# IV. Implementation

## IV-A Method 1: Direct Implementation

以 STFT 為例

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$\Delta_t = \frac{1}{f_s}$

Converting into the Discrete Form

$t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_t$

note: $d\tau \Rightarrow \Delta_t$

$$X(n\Delta_t, m\Delta_f)=\sum_{p=-\infty}^{\infty} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

$w(n\Delta_t) \cong 0$ $|n| > B/\Delta_t$

Suppose that $w(t) \cong 0$ for $|t| > B$, $B/\Delta_t = Q$

$$X(n\Delta_t, m\Delta_f)=\sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

$T$ $F$

$TF(2Q+1)$

$\cong 2TFQ$

$O(TFQ)$

Problem：對 scaled Gabor transform 而言，$Q = ?$ $\frac{1.9143}{\sqrt{\sigma}\,\Delta_t}$

- **Constraint for $\Delta_t$** (The only constraint for the direct implementation method)

To avoid the aliasing effect,

$\Delta_t < 1/2\Omega$, Ω is the bandwidth of ? $x(\tau)w(t-\tau)$ $\Omega_x + \Omega_w$

If the bandwidth of $x(\tau) = \Omega_x$

bandwidth of $w(\tau) = \Omega_w$, $w(t-\tau)$ also has the bandwidth of $\Omega_w$

$w(\tau) \to W(f)$, $w(-\tau) \to W(-f)$, $w(t-\tau) \to e^{-j2\pi ft} W(-f)$

There is no constraint for $\Delta_f$ when using the direct implementation method.

## Four Implementation Methods

(1) Direct implementation

Complexity: $O(TFQ)$

假設 $t$-axis 有 $T$ 個 sampling points, $f$-axis 有 $F$ 個 sampling points

unbalanced form

(2) FFT-based method

Complexity: $O(TN\log N)$

$O(\frac{T}{S} N\log N)$

(3) FFT-based method with recursive formula

Complexity: $O(TF)$

(4) Chirp-Z transform method

Complexity: $O(TN\log N)$

complexity

(A) Direct Implementation

4 Advantage： simple, flexible

the only constraint: $\Delta_t < \frac{1}{2(\Omega_x + \Omega_w)}$

Disadvantage ： higher complexity

(B) DFT-Based Method

2 Advantage ： lower complexity

$\Delta_t \Delta_f = \frac{1}{N}$, $N \geq 2Q+1$

Disadvantage ： with some constraints

(C) Recursive Method

1 Advantage ： least complexity

Disadvantage ： (i) only suitable for rectangular windows (ii) accumulation of error (iii) unable to be converted to the unbalanced form

(D) Chirp Z Transform

3 Advantage ： the only constraint: $\Delta_t < \frac{1}{2(\Omega_x + \Omega_w)}$

Disadvantage ： middle complexity

# IV-B Method 2: FFT-Based Method

Constraints : (i) $\Delta_t\Delta_f = 1/N$, N can be any integer

(ii) $N = 1/(\Delta_t\Delta_f) \geqq \underline{2Q+1}$: ($\Delta_t\Delta_f$是整數的倒數)

$$X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} \underbrace{w((n-p)\Delta_t)x(p\Delta_t)}_{x_1(q)} e^{-j\frac{2\pi pm}{N}}\Delta_t$$

$e^{-j\frac{2\pi}{N}(q+n-Q)m}$

Note that the input of the FFT has less than $N$ points (others are set to zero).

Standard form of the DFT $\quad Y[m] = \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi mn}{N}}$

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}}, \quad \underline{q = p-(n-Q)} \rightarrow p = (n-Q)+q$$

when $p = n-Q$, $q = 0$

$p = n+Q$, $q = 2Q$

$x_1(q) = w(k\Delta_t)x((n+k)\Delta_t)$

where $x_1(q) = w((Q-q)\Delta_t)x((n-Q+q)\Delta_t)$ for $\boxed{0 \le q \le 2Q,}$

$x_1(q) = 0$ for $2Q < q < N$.

$k = q-Q$ (suppose that $w(t) = w(-t)$)

$k = -Q \sim Q$ when $q = 0 \sim 2Q$

$n\text{-}Q \le n\text{-}Q\text{+}q \le n\text{+}Q$

$Q \ge Q - q \ge -Q$

注意：

(1) 可以使用 Matlab 的 FFT 指令來計算 $\sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}$

(2) 對每一個固定的 $n$，都要計算一次下方的式子

$$X\left(n\Delta_t, m\Delta_f\right) = \underbrace{\Delta_t e^{j\frac{2\pi(Q-n)m}{N}}}_{\text{Step 5}} \underbrace{\sum_{q=0}^{N-1} \underbrace{x_1(q)}_{\text{Step 3}} e^{-j\frac{2\pi qm}{N}}}_{\text{Step 4}}$$

T F

(fixed $n$)

N-point DFT

$O(N\log N)$

$O(TN\log N)$

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假設 $t = n_0\Delta_t,\ (n_0+1)\Delta_t,\ (n_0+2)\Delta_t,\ \cdots\cdots,\ (n_0+T\text{-}1)\Delta_t$

$f = m\Delta_f$

$f = m_0\Delta_f,\ (m_0+1)\Delta_f,\ (m_0+2)\Delta_f,\ \cdots\cdots,\ (m_0+F\text{-}1)\Delta_f$

$f = -5 \sim 5,\ \Delta_f = 0.1$

$f = m\Delta_f,\ m = -50 \sim 50$

$t = 0 \sim 30 \quad \Delta_t = 0.1$

$n = 0 \sim 300 \quad (t = n\Delta_t)$

$T = 301,\ F = 101$

$N = \frac{1}{\Delta_t \Delta_f}$

$\frac{B}{\Delta_t}$

Step 1: Calculate $n_0$, $m_0$, $T$, $F$, $N$, $Q$

0 -50 301 101 100 10

Step 2: $n = n_0$

0

Step 3: Determine $x_1(q)$

Step 4: $X_1(m) = \text{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(n\Delta_t, m\Delta_f)$

page 99

$m = f/\Delta_f$

$m_1 = \text{mod}(m, N)+1$

$X_1[-1] = X_1[99]$

$X_1[-48] = X_1[52]$

$X_1[-49] = X_1[51]$

$X_1[-50] = X_1[50]$

$$X(n\Delta_t, m\Delta_f) = X_1(?)\times ?$$

$X(n\Delta_t, m\Delta_f) = X_1(((m))_N)\, e^{j\frac{2\pi}{N}(Q-n)m}\,\Delta_t$

$((m))_N$ : m 除以 N 的餘數

X1(mod(m,N)+1)

$$X_1[m] = \sum_{q=0}^{N-1} x_1(q)\, e^{-j\frac{2\pi qm}{N}}$$

$$X_1[m] = X_1[m+N]$$

$m = 0 \sim N-1$ (However, m may be negative)

Step 6: Set $n = n+1$ and return to Step 3 until $n = n_0+T$-1.

iteration T times

ex: $m = -100,\ N = 400$

$((m))_N = 300$

# IV-C Method 3: Recursive Method

- A very fast way for implementing the rec-STFT

($n$ 和$n$–1有recursive的關係)

$$X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} x\left(p\Delta_t\right) e^{-j\frac{2\pi pm}{N}} \Delta_t$$

$$X\left((n-1)\Delta_t, m\Delta_f\right) = \sum_{p=n-1-Q}^{n-1+Q} x(p\Delta_t) e^{-j\frac{2\pi pm}{N}} \Delta_t$$

(1) Calculate $X(\min(n)\Delta_t, m\Delta_f)$ by the $N$-point FFT

$N \log N$

$$X\left(n_0\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n_0)m}{N}} \sum_{q=0}^{N-1} x_1\left(q\right) e^{-j\frac{2\pi qm}{N}},\quad n_0 = \min(n),$$

$$x_1\left(q\right) = x\left((n-Q+q)\Delta_t\right) \quad \text{for } q \le 2Q, \qquad x_1(q) = 0 \quad \text{for } q > 2Q$$

(2) Applying the recursive formula to calculate $X(n\Delta_t, m\Delta_f)$,

$(T-1)(2F)$

$n = n_0 + 1 \sim \max(n)$

$2F(T-1) + N\log N$

$\cong 2FT$

$O(FT)$

$p = n-Q-1$

$$X\left(n\Delta_t, m\Delta_f\right) = X\left((n-1)\Delta_t, m\Delta_f\right) - x\left((n-Q-1)\Delta_t\right) e^{-j2\pi(n-Q-1)m/N} \Delta_t$$

$T$點 $F$點

$$+ x\left((n+Q)\Delta_t\right) e^{-j2\pi(n+Q)m/N} \Delta_t$$

F-point vectors

$p = n+Q$

# IV-D Method 4: Chirp Z Transform

$$\exp\left(-j2\pi\, pm\Delta_t\Delta_f\right) = \exp\left(-j\pi\, p^2\Delta_t\Delta_f\right)\exp\left(j\pi\,(p-m)^2\Delta_t\Delta_f\right)\exp\left(-j\pi m^2\Delta_t\Delta_f\right)$$

## For the STFT

$$X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} w\left((n-p)\Delta_t\right)x\left(p\Delta_t\right)e^{-j2\pi\, pm\Delta_t\Delta_f}\Delta_t$$

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{-j\pi m^2\Delta_t\Delta_f}\sum_{p=n-Q}^{n+Q} w\left((n-p)\Delta_t\right)x\left(p\Delta_t\right)e^{-j\pi\, p^2\Delta_t\Delta_f}e^{j\pi\,(p-m)^2\Delta_t\Delta_f}$$

T F

Step 1 multiplication $2Q+1$

Step 2 convolution = 3 DFTs $3N\log N$

Step 3 multiplication $F$

$T(2Q+1+3N\log N+F)$

$\cong 3TN\log N$

$O(TN\log N)$

Step 1 $x_1[p] = w((n-p)\Delta_t)x(p\Delta_t)e^{-j\pi p^2\Delta_t\Delta_f}$ $\qquad n\text{-}Q \le p \le n\text{+}Q$

Step 2 $X_2[n,m] = \sum_{p=n-Q}^{n+Q} x_1[p]c[m-p]$ $\qquad c[m] = e^{j\pi m^2\Delta_t\Delta_f}$

Step 3 $X(n\Delta_t, m\Delta_f) = \Delta_t\, e^{-j\pi m^2\Delta_t\Delta_f} X_2[n,m]$

$\mathrm{DFT}(X_2) = \mathrm{DFT}(x_1)\,\mathrm{DFT}(c)$

$X_2 = \mathrm{IDFT}[\mathrm{DFT}(x_1)\,\mathrm{DFT}(c)]$

Step 2 在計算上，需要用到 linear convolution 的技巧

Question: <u>Step 2 要用多少點的 DFT</u>?

