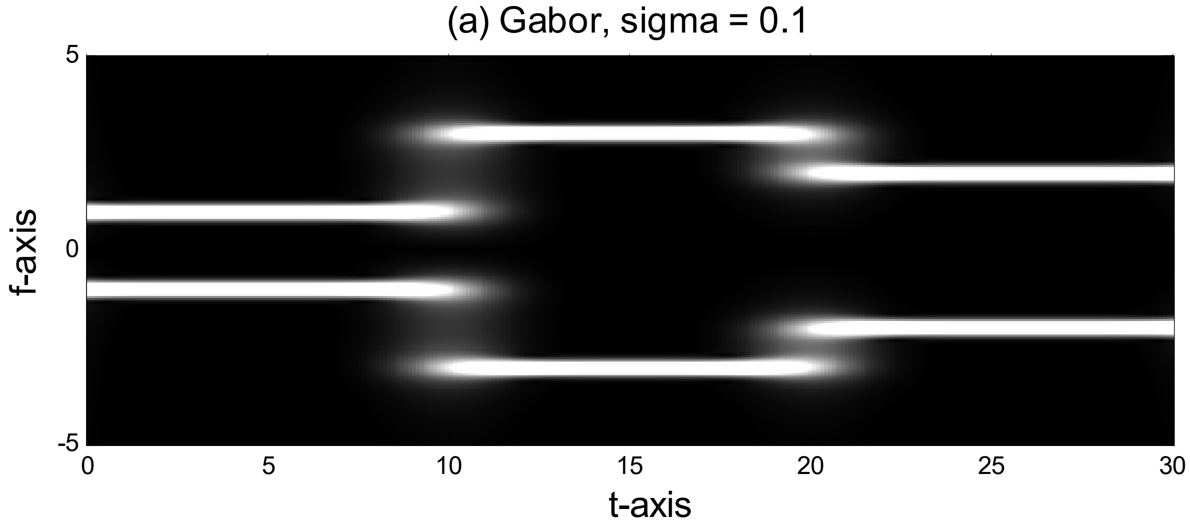
Original spectrogram: $w_1(t) = w_2(t)$

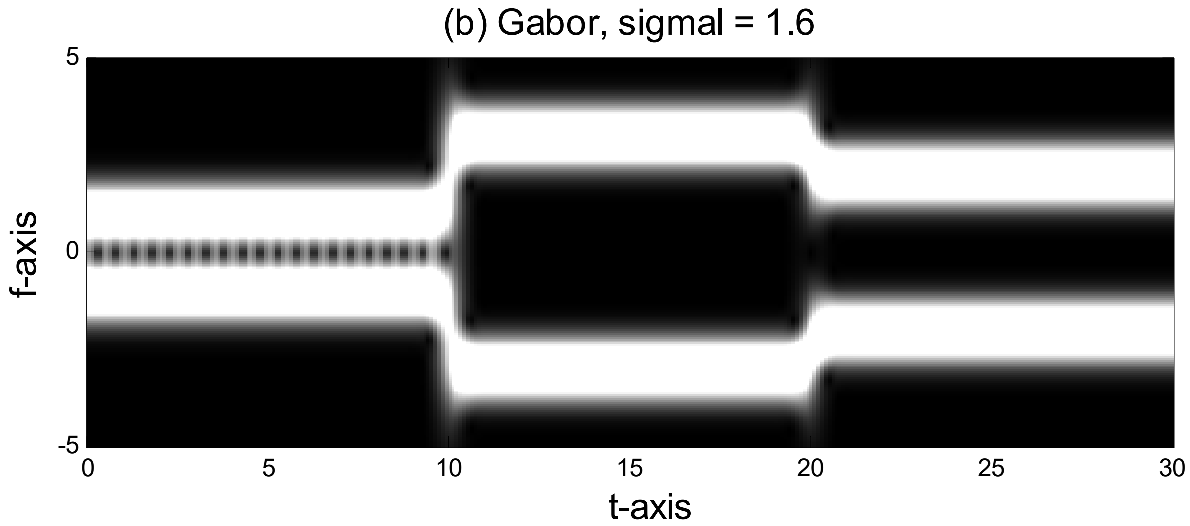
To achieve better clarity, $w_1(t)$ can be chosen as a wider window,
$w_2(t)$ can be chosen as a narrower window.