# • Illustration for the Question on Page 104

$$y[n] = \sum_k x[n-k]h[k]$$

• Case 1

When length($x[n]$) = $N$,  length( $h[n]$) = $K$,  $N$ and $K$ are finite,

$\longrightarrow$ length($y[n]$) = $N$+$K$–1,

Using the ($N$+$K$–1)-point DFTs (學信號處理的人一定要知道的常識)

• Case 2

$x[n]$ has finite length but $h[n]$ has infinite length ????

$$y[n] = \sum_{k} x[n-k]h[k]$$

- Case 2

$x[n]$ has finite length but $h[n]$ has infinite length

$\boxed{x[n]\text{ 的範圍為 } n \in [n_1, n_2]\text{，範圍大小為 } N = n_2 - n_1 + 1}$

$h[n]$ 無限長

$$y[n] = \sum_{k} x[n-k]h[k]$$

$y[n]$ 每一點都有值 (範圍無限大)

但我們只想求出 $y[n]$ 的其中一段

$\boxed{\text{希望算出的 } y[n]\text{ 的範圍為 } n \in [m_1, m_2]\text{，範圍大小為 } M = m_2 - m_1 + 1}$

$h[n]$ 的範圍？

要用多少點的 FFT？

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

改寫成 $y[n]=x[n]*h[n]=\sum_{s=n_1}^{n_2}x[s]h[n-s]$

$$y[n]=x[n_1]h[n-n_1]+x[n_1+1]h[n-n_1-1]+x[n_1+2]h[n-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[n-n_2]$$

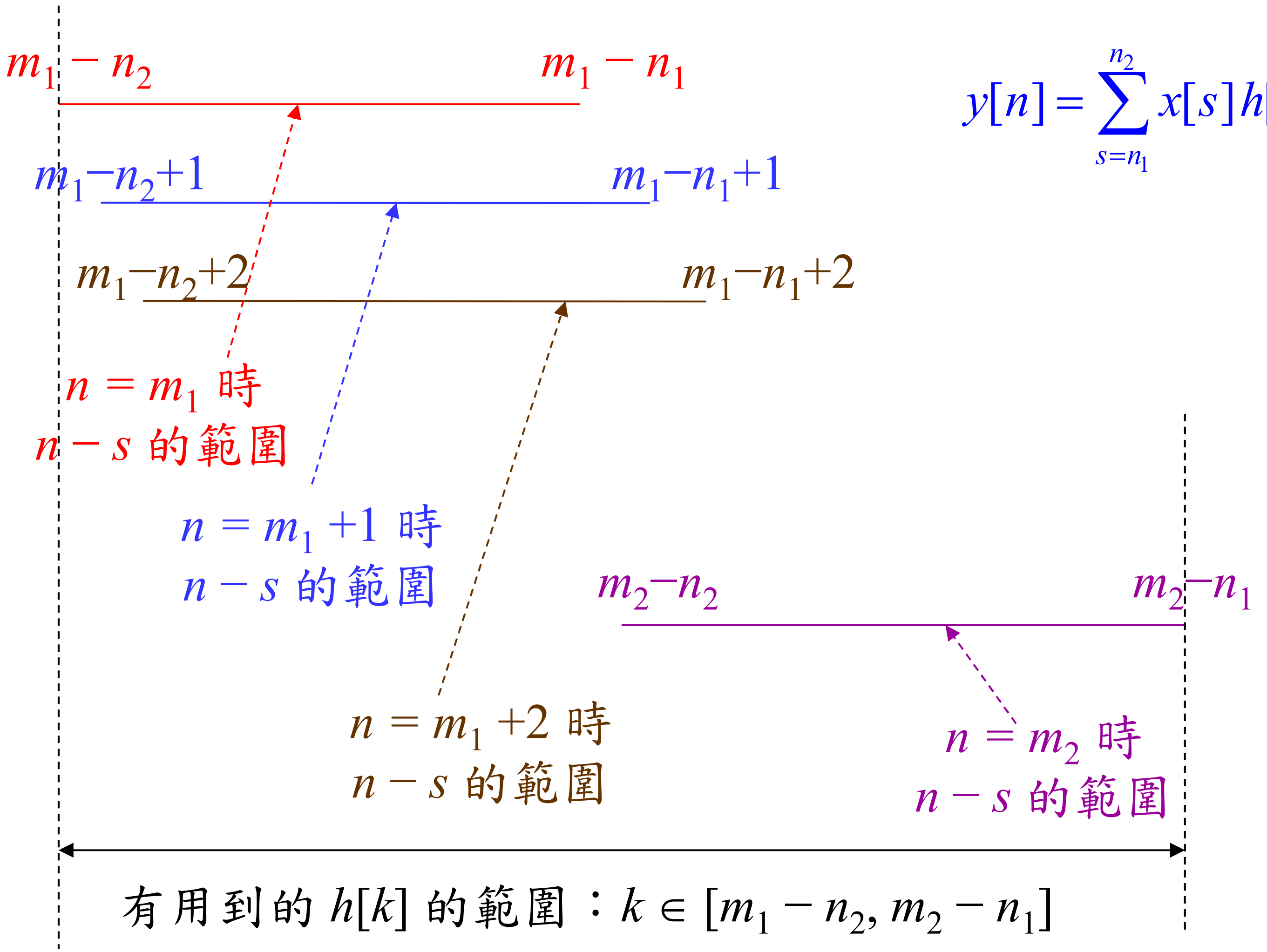
當$n=m_1$

$$y[m_1]=x[n_1]h[m_1-n_1]+x[n_1+1]h[m_1-n_1-1]+x[n_1+2]h[m_1-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_1-n_2]$$

當$n=m_2$

$$y[m_2]=x[n_1]h[m_2-n_1]+x[n_1+1]h[m_2-n_1-1]+x[n_1+2]h[m_2-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_2-n_2]$$

$$y[n]=\sum_{s=n_1}^{n_2} x[s]h[n-s]$$



**所以，有用到的 $h[k]$ 的範圍是 $k \in [m_1 - n_2\ , m_2 - n_1\ ]$**

範圍大小為 $m_2 - n_1 - m_1 + n_2 + 1 = N + M - 1$

FFT implementation for Case 2

$x_1[n] = x[n+n_1]$ for $n = 0, 1, 2, \dots, N-1$

$x_1[n] = 0$ for $n = N, N+1, N+2, \dots\dots, L-1$ $\quad L = N + M - 1$

$h_1[n] = h[n+m_1-n_2]$ for $n = 0, 1, 2, \dots, L-1$

$$y_1[n] = IFFT_L\left(FFT_L\{x_1[n]\}\,FFT_L\{h_1[n]\}\right)$$

$y[n] = y_1[n - m_1 + N - 1]$ for $n = m_1, m_1+1, m_1+2, \dots, m_2$

# IV-E Unbalanced Sampling for STFT and WDF

將 pages 95 and 99 的方法作修正

$w(n\Delta_t - p\Delta_\tau)$
$= w(nS\Delta_\tau - p\Delta_\tau)$
$= w((nS - p)\Delta_\tau)$

$$X(t,f) = \int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$\int_{t-B}^{t+B} \Rightarrow \int_{n\Delta_t - Q\Delta_\tau}^{n\Delta_t + Q\Delta_\tau}$

$n\Delta_t + Q\Delta_\tau$
$= nS\Delta_\tau + Q\Delta_\tau$
$= (nS+Q)\Delta_\tau$

$$X(n\Delta_t, m\Delta_f) = \sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j2\pi pm\Delta_\tau\Delta_f}\Delta_\tau$$

where $t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_\tau$, $B = Q\Delta_\tau$ (假設 $w(t) \cong 0$ for $|t| > B$),

$S = \Delta_t / \Delta_\tau$ $\quad\quad$ $\Delta_t \neq \Delta_\tau$

ex: $\Delta_\tau = \frac{1}{44100}$ $\quad \Delta_t = \frac{1}{100}$
$S = 441$

註：$\Delta_\tau$ (sampling interval for the input signal)

$\Delta_t$ (sampling interval for the output $t$-axis) can be different.

However, it is better that $S = \Delta_t / \Delta_\tau$ is an integer.

不同

When (1) $\Delta_\tau \Delta_f$ = **1/*N***, (2) $N = 1/(\Delta_\tau\Delta_f) > 2Q+1$: ($\Delta_\tau\Delta_f$只要是整數的倒數即可)

(3) $\Delta_\tau < 1/2\Omega$, $\Omega$ is the bandwidth of $w(\tau - t)x(\tau)$

i.e., $|FT\{w(\tau-t)x(\tau)\}| = |X(t,f)| \approx 0$ when $|f| > \Omega$

$$\boxed{X(n\Delta_t, m\Delta_f) = \sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j\frac{2\pi pm}{N}}\Delta_\tau}$$

令 $q = p-(nS-Q) \rightarrow p = (nS-Q)+q$

比較 page 99

$x_1(q) = w((Q-q)\Delta_\tau)x(\underline{(nS}-Q+q)\Delta_\tau)$ for $0 \le q \le 2Q$,

不同

$x_1(q) = 0$ for $2Q < q < N$.

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_\tau e^{j\frac{2\pi(Q-nS)m}{N}}\sum_{q=0}^{N-1} x_1(q)e^{-j\frac{2\pi qm}{N}}}$$

假設 $t = c_0\Delta_t,\ (c_0+1)\,\Delta_t,\ (c_0+2)\,\Delta_t,\ \cdots\cdots,\ (c_0+C-1)\,\Delta_t$

$= c_0 S\Delta_\tau,\ (c_0S+S)\,\Delta_\tau,\ (c_0S+2S)\,\Delta_\tau,\ \cdots\cdots,\ [c_0S+(C-1)S]\,\Delta_\tau,$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F-1)\,\Delta_f$

$\tau = n_0\,\Delta_\tau,\ (n_0+1)\,\Delta_\tau,\ (n_0+2)\,\Delta_\tau,\ \cdots\cdots,\ (n_0+T-1)\,\Delta_\tau,$ $S = \Delta_t / \Delta_\tau$

Step 1: Calculate $c_0$, $m_0$, $n_0$, $C$, $F$, $T$, $N$, $Q$

Step 2: $n = c_0$

Step 3: Determine $x_1(q)$

Step 4: $X_1(m) = \mathrm{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(n\Delta_t, m\Delta_f)$

Step 6: Set $n = n+1$ and return to Step 3 until $n = c_0 + C - 1$.

**Complexity = ?**

# IV-F Non-Uniform $\Delta_t$

(A) 先用較大的 $\Delta_t$

(B) 如果發現 $\left|X\left(n\Delta_t, m\Delta_f\right)\right|$ 和 $\left|X\left((n+1)\Delta_t, m\Delta_f\right)\right|$ 之間有很大的差異

則在 $n\Delta_t$, $(n+1)\,\Delta_t$ 之間選用較小的 sampling interval $\Delta_{t1}$

($\Delta_\tau < \Delta_{t1} < \Delta_t$, $\Delta_t / \Delta_{t1}$ 和 $\Delta_{t1} / \Delta_\tau$ 皆為整數)

再用 page 112 的方法算出

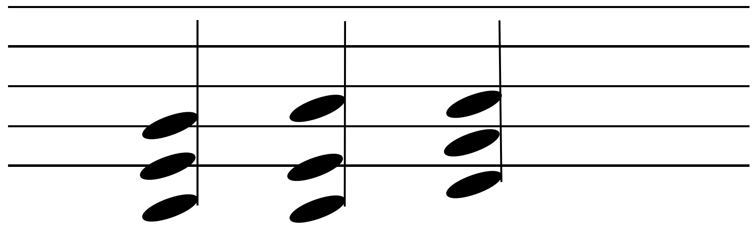
$$X\left(n\Delta_t + \Delta_{t1}, m\Delta_f\right),\quad X\left(n\Delta_t + 2\Delta_{t1}, m\Delta_f\right),\quad \cdots\cdots,\quad X\left((n+1)\Delta_t - \Delta_{t1}, m\Delta_f\right)$$