$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$



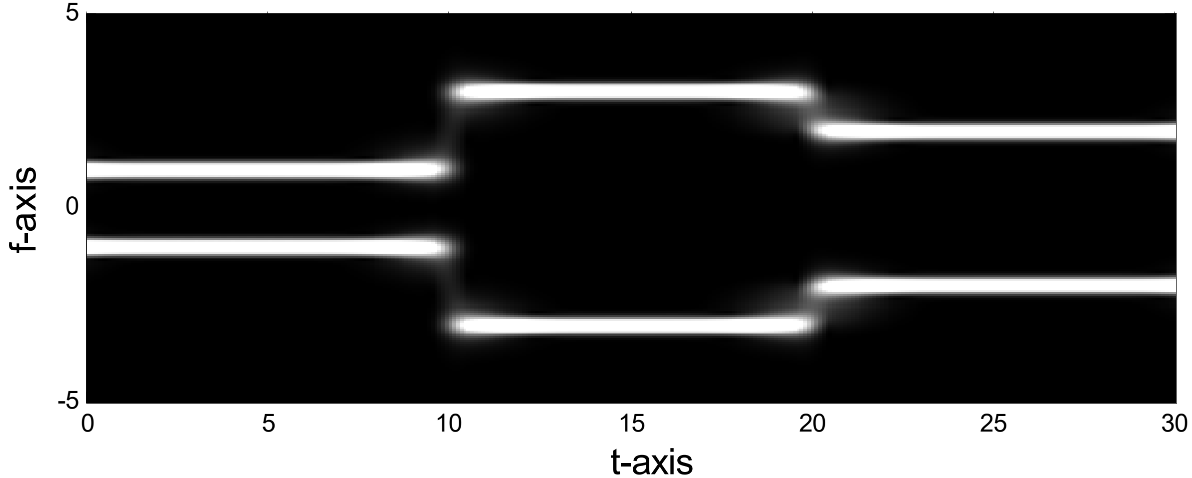


$x(t) = \cos(2\pi t)$ when $t < 10$,

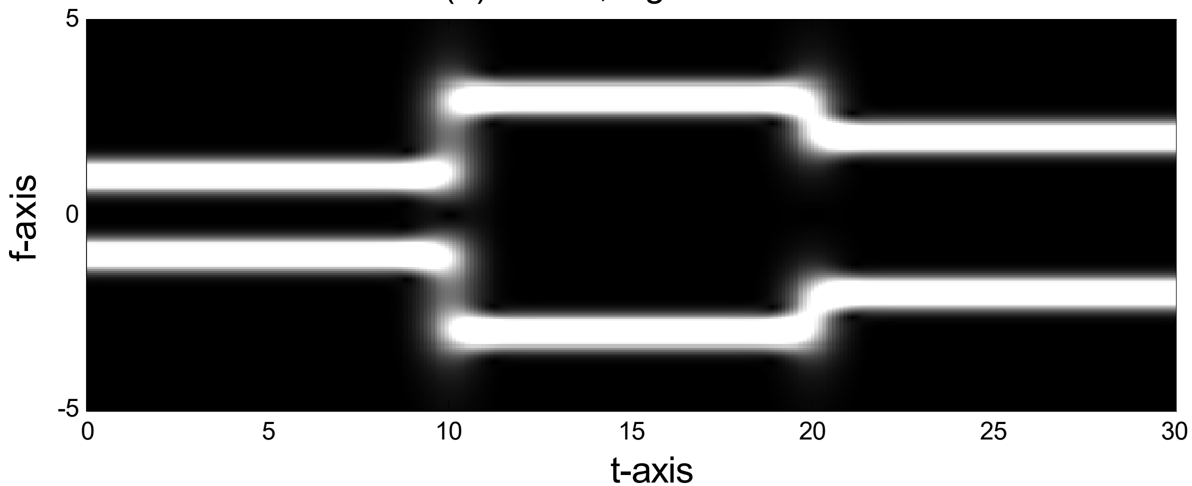
$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$

(c) Generalized spectrogram



(d) Gabor, sigmal = 0.4



Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^{*}_{x,w_2}(t,f)$

Further Generalization for the spectrogram:

$$\boxed{SP_{x,w_1,w_2}(t,f) = G^{\alpha}_{x,w_1}(t,f)\,\overline{G^{\beta}_{x,w_2}(t,f)}}$$

or

$$\boxed{SP_{x,w_1,w_2}(t,f) = \left|G_{x,w_1}(t,f)\right|^{\alpha}\left|G_{x,w_2}(t,f)\right|^{\beta}}$$

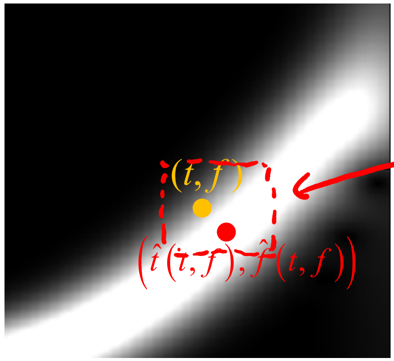
VII-C Reassignment Method (post-processing)

After computing the time-frequency distribution, we can use the following way to make the energy even more concentrated.

v

(t,f) u

(1) First, estimate the offset. patch



patch

$$\hat{t}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}$$ = expected value of u

$$\hat{f}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v \cdot \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}$$ = expected value of v

$X(t, f)$: time-frequency analysis (STFT, WDF…) of $x(t)$,

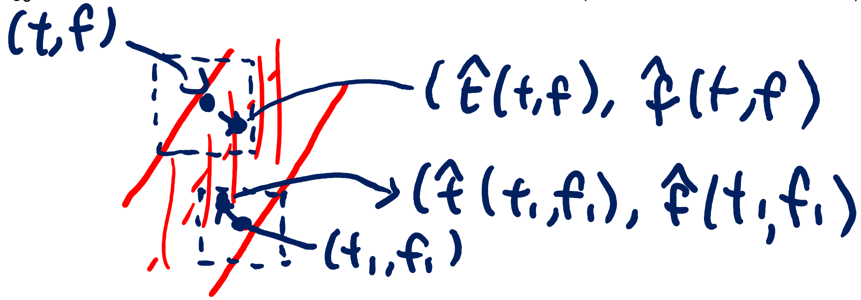
$\varphi(u, v) = 1$ when $|u|$, $|v|$ are small, $\varphi(u, v) = 0$ otherwise

$\varphi(u,v)=1$ for $|u|, |v| < \frac{L}{2}$ $\varphi(u,v)=0$ otherwise

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}(t,f), \hat{f}(t,f)\right)$

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}\left(t, f\right), \hat{f}\left(t, f\right)\right)$

$$\hat{X}\left(t, f\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X\left(t_1, f_1\right)\delta\left(t - \hat{t}\left(t_1, f_1\right)\right)\delta\left(f - \hat{f}\left(t_1, f_1\right)\right)dt_1 df_1$$



References

[1] F. Auger and P. Flandrin, “Improving the readability of time-frequency and time-scale representations by the reassignment method,” *IEEE Trans. Signal Processing*, vol. 43, issue 5, pp. 1068-1089, May 1995.

[2] F. Auger, P. Flandrin, Y.T. Lin, S. McLaughlin, S. Meignen, T. Oberlin, and H.T. Wu, “Time-frequency reassignment and synchrosqueezing: An overview,” *IEEE Signal Processing Magazine*, vol. 30, issue 6, pp. 32-41, 2013.

PS: 感謝 2017 年修課的盧德晏同學

VII-D Basis Expansion Time-Frequency Analysis

就如同

exp(j2πfmt) always has the amplitude of 1
frequency of fm

• Fourier series: $\varphi_m(t) = \exp(j2\pi f_m t)$, $x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$

$f_m = m/T$

$$a_m = \frac{\langle x(t), \varphi_m^*(t) \rangle}{\langle \varphi_m(t), \varphi_m^*(t) \rangle} = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi f_m t)dt$$