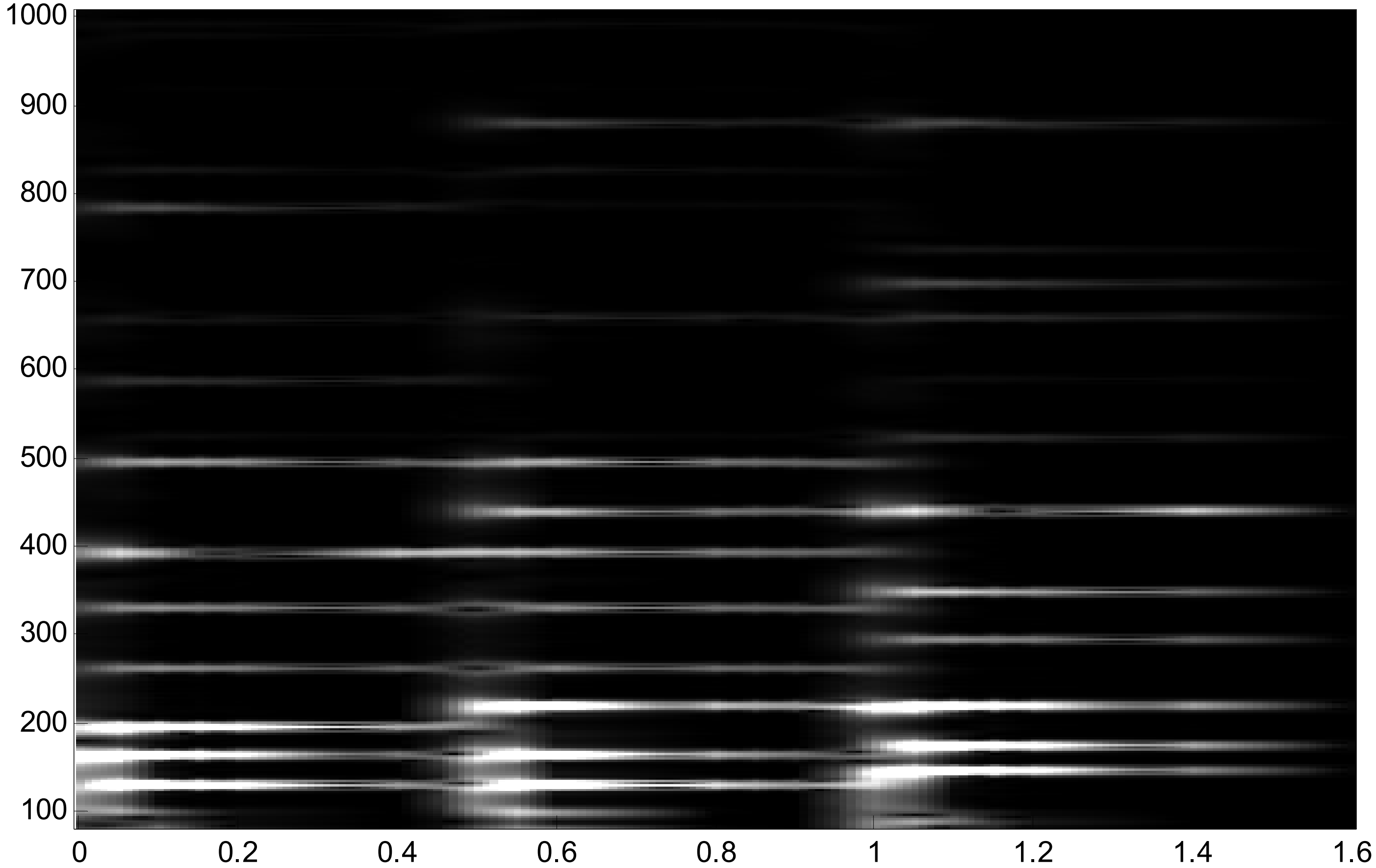
(C) 以此類推，如果 $\left|X\left(n\Delta_t + k\Delta_{t1}, m\Delta_f\right)\right|$, $\left|X\left(n\Delta_t + (k+1)\Delta_{t1}, m\Delta_f\right)\right|$

的差距還是太大，則再選用更小的 sampling interval $\Delta_{t2}$

($\Delta\tau < \Delta t_2 < \Delta t_1$, $\Delta t_1 / \Delta t_2$ 和 $\Delta t_2 / \Delta\tau$ 皆為整數)

Gabor transform of a music signal






$\Delta_\tau = 1/44100$ (總共有 $44100 \times 1.6077$ sec + 1 = 70902 點

(A) Choose $\Delta_t = \Delta_\tau$

running time = out of memory

(B) Choose $\Delta_t = 0.01 = 441\Delta_\tau$ (1.6/0.01 + 1 = 161 points)

running time = 1.0940 sec (2008年)

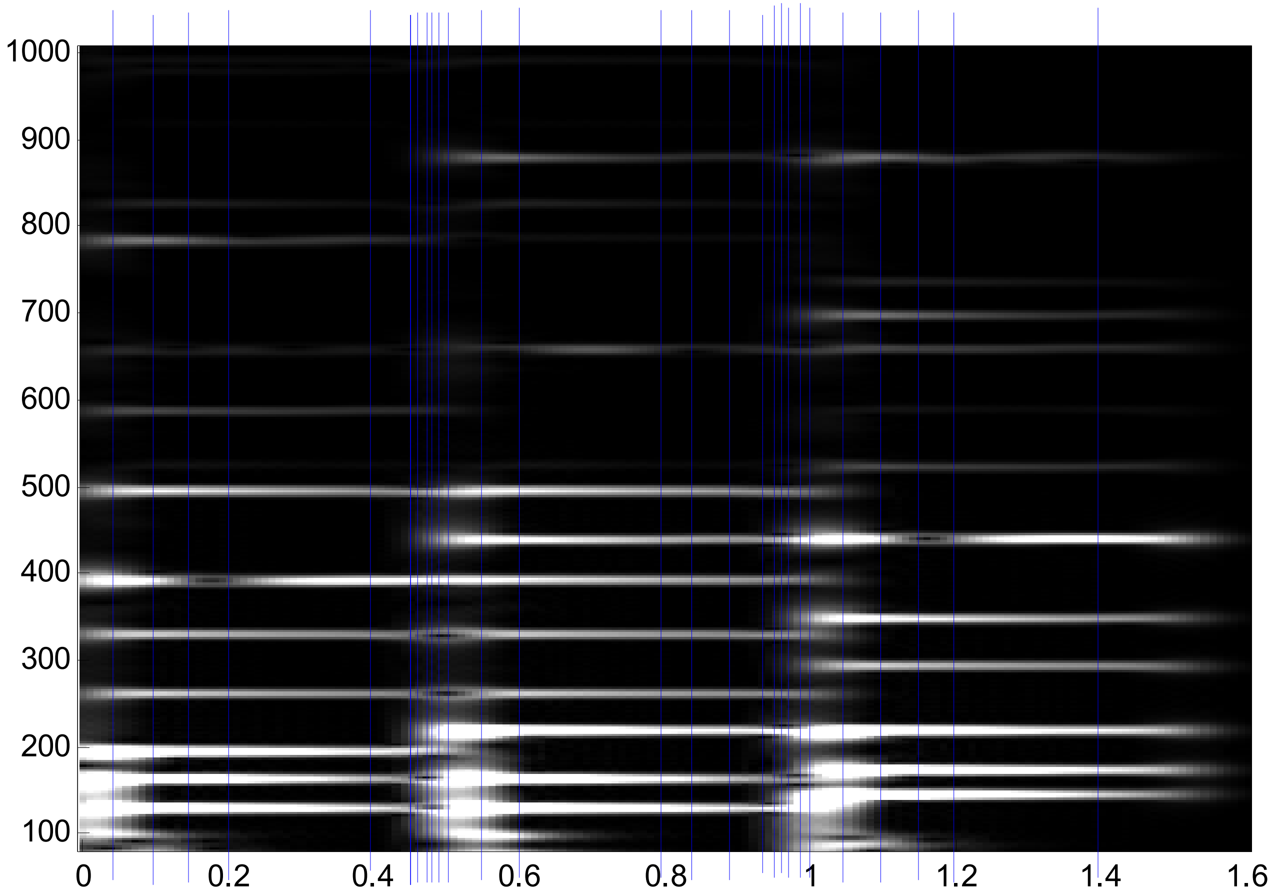
(C) Choose the sampling points on the *t*-axis as

$t$ = 0, 0.05, 0.1, 0.15, 0.2, 0.4, 0.45, 046, 0.47, 0.48, 0.49, 0.5, 0.55, 0.6, 0.8, 0.85, 0.9, 0.95, 0.96, 0.97, 0.98, 0.99, 1, 1.05, 1.1, 1.15, 1.2, 1.4, 1.6

(29 points)

running time = 0.2970 sec

with adaptive output sampling intervals



# 附錄四 和 Dirac Delta Function 相關的常用公式

(1) $\int_{-\infty}^{\infty} e^{-j2\pi t f}\,dt = \delta(f)$

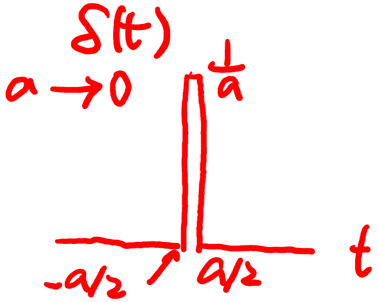
(2) $\delta(t) = |a|\,\delta(at)$ (scaling property)

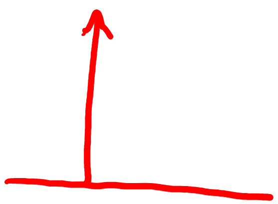
(3) $\int_{-\infty}^{\infty} e^{-j2\pi t g(f)}\,dt = \delta\big(g(f)\big) = \sum_{n} |g'(f_n)|^{-1}\,\delta(f - f_n)$

where $f_n$ are the zeros of $g(f)$

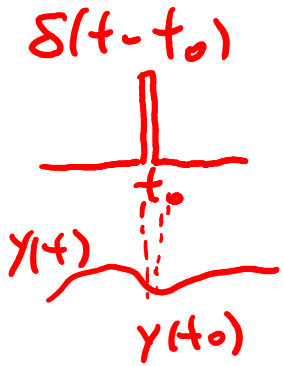
(4) $\int_{-\infty}^{\infty} \delta(t - t_0)\,y(t, \cdots\cdots)\,dt = y(t_0, \cdots\cdots)$ (sifting property I)

(5) $\delta(t - t_0)\,y(t, \cdots\cdots) = \delta(t - t_0)\,y(t_0, \cdots\cdots)$ (sifting property II)



$\int \delta(t)\,dt = 1$



# V. Wigner Distribution Function

韋格納

## V-A Wigner Distribution Function (WDF)

Fourier transform of the auto-correlation function

Definition 1: $W_x(t,f)=\int_{-\infty}^{\infty}x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}d\tau$

Definition 2: $W_x(t,\omega)=\int_{-\infty}^{\infty}x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j\omega\tau}d\tau$

比較: if $x(t)$ is a random process

$E(W_x(t,f))=\int_{-\infty}^{\infty}E(x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2}))e^{-j2\pi\tau f}d\tau$

expected value

$=\int_{-\infty}^{\infty}R_x(t,\tau)e^{-j2\pi\tau f}d\tau$

auto-correlation function

$=S_x(t,f)$

power spectral density (PSD)

Another way for computation from the frequency domain

eta/ ita/

Definition 1: $W_x(t,f)=\int_{-\infty}^{\infty}X(f+\eta/2)\cdot X^*(f-\eta/2)e^{j2\pi\eta t}d\eta$

where $X(f)$ is the Fourier transform of $x(t)$

Definition 2: $W_x(t,\omega)=\int_{-\infty}^{\infty}X(\omega+\eta/2)\cdot X^*(\omega-\eta/2)e^{j\eta t}d\eta$

## Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 5, Prentice Hall, N.J., 1996.

## Other References

[Ref] E. P. Wigner, "On the quantum correlation for thermodynamic equilibrium," *Phys. Rev.*, vol. 40, pp. 749-759, 1932.

[Ref] T. A. C. M. Classen and W. F. G. Mecklenbrauker, "The Wigner distribution–A tool for time-frequency signal analysis; Part I," *Philips J. Res.*, vol. 35, pp. 217-250, 1980.

[Ref] F. Hlawatsch and G. F. Boudreaux–Bartels, "Linear and quadratic time-frequency signal representation," *IEEE Signal Processing Magazine*, pp. 21-67, Apr. 1992.

[Ref] R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley-Interscience, NJ, 2004.