部分的 Time-Frequency Analysis 也是意圖要將 signal 表示成如下的型態

$$x(t) \approx \sum_{m=1}^{M} a_m \varphi_m(t)$$

並且要求在 M 固定的情形下，

$$\text{approximation error} = \int_{-\infty}^{\infty} \left| x(t) - \sum_{m=1}^{M} a_m \varphi_m(t) \right|^2 dt$$ 為最小

將 $\varphi_m(t)$ 一般化，不同的 basis 之間不只是有 frequency 的差異

(1) Three Parameter Atoms

not orthogonal

$$x(t) \approx \sum a_{t_0,f_0,\sigma}\varphi_{t_0,f_0,\sigma}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}}\exp(j2\pi f_0 t)\exp(-\frac{\pi(t-t_0)^2}{\sigma^2})$$

3 parameters: t_0 controls the central time

f_0 controls the frequency

σ controls the scaling factor

How to find t_0, f_0, σ ?

[Ref] S. G. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397-3415, Dec. 1993.

Since $\varphi_{t_0,f_0,\sigma}(t)$ are not orthogonal, $a_{t_0,f_0,\sigma}$ should be determined by a matching pursuit process.

(2) Four Parameter Atoms (Chirplet)

$$x(t) \approx \sum a_{t_0,f_0,\sigma,\eta} \varphi_{t_0,f_0,\sigma,\eta}(t)$$

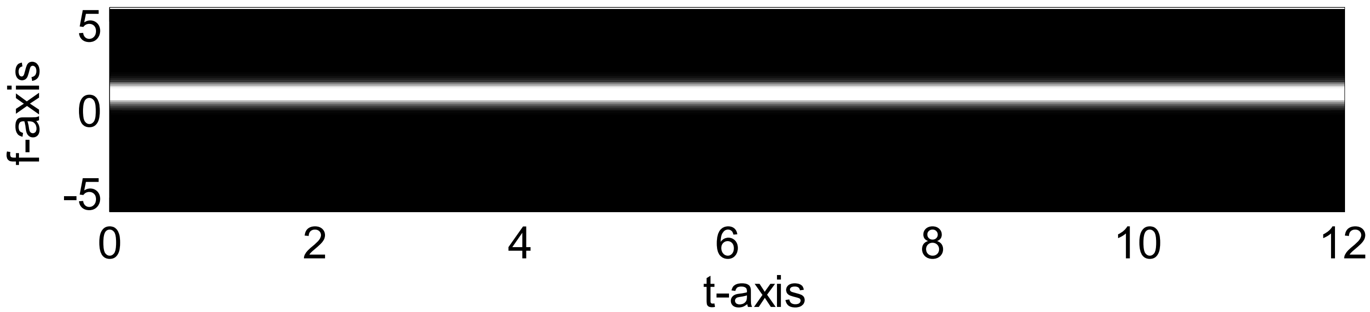
$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp(j2\pi(f_0 t + \frac{\eta}{2} t^2) - \frac{\pi(t-t_0)^2}{\sigma^2})$$

4 parameters: t_0 controls the central time
f_0 controls the initial frequency
σ controls the scaling factor
η controls the chirp rate

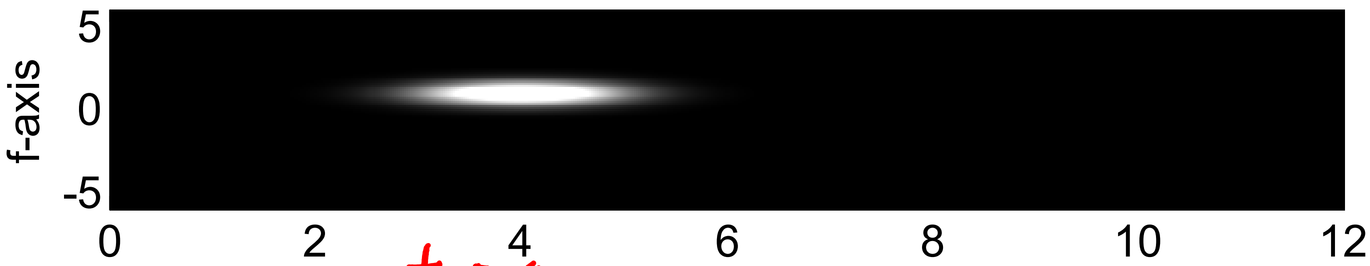
[Ref] A. Bultan, "A four-parameter atomic decomposition of chirplets," *IEEE Trans. Signal Processing*, vol. 47, no. 3, pp. 731–745, Mar. 1999.

[Ref] C. Capus, and K. Brown. "Short-time fractional Fourier methods for the time-frequency representation of chirp signals," *J. Acoust. Soc. Am*. vol. 113, issue 6, pp. 3253-3263, 2003.

(a) STFT of a Fourier basis

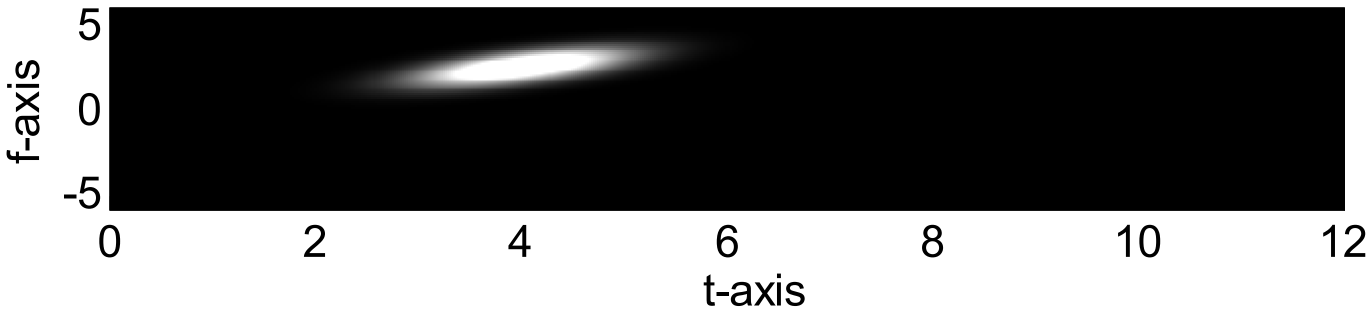


(b) STFT of a 3-parameter atom



$t_0 = 4$

(c) STFT of a chirplet (4-parameter atom)



(3) Prolate Spheroidal Wave Function (PSWF)

$$x(t) \cong \sum_{n,T,\Omega,t_0,f_0} a_{n,T,\Omega,t_0,f_0} \psi_{n,T,\Omega}(t-t_0)\exp(j2\pi f_0 t)$$

where $\psi_{n,T,\Omega}(t)$ is the prolate spheroidal wave function

[Ref] D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty-I," *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, 1961.

Concept of the prolate spheroidal wave function (PSWF):

- FT: $X(f) = \int_{-\infty}^{\infty} \exp(-j2\pi f t) x(t) dt$, $x, f \in (-\infty, \infty)$.

energy preservation property (Parseval's property)

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- finite Fourier transform (fi-FT):

$$X_{fi}(f) = \int_{-T}^{T} \exp(-j2\pi f t) x(t) dt$$

space interval: $t \in [-T, T]$, frequency interval: $f \in [-\Omega, \Omega]$

$$0 < \text{energy preservation ratio} = \frac{\int_{-\Omega}^{\Omega} |X_{fi}(f)|^2 df}{\int_{-T}^{T} |x(t)|^2 dt} < 1$$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega} |X_{fi}(f)|^2 df}{\int_{-T}^{T} |x(t)|^2 dt}$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$

$\psi_{1,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$ and$\psi_{1,T,\Omega}$

$\psi_{2,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

and so on.