The operators that are related to the WDF:

(a) Signal auto-correlation function:

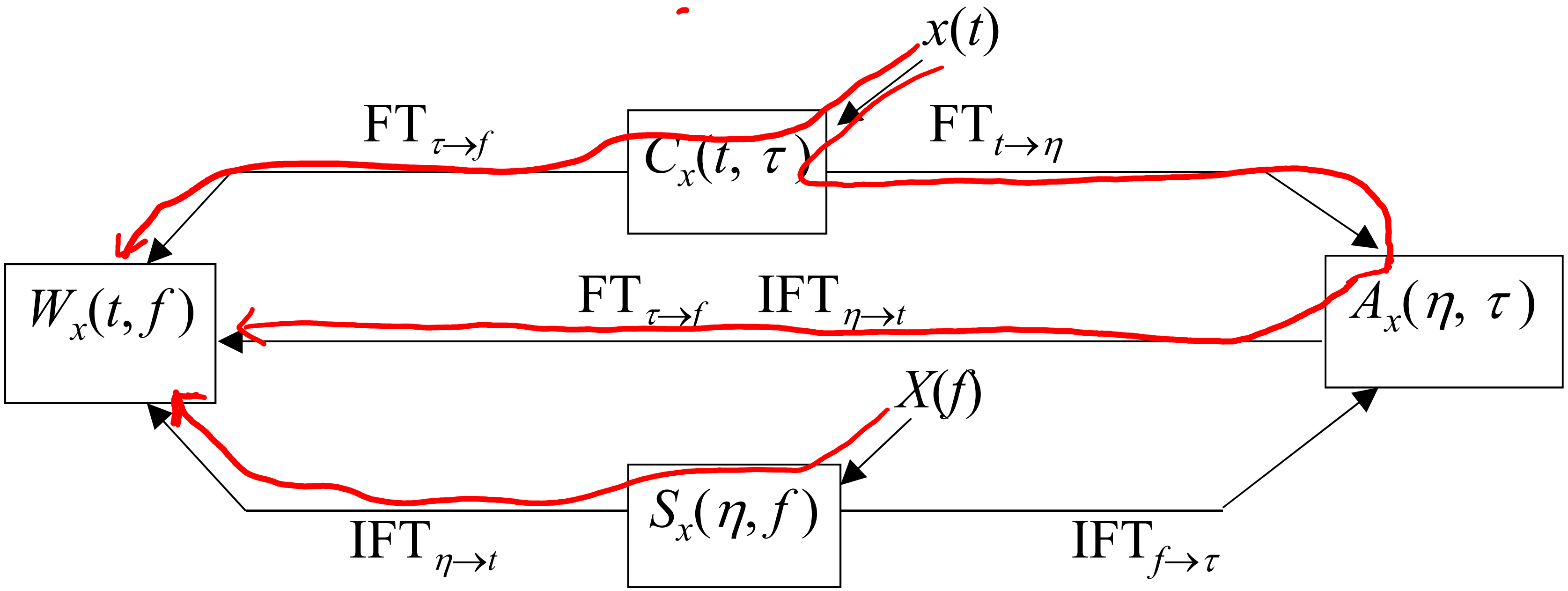
$$C_x(t,\tau) = x(t+\tau/2)\cdot x^*(t-\tau/2)$$

(b) Spectrum auto-correlation function:

$$S_x(\eta,f) = X(f+\eta/2)\cdot X^*(f-\eta/2)$$

(c) Ambiguity function (AF):

$$A_x(\eta,\tau) = \int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)\cdot e^{-j2\pi t\eta}\cdot dt$$



# V-B Why the WDF Has Higher Clarity?

Due to signal auto-correlation function

(1) If $x(t) = 1$ ← $h=0$

(2) If $x(t) = \exp(j2\pi h\, t)$

$x(t+\tau/2)\ x^*(t-\frac{\tau}{2})$

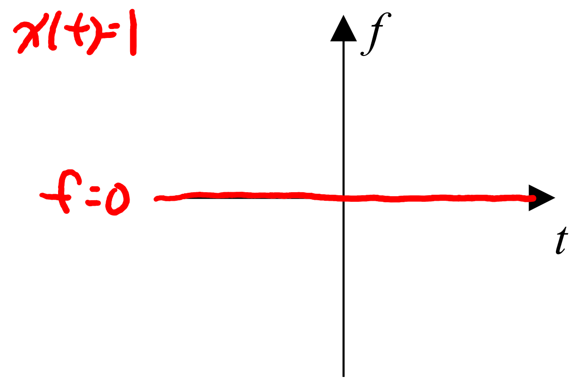
$$W_x(t,f) = \int_{-\infty}^{\infty} e^{j2\pi h(t+\tau/2)} e^{-j2\pi h(t-\tau/2)} \cdot e^{-j2\pi\tau f} d\tau$$

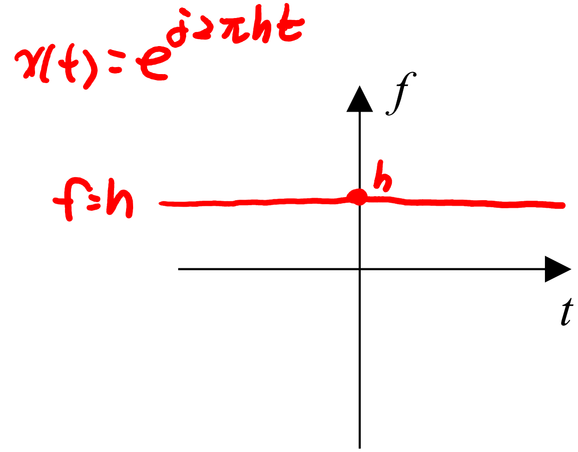
$$= \int_{-\infty}^{\infty} e^{j2\pi h\tau} \cdot e^{-j2\pi\tau f} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi\tau(f-h)} d\tau$$

$$= \delta(f-h)$$

page 117,(1)




Comparing: for the case of the STFT

$x(t+\frac{\tau}{2})$ $x^*(t-\frac{\tau}{2})$

(3) If $x(t) = \exp(j2\pi k\, t^2)$

$W_x(t,f) = \int_{-\infty}^{\infty} e^{j2\pi k(t+\frac{\tau}{2})^2} e^{-j2\pi k(t-\frac{\tau}{2})^2} e^{-j2\pi f\tau} d\tau$

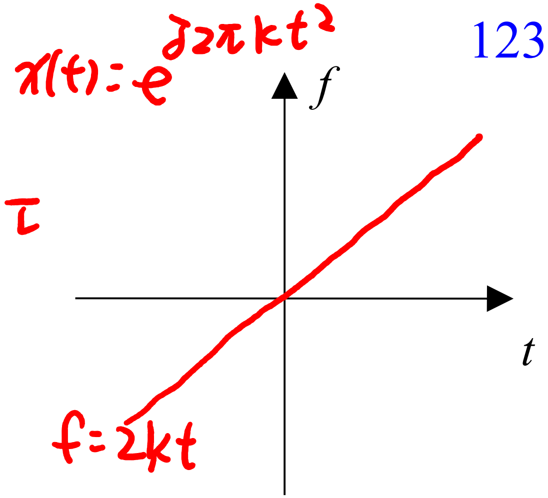
$= \int_{-\infty}^{\infty} e^{j2\pi k(2t\tau)} e^{-j2\pi f\tau} d\tau$

$= \int_{-\infty}^{\infty} e^{-j2\pi\tau(f-2kt)} d\tau$

$= \delta(f-2kt)$ (page 117 (1))

$\phi(t) = 2\pi k t^2$

$\frac{\phi'(t)}{2\pi} = 2kt$




(4) If $x(t) = \delta(t)$

$$W_x(t,f) = \int_{-\infty}^{\infty} \delta(t+\tau/2)\cdot\delta(t-\tau/2)e^{-j2\pi\tau f} d\tau$$

Page 117 公式(2)

$$= 4\int_{-\infty}^{\infty} \delta(2t+\tau)\cdot\delta(2t-\tau)e^{-j2\pi\tau f} d\tau$$

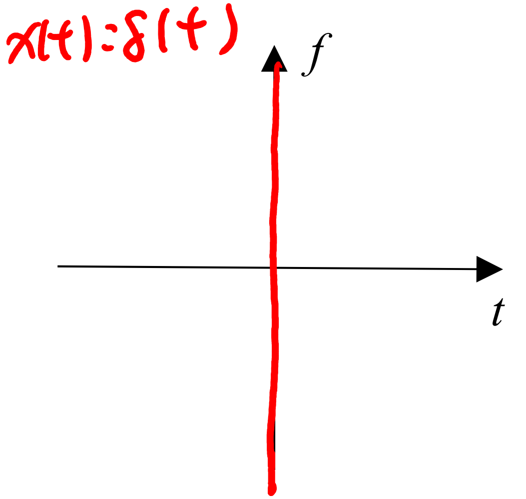
$$= 4\delta(4t)e^{j4\pi t f} = \delta(t)e^{j4\pi t f} = \delta(t)$$

Page 117 公式(4) Page 117 公式(2) Page 117 公式(5), $t_0 = 0$

$\delta(\tau+\frac{t}{2})$ $t \to \tau$

$t_0 \to -2t$

$a=4$



# V-C The WDF is not a Linear Distribution

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

linear

if $g \to G$

$s \to S$

$\alpha g+\beta s \to \alpha G+\beta S$

If $h(t) = \alpha\, g(t) + \beta\, s(t)$

$$W_h(t,f)=\int_{-\infty}^{\infty} h(t+\tau/2)\cdot h^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty}\left[\alpha\, g(t+\tau/2)+\beta\, s(t+\tau/2)\right]\left[\alpha^* g^*(t-\tau/2)+\beta^* s^*(t-\tau/2)\right]e^{-j2\pi\tau f}\, d\tau$$

$$=\int_{-\infty}^{\infty}\Big[|\alpha|^2 g(t+\tau/2)g^*(t-\tau/2)+|\beta|^2 s(t+\tau/2)s^*(t-\tau/2)$$

$$+\alpha\beta^* g(t+\tau/2)s^*(t-\tau/2)+\alpha^*\beta\, g^*(t-\tau/2)s(t+\tau/2)\Big]e^{-j2\pi\tau f}\, d\tau$$

$$=\ |\alpha|^2 W_g(t,f)+|\beta|^2 W_s(t,f)$$

$$+\underbrace{\int_{-\infty}^{\infty}\left[\alpha\beta^* g(t+\tau/2)s^*(t-\tau/2)+\alpha^*\beta\, g^*(t-\tau/2)s(t+\tau/2)\right]e^{-j2\pi\tau f}d\tau}_{\text{cross terms}}$$

# V-D Examples of the WDF

## Simulations

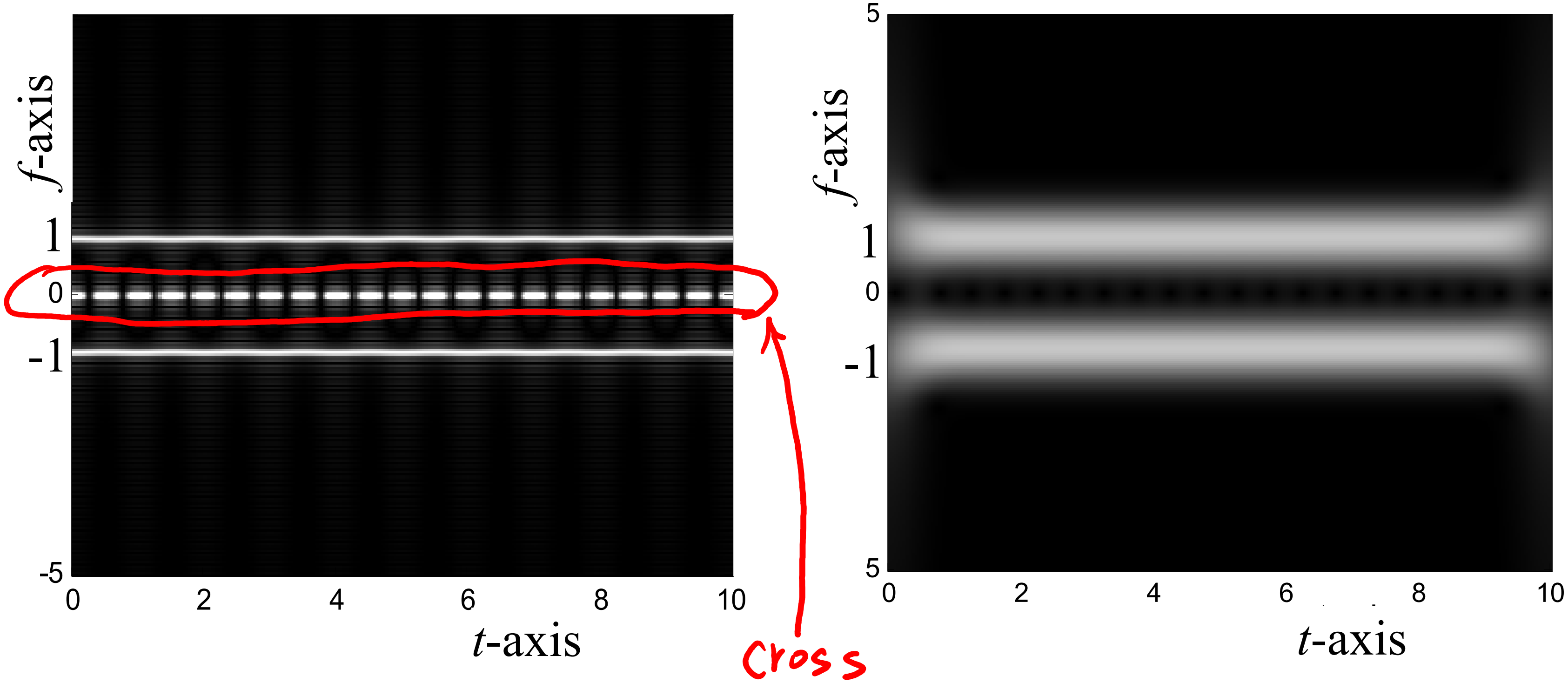
$$x(t) = \cos(2\pi t) = 0.5[\exp(j2\pi t) + \exp(-j2\pi t)]$$

$f = \pm 1$

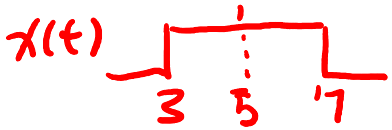
windowed WDF

by the WDF

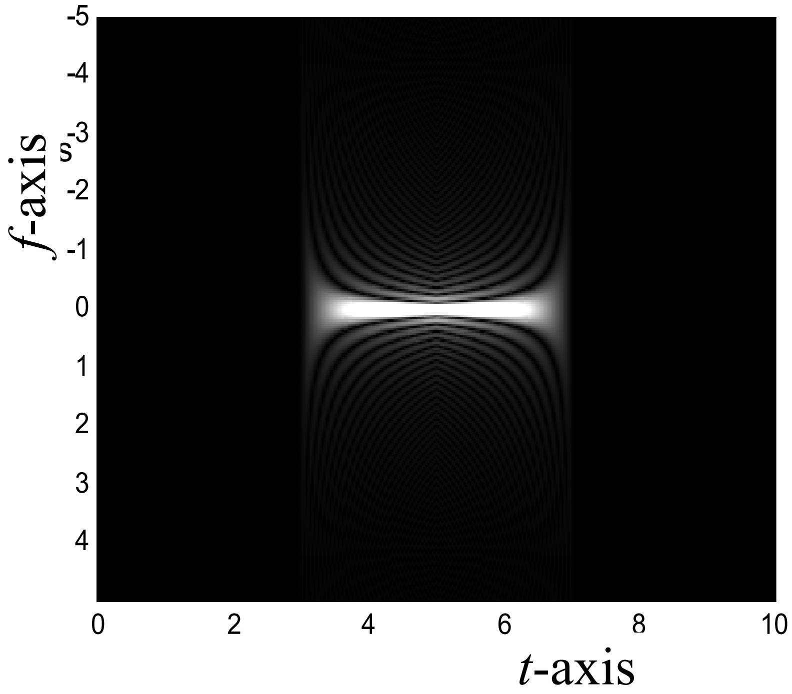
by the Gabor transform



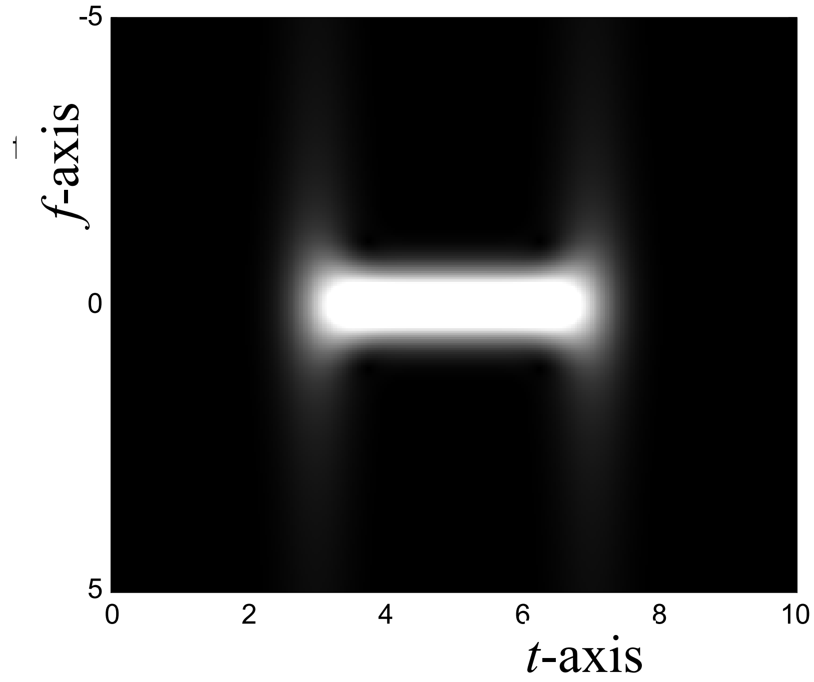
$x(t) = \Pi((t-5)/4)$ $\Pi$: rectangular function




by the WDF

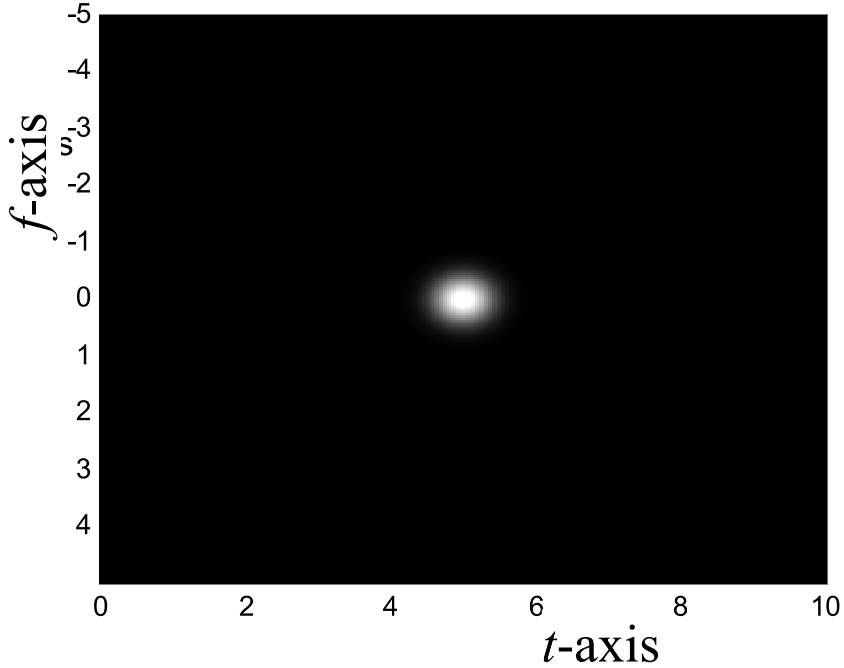



by the Gabor transform

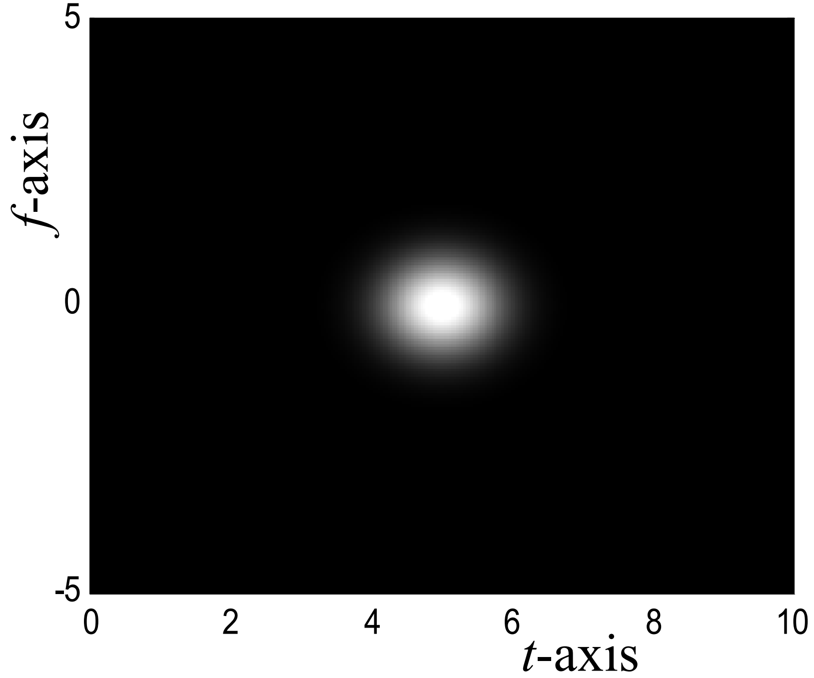


$$x(t) = \exp\left[-\pi(t-5)^2\right]$$

by the WDF




by the Gabor transform




Gaussian function: $e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

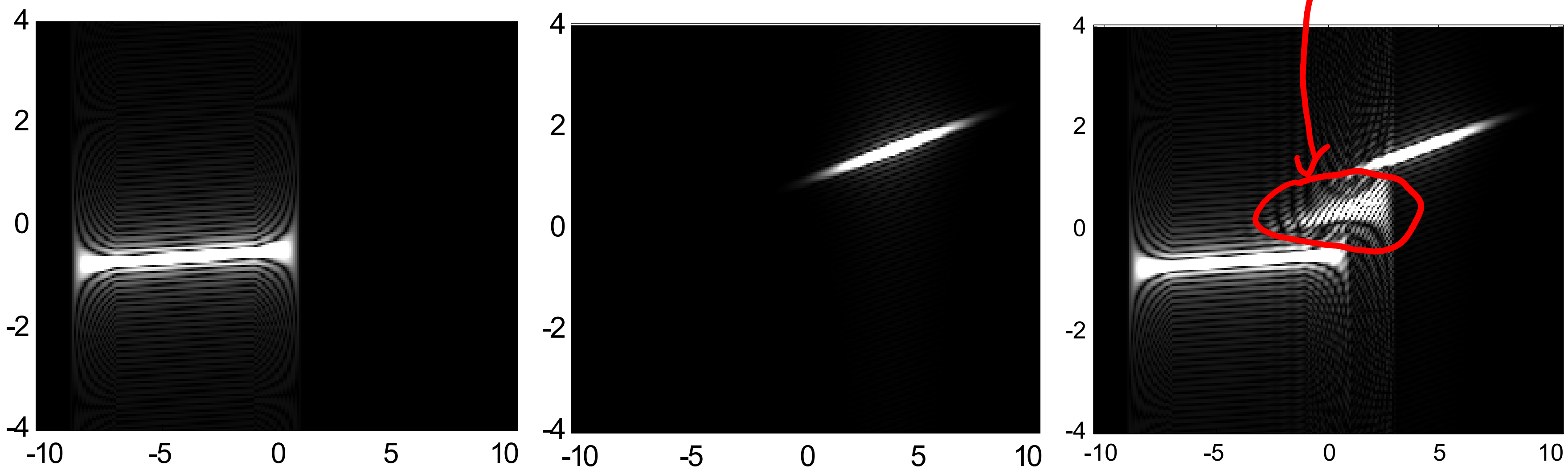
Gaussian function's T-F area is minimal.