- Prolate spheroidal wave functions (PSWFs) are the continuous functions that satisfy: $\int_{-T}^{T} K_{F,\Omega}(t_1,t)\psi_{n,T,\Omega}(t)\,dt = \lambda_{n,T,\Omega}\psi_{n,T,\Omega}(t_1)$,

  where $$K_{F,\Omega}(t_1,t) = \frac{\sin\left[2\pi\Omega(t_1-t)\right]}{\pi(t_1-t)}$$

PSWFs are orthonornal and can be sorted according to the values of $\lambda_{n,T,\Omega}$'s:

$$\int_{-T}^{T}\psi_{m,T,\Omega}(t)\psi_{n,T,\Omega}(t)\,dt = \delta_{m,n}$$

$1 > \lambda_{0,T,\Omega} > \lambda_{1,T,\Omega} > \lambda_{2,T,\Omega} > \ldots\ldots\ldots\ldots > 0.$ (All of $\lambda_{n,T,\Omega}$'s are real)

附錄七： Compressive Sensing and Matching Pursuit 的觀念

Different from orthogonal basis expansion, which applies a complete and orthogonal basis set, compressive sensing is to use an over-complete and non-orthogonal basis set to expand a signal.

Example:

Fourier series expansion is an orthogonal basis expansion method:

$$x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$$

$$\int \exp(j2\pi f_m t)\overline{\exp(j2\pi f_n t)}\, dt = 0 \qquad \text{if } f_m \neq f_n$$

Three-parameter atom expansion, Four-parameter atom (chirplet) expansion, and PSWF expansion are over-complete and non-orthogonal basis expansion methods.

$$x(t) \approx \sum a_{t_0, f_0, \sigma} \varphi_{t_0, f_0, \sigma}(t)$$

$\varphi_{t_0,f_0,\sigma}(t)$ do not form a complete and orthogonal set.

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ ………….. form an over-complete and non-orthogonal basis set.

(Problem 1) We want to minimize $||c||_0$ ($||\ ||_0$ 是 zero-order norm，$||c||_0$ 意指 c_m 的值不為 0 的個數) such that

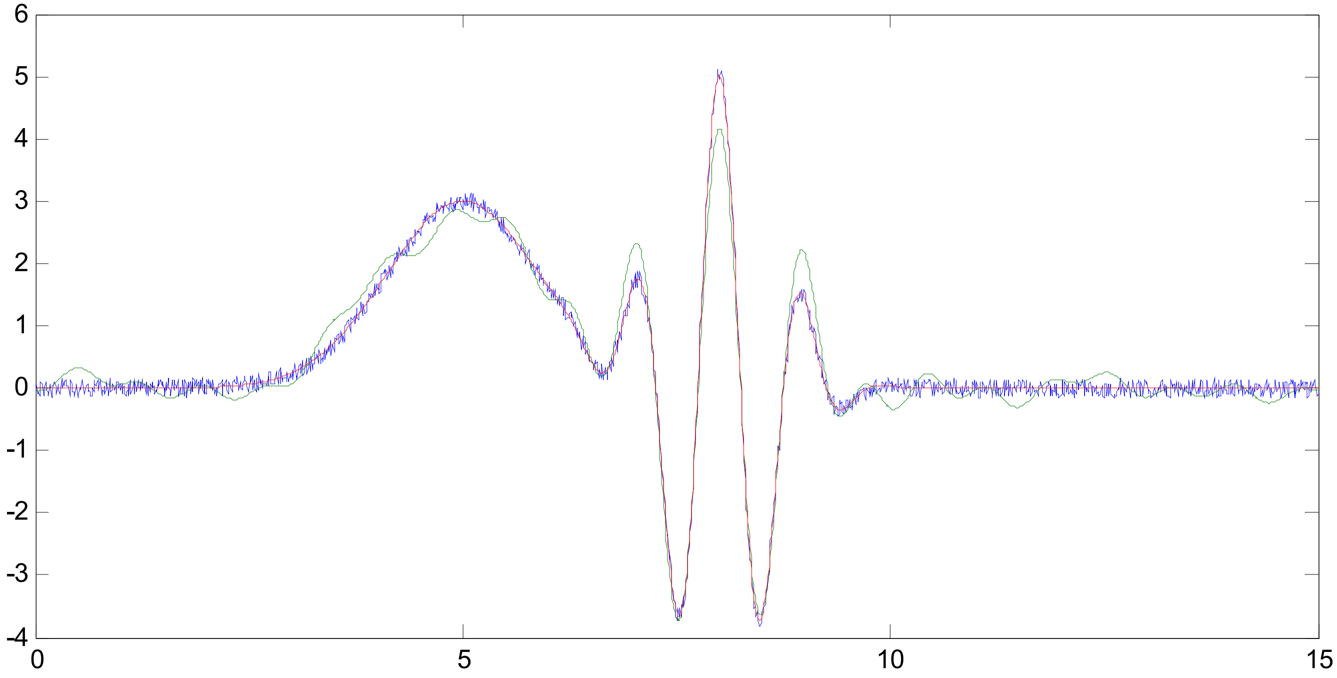
$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_0$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_0$ is limited to M, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$