$$s(t) = \exp\left(jt^2/10 - j3t\right) \quad \text{for } -9 \le t \le 1,\ s(t) = 0 \text{ otherwise,}$$

$$r(t) = \exp\left(jt^2/2 + j6t\right)\exp\left[-(t-4)^2/10\right]$$

$$f(t) = s(t) + r(t)$$

cross term




WDF of $s(t)$, WDF of $r(t)$, WDF of $s(t) + r(t)$

橫軸: $t$-axis, 縱軸: $f$-axis

instantaneous frequency

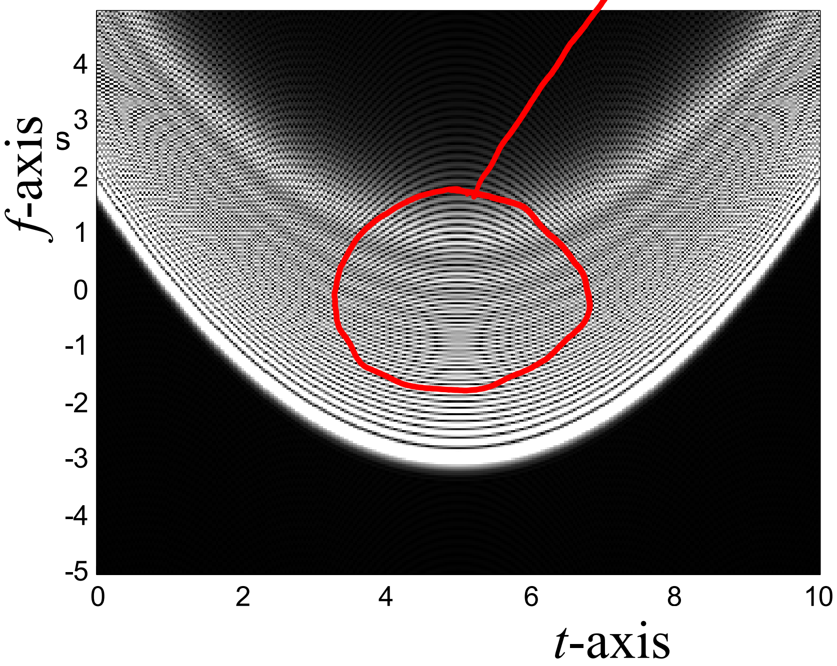
$$x(t) = \exp(j(t-5)^3 - j6\pi t)$$

$$: \frac{3}{2\pi}(t-5)^2 - 3$$

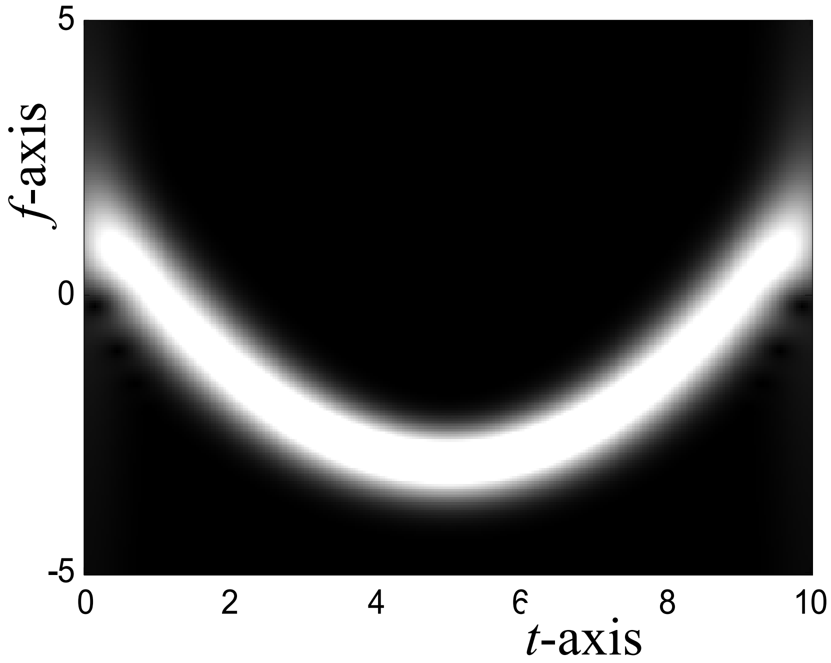
windowed WDF

by the WDF

cross term




by the Gabor transform



# V-E Digital Implementation of the WDF

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau \ ,$$

$$W_x(t,f) = 2\int_{-\infty}^{\infty} x(t+\tau')\cdot x^*(t-\tau')e^{-j4\pi\tau' f}\cdot d\tau' \quad \text{(using } \tau' = \tau/2 \text{)}$$

$\tau = 2\tau'$
$d\tau = 2d\tau'$

Sampling: $t = n\Delta_t$, $f = m\Delta_f$, $\tau' = p\Delta_t$

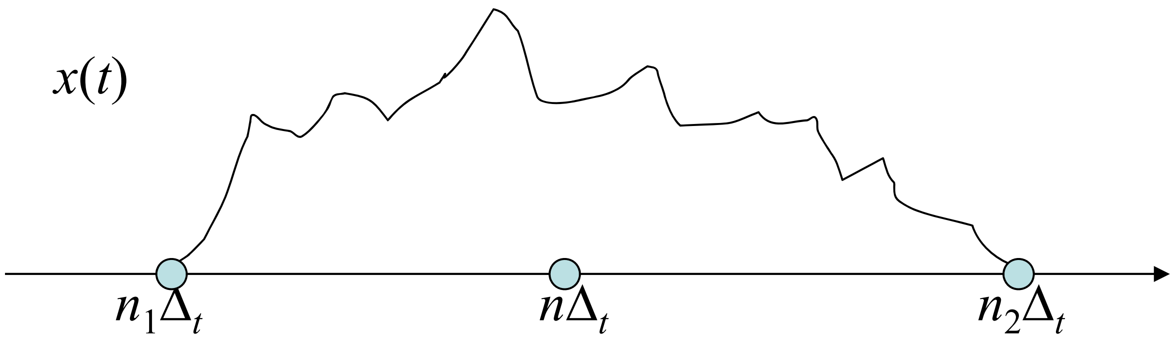
$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-\infty}^{\infty} x((n+p)\Delta_t)x^*((n-p)\Delta_t)\exp(-j4\pi mp\Delta_t\Delta_f)\Delta_t}$$

When $x(t)$ is not a time-limited signal, it is hard to implement.

Suppose that x(t) is time-limited



Suppose that $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$



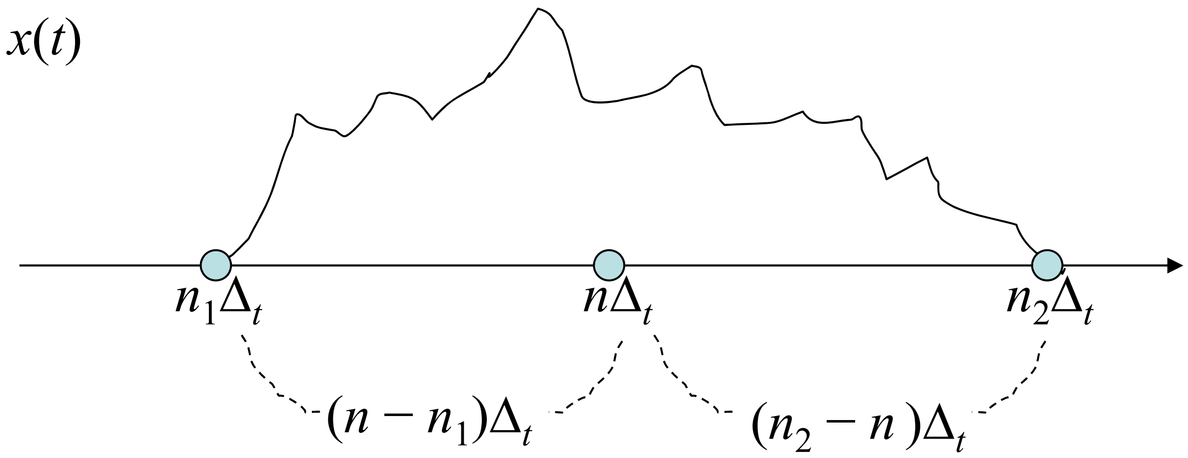

$$x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right) = 0 \qquad \text{if } n+p \notin [n_1, n_2] \text{ or } n-p \notin [n_1, n_2]$$

- $p$ 的範圍的問題 (當 $n$ 固定時)

$$\begin{cases} n_1 \le n+p \le n_2 \longrightarrow n_1 - n \le p \le n_2 - n \\ n_1 \le n-p \le n_2 \longrightarrow n_1 - n \le -p \le n_2 - n, \quad n - n_2 \le p \le n - n_1 \end{cases}$$

$$\max(n_1 - n\ , n - n_2) \ \le\ p \le \min(n_2 - n\ , n - n_1)$$

$$-\min(n_2 - n\ , n - n_1) \ \le\ p \le \min(n_2 - n\ , n - n_1)$$




$$\underbrace{-\min(n_2-n\,,\,n-n_1)}_{-Q} \le p \le \underbrace{\min(n_2-n\,,\,n-n_1)}_{Q}$$

$(n_2-n)\Delta_t$，$(n-n_1)\Delta_t$：離兩個邊界的距離

注意：當 $n>n_2$ 或 $n<n_1$ 時，

將沒有 $p$ 能滿足上面的不等式

If $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$

$$\boxed{W_x\big(\underset{T點}{n\Delta_t}, \underset{F點}{m\Delta_f}\big) = 2\sum_{p=-Q}^{Q} x\big((n+p)\Delta_t\big)x^*\big((n-p)\Delta_t\big)\exp\big(-j4\pi mp\Delta_t\Delta_f\big)\Delta_t}$$

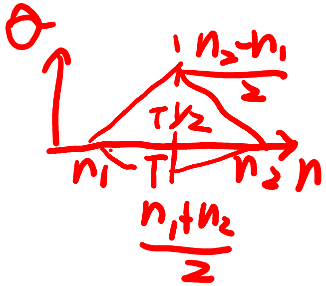
$\underline{Q} = \min(n_2-n, n-n_1)$. (varies with $n$)

$p \in [-Q, Q]$, $\quad n \in [n_1, n_2]$,

possible for implementation

When $n<n_1$, $n>n_2$
since $Q<0$, $W_x(n\Delta_t, m\Delta_f)=0$
When $n=n_1$ or $n=n_2$
$Q=0$
$Q$ is maximal when $n=\frac{n_1+n_2}{2}$

**Method 1: Direct Implementation (brute force method)**




$n_2-n_1+1=T$
$T\cong n_2-n_1$

唯一的限制條件？

$$F\sum_{n=n_1}^{n_2}\big(2Q(n)+1\big)$$

$$\cong 2F\sum_{n=n_1}^{n_2}Q(n) \cong 2F\cdot T\cdot\left(\frac{T}{2}\right)\frac{1}{2} = \frac{FT^2}{2} \qquad \underline{O(FT^2)}$$

## Method 2: Using the DFT

3 大限制條件

$N \geq 2\,\mathrm{Max}(Q)+1 = 2\left(\frac{n_2-n_1}{2}\right)+1 = n_2-n_1+1 = T$

When $\Delta_t \Delta_f = \frac{1}{2N}$ and $N \geq 2Q+1$

$$\boxed{W_x\left(\underset{T}{n\Delta_t}, \underset{F}{m\Delta_f}\right) = 2\Delta_t \sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right) x^*\left((n-p)\Delta_t\right) e^{-j\frac{2\pi mp}{N}}}$$

$q = p+Q \rightarrow p = q - Q$

$p=-Q,\ q=0$

$p=Q,\ q=2Q$

$O(T N \log N)$

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{2Q} x\left((n+q-Q)\Delta_t\right) x^*\left((n-q+Q)\Delta_t\right) e^{-j\frac{2\pi mq}{N}}$$

for each n

$$\boxed{W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{N-1} c_1(q) e^{-j\frac{2\pi mq}{N}}}$$

$Q = \min(n_2-n,\ n-n_1)$.

$n \in [n_1, n_2]$,

$c_1(q) = x\left((n+q-Q)\Delta_t\right) x^*\left((n-q+Q)\Delta_t\right)$ for $0 \leq q \leq 2Q$

$c_1(q) = 0$ for $2Q+1 \leq q \leq N-1$

假設 $t = n_0\Delta_t,\ (n_0+1)\,\Delta_t,\ (n_0+2)\,\Delta_t,\ \cdots\cdots,\ n_1\Delta_t$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ m_1\,\Delta_f$

Step 1: Calculate $n_0$, $n_1$, $m_0$, $m_1$, $N$

Step 2: $n = n_0$

Step 3: Determine $Q$

Step 4: Determine $c_1(q)$

Step 5: $C_1(m) = \mathrm{FFT}[c_1(q)]$

Step 6: Convert $C_1(m)$ into $C(\,n\Delta_t,\, m\Delta_f)$

Step 7: Set $n = n+1$ and return to Step 3 until $n = n_1$.

## Method 3: Using the Chirp Z Transform

$- 2mp : -m^2 - p^2$
$+ (m-p)^2$

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)\exp\left(-j4\pi\, mp\Delta_t\Delta_f\right)\Delta_t$$

T F

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t\, e^{-j2\pi m^2\Delta_t\Delta_f} \sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j2\pi\, p^2\Delta_t\Delta_f} e^{j2\pi(p-m)^2\Delta_t\Delta_f}$$

$O(TN\log N)$

Step 1 $x_1(n, p) = x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j2\pi\, p^2\Delta_t\Delta_f}$

Step 2 $X_2[n, m] = \sum_{p=-Q}^{Q} x_1[n, p]\,c[m-p]$ $\qquad c[m] = e^{j2\pi m^2\Delta_t\Delta_f}$

Step 3 $X\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t\, e^{-j2\pi\, m^2\Delta_t\Delta_f} X_2[n, m]$

思考：Method 1 的複雜度為多少

思考：Method 2 的複雜度為多少

思考：Method 3 的複雜度為多少

The computation time of the WDF is more than those of the rec-STFT and the Gabor transform.

# V-F Properties of the WDF

| | |
|---|---|
| (1) Projection property | $\lvert x(t)\rvert^2 = \int_{-\infty}^{\infty} W_x(t,f)\,df$ $\quad \lvert X(f)\rvert^2 = \int_{-\infty}^{\infty} W_x(t,f)\,dt$ |
| (2) Energy preservation property | $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2\,df$ |
| (3) Recovery property | $\int_{-\infty}^{\infty} W_x(t/2,f)e^{j2\pi ft}\,df = x(t)\cdot x^*(0)$ $\quad x^*(0)$ 已知 <br> $\int_{-\infty}^{\infty} W_x(t,f/2)e^{-j2\pi ft}\,dt = X(f)\cdot X^*(0)$ |
| (4) Mean condition frequency and mean condition time | If $\quad x(t)=\lvert x(t)\rvert\cdot e^{j2\pi\phi(t)}$ , $\quad X(f)=\lvert X(f)\rvert\cdot e^{j2\pi\Psi(f)}$ <br> then $\quad \phi'(t)=\lvert x(t)\rvert^{-2}\cdot\int_{-\infty}^{\infty} f\cdot W_x(t,f)\cdot df$ <br> $-\Psi'(f)=\lvert X(f)\rvert^{-2}\int_{-\infty}^{\infty} t\cdot W_x(t,f)\cdot dt$ |
| (5) Moment properties | $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} t^n \lvert x(t)\rvert^2\,dt$ , <br> $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^n W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} f^n \lvert X(f)\rvert^2\,df$ |

| | |
|---|---|
| (6) $W_x(t,f)$ is real | $\overline{W_x(t,f)} = W_x(t,f)$ |
| (7) Region properties | If $x(t) = 0$ for $t > t_2$ then $W_x(t,f) = 0$ for $t > t_2$<br>If $x(t) = 0$ for $t < t_1$ then $W_x(t,f) = 0$ for $t < t_1$ |
| (8) Multiplication theory | If $y(t) = x(t)h(t)$, then<br>$W_y(t,f) = \int_{-\infty}^{\infty} W_x(t,\rho) W_h(t, f-\rho) \cdot d\rho$ |
| (9) Convolution theory | If $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$, then<br>$W_y(t,f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(t-\rho, f) \cdot d\rho$ |
| (10) Correlation theory | If $y(t) = \int_{-\infty}^{\infty} x(t+\tau)h^*(\tau)d\tau$, then<br>$W_y(t,f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(-t+\rho, f) \cdot d\rho$ |

| | |
|---|---|
| (11) Time-shifting property | If $y(t) = x(t - t_0)$, then $W_y(t, f) = W_x(t - t_0, f)$ |
| (12) Modulation property | If $y(t) = \exp(j2\pi f_0 t)x(t)$, then $W_y(t, f) = W_x(t, f - f_0)$ |

The STFT (including the rec-STFT, the Gabor transform) does not have real region, multiplication, convolution, and correlation properties.

Note: The WDF of $e^{j\phi}x(t)$ is all the same as the WDF of $x(t)$

- Why the WDF is always real?

What are the advantages and disadvantages it causes?

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})e^{-j2\pi f\tau}d\tau$$

$$W_x^*(t,f) = \int_{-\infty}^{\infty} x^*(t+\tfrac{\tau}{2})x(t-\tfrac{\tau}{2})e^{j2\pi f\tau}d\tau \qquad \tau' = -\tau \quad d\tau = -d\tau'$$

$$= -\int_{\infty}^{-\infty} x^*(t-\tfrac{\tau'}{2})x(t+\tfrac{\tau'}{2})e^{-j2\pi f\tau'}d\tau'$$

$$= \int_{-\infty}^{\infty} x(t+\tfrac{\tau'}{2})x^*(t-\tfrac{\tau'}{2})e^{-j2\pi f\tau'}d\tau' = W_x(t,f)$$

(Note: if $y(t) = y^*(-t)$, then $FT(y(t))$ is real) If $y(\tau) = x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})$

$y(\tau) = y^*(-\tau)$

- Try to prove of the projection and recovery properties

projection

$$\int W_x(t,f)df = \iint x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})e^{-j2\pi f\tau}d\tau\, df$$

$$= \int x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})\left(\int e^{-j2\pi f\tau}df\right)d\tau \quad (\text{page } 117\ (\text{I}))$$

$$= \int x(t+\tfrac{\tau}{2})x^*(t-\tfrac{\tau}{2})\delta(\tau)d\tau \quad (\text{page } 117\ (\text{IV}))$$

$$= x(t)x^*(t) = |x(t)|^2$$

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

- Proof of the region properties

If $x(t) = 0$ for $t < t_0$,

$x(t+\tau/2) = 0$ for $\tau < (t_0 - t)/2 = -(t - t_0)/2$,

$x(t-\tau/2) = 0$ for $\tau > (t - t_0)/2$,

Therefore, if $t - t_0 < 0$, the nonzero regions of $x(t+\tau/2)$ and $x(t-\tau/2)$ does not overlap and $x(t+\tau/2)\, x^*(t-\tau/2) = 0$ for all $\tau$.

The importance of region property

## V-G Advantages and Disadvantages of the WDF

Advantages: clarity

many good properties

suitable for analyzing the random process

Disadvantages: cross-term problem

more time for computation, especial for the signal with long time duration

not one-to-one

not suitable for $\exp\left(jt^n\right),\ n \neq 0,1,2$

# V-H Windowed Wigner Distribution Function

When $x(t)$ is not time-limited, its WDF is hard for implementation

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$\Big\downarrow \text{ with mask}$$

$$W_x(t,f)=\int_{-\infty}^{\infty} \underline{w(\tau)}\,x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$w(\tau)$ is real and time-limited

Advantages: (1) reduce the computation time

(2) may reduce the cross term problem

Disadvantages:

$$W_x(t,f) = 2\int_{-\infty}^{\infty} w(2\tau')x(t+\tau')\cdot x^*(t-\tau')e^{-j4\pi\tau' f}\cdot d\tau'$$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-\infty}^{\infty} w(2p\Delta_t)x((n+p)\Delta_t)x^*((n-p)\Delta_t)e^{-j4\pi mp\Delta_t\Delta_f}\Delta_t}$$

Suppose that $w(t) = 0$ for $|t| > B$

$$w(2p\Delta_t) = 0 \quad \text{for } p < -Q \text{ and } p > Q$$

$$Q = \frac{B}{2\Delta_t}$$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-Q}^{Q} w(2p\Delta_t)x((n+p)\Delta_t)x^*((n-p)\Delta_t)e^{-j4\pi mp\Delta_t\Delta_f}\Delta_t}$$

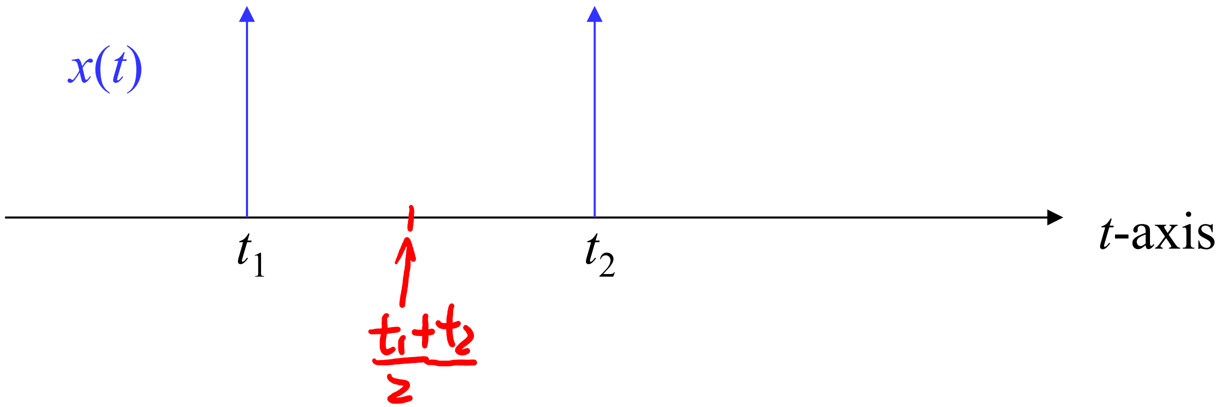
當然，乘上 mask 之後，有一些數學性質將會消失

## (B) Why the cross term problem can be avoided ?

$$W_x(t,f) = \int_{-\infty}^{\infty} w(\tau)\, x(t+\tau/2) \cdot x^*(t-\tau/2)\, e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$ is real