For example, in the above figure, the blue line is the original signal

- When using three-parameter atoms, the expansion result is the red line

$$x(t) = 3e^{-0.2\pi(t-5)^2} + 2.5e^{-0.4\pi(t-8)^2 + j2\pi t} + 2.5e^{-0.4\pi(t-8)^2 - j2\pi t}$$

Only 3 terms are used and the normalized root square error is 0.39%

- When using Fourier basis, if 31 terms are used, the expansion result is the green line and the normalized root square error is 3.22%

Question: How do we solve the optimization problems on page 205?

Method 1: Matching Pursuit (Greedy Algorithm)

atoms: $b_m(t)$, $m = 1, 2, 3, \ldots$.

$$\left|\int b_m(t) b_m^*(t)\,dt\right| = 1$$

(normalized)

Input: $x(t)$

Initial: $n = 0$, $x_r(t) = x(t)$

$\downarrow$

Find m such that $\left|\int x_r(t) b_m^*(t)\,dt\right|$ is maximal

$\downarrow$

Set $\phi_n(t) = b_m(t)$, $\mu_n = \int x(t) b_m^*(t)\,dt$

$\downarrow$

$x_r(t) = x_r(t) - \mu_n \phi_n(t)$

$\downarrow$

$n = n+1$

$\downarrow$

For Problem 1: Whether $x_r(t) = 0$

For Problem 2: Whether $\int x_r^2(t)\,dt < threshold$

For Problem 3: Whether $n = M$

No

Yes $\rightarrow$ $x(t) \cong \sum_m \mu_m \phi_m(t)$

Method 2: Basis Pursuit

Change the zero-order norm into the first order norm

$$||c||_1 = |c_0| + |c_1| + |c_2| + \ldots\ldots$$

(Problem 1) We want to minimize $||c||_1$ such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_1$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_1 \le M$, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Norm (L_α norm): $\|x[n]\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$

$\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha = K$ where K is the number of points such that $x[n] \neq 0$

(Physical meaning: The number of nonzero points)

L_1 norm: $\|x[n]\|_1 = \sum_{n=0}^{N-1} |x[n]|$

(Physical meaning: Sum of Amplitudes)

L_2 norm: $\|x[n]\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$

(Physical meaning: Distance)

Matching Pursuit: Zero order norm $\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha$

Basis Pursuit: First order norm L_1 norm

- D. L. Donoho, “Compressed sensing,” *IEEE Trans. Inf. Theory*, vol. 52, issue 4, pp. 1289–1306, 2006. (被視為最早提出 compressive sensing 概念的論文)
- E. J. Candès and M. B. Wakin, “An introduction to compressive sampling,” *IEEE Signal Processing Magazine*, vol. 25, issue 2, pp. 21-30, 2008. (對 compressive sensing 做 tutorial 式的介紹)
- S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*, Birhauser, Basel, 2013. (以數學的方式介紹 compressive sensing)
- S. G. Mallat and Z. Zhang. “Matching pursuits with time-frequency dictionaries,” *IEEE Trans. Signal Processing*, vol. 41, issue 12, pp. 3397-3415, 1993. (最早提出 matching pursuit)
- S. S. Chen, D. L. Donoho, and M. A. Saunders, “Atomic decomposition by basis pursuit,” *SIAM Journal on Scientific Computing*, vol. 20, issue 1, pp. 33-61, 1998. (最早提出 basis pursuit)
- S. Kunis and H. Rauhut, “Random sampling of sparse trigonometric polynomials, II. Orthogonal matching pursuit versus basis pursuit,” *Foundations of Computational Mathematics*, vol. 8, issue 6, pp. 737-763, 2008. (將 orthogonal expansion 以及 matching pursuit, basis pursuit 的概念做綜合)