Viewing the case where $x(t) = \delta(t - t_1) + \delta(t - t_2)$



理想情形：　$W_x(t,f)=0$　　for $t \neq t_1, t_2$

然而，當 mask function $w(\tau) = 1$ 時　(也就是沒有使用 mask function)

$$y(t,\tau)=x(t+\tau/2) \qquad y^*(t,-\tau)=x^*(t-\tau/2)$$

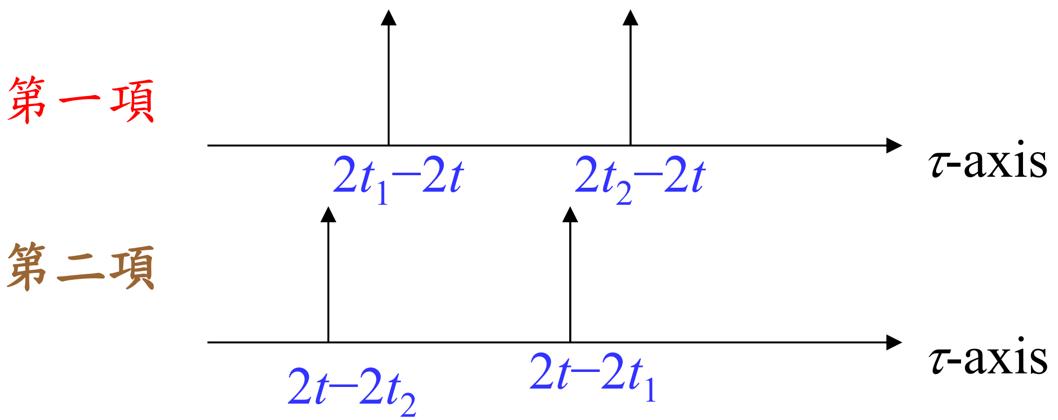
$x(t)=\delta(t-t_1)+\delta(t-t_2)$

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

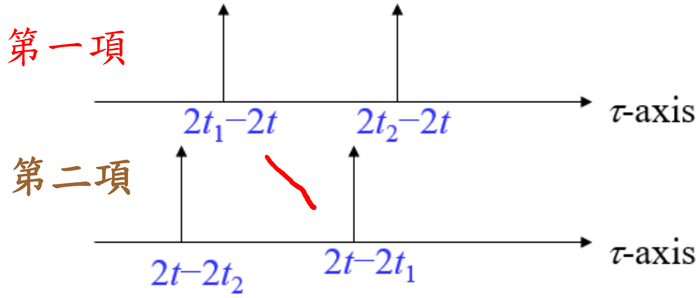
$$=\int_{-\infty}^{\infty}\left[\delta\left(t+\frac{\tau}{2}-t_1\right)+\delta\left(t+\frac{\tau}{2}-t_2\right)\right]\left[\delta\left(t-\frac{\tau}{2}-t_1\right)+\delta\left(t-\frac{\tau}{2}-t_2\right)\right]e^{-j2\pi\tau f}\cdot d\tau$$

from page 117, property 2

$$=4\int_{-\infty}^{\infty}\underbrace{[\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)]}_{\text{第一項} \quad \times 2}\underbrace{[\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)]}_{\text{第二項} \quad \times(-2)}e^{-j2\pi\tau f}\cdot d\tau$$

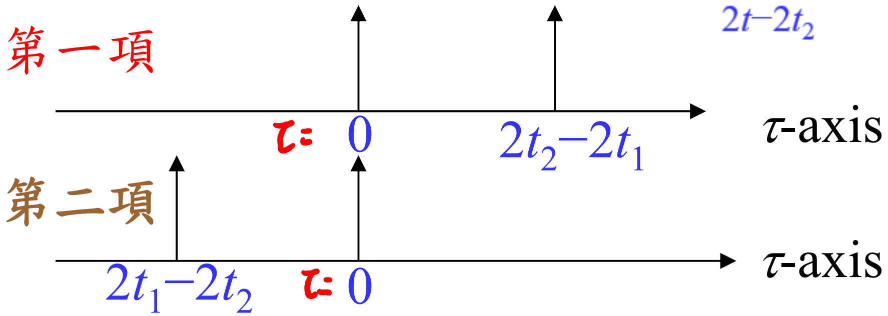


3種情形 $W_x(t,f)\neq 0$




(1) If $t = t_1$

auto term



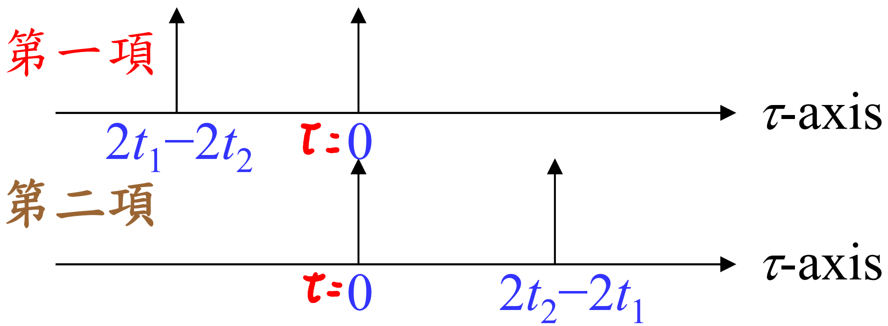

$2t_1-2t = 2t-2t_1$

$4t_1 = 4t$

$t = t_1$

(2) If $t = t_2$

auto term




$2t_2-2t = 2t-2t_2$

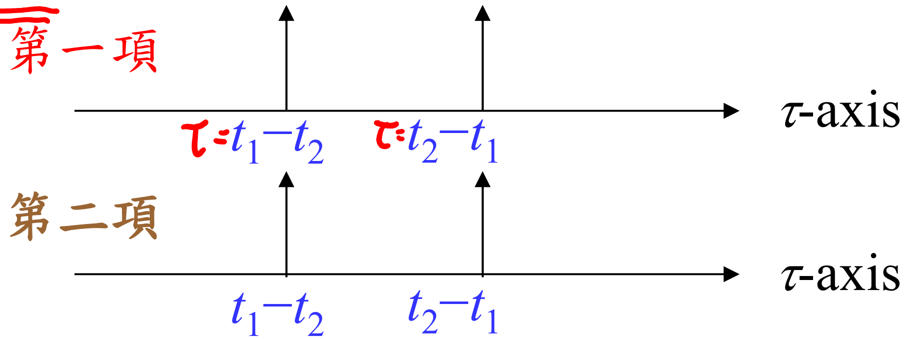
$4t = 4t_2$, $t = t_2$

(3) If $t = (t_1 + t_2)/2$

$2t_1-2t$
$= t_1-t_2$

$2t_2-2t$
$= t_2-t_1$

cross term




$2t_1-2t = 2t-2t_2$

$4t = 2(t_1+t_2)$

$t = \frac{1}{2}(t_1+t_2)$

With mask function

$$W_x(t,f)=\int_{-\infty}^{\infty} w(\tau)\,x(t+\tau/2)\,x^*(t-\tau/2)\,e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty} \underline{w(\tau)}\left[\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)\right]$$

$$\times\left[\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)\right]e^{-j2\pi\tau f}\cdot d\tau$$

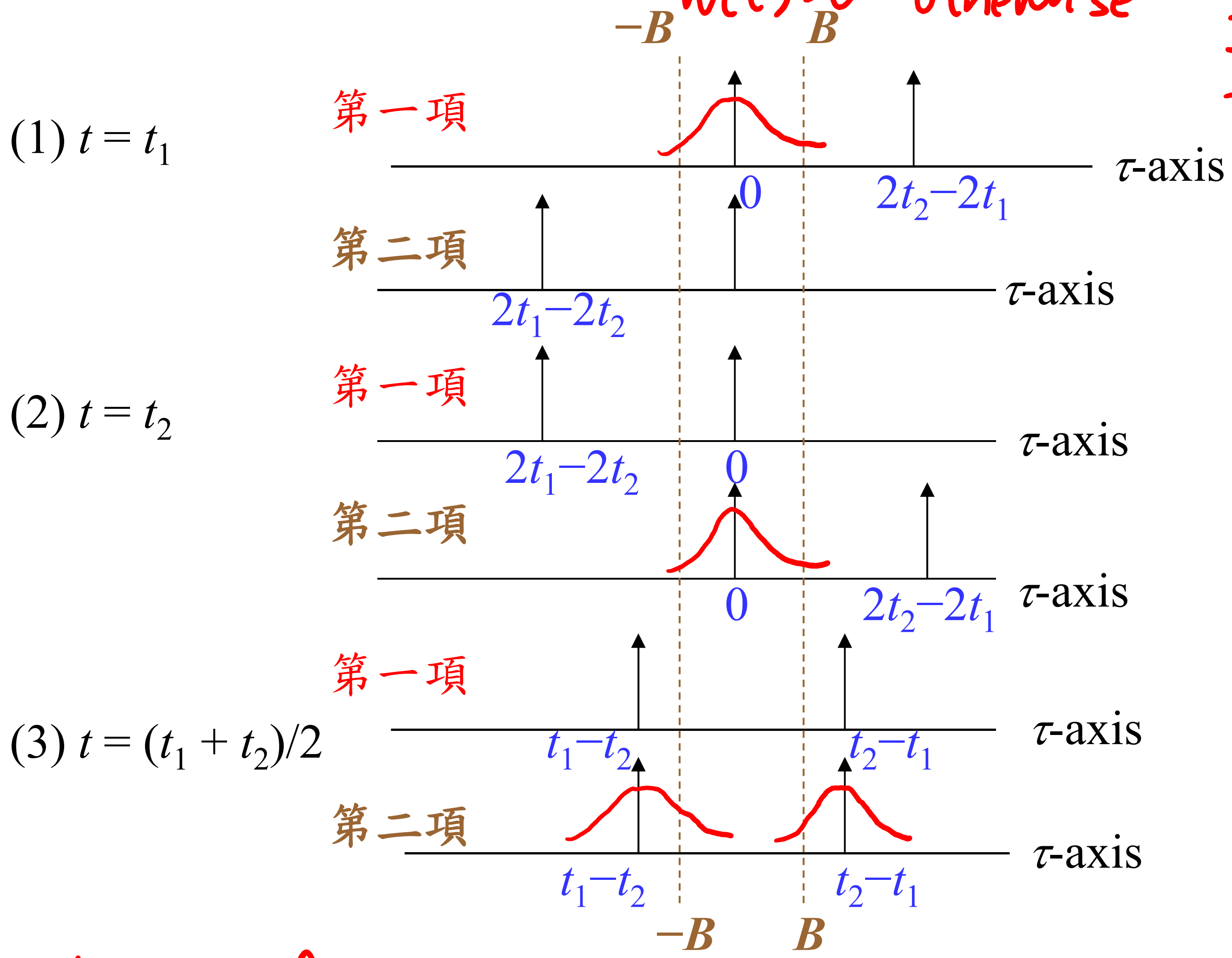
Suppose that $w(\tau)=0$ for $|\tau|>B$, $B$ is positive.

If $B<t_2-t_1$

$W(\tau) = 1$ for $-B < \tau < B$

$W(\tau) = 0$ otherwise

If $B < t_2 - t_1$, the cross term is removed




However, if $t_2 - t_1 < B$

or the components are not narrow

the windowed WDF may not remove the cross term well.

# 附錄五： 研究所學習新知識把握的要點

(1) Concepts: 這個方法的核心概念、基本精神是什麼

(2) Comparison: 這方法和其他方法之間，有什麼相同的地方？
有什麼相異的地方

(3) Advantages: 這方法的優點是什麼

(3-1) Why? 造成這些優點的原因是什麼

(4) Disadvantages: 這方法的缺點是什麼

(4-1) Why? 造成這些缺點的原因是什麼

(5) Applications: 這個方法要用來處理什麼問題，有什麼應用

(6) Innovations: 這方法有什麼可以改進的地方
或是可以推廣到什麼地方

看過一篇論文或一個章節之後，若能夠回答 (1)-(5) 的問題，就代表你已經學通了這個方法

如果你的目標是發明創造出新的方法，可試著回答 (3-1), (4-1), 和 (6) 的問題